MODULI SPACES OF QUADRATIC DIFFERENTIALS

By

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Abstract. The cotangent bundle of $\mathcal{T}(g, n)$ is a union of complex analytic subvarieties, $V(\pi)$, the level sets of the function "singularity pattern" of quadratic differentials. Each $V(\pi)$ is endowed with a natural affine complex structure and volume element. The latter contracts to a real analytic volume element, μ_π, on the unit hypersurface, $V_1(\pi)$, for the Teichmüller metric. μ_π is invariant under the pure mapping class group, $\Gamma(g, n)$, and a certain class of functions is proved to be $L^p(\mu_\pi)$, $0 < p < 1$, over the moduli space $V_1(\pi)/\Gamma(g, n)$. In particular, $\mu_\pi(V_1(\pi)/\Gamma(g, n)) < \infty$, a statement which generalizes a theorem by H. Masur.

0. Introduction

$\mathcal{T}(g, n)$, $3g - 3 + n > 0$, denotes the Teichmüller space of marked closed Riemann surfaces of genus g with n punctures. The pure mapping class group, $\Gamma(g, n)$, acts as a discrete group of biholomorphisms on both $\mathcal{T}(g, n)$ and its cotangent bundle $Q(g, n) \xrightarrow{\rho} \mathcal{T}(g, n)$. Each $q \in Q(g, n)$ corresponds to a meromorphic quadratic differential σ_X on any $X \in \rho(q)$. The poles of σ, if any, are simple and occur in the puncture set. It makes sense to define the *symbol*, $\pi(q) = \pi(\sigma)$, to be the triple $\pi(q) = (k, \nu(\cdot), \varepsilon)$, where $k \leqq n$ is the number of poles of σ, $\nu(l)$ is the number of zeros of σ of order $l \geqq 1$, and $\varepsilon = +1$ or -1 as σ is or is not the square of a holomorphic 1-form.

If $\pi = \pi(q)$ is the symbol of some $q \in Q(q, n)$, define $V(\pi) = \{q \in Q(g, n) \mid \pi(q) = \pi\}$. $V(\pi)$ is a complex analytic subvariety of $Q(g, n)$, of dimension

$$\dim_{\mathbf{C}} V(\pi) = 2g + k + \frac{\varepsilon - 3}{2} + \sum_{l=1}^{\infty} \nu(l) \tag{0.1}$$

(cf. [V86]). $V_1(\pi)$ denotes the norm one hypersurface of $V(\pi)$ (for the Teichmüller metric).

The principal goals of the present paper may now be stated. First, we describe for each symbol π an atlas $\mathscr{E}(\pi)$ on $V(\pi)$ with the properties (1) $\mathscr{E}(\pi)$-transitions are locally complex affine and (2) $\mathscr{E}(\pi)$-transitions are euclidean measure preserving (Sections 1–8). Secondly, we describe a procedure for reducing (a cover of) $V(\pi)$ to (a finite cover of) $V(\pi)/\Gamma(g, n)$ (Sections 9, 12–13). Finally, we note that $\mathscr{E}(\pi)$ endows $V(\pi)$ with a real analytic volume element whose contraction,

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denoted μ_π, on $V_1(\pi)$ is $\Gamma(g, n)$-invariant. With the help of topological "basic lemmas" (Sections 10, 11) we establish in Section 14 the finiteness of a class of integrals over $(V_1(\pi)/\Gamma(g, n), \mu_\pi)$. The main result is

0.2. Theorem. *Define $\phi(q)$ to be the minimum length of a homotopically nontrivial path connecting singularities (zeros or poles) of q (i.e., of any representative σ of q) with respect to the metric $|q|$ (i.e. $|\sigma|$). Then*

$$\int_{V_1(\pi)/\Gamma(g,n)} \frac{1}{\phi(q)^p}\mu_\pi(dq) < \infty \qquad (0 < p < 1). \tag{0.3}$$

We obtain (0.3) as a consequence of a modestly stronger result which is critical to the analysis of cocycles in [V86] (as is the reduction procedure itself).

It follows from (0.3) that

$$\mu_\pi(V_1(\pi)/\Gamma(g, n)) < \infty. \tag{0.4}$$

The finiteness of the volume in (0.4) generalizes the corresponding statement by Masur [M82] in the cases which correspond to $\pi = (0, (4g-4, 0, \dots), -1)$ and $\pi = (0, \nu, +1)$ in the present notation.

The moduli spaces $V(\pi)/\Gamma(g, n)$ are finitely connected but not, in general, connected (Section 13). If $W \subseteq V_1(\pi)/\Gamma(g, n)$ is a component of the moduli space, μ_π is uniquely determined, up to a scale factor, as a finite absolutely continuous measure which is invariant under the "Teichmüller geodesic flow." (See [V86] for details and references.) The first instance of this fact is due to Masur ([M82]) for $\pi = (0, (4g-4, 0, \dots), -1)$.

It is a classical fact that $V_1(\pi)$ is foliated by copies of the unit tangent bundle of the Poincaré upper half-plane, and this foliation is the orbit foliation of an action of $G = SL(2, \mathbf{R})$. This (real analytic) action is most easily described directly, i.e., without reference to Beltrami differentials (Section 1), and the description makes it obvious that μ_π is preserved by G (Sections 7, 8, 14). The significance of the G-action is that its *existence* implies, for any noncompact one parameter subgroup H, if μ_π is ergodic for G on $V_1(\pi)/\Gamma(g, n)$, then it is automatically mixing [Mo66] for H. In particular, the Teichmüller horocycle flow

$$\left(H = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| t \in \mathbf{R} \right\}\right)$$

is mixing (of all orders) on each component of the moduli space. Ergodicity of the Teichmüller flow when $\pi = (0, (4g-4, 0, \dots), -1)$ is due to Masur [M84]. See [V86] for the ergodic theory of $(V_1(\pi)/\Gamma(g, n), \mu_\pi, G)$.

Most of the results of the present paper have first appeared in the preprints [V84, I, II] which were circulated in 1984.

1. F-structures

Fix $g \geqq 0$, and let M_g^2 be a closed oriented surface of genus g. $\mathscr{C}_g$ denotes the set of all (maximal) complex atlases compatible with the orientation. If $n \geqq 0$, S_n denotes a set in M_g^2 with n elements. Set $M_{g,n}^2 = S_n^c$, and let $H(g, n)$ be the group of orientation preserving homeomorphisms ϕ such that $\phi|_{S_n} = \mathrm{Id}$. $H_0(g, n)$ denotes the identity component of $H(g, n)$, and $\Gamma(g, n) = H(g, n)/H_0(g, n)$ is the pure mapping class group. $H(g) = H(g, 0)$ acts upon $\mathscr{C}_g$ by $\mathscr{U} \to \mathscr{U}\phi^{-1} = \{(\phi U, f\cdot\phi^{-1}) \mid (U, f)\in\mathscr{U}\}$. If $3g - 3 + n > 0$, the Teichmüller space is defined as $\mathscr{T}(g, n) = \mathscr{C}_g/H_0(g, n)$.

If $\mathscr{U} \in \mathscr{C}_g$, define $Q_n(\mathscr{U})$ to be the set of $\mathscr{U}$-meromorphic quadratic differentials σ whose poles, if any, are simple and contained in S_n. If $\phi\in H(g, n)$, then $(M_g^2, \mathscr{U}) \xrightarrow{\phi} (M_g^2, \mathscr{U}\phi^{-1})$ is holomorphic, and $\phi^* Q_n(\mathscr{U}\phi^{-1}) = Q_n(\mathscr{U})$. When $3g - 3 + n > 0$, it is true that $\mathscr{U}\phi^{-1} \neq \mathscr{U}$, $\mathscr{U} \in \mathscr{C}_g$, $1 \neq \phi\in H_0(g, n)$. This means $p^{-1}([\mathscr{U}])$ is naturally identified with $Q_n(\mathscr{U}')$ for any $\mathscr{U}'\in[\mathscr{U}] = \mathscr{U}H_0(g, n)$.

For convenience of notation, we shall deal only with symbols $\pi = (n, \nu, \varepsilon)$, i.e. with $k = n$ in terms of earlier notation. The study of $V(\pi)$ is facilitated by use of a cover. Define $m(\pi) = \Sigma_{l=1}^{\infty} \nu(l)$, and let $N(\pi) = n + m(\pi)$. We may and shall suppose $S_{N(\pi)} = S_n \cup T$, where T is partitioned into sets T_l with $\nu(l)$ elements. Define $\tilde{V}(\pi)$ to be the set of $q \in Q(g, N(\pi))$ such that (1) $\pi(q) = \pi$, (2) if $X \in \rho(q)$, then $\sigma_X(q)$ has a pole at each point of S_n, and (3) for each $l \geqq 1$, $\sigma_X(q)$ has a zero of order l at each point of T_l.

Let $\tilde{V}(\pi) \xrightarrow{a} V(\pi)$ be the canonical map. If $aq_1 = aq_2$, there exist $\mathscr{U}_1, \mathscr{U}_2 \in \mathscr{C}_g$, $\sigma_j \in Q_{N(\pi)}(\mathscr{U}_j)$, $j = 1, 2$ and $\phi\in H_0(g, n)$ such that $\mathscr{U}_2 = \mathscr{U}_1\phi^{-1}$ and $\sigma_1 = \phi^*\sigma_2$. It follows that ϕ is unique $(3g - 3 + n > 0)$, and moreover $\phi T_l = T_l$, $l \geqq 1$. If $H(g, \pi) = \{\phi\in H_0(g, n) \mid \phi T_l = T_l, l \geqq 1\}$, then $\Gamma(g, \pi) = H(g, \pi)/H_0(g, N(\pi))$ is the group of the cover $\tilde{V}(\pi) \xrightarrow{a} V(\pi)$. The finite group $\Gamma(g, n)/\Gamma(g, N(\pi))$ is the group of the (branched) cover $\tilde{V}((\pi)/\Gamma(g, N(\pi)) \to V(\pi)/\Gamma(g, n)$.

By an *F-structure* $\xi = (Y, \mathscr{U})$ we understand (1) an oriented connected surface Y of finite topological type and (2) an atlas $\mathscr{U}$ for Y with transitions locally of the form $z \to \pm z + c$. $\mathscr{U}$ should be compatible with the fixed orientation and maximal with respect to its defining property. Associate to the F-structure ξ (a) a complex structure $\mathscr{U}(\xi) \supseteq \mathscr{U}$, (b) a nowhere zero holomorphic quadratic differential σ_ξ, $\sigma_\xi|_U = f^*dz^2$, $(U, f)\in\mathscr{U}$, (c) a Riemannian flat metric h_ξ, $h_\xi|_U = f^*|dz|^2$, $(U, f)\in\mathscr{U}$, and (d) the h_ξ-volume form θ_ξ, $\theta_\xi|_U = f^*(\frac{1}{2}idz \wedge d\bar{z})$, $(U, f)\in\mathscr{U}$.

The assumption (1) above implies $Y \cong M_{g,n}^2$ for some g, n. The F-structure ξ is said to be *admissible* if the completion of Y with respect to the h_ξ-distance function is compact and contains only a finite set of ideal points. It is easy to show ([V84]) that if ξ is admissible, then (i) M_g^2 is the natural h_ξ-completion of $M_{g,n}^2$, (ii) $\mathscr{U}(\xi)$ extends to $\mathscr{U}^*(\xi) \in \mathscr{C}_g$, (iii) σ_ξ extends to be $\mathscr{U}^*(\xi)$ meromorphic with at worst simple poles, and (iv) the norm of σ_ξ is

$$\| \sigma_\xi \| = \int_{M_j^2} \theta_\xi .$$

Conversely, if $\mathscr{U}^* \in \mathscr{C}_g$, and if $\sigma \in Q_n(\mathscr{U}^*)$ has both its poles and its zeros contained in S_n, the atlas $\mathscr{U} = \mathscr{U}(\sigma)$ of natural parameters of σ determines an admissible F-structure on $Y = M_{g,n}^2$. We write $\xi = \xi(\sigma)$.

Define $\Omega(g, n)$ to be the set of admissible F-structures $\xi = (M_{g,n}^2, \mathscr{U})$. $H(g, n)$ acts on $\Omega(g, n)$, and we define a "Teichmüller space"

$$\mathscr{M}(g, n) = \Omega(g, n)/H_0(g, n).$$

Define $\pi(\xi) = \pi(\sigma_\xi)$, and let $\mathscr{M}(g, n, \pi)$ be the corresponding stratum of $\mathscr{M}(g, n)$. Notice that $[\xi] = m \in \mathscr{M}(g, n)$ defines a function $O_m : S_n \to \{-1, 0, 1, \dots\}$ giving the order of σ_ξ at each point of S_n. Thus $\pi(m) = (k, \nu, \varepsilon)$, where $k = |O_m^{-1}(-1)|$ and $\nu(l) = |O_m^{-1}(l)|$, $l \geqq 1$.

It is now possible to describe $\tilde{V}(\pi)$ above in terms of $\mathscr{M}(g, N(\pi))$. Define

$$W(\pi) = \{ m \in \mathscr{M}(g, N(\pi)) \mid \pi(m) = \pi, \text{ and } \quad O_m(s) = -1 \text{ on } S_n, O_m(s) = l \text{ on } T_l \}. \tag{1.1}$$

(Recall that $\pi = (n, \nu, \varepsilon)$, by assumption. The requirement $\pi(m) = \pi$ is redundant in (1.1) if $n > 0$ or if $\nu(2j+1) > 0$ for some j. However, knowing $n = 0$ and $\nu(2j+1) = 0$, all j does not determine ε in general.) Now $W(\pi)$ and $\tilde{V}(\pi)$ are canonically identified ($[\sigma] \sim [\xi(\sigma)]$, $[\xi] \sim [\sigma_\xi]$).

If $\xi_1, \xi_2 \in \Omega(g, n)$, define a "distance" $\alpha(\xi_1, \xi_2)$ by

$$\alpha(\xi_1, \xi_2) = \sup_{x \in M_{g,n}^2} \sup_{\substack{(U_j, f_j) \in \xi_j \\ x \in U_1 \cap U_2}} \limsup_{x' \to x} L\left(\left(\frac{f_1(x') - f_1(x)}{f_2(x') - f_2(x)} \right)^2 \right)$$

where $L(z)$ is the branch of $\log z$ with $-\pi \leqq \arg z < \pi$. If $K \subseteq H(g, n)$ is a closed subgroup, define $D_K(\xi_1, \xi_2) = \inf_{\phi \in K} \alpha(\xi_1, \xi_2 \phi^{-1})$. It is proved in [V84] that if $D_K < \infty$ on $\Omega(g, n) \times \Omega(g, n)$, then D_K is a complete K-invariant pseudometric. Moreover, each orbit ξK is D_k-closed, and therefore D_K determines a complete metric on $\Omega(g, n)/K$. We use D_0 to denote the metric/pseudometric which arises from the choice $K = H_0(g, n)$. The functions $m \to O_m(s)$, $s \in S_n$ and $m \to \pi(m)$ are D_0 continuous, and therefore if we replace n by $N(\pi)$, the space $W(\pi)$ in (1.1) is both open and closed in $\mathscr{M}(g, N(\pi))$.

1.2. Remark. The group $\Gamma(g, \pi)$, defined earlier in this section, acts naturally upon $W(\pi)$, e.g., because it already acts upon $\tilde{V}(\pi)$. $\Gamma(g, \pi)$ acts as a properly discontinuous group of isometries, and therefore $V(\pi)$ inherits a metric from its realization as $W(\pi)/\Gamma(g, \pi)$. It is also possible to endow $V(\pi)$ with a metric directly, using $H_0(g, n) = K$ and the realization of $V(\pi)$ as a space of classes of F-structures on $M^2_{g,n}$ with singularities (i.e., σ_ξ is allowed to have zeros in $M^2_{g,n}$). The latter approach results in the same metric on $V(\pi)$. It is possible to relate D_0 to the circular dilatation [LV73] and to prove $V(\pi) \xrightarrow{\rho} \mathscr{T}(g, n)$ is contractive with respect to the Teichmüller metric on $\mathscr{T}(g, n)$ ([V84]).

We conclude this section with a description of the action of $G = \mathrm{SL}(2, \mathbf{R})$ on $\mathscr{M}(g, n)$. The elementary details are found in [V84].

If $\xi = (Y, \mathscr{U})$ is any F-structure, and if $A \in G$, define $A\xi = (Y, A\mathscr{U})$, where $A\mathscr{U} = \{(U, A \circ f) \mid (U, f) \in \mathscr{U}\}$. The notation $A \circ f$ is meaningful if A is understood to be an $\mathbf{R}$-linear transformation of $\mathbf{R}^2$. Because G normalizes the group $z \to \pm z + c$, $A\xi$ is again an F-structure. As Id is uniformly Lipschitz relative to the metrics h_ξ and $h_{A\xi}$, $\xi \to A\xi$ preserves admissibility. Also, $A \to \alpha(\xi, A\xi)$ is continuous, and because G is connected $\pi(\xi) = \pi(A\xi)$ and $O_\xi(\cdot) = O_{A\xi}(\cdot)$ (in the case $\xi \in \Omega(g, n)$). It is clear that $A(\xi\phi) = (A\xi)\phi$, $A \in G$, $\xi \in \Omega(g, n)$, $\phi \in H(g, n)$, and therefore G acts upon $\mathscr{M}(g, n)$, $W(\pi) \cong \tilde{V}(\pi)$, and $V(\pi) = \tilde{V}(\pi)/\Gamma(g, \pi) \cong W(\pi)/\Gamma(g, \pi)$.

1.3. Remark. Suppose $q_1, q_2 \in V(\pi)$ are such that $q_2 = Aq_1$ for some $A \in \mathrm{SL}(2, \mathbf{R})$. Let $A = R(\theta_2)\mathrm{Diag}(e^t, e^{-t})R(\theta_1)$, where $R(\theta_j)$, $\theta_j \in \mathbf{R}$, denotes rotation matrix and $t \geqq 0$. Then $\rho(q_2) = \rho(R(-2\theta_2)q_2)$ and $\rho(q_1) = \rho(R(2\theta_1)q_1)$, and therefore Id is a Teichmüller map relative to $\mathscr{U}_1 \in \rho(q_1)$, $\mathscr{U}_2 \in \rho(q_2)$. The associated quadratic differentials are $e^{-2i\theta_2}\sigma_{\mathscr{U}_2}(q_2)$ and $e^{2i\theta_1}\sigma_{\mathscr{U}_1}(q_1)$, and the Teichmüller distance is t. Therefore, $\rho(q_2)$ lies in the Teichmüller disc through $\rho(q_1)$ determined by q_1. Conversely, any element of the same disc has the form $\rho(AR(\theta_1)q_1)$ for some θ_1, and diagonal $A \in G$ (this is virtually the definition of Teichmüller map).

When $\pi = (n, \nu, -1)$, i.e. when $\varepsilon = -1$, $W(\pi)$ will be replaced by a similar object on a higher genus surface with $\varepsilon = +1$. For the next several sections it will be assumed that $\varepsilon = +1$. We return to $\varepsilon = -1$ in Section 8.

2. Trajectories of holomorphic 1-forms

The next several sections (2–7) will be concerned with admissible *positive* F-structures. In these sections $\xi = (Y, \mathscr{U})$ denotes an admissible F-structure such that (a) $\mathscr{U}$-transitions are locally translations, (b) Y is a subset of a closed oriented surface X, and $S(\xi) = Y^c$ is finite, and (c) $\mathscr{U}$ is compatible with the orientation of X. $\mathscr{U}$ is assumed to be maximal for (a) instead of for the property of having transitions $z \to \pm z + c$. In effect, we are working with a double cover of the space of "orientable" admissible F-structures.

Suppose now X is a closed Riemann surface of genus $g > 0$, and let ω be a nontrivial holomorphic 1-form on X. If $S^*(\omega)$ is the zero set of ω, we suppose fixed a finite set S which contains $S^*(\omega)$. Let $\mathscr{U}(\omega)$ be the atlas of natural parameters of ω on $S^c = Y$. For any $\theta \in \mathbf{R}$ the foliation of $\mathbf{C}$ by oriented lines making an angle θ with the horizontal lifts, via $\mathscr{U}$-charts, to an oriented foliation $\mathscr{F}(\theta)$ of Y. Leaves of $\mathscr{F}(\theta)$ are geodesics for the metric $|\omega|$.

If $s \in S$ is a zero of ω of order $l = l(s)$, then for each θ there exist $2(l+1)$ leaves of $\mathscr{F}(\theta)$ which terminate or originate at s. These separatrices will be referred to as incoming or outgoing. Separatrices from either class partition a disc neighborhood of s into $l+1$ sectors, and every sector contains a separatrix from the opposite class. Recall that

$$\sum_{s \in S} l(s) = 2g - 2. \tag{2.1}$$

A leaf $L \subseteq \mathscr{F}(\theta)$ is a saddle connection if it has finite $|\omega|$ length or, what is the same, if $\partial L = s_1 - s_2$, $s_1, s_2 \in S$. $C(\theta)$ denotes the compact set which is the union of S and all saddle connections $L \subseteq \mathscr{F}(\theta)$. In a dynamical context the following well-known fact goes back to A. G. Maier [M43]; see also [S84].

2.2. Theorem. *Let notations be as above. If W is a component of $C(\theta)^c$, then either* (1) $(W, \mathscr{F}(\theta)|_W)$ *is minimal, i.e. every leaf $L \subseteq W$ is dense in W or* (2) $(W, \mathscr{F}(\theta)|_W)$ *is holomorphically equivalent to the foliation of an annulus by concentric circles.*

It will be our continuing assumption that $S \neq \varnothing$, an assumption which is redundant unless $g = 1$. When $S \neq \varnothing$ and $C(\theta) = S$, the theorem implies every leaf of $\mathscr{F}(\theta)$ is dense. If $L \subseteq \mathscr{F}(\theta)$ is a saddle connection, then

$$\int_L \omega = e^{i\theta} |L| \tag{2.3}$$

where $|L|$ is the $|\omega|$ length of L. The countability of $H_1(X, S)$ (integer coefficients) implies that $C(\theta) = S$ for all but a countable set of θ, and for these same θ every leaf of $\mathscr{F}(\theta)$ is dense in X.

2.4. Definition. Let $\xi = (Y, \mathscr{U})$ be an admissible positive F structure, and let $F(\xi, \theta)$ be the foliations associated to ω_ξ, $\theta \in \mathbf{R}$. ξ is said to be in *general position* if $C(0) = S = C(\pi/2)$. (That is, the horizontal and vertical foliations have all leaves dense.)

2.5. Remark. If $A(\theta) \in SL(2, \mathbf{R}) = G$ is the matrix of rotation by θ, then for all but a countable set of θ $A(\theta)\xi$ is general position.

3. Weavings

We suppose fixed $\xi = (Y, \mathscr{U})$ as in the first paragraph of Section 2, and we suppose X, $\omega = \omega_\xi$, and $S \neq \emptyset$ are as in the second paragraph of Section 2. Later on, it will be assumed that ξ is in general position.

3.1. Definition. An $|\omega_\xi|$ geodesic, $\gamma = ((a, b), \psi)$, in the direction θ is an *end* at $s \in S$ if $\partial\gamma = \pm(x - s)$, as a 0-chain, where $x \notin S$. The point $x (= \psi(a^+)$ or $\psi(b^-))$ is called the *terminus* of γ.

The notations $\mathscr{F}_h$ and $\mathscr{F}_v$, h for "horizontal" and v for "vertical," are reserved for the foliations $\mathscr{F}(0)$ and $\mathscr{F}(\pi/2)$, respectively. Accordingly, we speak of *horizontal* or *vertical ends*. Let γ be a separatrix for, say, $\mathscr{F}_h$. If γ is a saddle connection, choose an end, γ_0, of γ. γ or γ_0 bisect a sector of $\mathscr{F}_v$, and this sector is determined by an incoming separatrix, denoted $\delta^-(\gamma)$ or $\delta^-(\gamma_0)$, and an outgoing separatrix, denoted $\delta^+(\gamma)$ or $\delta^+(\gamma_0)$. The same notations, $\delta^\pm(\gamma)$ or $\delta^\pm(\gamma_0)$, are associated to vertical separatrix, γ, or vertical end, γ_0.

3.2. Definition. A closed set $\Lambda \subseteq X$ shall be called a *weaving* (of ξ or of $\mathscr{F}_h$ and $\mathscr{F}_v$) if

(a) Λ is a finite union of horizontal and vertical ends,

(b) if γ is a maximal horizontal (resp. vertical) end which is contained in Λ, and if x is the terminus of γ, then x is interior to a vertical (resp. horizontal) end in Λ, and

(c) if $s \in S$, and if γ_0 is a horizontal or vertical end at s, no subend of which lies in Λ, then both $\delta^\pm(\gamma_0)$ have an end at s which is contained in Λ.

If ξ is not in general position, there is an ambiguity in (c). If γ_0 is, say, a horizontal end at x, and if one or both of $\delta^\pm(\gamma_0)$ are saddle connections, the ends to be chosen are the ends which bound the sector of $\mathscr{F}_v$ which γ_0 determines (bisects).

The following is a consequence of the fact quadratic differentials determine decompositions of surfaces into rectangles. See Strebel [S84].

3.3. Proposition. *Assume ξ is in general position. There exists a weaving Λ.*

3.4. Remark. The conclusion of the proposition is also true for ξ if ξ is not in general position. We do not require the statement, however.

3.5. Proposition. *Assume ξ is in general position. If Λ is a weaving, and if U is a component of Λ^c, there is a chart, (U, f), such that $R = fU$ is a rectangle with sides parallel to the axes and $f^*dz = \omega_\xi|_U$.*

Proof. $\mathscr{F}_h|_U$ is without singularities, and therefore U is an annulus or a disc. Recurrent non-periodicity and transversality of $\mathscr{F}_h$ and $\mathscr{F}_v$ preclude the possibility that U is an annulus. Now let $f: U \to \mathbf{C}$ be a local ξ-chart function. f extends to ∂U, and $f, \partial U)$ is a polygonal path with all interior angles $\pi/2$; i.e. $(f, \partial U)$ defines a rectangle R. As $f: \partial U \to R$ is of degree 1, f is a homeomorphism on U.

3.6. Remark. If ξ is arbitrary, perhaps not in general position, and if U is a component of Λ^c for some weaving, Λ, then U satisfies one of (a) the conclusion of Proposition 3.5, or (b) U is a cylinder of closed leaves of $\mathscr{F}_h$ or $\mathscr{F}_v$.

4. Weavings and cohomology

Notations are as in the first paragraph of Section 3. It is assumed that ξ is in general position, and we fix a weaving, λ, of ξ.

Λ determines a chain complex, $(\mathscr{C}, \partial)$. If $0 \leqq j \leqq 2$, $\mathscr{C}_j = \mathscr{C}_j(\Lambda)$ is the $\mathbf{R}$-vector space of j chains with the natural basis (a) if $j = 0$ the set of vertices of Λ, (b) if $j = 1$ the set of horizontal and vertical edges of Λ, carrying the induced orientation from $\mathscr{F}_h$ and $\mathscr{F}_v$, and (c) if $j = 2$ the set of components, $U \subseteq \Lambda^c$, carrying the orientation induced by X.

By Proposition 3.5 Λ defines a cellular decomposition of X, and therefore the real homology of X is the homology of the complex

$$\mathscr{C}_2 \xrightarrow{\partial} \mathscr{C}_1 \xrightarrow{\partial} \mathscr{C}_0. \tag{4.1}$$

Let $\langle \cdot, \cdot \rangle$ be the natural inner product on $\mathscr{C}_j$ determined by the basis in (a)–(c) above. If $\mathscr{C}_j$ is identified with its dual, $\mathscr{C}_j^*$, by means of $\langle \cdot, \cdot \rangle$, there is the cochain complex dual to (4.1),

$$\mathscr{C}_0 \xrightarrow{\partial^*} \mathscr{C}_1 \xrightarrow{\partial^*} \mathscr{C}_2. \tag{4.2}$$

If σ is a smooth closed 1-form on X, σ determines a 1-cocycle, $\hat{\sigma}$, $\hat{\sigma} = \Sigma\, \hat{\sigma}(e)e$, where the sum extends over all edges of Λ, and

$$\hat{\sigma}(e) = \int_e \sigma. \tag{4.3}$$

Let $\mathscr{Z}^1 = \mathscr{Z}^1(\Lambda) = \{\alpha \in \mathscr{C}_1 \mid \partial^* \alpha = 0\}$. We observe each $\alpha \in \mathscr{Z}^1$ has a representation $\alpha = \hat{\sigma}_\alpha$ as above. Indeed, by de Rham's Theorem it is necessary only to establish this claim for $\alpha \in \partial^* \mathscr{C}_0$. If $u = \Sigma\, u(v)v$, the sum extending over all vertices of Λ, and if $\alpha = \partial^* u$, then $\alpha = \hat{df}$, where f is any smooth extension of $v \to u(v)$ to X.

If $\alpha, \beta \in \mathscr{Z}^1$, choose $\sigma_\alpha, \sigma_\beta$, so that $\hat{\sigma}_\alpha = \alpha$ and $\hat{\sigma}_\beta = \beta$, and define the intersection form, $[\alpha, \beta]$ by

$$[\alpha, \beta] = \int_X \sigma_\alpha \wedge \sigma_\beta. \tag{4.4}$$

Of course, it is true that $[\alpha, \cdot] \equiv 0$ if $\alpha \in \partial^* \mathscr{C}_0$.

With all notations as above, let $\mathscr{C}_h$ and $\mathscr{C}_v$ be the linear spans in $\mathscr{C}_1$ of the horizontal and vertical edges, respectively. Set $\mathscr{Z}_h^1 = \mathscr{Z}^1 \cap \mathscr{C}_h$ and $\mathscr{Z}_v^1 = \mathscr{Z}^1 \cap \mathscr{C}_v$. Let F be the dimension of $\mathscr{C}_2(\Lambda)$.

4.5. Lemma. *With notations as above*

$$\dim \mathscr{C}_h = \dim \mathscr{C}_v = 2g - 2 + |S| + F. \tag{4.6}$$

Proof. Let e be a horizontal edge. e lies on the boundary of a component, U, of Λ^c such that if (U, f) is the chart whose existence is guaranteed by Proposition 3.5, the $fU = R$ has fe on its lower side. Set $U = U(e)$. If $\tau(U)$ is the number of vertical termini and/or elements of S which are interior to the lower side of U, i.e. to f^{-1} (lower side R), the lower side of U contains $1 + \tau(U)$ edges, and

$$\begin{aligned} \dim \mathscr{C}_h &= \sum_U (1 + \tau(U)) \\ &= F + \sum_U \tau(U). \end{aligned} \tag{4.7}$$

Because ξ is assumed to be in general position, the sum on the right-hand side of (4.7) represents the total number of outgoing vertical separatrices. The latter number equals $2g - 2 + |S|$, and (4.6) follows for $\mathscr{C}_h$. The argument for $\mathscr{C}_v$ is the same, and the lemma is proved.

4.8. Lemma. *If* $\alpha \in \mathscr{C}_h \cup \mathscr{C}_v$, *and if* $\partial\alpha = 0$, *then* $\alpha = 0$.

Proof. Suppose $\alpha \in \mathscr{C}_h$ and $\partial\alpha = 0$. If γ is a maximal horizontal end in Λ, and if α_γ is the contribution to α coming from edges of Λ on γ, then $\partial\alpha = 0$ implies $\partial\alpha_\gamma \subseteq S$. It must be that $\alpha_\gamma = 0$, and therefore $\alpha = 0$. The lemma is proved.

Denote by Q_h and Q_v the orthogonal projections on $\mathscr{C}_h$ and $\mathscr{C}_v$, respectively, relative to $\langle \cdot, \cdot \rangle$. Lemma 4.8 implies Q_h and Q_v are each one-to-one on $\partial\mathscr{C}_2$, because $Q_h\alpha = 0$ implies $\alpha \in \mathscr{C}_v$, and $\alpha = \partial r$ implies $\partial\alpha = 0$. We have

4.9. Lemma. Q_h *and* Q_v *are injective on* $\partial\mathscr{C}_2$.

4.10. Lemma. *With notations as above*

$$\dim \mathscr{Z}_v^1 = \dim \mathscr{Z}_h^1 = 2g - 1 + |S|. \tag{4.11}$$

Proof. Use $\perp$ to denote orthogonal complement relative to $\langle \cdot, \cdot \rangle$. The definitions imply

$$\mathscr{X}_h^1 = \mathscr{X}^1 \cap \mathscr{C}_h$$
$$= (\partial \mathscr{C}_2)^\perp \cap \mathscr{C}_h$$
$$= (Q_h \partial \mathscr{C}_2)^\perp \cap \mathscr{C}_h.$$

As $\dim \partial \mathscr{C}_2 = F - 1$, (4.11) follows from the above and Lemmas 4.5 and 4.9. The lemma is proved.

4.12. Lemma. *Let V be the set of vertices of Λ, and suppose $x = \Sigma_{v \in V} x(v)v \in \mathscr{C}_0$ satisfies $\partial^* x \in \mathscr{C}_h$ (resp. $\mathscr{C}_v$). If e is a vertical (resp. horizontal) edge such that $\partial e = v_1 - v_2$, then $x(v_1) = x(v_2)$. Conversely, any x with this latter property satisfies $\partial^* x \in \mathscr{C}_h$ (resp. $\mathscr{C}_v$).*

Proof. If $\partial^* x \in \mathscr{C}_h$, and if e is a vertical edge such that $\partial e = v_1 - v_2$, then

$$0 = \langle e, \partial^* x \rangle$$
$$= \langle \partial e, x \rangle$$
$$= x(v_1) - x(v_2).$$

The first assertion follows. The second is obvious, and the lemma is proved.

Define $B^1 = \partial^* \mathscr{C}_0$, and also set $B_h^1 = B^1 \cap \mathscr{C}_h$ and $B_v^1 = B^1 \cap \mathscr{C}_v$. If $s \in S$, define $x_h(s) \in \mathscr{C}_0$ by

$$x_h(s) = \sum_{q \in V} x_h(s, q) q$$

where $x_h(s, q) = 1$ or 0 as q lies on a vertical end at s or not. Similarly, define $x_v(s)$ in terms of the horizontal ends at s. If $\gamma_h(s) = \partial^* x_h(s)$, $\gamma_v(s) = \partial^* x_v(s)$, Lemma 4.12 implies B_h^1 and B_v^1 are the linear spans of $\{\gamma_h(s) \mid s \in S\}$ and $\{\gamma_v(s) \mid s \in S\}$, respectively. The kernel of ∂^* on $\mathscr{C}_0$ is one dimensional, spanned by $\Sigma_{s \in S} x_h(s) = \Sigma_{s \in S} x_v(s)$ and therefore both B_h^1 and B_v^1 have dimension $|S| - 1$.

4.13. Lemma. *With notations as above*

$$\dim B_h^1 = |S| - 1 = \dim B_v^1. \tag{4.14}$$

Let $\mathscr{C}_0(S)$ be the linear span of S in $\mathscr{C}_0$, and set up the relative cochain complex

$$\mathscr{C}_0(S)^\perp \xrightarrow{\partial^*} \mathscr{C}_1 \xrightarrow{\partial^*} \mathscr{C}_2. \tag{4.15}$$

Because Λ is cellular decomposition of X, the relation

$$H_{\mathbf{R}}^1(X, S) = \ker \partial^* \cap \mathscr{C}_1 / \partial^* \mathscr{C}_0(S)^\perp$$

obtains. Let $\mathscr{X}^1 \xrightarrow{R} H_{\mathbf{R}}^1(X, S)$ be the canonical projection.

4.16. Proposition. *With notations as above the restrictions.*

$$\mathscr{X}_h^1 \xrightarrow{R} H^1_{\mathbf{R}}(X, S)$$
(4.17)
$$\mathscr{X}_v^1 \xrightarrow{R} H^1_{\mathbf{R}}(X, S)$$

are onto isomorphisms.

Proof. Proposition 4.10 implies the three vector spaces in (4.17) have the same dimension. Suppose $\alpha \in \mathscr{X}_h^1$, say, and $R\alpha = 0$. There exists $x \in \mathscr{C}_0(S)^\perp$ such that $\alpha = \partial^* x$. Since $x(s) = 0$, $s \in S$, and since $\alpha \in \mathscr{C}_h$, Lemma 4.12 implies $x = 0$. Therefore, $\alpha = 0$, and R is injective on $\mathscr{X}_h^1$, and the proposition is proved.

In preparation for our final proposition we shall make a construction. Let $\alpha \in \mathscr{X}_h^1$, and select a smooth closed 1-form, σ_α, such that $\hat{\sigma}_\alpha = \alpha$. Let $\Lambda_v \subseteq \Lambda$ be the union of the vertical ends in Λ, and define F_α on Λ_v by

(4.18)
$$F_\alpha(x) = \int_s^x \sigma_\alpha.$$

In this formula $s \in S$, and x lies on a vertical end at s. Since σ_α is closed, F_α extends to a smooth function, satisfying $dF_\alpha = \sigma_\alpha$, on an open set which contains Λ_v. We multiply the extended F_α by a smooth cutoff function which is one on a neighborhood of Λ_v, and thus assume F_α is defined on X. Define $\tau_\alpha = \sigma_\alpha - dF_\alpha$. We claim $\hat{\tau}_\alpha = \alpha$. This follows because $\hat{\sigma}_\alpha(e) = 0$ for every vertical edge e, and then $F_\alpha|_V = 0$ by (4.18). That is, $\hat{\tau}_\alpha = \hat{\sigma}_\alpha = \alpha$, as claimed. Observe that by construction $\tau_\alpha = 0$ on Λ_v.

If U is a component of Λ^c, represent U as a rectangle, R, by Proposition 3.5. Let $\alpha \in \mathscr{X}_h^1$, and define $\alpha(U)$ to be the sum of $\alpha(e)$ over all horizontal edges which lie on the bottom side of U with respect to its rectangular representation. If $\beta \in \mathscr{X}_v^1$, define $\beta(U)$ using vertical edges on one side of U. We shall prove

4.19. Proposition. *If* $(\alpha, \beta) \in \mathscr{X}_h^1 \times \mathscr{X}_v^1$, *then*

(4.20)
$$[\alpha, \beta] = \sum \alpha(U)\beta(U)$$

the sum on the right in (4.20) *extending over all components*, $U \subseteq \Lambda^c$.

Proof. Let τ_α and τ_β be smooth closed forms such that (1) $\hat{\tau}_\alpha = \alpha$ and $\hat{\tau}_\beta = \beta$, and (2) $\tau_\alpha \equiv 0$ on Λ_v and $\tau_\beta \equiv 0$ on (the corresponding set) Λ_h. By definition

$$[\alpha, \beta] = \int_X \tau_\alpha \wedge \tau_\beta$$

(4.21)

$$= \sum_U \int_U \tau_\alpha \wedge \tau_\beta.$$

Fix a component $U \subseteq \Lambda^c$. Since τ_α is closed, there exists a smooth function, f_α, on $\bar{U}$ such that $df_\alpha = \tau_\alpha$, and then $d(f_\alpha \tau_\beta) = \tau_\alpha \wedge \tau_\beta$ on U. By Stokes' Theorem

$$\int_U \tau_\alpha \wedge \tau_\beta = \int_{\partial U} f_\alpha \tau_\beta. \tag{4.22}$$

Because $\tau_\alpha \equiv 0$ on the vertical boundary of U, f_α is constant on each vertical side of U. The values are c on the left side of U and $c + \alpha(U)$ on the right side of U. Since $\tau_\beta = 0$ on the horizontal sides of U, and since τ_β integrates to $\beta(U)$ over the right side of U, the integral (4.22) has the value $\alpha(U)\beta(U)$. Now (4.20) follows from (4.21), and the proposition is proved.

5. Weavings and variation of F-structure

Let ξ and Λ be as in the first paragraph of Section 4. The spaces $\mathscr{X}_h^1$ and $\mathscr{X}_v^1$ are defined in Section 4, and Proposition 4.16 establishes a canonical isomorphism between each of them and $H^1_{\mathbf{R}}(X, S)$. To avoid confusion the ξ which has been fixed above will be denoted ξ_0.

Let $e \subseteq \Lambda$ be a horizontal edge, and define $\alpha_0(e)$ to be the length of e relative to the metric $|\omega_{\xi_0}|$. (Recall that e is a geodesic in the direction 0.) If U is a component of Λ^c such that $e \subseteq \partial U$, and if (U, f) is the chart determined in Proposition 3.5, then $\alpha_0(e)$ is the euclidean length of $fe \subseteq \partial R = \partial(fU)$. Extend α_0 to all edges by setting $\alpha_0(e) = 0$ when e is vertical. The statement that the horizontal sides of R have the same length implies $\partial^*\alpha_0 = 0$ ($\langle U, \partial^*\alpha_0\rangle = \langle \partial U, \alpha_0\rangle$). That is, $\alpha_0 \in \mathscr{X}_h^1$.

Observe that α_0 is positive in the sense that $\alpha_0(e) > 0$ for every *horizontal* edge e. We define $P_h^1 = P_h^1(\Lambda)$ to be the set of $\alpha \in \mathscr{X}_h^1$ which are positive in this sense.

By an entirely analogous procedure we define $\beta_0 \in \mathscr{X}_v^1$, in terms of ξ_0. Define $P_v^1 = P_v^1(\Lambda)$ to be the set of $\beta \in \mathscr{X}_v^1$ which are positive on vertical edges. Finally, define $P^1 = P^1(\Lambda)$ by

$$P^1 = P_h^1 \times P_v^1 \qquad (P^1 = P^1(\Lambda)). \tag{5.1}$$

The cones P_h^1 and P_v^1 are open in $\mathscr{X}_h^1$ and $\mathscr{X}_v^1$, respectively, and the constructions of α_0, β_0 imply each is nonempty. We define $\hat{\xi}_0 = (\alpha_0, \beta_0) = \lambda_0 \in P^1$.

In this section Ω_0^+ is the set of admissible positive maximal F-structures, $\xi = (Y, \mathscr{U})$, such that $Y^c \subseteq S$ (in particular, $Y \subseteq X$). The subscript 0 is to remind that S is allowed to contain properly the set $S^*(\omega_\xi)$ of zeros of ω_ξ, $\xi \in \Omega_0^+$.

The goal of this section is to construct a map, $\xi : P^1(\Lambda) \to \Omega_0^+$, satisfying the following properties:

(a) $\alpha(\xi(\lambda_1), \xi(\lambda_2))$ is continuous on $P^1(\Lambda) \times P^1(\Lambda)$, where $\alpha(\cdot, \cdot)$ is defined in Section 1.

(b) $\xi(\lambda_0) = \xi_0$, $\lambda_0 = (\alpha_0, \beta_0)$ as above (ξ_0 extended to be maximal).

(c) Λ is a weaving for $\xi(\lambda)$, all $\lambda \in P^1(\Lambda)$.

(d) $\widehat{\xi(\lambda)} = \lambda$, $\lambda \in P^1(\Lambda)$.

(e) $\xi(\cdot)$ modulo $H_0(X, S)$ is injective.

In (e) $H_0(X, S)$ denotes the set of $\phi \in H(X)$ such that $\phi|_S = \mathrm{Id}$ and $\phi \sim \mathrm{Id}$ by an isotopy which fixes S. If D_0 is the pseudometric determined by $H_0(X, S)$, (a) implies $\xi(\cdot)$ is D_0 continuous, and (a), (e) imply $\lambda \to \xi(\lambda)H_0(X, S)$ is continuous and injective from $P^1(\Lambda)$ into Ω_0^+/H_0. In Section 6 we will prove this map is also *open*.

In preparation of the construction of $\xi(\cdot)$ we introduce some notation. $\mathscr{R}$ denotes the set of pairs, $r = (R, \Sigma)$, such that (1) $R \subseteq C$ is a rectangle with sides parallel to the axes and having 0 for its lower left vertex, and (2) Σ is a finite subset of ∂R which contains (at least) the four vertices of R. Σ partitions ∂R into edges; Σ determines a triangulation of R by joining each point of Σ to the center of R by a straight line segment.

If $r_j = (R_j, \Sigma_j) \in \mathscr{R}$, $j = 1, 2$, we say $r_1 \sim r_2$ if corresponding sides of R_1 and R_2 have the same number of points of Σ_1 and Σ_2. When $r_1 \sim r_2$, there is a uniquely determined PL map, $\phi : R_1 \to R_2$. ϕ sends the center of R_1 to the center of R_2, and ϕ maps a triangle of the canonical triangulation of R_1 linearly onto the corresponding triangle of the canonical triangulation of R_2.

Continuing with $r_1, r_2 \in \mathscr{R}$, $r_1 \sim r_2$, and ϕ, ϕ is linear on each edge (not side) of R_1, and ϕ has a well defined locally constant arc length derivative, $d\phi/ds > 0$, on $\partial R_1 - \Sigma_1$. Define $\delta(r_1, r_2)$ by

$$\delta(r_1, r_2) = \max \left| \ln \frac{d\phi}{ds} \right|.$$

It is easily checked that $\delta(\cdot, \cdot)$ defines a metric on each equivalence class for $\sim$. If $r \in R$, and if $[r]$ is the equivalence class of r, then $([r], \delta(\cdot, \cdot))$ is a locally compact metric space.

One can also define $\delta^*(r_1, r_2)$ by

$$\delta^*(r_1, r_2) = \sup_{z \in R_1} \limsup_{z' \to z} L\left(\frac{\phi z' - \phi z}{z' - z}\right).$$

Here $Lw = |\log w|$ with $-\pi \leqq \arg w < \pi$. An elementary calculation reveals there exists for each $r_1 \in \mathscr{R}$ a constant $C(r_1) < \infty$ such that

$$\delta(r_1, r_2) \leqq \delta^*(r_1, r_2) \leqq C(r_1)\delta(r_1, r_2)(r_2 \in [r_1]). \tag{5.2}$$

See [V86], (15.14).

We now construct the map $\xi(\cdot)$. If U is a component of Λ^c, choose a chart (U, f) by Proposition 3.5, and arrange that $R = fU$, which already has sides parallel to the axes, have lower left vertex 0. Let $\Sigma \subseteq \partial R$ be the image of the set of vertices of Λ on ∂U under f. Set $r(U) = (R, \Sigma) \in \mathscr{R}$.

Let $\lambda = (\alpha, \beta) \in P^1(\Lambda)$. We construct $r(U, \lambda) = (R(\lambda), \Sigma(\lambda))$ as follows. If e is a horizontal edge on ∂U, assign fe a new length, $\alpha(e) > 0$. (It is possible e occurs on both the "top" and the "bottom" of U, in which case fe is not well defined but occurs naturally as an edge on the top and bottom of R.) The assumptions $\alpha \in \mathscr{C}_h$ and $\partial^*\alpha = 0$ imply the top and bottom of $R(\lambda)$ have the same length. Similarly, use β to assign new lengths to the vertical edges on ∂U and ∂R. The result is an element, $r(\lambda) = (R(\lambda), \Sigma(\lambda)) \in \mathscr{R}$, and $r(\lambda) \sim r(U)$. We shall also write $r(U, \lambda)$ and $R(U, \lambda)$. Observe that by construction if $\lambda = \lambda_0 = \xi_0$, then $r(U, \lambda_0) = r(U)$.

If $\lambda_1, \lambda_2 \in P^1(\Lambda)$, define

$$\delta_1(\lambda_1, \lambda_2) = \max_e \left| \ln \frac{\alpha_1(e) + \beta_1(e)}{\alpha_2(e) + \beta_2(e)} \right|$$

the maximum being taken over all edges, e, of Λ. If $\delta(\cdot, \cdot)$ is defined as earlier on $[r(U)]$, then

$$\delta(r(U, \lambda_1), r(U, \lambda_2)) \leqq \delta_1(\lambda_1, \lambda_2). \tag{5.3}$$

Let $\phi^U_{\lambda_1, \lambda_2} : r(U, \lambda_1) \to r(U, \lambda_2)$ be the canonical PL map.

If U is a component of Λ^c, denote the chart map which defines $r(U)$ by f^U. Introduce the charts (U, F^U_λ), $\lambda \in P^1(\Lambda)$, by

$$F^U_\lambda = \phi^U_{\lambda_0 \lambda} \circ f^U \qquad (U \subseteq \Lambda^c) \tag{5.4}$$

where, as before, $\lambda_0 = \xi_0$.

Let e be an edge which is common to components, U_1 and U_2, of Λ^c. If $U_1 \neq U_2$, or if $U_1 = U_2$ but e lies on two sides of $U_1 = U_2$, then along e $F^{U_j}_\lambda$, $j = 1, 2$ are related by

$$F^{U_1}_\lambda = F^{U_2}_\lambda + c \tag{5.5}$$

where $c \in C$ is a constant. Let $\mathscr{U}'(\lambda)$ be the atlas for Λ^c whose charts are defined by (5.4). $\mathscr{U}'(\lambda)$ has no coordinate transitions, and so $\xi'(\lambda) = (\Lambda^c, \mathscr{U}'(\lambda))$ is trivially a positive F-structure.

Let $e \subseteq \Lambda$ be an edge. The relation (5.5) implies e is contained in an open set, U_e (e.g., $U_e = U_1 \cup U_2$ if $U_1 \neq U_2$, $U_e = U_1 = U_2$ otherwise), which is the domain of an $\xi'(\lambda)$-compatible chart function, F_e. Let

$$\mathscr{U}''(\lambda) = \mathscr{U}'(\lambda) \cup \{(U_e, F_e) \mid e \subseteq \Lambda \text{ an edge}\}.$$

If V is the set of vertices of Λ, then $(V^c, \mathscr{U}''(\lambda)) = \xi''(\lambda)$ is a positive F-structure.

If $v \in V$, there is defined an integer, $n(v) \geqq 0$, such that $2\pi n(v)$ is the total angle at v determined by rectangles, $R(U, \lambda)$, such that $v \in \partial U$. Each summand determining total angle is π or $\pi/2$, and the summands remain the same as λ varies. If $n(v) = 1$, then v is contained in an open set U_v, which is the domain of a $\xi''(\lambda)$-compatible chart function, F_v. If $n(v) > 1$, then $v \in S$. Therefore, we may define $\mathscr{U}(\lambda) = \mathscr{U}''(\lambda) \cup \bigcup_{v \notin S}(U_v, F_v)$ to obtain a positive F-structure, $\xi(\lambda) = (S^c, \mathscr{U}(\lambda))$. It is evident that $\xi(\lambda)$ is admissible, and therefore $\xi(\lambda) \in \Omega_0^+$. Observe that $\pi(\xi(\lambda)) = \pi(\xi(\lambda_0)) = \pi(\xi_0)$; indeed, by the construction $\omega_{\xi(\lambda)}$ has a zero of order $n(v) - 1$ at $v \in S$.

5.6. Proposition. *The map $\lambda \to \xi(\lambda)$ satisfies the inequality*

$$\delta(\lambda_1, \lambda_2) \leqq \alpha(\xi(\lambda_1), \xi(\lambda_2)) \leqq C(r(\lambda_1))\delta(\lambda_1, \xi_2) \tag{5.7}$$

where $C(\cdot)$ is the constant in (5.2) *and $\alpha(\cdot, \cdot)$ is as in Section* 1.

Proof. Immediate from (5.2) and the definition of $\alpha(\cdot, \cdot)$.

5.8. Proposition. With notations as above the map $\lambda \to \xi(\lambda)$ *satisfies conditions* (a)–(d) *from the beginning of this section.*

Proof. Proposition 5.6 implies $\alpha(\xi(\lambda_1), \xi(\lambda_2))$ is continuous (recall that $\alpha(\cdot, \cdot)$ is a pseudometric), and (a) is established. (b) is immediate, in the sense that $\xi_0 \xrightarrow{\text{Id}} \xi(\lambda_0)$ is an equivalence with derivative 1. The construction implies each horizontal or vertical maximal end in Λ is a horizontal or vertical geodesic for $\xi(\lambda)$, $\lambda \in P^1(\Lambda)$. Therefore, Λ is a weaving for $\xi(\lambda)$, and (c) is established. Finally, (d) is obvious because $\alpha(e) + \beta(e)$ is the $\xi(\lambda)$ length of e for each edge, e, of Λ. The proposition is proved.

The purpose of the discussion to follow is to establish that $\xi(\cdot)$ is injective modulo $H_0(X, S)$. We begin with a construction.

Let e be a horizontal edge of Λ, and set $\partial e = q_2 - q_1$. Let $s_j \in S$ be such that q_j lies on a vertical end (in Λ) at s_j. Denote by γ_j the corresponding vertical geodesic from s_j to q_j, and define a path, $\gamma(e)$, by

$$\gamma(e) = \gamma_1 + e - \gamma_2. \tag{5.9}$$

$\gamma(e)$ is a path from s_1 to s_2, and therefore $\gamma(e)$ is a cycle modulo S. ($\partial\gamma(e) = s_2 - s_1 \in \mathscr{C}_0(S)$.) A similar construction associates a relative cycle to each vertical edge. Observe that by (5.9) and the definition of $\mathscr{C}_h$

$$\alpha(e) = \langle \gamma(e), \alpha \rangle \qquad (\alpha \in \mathscr{C}_h). \tag{5.10}$$

5.11. Proposition. *With notations as above suppose $\lambda_1, \lambda_2 \in P^1(\Lambda)$ and $\phi \in H_0(X, S)$ are such that $\xi(\lambda_1) = \xi(\lambda_2)\phi$. Then $\phi = \mathrm{Id}$ and $\lambda_1 = \lambda_2$.*

Proof. Define $\Lambda^\phi = \phi\Lambda$, and observe that Λ^ϕ is a weaving associated to $\xi(\lambda_2)$. Define $\lambda_2^\phi = \widehat{\xi(\lambda_2)} \in P^1(\Lambda^\phi)$, and set $\lambda_2^\phi = (\alpha_2^\phi, \beta_2^\phi)$. If e is a horizontal edge, form $\gamma(e)$ as above, and let $e^\phi = \phi e$ and $\gamma(e)^\phi = \phi\gamma(e)$ be the corresponding objects for Λ^ϕ. We have by the assumption on λ_1, λ_2 and ϕ

$$\alpha_1(e) = \alpha_2^\phi(e^\phi) = \langle \gamma(e)^\phi, \alpha_2^\phi \rangle. \tag{5.12}$$

Λ and Λ^ϕ are weavings of $\xi(\lambda_2)$, and therefore $\Lambda' = \Lambda \cup \Lambda^\phi$ is also a weaving of $\xi(\lambda_2)$. Let $\lambda_2' = (\alpha_2', \beta_2') \in P^1(\Lambda')$ be defined accordingly (i.e., $\lambda_2' = \widehat{\xi(\lambda_2)}$). The right hand side of (5.12) can be rewritten as

$$\langle \gamma(e)^\phi, \sigma_2^\phi \rangle = \langle \gamma(e)^\phi, \alpha_2' \rangle \tag{5.13}$$

because $\gamma(e)^\phi$ is also a sum of edges of Λ'. Now $\mathscr{Z}_h^1(\Lambda')$ is canonically identified with $H_{\mathbf{R}}^1(X, S)$, and since $\phi \in H_0(X, S)$, ϕ induces $\phi^* = \mathrm{Id}$ on cohomology. The right hand side of (5.13) becomes

$$\begin{aligned} \langle \gamma(e)^\phi, \alpha_2' \rangle &= \langle \phi\gamma(e), \alpha_2' \rangle \\ &= \langle \gamma(e), \phi^*\alpha_2' \rangle \\ &= \langle \gamma(e), \alpha_2' \rangle \\ &= \langle \gamma(e), \alpha_2 \rangle \\ &= \alpha_2(e). \end{aligned} \tag{5.14}$$

Together, (5.14) and (5.12) imply $\alpha_1 = \alpha_2$. Similarly, $\beta_1 = \beta_2$. Therefore, $\lambda_1 = \lambda_2$.

We have now that $\xi(\lambda_1) = \xi(\lambda_1)\phi$. This implies ϕ is a $T(x_1)$ biholomorphism of X. If $g > 1$, then $\phi = \mathrm{Id}$ because $\phi \sim \mathrm{Id}$. If $g = 1$, then ϕ is a complex translation; because $S \neq \phi$ and $\phi|_S = \mathrm{Id}$, $\phi = \mathrm{Id}$. The proposition is proved.

In what follows $H(X, S) = \{\phi \in H(X) \mid \phi|_S = \mathrm{Id}\}$. Let $\Gamma(X, S) = H(X, S)/H_0(X, S)$.

5.15. Proposition. *With notations as above let $\lambda \in P^1(\Lambda)$. There exist a neighborhood, E, of λ and an $\varepsilon > 0$ such that the following holds: if $\lambda_1, \lambda_2 \in E$ and $\phi \in H(X, S)$ satisfy $\alpha(\xi(\lambda_1), \xi(\lambda_2)\phi) < \varepsilon$, then $\phi H_0(H, S)$ has finite order in $\gamma(X, S)$.*

Proof. Let D_0 be the pseudometric defined by $H_0(X, S)$ on Ω_0^+. Suppose there are sequences $\{\phi_n\} \subseteq H(X, S)$ and $\{\lambda_j^n\} \subseteq P^1(\Lambda), j = 1, 2$ such that (1) $\lim_n \lambda_j^n = \lambda$, $j = 1, 2$, and (2) $\lim_n \alpha(\xi(\lambda_1^n), \xi(\lambda_2^n)\phi_n) = 0$. Proposition 5.6 implies that $\lim_n D_0(\xi(\lambda_1^n), \xi(\lambda)) = 0$. It follows readily that we may suppose ϕ_n converges in $H(X, S)$ to an element ϕ. Moreover, $\xi(\lambda) = \xi(\lambda)\phi$. As $g \geqq 1$ and $S \neq \phi$, $\phi H_0(X, S)$ has finite order in $\Gamma(X, S)$. For large n the same must be true of $\phi_n H_0(X, S)$. The proposition follows.

5.16. Remark. The proof of Proposition 5.15 implies that if for every neighborhood, E, of λ and every $\varepsilon > 0$ there exist $\lambda_1, \lambda_2 \in E$ and $\phi \in H(X, S) \cap H_0(X, S)^c$ such that $\alpha(\xi(\lambda_1), \xi(\lambda_2)\phi) < \varepsilon$, then there exists a nontrivial equivalence of F-structures, $\xi(\lambda) \xrightarrow{\phi} \xi(\lambda)$. Of course, $\phi H_0(X, S)$ has finite order (> 1) in $\Gamma(X, S)$.

6. Openness of the map $\lambda \to \xi(\lambda)H_0$

Let $\xi_0 \in \Omega_0^+$ and Λ be fixed as in Section 5. Λ is a weaving for $\xi(\lambda)$, $\lambda \in P^1(\Lambda)$, and therefore we can imagine that $\xi_0 = \xi(\lambda_0)$ for any fixed λ_0. $D_0(\cdot, \cdot)$ and $\alpha(\cdot, \cdot)$ are as in Section 1.

If $\varepsilon > 0$, Λ_ε denotes the set obtained by extending each end $\gamma \subseteq \Lambda$ to an end $\gamma(\varepsilon)$, relative to ξ_0, of length $|\gamma|_{\xi_0} + \varepsilon$. Assume ε is small enough that Λ_ε^c and Λ^c are homotopically the same.

Let $\xi_n \in \Omega_0^+$ be a sequence such that $\lim_n D_0(\xi_n, \xi_0) = 0$. It is no loss to suppose then that $\lim_n \alpha(\xi_n, \xi_0) = 0$. For large n each $\gamma(\varepsilon)$ above which is a vertical (resp. horizontal) end lies in a sector determined by $F_h(\xi_n)$ (resp. $F_v(\xi_n)$). In the same sector let $\gamma_n(\varepsilon)$ be the vertical (resp. horizontal) end of ξ_n of length $|\gamma(\varepsilon)|_{\xi_0} = |\gamma_n(\varepsilon)|_{\xi_n}$, and let Λ_ε^n be the union of the ends $\gamma_n(\varepsilon)$, $\gamma \subseteq \Lambda$. For large n Λ_ε^n may be "trimmed" to become a weaving Λ_n of ξ_n. Moreover, there exists $\phi_n \in H(X, S)$ such that $\phi_n \Lambda = \Lambda_n$, and ϕ_n is uniformly close to Id, meaning $\phi_n \in H_0(X, S)$, large n. Now $\xi_n \phi_n$ determines $\lambda_n \in P^1(\Lambda)_1$, and $\lim_n \lambda_n = \lambda_0$. As $\xi(\lambda_n)H_0(X, S) = \xi_n H_0(X, S)$, large n, we have

6.1. Proposition. *With notation as above, the map $J\lambda = \xi(\lambda)H_0(X, S)$ is open from $P^1(\Lambda)$ to $\mathcal{M}_0^+(S) = \Omega_0^+/H_0(X, S)$.*

7. The manifold structure of $\mathcal{M}_0^+(S)$

Let the notations be as in Section 6. We use $\mathcal{W} = \mathcal{W}(S)$ to denote the set of weavings Λ which arise from elements $\xi_0 \in \Omega_0^+$ which are in general position. Proposition 6.1 sets up a correspondence between elements of Λ and open sets, $E(\Lambda) \subseteq \mathcal{M}_0^+$, where $E(\Lambda) = JP^1(\Lambda)$. If $F_\Lambda = J^{-1}$ on $E(\Lambda)$, then

$$F_\Lambda : E(\Lambda) \to P^1(\Lambda) \tag{7.1}$$

is a homeomorphism.

7.2. Definition $\mathscr{E}_0 = \mathscr{E}_0(S)$ denotes the set of charts, $(E(\Lambda), F_\Lambda)$, in (7.1). Also, define $(\mathscr{M}_0^+)^*$ to be the union of the domains,

$$(\mathscr{M}_0^+)^* = \bigcup_{\Lambda \in \mathscr{W}} E(\Lambda). \tag{7.3}$$

$(\mathscr{M}_0^+)^*$ contains every class $\xi H_0(X, S)$ such that $\xi \in \Omega_0^+$ is in general position, but of course it contains other classes as well. On the other hand, there will be elements of $\mathscr{M}_0^+$ which do not occur in $(\mathscr{M}_0^+)^*$.

If $\theta \in \mathbf{R}$, define $A(\theta) \in G = \mathrm{SL}(2, \mathbf{R})$ to the matrix of rotation by θ,

$$A(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

If $\xi \in \Omega_0^+$, Proposition 2.8 implies $A(\theta)\xi$ is in general position for all but a countable set of θ. It follows that

$$\mathscr{M}_0^+ = \bigcup_{\theta \in \mathbf{R}} A(\theta)(\mathscr{M}_0^+)^*$$

and therefore the atlas, $\mathscr{E} = \mathscr{E}(S)$, defined by

$$\mathscr{E} = \{A^{-1}(\theta)E(\Lambda), F_\Lambda \cdot A(\theta)) \mid \theta \in \mathbf{R}, \Lambda \in \mathscr{W}\} \tag{7.4}$$

endows $\mathscr{M}_0^+$ with the structure of a topological manifold. The discussion to follow will enable us to calculate the transitions associated to $\mathscr{E}$ and to determine the action of G on $\mathscr{M}_0^+$ in local coordinates.

Suppose $\Lambda_1, \Lambda_2 \in \mathscr{W}$. Proposition 4.16 implies there are canonical isomorphisms,

$$\begin{aligned} \rho^h_{\Lambda_1\Lambda_2} &: \mathscr{X}^1_h(\Lambda_1) \to \mathscr{X}^1_h(\Lambda_2) \\ \rho^v_{\Lambda_1\Lambda_2} &: \mathscr{X}^1_v(\Lambda_1) \to \mathscr{X}^1_v(\Lambda_2) \\ \rho^v_{\Lambda_1} &: \mathscr{X}^1_v(\Lambda_1) \to \mathscr{X}^1_h(\Lambda_1) \end{aligned} \tag{7.5}$$

and that these are related by

$$r_{\Lambda_1\Lambda_3} = r_{\Lambda_2\Lambda_3} \circ r_{\Lambda_2\Lambda_2} \qquad (r = \rho^h \text{ or } \rho^v) \tag{7.6}$$

and

$$\rho_{\Lambda_2} \circ \rho^v_{\Lambda_1\Lambda_2} = \rho^h_{\Lambda_1\Lambda_2} \circ \rho_{\Lambda_1}. \tag{7.7}$$

In what follows we adopt the notation

$$\beta^* = \rho_\Lambda \beta \in \mathscr{X}^1_h(\Lambda) \qquad (\beta \in \mathscr{X}^1_v(\Lambda)). \tag{7.8}$$

Let $\Lambda_1, \Lambda \in \mathscr{W}$ and $A \in G$, and define

$$K(\Lambda_1, \Lambda_2, A)) = A^{-1}E(\Lambda_2) \cap E(\Lambda_1). \tag{7.9}$$

The composition

$$H^A_{\Lambda_1\Lambda_2} = F_{\Lambda_2} \circ A \circ F_{\Lambda_1}^{-1} \tag{7.10}$$

has the domain $F_{\Lambda_1}K(\Lambda_1, \Lambda_2, A) \subseteq P^1(\Lambda_1)$ and image contained in $P^1(\Lambda_2)$. We shall compute (7.10) in these coordinates.

Let $\Lambda \in \mathscr{W}$ and $\lambda \in P^1(\Lambda)$. If $\lambda \in (\alpha, \beta)$, and if $\omega_{\xi(\lambda)} = u_\lambda + iv_\lambda$ in real and imaginary parts, condition (c) at the beginning of Section 5 implies $\alpha = \widehat{u_\lambda}$ and $\beta = \widehat{v_\lambda}$. If $A \in G$,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and if (U, f) is a compatible chart for $\xi(\lambda)$, then by definition $(U, a \circ f)$ is a compatible chart for $A\xi(\lambda)$. In this coordinate $\omega_{A\xi(\lambda)} = f^*A^*dz$, or

$$\omega_{A\xi(\lambda)} = (au_\lambda + bv_\lambda) + i(cu_\lambda + dv_\lambda). \tag{7.11}$$

We use (7.8) to determine the elements of $\mathscr{L}^1_h$ which correspond to Re $\omega_{A\xi(\lambda)}$ and Im $\omega_{A\xi(\lambda)}$. They are

$$\begin{aligned} au_\lambda + bv_\lambda &\cong a\alpha + b\beta^*, \\ cu_\lambda + dv_\lambda &\cong c\alpha + d\beta^*. \end{aligned} \tag{7.12}$$

Suppose now $AJ(\lambda) \in E(\Lambda_1)$, and write $J\lambda_1 = AJ(\lambda)$, where $\lambda_1 = (\alpha_1, \beta_1) \in P^1(\Lambda_1)$. It follows from (7.6)–(7.7) that

$$\begin{aligned} \alpha_1 &= \rho^h_{\Lambda\Lambda_1}(a\alpha + b\beta^*), \\ \beta_1^* &= \rho^h_{\Lambda\Lambda_1}(c\alpha + d\beta^*). \end{aligned} \tag{7.13}$$

Introduce complex coordinates $\zeta = \alpha + i\beta^*$ and $\zeta_1 = \alpha_1 + i\beta_1^*$. If $A = A(\theta)$, $\theta \in \mathbf{R}$, then (7.13) implies

$$\zeta_1 = e^{-i\theta}\rho^h_{\Lambda\Lambda_1}\zeta \tag{7.14}$$

where $\rho^h_{\Lambda\Lambda_1}$ is $\mathbf{C}$ linear on $\mathscr{L}^1_h \otimes \mathbf{C}$. We have

7.15. Theorem. *Let $N = \dim H^1_{\mathbf{R}}(X, S)$. The atlas $\mathscr{E}$ in* (7.4) *gives $\mathscr{M}_0^+$ the structure of a complex N-manifold. The coordinate transitions for $\mathscr{E}$ are representable in the form* (7.14).

Proof. If $\theta_1, \theta_2 \in \mathbf{R}$ and $A = A(\theta_1 - \theta_2)$, a typical transition for $\mathscr{E}$ has the form (7.10) on the domain $F_{\Lambda_1}K$, $K = (\Lambda_1, \Lambda_2, A)$ as in (7.9). By (7.14) this function is $\mathbf{C}$ linear, and the theorem is proved.

The isomorphisms (7.5) arise as the maps induced on various realizations of $H^1_{\mathbf{R}}(X, S)$ by the identity map of X. These induced maps also are defined on the level of the integer cohomology, $H^1_{\mathbf{Z}}(X, S)$; this means each map in (7.5) sends the integer lattice in its domain (using the natural coordinate systems of the base comprised of edges) onto the integer lattice in its range. The significance of this fact is the following: If μ_Λ is the additive Haar measure on $\mathscr{X}^1_h$ which assigns covolume 1 to the integer lattice, then

$$\rho^h_{\Lambda_1\Lambda_2}\mu_{\Lambda_1} = \mu_{\Lambda_2}. \tag{7.16}$$

It follows from (7.14) that if $\theta \in \mathbf{R}$, $\Lambda \in \mathscr{W}$, and if $\nu_{\theta,\Lambda} = A^{-1}(\theta)F_\Lambda^{-1}(\mu_\Lambda \times \mu_\Lambda)$, then $\{\nu_{\theta,\Lambda} \mid \theta \in \mathbf{R}, \Lambda \in \mathscr{W}\}$ determines an everywhere positive real analytic measure, μ, on $\mathscr{M}_0^+$.

7.17. Theorem. *Let μ be defined as above on $\mathscr{M}_0^+ = \mathscr{M}_0^+(S)$. μ is invariant under the actions of G and $\Gamma(X, S) = H(X, S)/H_0(X, S)$.*

Proof. Invariance under $\Gamma(X, S)$ is tautological and will not be proved. As for G, it is necessary to calculate with (7.10). To this end we use the notations of (7.12) and (7.13) to realize $H^A_{\Lambda\Lambda_1}$ as a 2×2 matrix of operators,

$$H = \begin{pmatrix} ap & bp \\ cp & dp \end{pmatrix} \tag{7.18}$$

where $p = \rho^h_{\Lambda\Lambda_1}$ and H acts on column vectors

$$\begin{pmatrix} \alpha \\ \beta^* \end{pmatrix}, \quad a \in \mathscr{X}^1_h(\lambda), \quad \beta \in \mathscr{X}^1_v(\Lambda).$$

Because $\rho\mu_\Lambda = \mu_{\Lambda_1}$ and $ad - bc = 1$, (7.18) is measure preserving from $\mathscr{X}^1_z(\Lambda) \times \mathscr{X}^1_h(\Lambda)$ to $\mathscr{X}^1_h(\Lambda_1) \times \mathscr{X}^1_h(\Lambda_1)$. The theorem follows.

If $\xi \in \Omega_0^+$, $\Lambda \in \mathscr{W}$, and $\lambda \in P^1(\Lambda)$ are such that $\xi(\lambda) = \xi$, and if $\lambda = (\alpha, \beta)$, then by definition

$$[\alpha, \beta] = \frac{i}{2} \int_X \omega_\xi \wedge \bar{\omega}_\xi. \tag{7.19}$$

The maps (7.5) are isometries relative to the intersection form, and therefore (7.19) is well defined and real analytic on $\mathscr{M}_0^+$. (Real analyticity in local coordinates is a consequence of Proposition 4.19.) The representation (7.12) implies

(7.20)
$$V(\xi H_0) = \frac{i}{2} \int_X \omega_\xi \wedge \bar{\omega}_\xi$$

is G invariant, and we have

7.21. Theorem. *Let $m = \xi H_0 \in \mathcal{M}_0^+(S)$, and define $V(m)$ by (7.20). Then*
(1) *$V(\cdot)$ is real analytic and positive on $\mathcal{M}_0^+$,*
(2) *for each $t > 0$ the level set V_t^{-1} is a real analytic embedded submanifold of $\mathcal{M}_0^+(S)$, and*
(3) *V_t^{-1} is both G and $\Gamma(X, S)$ invariant.*

Proof. Statement (2) is easily proved in local coordinates, thanks to Proposition 4.19. Statements (1) and (3), save the tautological ($\Gamma(X, S)$) part of (3), have been proved.

7.22. Remark. Define $\Phi : \mathcal{M}_0^+(S) \to H_{\mathbf{C}}^1(X, S)$ by $\Phi(m) = \hat{\omega}_\xi$, any $\xi \in m$. In the notation of (4.17) and (7.1), $\Phi = R^{\mathbf{C}} \circ F_\Lambda$ on $E(\Lambda)$, where $R^{\mathbf{C}}(\alpha, \beta) = R\alpha + iR\beta = R(\alpha + i\beta^*)$. It follows Φ is a local biholomorphism and $\mathcal{M}_0^+(S)$ is a Riemann domain.

8. $\mathcal{M}^+(S, \tau)$

Let X, S and $\Omega_0^+(S)$ be as in previous sections. In this section we suppose there is given in addition an element $\tau \in H(X, S)$ such that (a) $\tau^2 = \mathrm{Id}$, (b) $\tau S = S$, and (c) $\mathrm{Fix}(\tau) \subseteq S$. In this situation $l(\tau) = \mathrm{Card}(\mathrm{Fix}(\tau))$ is the Lefschetz number of τ. It is a fact which goes back to Neilsen that if τ_1 satisfies (a)–(c), and if $l(\tau_1) = l(\tau)$, then $\tau_1 = \phi^{-1}\tau\phi$ for some orientation preserving homeomorphism ϕ, $\phi S = S$. (If $\tau|_S = \tau_1|_S$, one may assume $\phi \in H(X, S)$.) Note that this also implies the existence of a commutative diagram

(8.1)
$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & X \\ \Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle c} \\ X/\tau & \xrightarrow{\ \psi\ } & X/\tau \end{array}$$

where c denotes canonical projection. (A proof of (8.1) is given in [V84].)

Introduce the space $\Omega^+(S, \tau) \subseteq \Omega_0^+(S)$ as

(8.2)
$$\Omega^+(S, \tau) = \{\xi \in \Omega_0^+(S) \mid \tau^* \omega_\xi = -\omega_\xi\}.$$

Define $Z(\tau)$ to be the centralizer of τ in $H(X, S)$, and let $L_0(\tau) = Z(\tau) \cap H_0(X, S)$. Introduce the objects

$$\mathcal{M}^+(S, \tau) = \Omega^+(S, \tau)/L_0(\tau),$$

$$\Gamma(\tau) = Z(\tau)/L_0(\tau).$$

The diagram (8.1) implies $\mathcal{M}^+(S, \tau)$ and $\mathcal{M}^+(S, \tau_1)$ are isomorphic when $l(\tau_1) = l(\tau)$; the isomorphism is determined up to elements of $\Gamma(\tau)$ and $\Gamma(\tau_1)$. In particular, the moduli spaces are canonically isomorphic,

$$\mathcal{M}^+(S, \tau)/\Gamma(\tau) \cong \mathcal{M}^+(S, \tau_1)/\Gamma(\tau_1) \qquad (l(\tau) = l(\tau_1)). \tag{8.3}$$

Let $X_\tau = X/\tau$ in (8.1), and let $c = c_\tau$. If ξ is an admissible F-structure on $X_\tau - c_\tau S$, then ξ determines an admissible F-structure, denoted (by abuse of notation) ξc_τ, on $X - S$. If ξ is not positive, i.e. if $\pi(\xi) = (k, \nu, -1)$, and if $\pi(\xi c_\tau) = (0, \tilde{\nu}, +1)$, then ξc_τ determines two elements of $\Omega^+(S, \tau)$, corresponding to $\pm\omega_{\xi c_\tau}$. Conversely, if $\eta \in \Omega^+(S, \tau)$, there is a second element $-\eta \in \Omega^+(S, \tau)$, $\omega_{-\eta} = -\omega_\eta$, and the pair $\{\eta, -\eta\}$ determines an element ξ on $X_\tau - c_\tau S$ such that $\eta \cup -\eta \subseteq \xi c_\tau$.

While our analysis will apply to $\mathcal{M}^+(S, \tau)$, only a portion of $\mathcal{M}^+(S, \tau)$ is related to the spaces $V(\pi)$ and $\tilde{V}(\pi)$ in Section 1. To understand the connection the set $S_{N(\pi)} = S_n \cup T$ from Section 1 will play the role of $c_\tau S$. It will be assumed of π that $\pi = (n, \nu, -1)$, i.e. $\varepsilon = -1$. If $M(\pi) = n + \Sigma_{l=0}^\infty \nu(2l+1)$, then for each $[\xi] \in W(\pi)$ (see (1.1)) there exists a 2-sheeted branched cover with $M(\pi)$ branch points, $X \xrightarrow{c_\tau} X_\tau$, and two elements of $\mathcal{M}^+(S, \tau)$ which project to $[\xi]$. If $[\pm\eta]$ project to $[\xi]$, the order of $\pm\omega_\eta$ at points of $\tau^{-1}S_{N(\pi)} = S$ are determined by π and (1.1). Conversely, any $[\eta] \in \mathcal{M}^+(S, \tau)$ which has the prescribed orders at points of S projects to $[\xi] \in W(\pi)$. Define $W_\tau(\pi) \subseteq W(\pi)$ to be the set of $[\xi]$ which arise from some $[\eta] \in \mathcal{M}^+(S, \tau)$, and let $\mathcal{M}^+(S, \tau, \pi)$ be the corresponding subset of $\mathcal{M}^+(S, \tau)$. Now $\Gamma(\tau)$ projects to a subgroup, $\Gamma_\tau(g, N(\pi))$, of finite index in $\Gamma(g, N(\pi))$. As (8.1) implies $W_\tau(\pi)\Gamma(g, N(\pi)) = W(\pi)$, it is true that the moduli spaces involved are canonically isomorphic, i.e.,

$$\begin{aligned} \tilde{V}(\pi)/\Gamma(g, N(\pi)) &\cong W(\pi)/\Gamma(g, N(\pi)) \\ &\cong W_\tau(\pi)/\Gamma_\tau(g, N(\pi)) \cong \mathcal{M}^+(S, \tau, \pi)/\Gamma(\pi). \end{aligned} \tag{8.4}$$

Because of (8.4), we will study $\mathcal{M}^+(S, \tau)$ and its moduli space $\mathcal{M}^+(S, \tau)/\Gamma(\tau)$. There is no further need for the symbol π.

Let $\mathcal{W}(S, \tau)$ be the set of weavings $\Lambda \in \mathcal{W}(S)$ such that (a) Λ arises from some $\xi \in \Omega^+(S, \tau)$ in general position and (b) $\tau\Lambda = \Lambda$. Recalling that $\mathrm{Fix}(\tau) \subseteq S$, it is true that $\tau a \cap a = \varnothing$ when a is an open edge or face of $\Lambda \in \mathcal{W}(S, \tau)$.

If $\Lambda \in \mathscr{W}(S, \tau)$, define $\mathscr{Z}^1_{*,\tau} = \mathscr{Z}^1_{*,\tau}(\Lambda)$ for $* = h$ or v to be the -1 eigenspace of τ^* in $\mathscr{Z}^1_*(\Lambda)$. If Λ arises from $\xi \in \Omega^+(S, \tau)$, ξ determines an element $\hat{\xi} \in P^1_\tau(\Lambda) = P^1_{h,\tau} \times P^1_{v,\tau}$, where $P^1_{*,\tau} = P^1_* \cap \mathscr{Z}^1_{*,\tau}$, $* = h$ or v. Conversely, if $\lambda \in P^1_\tau(\Lambda)$, then $\xi(\lambda) \in \Omega^+(S, \tau)$.

In what follows the group $L(\tau)$ is used to determine a metric on $\mathscr{M}^+(S, \tau)$ (i.e. use $K = L(\tau)$ in the discussion preceding Remark 1.2). Corresponding to Proposition 6.1 we have

8.5. Proposition. *Let $\Lambda \in \mathscr{W}(S, \tau)$. The map $J_\tau \lambda = \xi(\lambda)L(\tau)$ is a homeomorphism between $P^1_\tau(\Lambda)$ and an open set $E_\tau(\Lambda) = J_\tau P^1_\tau(\Lambda) \subseteq \mathscr{M}^+(S, \tau)$.*

Notice that

$$\dim_{\mathbf{R}} \mathscr{Z}^1_{*,\tau}(\Lambda) = g(X) - 1 + \tfrac{1}{2}\operatorname{Card} S \quad (* = h \text{ or } v). \tag{8.6}$$

The reason is that $l(\tau)$ enters in two formulas, (i) $\dim(B^1_* \cap \mathscr{Z}^1_{*,\tau}) = \frac{1}{2}(|S| - l(\tau))$, $* = h$ or v, and (ii) $l(\tau) = 2 + \nu^- - \nu^+$, where $\nu^\pm$ are the dimensions of the ± 1 eigenspaces of τ^* on $H^1_{\mathbf{R}}(X)$, $\nu^+ + \nu^- = 2g$. When (X, S, τ) arise from $V(\pi)$, then $l(\tau) = M(\pi)$ above, $\operatorname{Card} S = 2N(\pi) - M(\pi)$, and

$$\dim Z^1_{*,\tau}(\Lambda) = 2g(X_\tau) + N(\pi) - 2. \tag{8.7}$$

8.8. Remark. Note that (8.7) is 1 less than $\dim_{\mathbf{R}} H^1_{\mathbf{R}}(X_\tau, S_{N(\pi)})$. We will see that (8.7) is the complex dimension of $V(\pi)$ when $\varepsilon(\pi) = -1$, so in all cases

$$\dim_{\mathscr{C}} V(\pi) = 2g + N(\pi) + \frac{\varepsilon - 3}{2} \tag{8.9}$$

where g is the genus of the surface carrying $V(\pi)$ $(= g(X_\tau)$ above when $\varepsilon = -1)$.

8.10. Theorem. *Let*

$$\mathscr{E}(S, \tau) = \{A^{-1}(\theta)E_\tau(\Lambda), F^\tau_\Lambda \circ A(\theta)) \mid \theta \in \mathbf{R}, \Lambda \in \mathscr{W}(S, \tau)\},$$

where $F^\tau_\Lambda = J^{-1}_\tau$ in Proposition 8.5. Introducing complex coordinates $(\alpha + i\beta^* \in \mathscr{C} \otimes \mathscr{Z}^1_{h,\tau}(\Lambda))$ as in Section 7, $\mathscr{E}(S, \tau)$ gives $\mathscr{M}^+(S, \tau)$ the structure of an affine complex manifold. Moreover, $\mathscr{E}(S, \tau)$ coordinate transitions are euclidean measure preserving.

The last statement in Theorem 8.10 follows as in (7.16). (The integer lattices of the ± 1-eigenspaces of τ^* correspond under $\rho^h_{\Lambda\Lambda_1}$.) It follows that $\mathscr{E}(S, \tau)$ determines an everywhere positive real analytic volume element μ and that μ is invariant under both G and $\Gamma(\tau)$.

9. Special weavings

X, S, and $\Omega_0^+ = \Omega_0^+(S)$ have the usual meanings (Section 5). In this and the next two sections we shall establish topological lemmas, Lemmas 10.26 and 11.22, which will play a critical role in establishing the finiteness of certain integrals. Let $\mathscr{W} = \mathscr{W}(S)$ be the set of weavings defined in the first paragraph of Section 7.

9.1a. Λ contains a single horizontal end, γ, and γ is outgoing.

9.1b. Λ contains a vertical edge, e_1, such that $\partial e_1 = s_1 - x_1$ with $s_1 \in S$ and x_1 the terminus of γ.

9.1c. If x is a vertex of Λ, then either $x = x_1$ in 9.1b, $x \in S$, or x is a vertical terminus.

9.2. Notation. If Λ is a special weaving, the notations 9.1a–c will be used in reference to Λ. In addition, we write

(i) s_0 for the element of S at which γ is a (horizontal) end,

(ii) e_T (T for terminal) for the edge of Λ on γ which has x_1 (9.1b) for its right-hand endpoint,

(iii) e_0 for the vertical edge whose upper endpoint is x_1 and y_1 for the other endpoint of e_0, $\partial e_0 = x_1 - y_1$ (note y_1 is interior to γ because $e_0 + e_1$ is an end at s_1 (9.1b)), and

(iv) e_R for the edge on γ which has y_1 for right-hand endpoint.

9.3. Lemma. *Let Λ be a special weaving. If x is a vertex of Λ which is interior to γ, and if e_x is the unique vertical edge which has x as one endpoint, then either $x = y_1$ and $e_x = e_0$ (9.2(iii)) or there exist $s_x \in S$ and $\varepsilon(x) = \pm 1$ such that $\partial e_x = \varepsilon(x)(s_x - x)$.*

Proof. Let s_x be the endpoint of e_x which is different from x. If $x \neq y_1$, then $s_x \neq x_1$. If $s_x \notin S$, then by 9.1c both x and s_x are vertical termini. This is not possible because e_x, being an edge of Λ, has no point of S in its interior. The lemma is proved.

9.4. Proposition. *Let $\xi \in \Omega_0^+(S)$ be in general position. There exists a special weaving for ξ.*

Proof. It is our continuing assumption that $S \neq \phi$ (Section 2, paragraph 2). Fix $s_0 \in S$, and let γ_0 be an outgoing horizontal end at s_0. Select $s_1 \in S$, perhaps $s_1 = s_0$, and let δ_0 be an incoming vertical separatrix at s_1. Define x_1 to be the final point of intersection between γ_0 and δ_0, before δ_0 reaches s_1. Then define γ to be the horizontal end obtained by truncating γ_0 at x_1. δ_0 intersects γ infinitely often. In particular there is a point $y_1 \in \delta_0 \cap \gamma$ such that if δ is the vertical end obtained by truncating δ_0 at y_1, then $\delta \cap \gamma = \{y_1, x_1\}$ (plus possibly $s_0 = s_1$). Finally, for each $s \in S$ and vertical separatrix, θ, at s such that $(s, \theta) \neq (s_1, \delta_0)$, let x_θ be the final point of intersection of θ and γ, before s. Define e_{x_θ} to be θ truncated at x. $\Lambda \in \gamma \cup \delta \cup \bigcup e_{x_\theta}$ is the desired special weaving.

9.5. Remark. The convention that e_1 be an incoming end at s_1, in 9.1b, is purely for notational convenience in this section. In later sections e_1 will be allowed to be outgoing ($\partial e_1 = x_1 - s_1$), and with obvious modifications in the definition and notations, the weaving will still be called special. One may also define "special weaving" by interchanging the occurrences of the words "horizontal" and "vertical" in Definition 9.1. This will not arise in the present chapter.

The two lemmas which follow are immediate consequences of the definition and Lemma 9.3. Proofs are omitted.

9.6. Lemma. *Let Λ be a special weaving, and let U be a component of Λ^c. If l is the left side of U (i.e., corresponding to the left side of the associated rectangle $R(U)$), there are three possibilities for l:*

(i) *$l = e_{s_0}$ where e_{s_0} is a vertical end at s_0* (see 9.2(i)),

(ii) *l is the union of two vertical edges which are ends at some $s \in S$,*

(iii) *l is the union of e_{y_1}, e_1, and possibly a third vertical edge, e_1', such that $\partial e_1' = x_1' - s_1$, x_1' interior to γ.*

Lemma 9.6 implies in all cases that l contains an element of s.

9.7. Lemma. *Let Λ and U be as in Lemma* 9.6. *If r is the right-hand side of U, there are two possibilities for r:*

(i) *$r = e_{y_1}$,*

(ii) *r is the union of two vertical edges which are ends at some $s \in S$.*

9.8. Corollary. *Let Λ, U, r, and l be as in Lemmas* 9.6–9.7. *Then l contains an element of s, and if $r \neq e_{y_1}$, r contains an element of S.*

9.9. Notation. U_1 will denote the component of Λ^c whose right-hand side is e_{y_1} (as in (i) above).

10. Basic Lemma — Part I

In this section Λ denotes a special weaving associated to some $\xi \in \Omega_0^+(S)$ in general position. Let there be fixed linearly independent sets, $A = \{\alpha_1, \ldots, \alpha_N\} \subseteq \mathscr{X}_h^1$ and $B = \{\beta_1, \ldots, \beta_N\} \subseteq \mathscr{X}_v^1$, such that

(1) $A \subseteq \overline{P_h^1}$ and $B \subseteq \overline{P_v^1}$,

(2) $\Sigma_{i=1}^N \alpha_i \in P_h^1$, and

(3) for each i, $1 \leqq i \leqq N$, α_i is an extremal of the cone $\overline{P_h^1}$.

We assume p, q are integers such that $a \leqq p, q < N$ and

$$(10.1) \qquad [\alpha_i, \beta_j] = 0 \qquad (1 \leqq j \leqq p, q < i \leqq N).$$

V and W shall denote the linear spans of $\{\alpha_{q+1}, \ldots, \alpha_N\}$ and $\{\beta_1, \ldots, \beta_p\}$. Finally, define $\alpha^* \in V \cap \overline{P_h^1}$ and $\beta^* \in W \cap \overline{P_v^1}$ by

$$(10.2) \qquad \alpha^* = \sum_{i=q+1}^{N} \alpha_i \quad \text{and} \quad \beta^* = \sum_{j=1}^{p} \beta_j.$$

Observe that if $\alpha^*(e) = 0$, then $\alpha(e) = 0$ for all $\alpha \in V$. A corresponding statement holds for β^* and $\beta \in$ W. This section is devoted to proving $p \leqq q$; moreover $p = q$ only if $\beta^*(e_1) > 0$ (Lemma 10.26).

10.3. Lemma. *With notations and assumptions fixed as above, the condition* (10.1) *is equivalent to each of the conditions* (10.4) *and* (10.5) *below*:

$$(10.4) \qquad [\alpha^*, \beta^*] = 0,$$

$$(10.5) \qquad \alpha^*(U)\beta^*(U) = 0 \qquad (\textit{all components } U \subseteq \Lambda^c).$$

Proof. Assumption (1) of the first paragraph and Proposition 4.19 imply $[\alpha_i, \beta_j] \geqq 0$ for all i, j. Therefore, (10.1) and (10.4) are equivalent. Assumption (1) also implies the left side of (10.5) is nonnegative for all components $U \subseteq \Lambda^c$, and Proposition 4.19 implies (10.4) and (10.5) are equivalent. The lemma is proved.

10.6. Lemma. *With notations and assumptions fixed as above, suppose* $\alpha \in \mathscr{X}_h^1$, $\beta \in W$, *and* $f \in \mathscr{C}_0(S)^{\perp}$ *are such that*

$$(10.7) \qquad \alpha + \beta = \partial^* f.$$

If U is a component of Λ^c distinct from U_1 (see 9.9), *and if $\beta^*(U) = 0$, then f vanishes at the four corners of U. In particular*,

$$(10.8) \qquad \alpha(U) = 0 \qquad (\beta^*(U) = 0,\ U \neq U_1).$$

Proof. Let e be a vertical edge of Λ on ∂U. If β^*(U) $= 0$, then $\beta^*(e) = 0$, and by the remark preceding the lemma $\beta(e) = 0$. Since e is vertical, $\alpha(e) = 0$ automatically, and (10.7) implies $\partial^* f(e) = 0$. That is, f assumes the same value at each endpoint of e. It follows f is constant (on the set of vertices) along each side of U. If $U \neq U_1$, the fact $f|_S = 0$ combines with Corollary 9.8 to imply the constants are both zero. The lemma is proved.

In the case that $\beta^*(U_1) = 0$, the argument above does imply f is zero along the left side of U and also that $\partial^* f(e_{y_1}) = f(x_1) - f(y_1) = 0$. If it is assumed or otherwise known that $\beta^*(e_1) = 0$, then $f(y_1) = f(x_1) = f(s_1) = 0$. We can state

10.9. Lemma. *Let notations and assumptions be fixed as above. If β^* and e_1 satisfy*

$$\beta^*(e_1) = 0 \tag{10.10}$$

and if α, β, and f satisfy the hypotheses of Lemma 10.6, *then* (10.8) *holds for all U, including $U = U_1$ (assuming $\beta^*(U_1) = 0$).*

10.11. Lemma. *Let notations and assumptions be fixed as above. If $\beta^*(e_1) = 0$, then V and W are linearly independent modulo $\partial^* \mathscr{C}_0(S)^\perp$.*

Proof. Let $\alpha \in V$, $\beta \in W$, $f \in \mathscr{C}_0(S)^\perp$ be such that (10.7) holds. It is to be proved that $\alpha = 0 = \beta$. To this end, let $e \subseteq \Lambda$ be a horizontal edge such that $\alpha^*(e) > 0$. Set $\partial e = a_1 - a_2$. Because Λ is a special weaving, there exists for $j = 1$ or 2 a component, $U^j \subseteq \Lambda^c$, such that a_j is a corner of U^j and $e \subseteq \partial U^j$. Since $\alpha^*(U^j) \geqq \alpha^*(e) > 0$, (10.5) implies $\beta^*(U^j) = 0$. Since $\beta^*(e_1) = 0$ by assumption, Lemma 10.9 implies $f(a_j) = 0$, $j = 1, 2$. Now (10.7) implies $\alpha(e) = 0$. Since $\alpha(e) = 0$ is automatic when $\alpha^*(e) = 0$, we have $\alpha = 0$. Lemma 4.12 implies $\mathscr{Z}_v^1 \cap \partial^* \mathscr{C}_0(S)^\perp = \{0\}$, and $\beta = 0$ follows from (10.7) and the fact $\alpha = 0$. The lemma is proved.

If we do not assume $\beta^*(e_1) = 0$, the argument used in Lemma 10.11 applies to every horizontal edge except $e = e_R$. (If $e = e_T$, choose U^j above to be distinct from U_1. That is, if $U^1 = U_1$, select U^2.) The conclusion is that if $\alpha \in V$, $\beta \in W$, and $f \in \mathscr{C}_0(S)^1$ satisfy (10.7), then $\alpha = ce_R$ for some $c \in \mathbf{R}$. We have

10.12. Lemma. *Let notations and assumptions be fixed as above. Then $V \cap (W + \partial^* \mathscr{C}_0(S)^\perp) \subseteq \mathbf{R}e_R$. That is, if $\alpha \in V$, $\beta \in W$, and $f \in \mathscr{C}_0(S)^\perp$ satisfy* (10.7), *then there exists $c \in \mathbf{R}$ such that*

$$\alpha = ce_R. \tag{10.13}$$

Observe that if $\partial^* e_R \neq 0$, then $c = 0$ in (10.13). The necessary and sufficient conditions that $\partial^* e_R = 0$ is that e_R lie on both top and bottom of U_1, when U_1 is realized as a rectangle in $\mathbf{C}$, using a chart which is compatible with ξ. It is equivalent to require that e_1 is interior to $\overline{U_1}$.

10.14. Lemma. *Let notations and assumptions be fixed as above. Suppose i, $1 \leqq i \leqq q$, is such that there exists a solution $(v_i, w_i, f_i) \in V \times W \times \partial^* \mathscr{C}_0(S)^\perp$ to*

$$\alpha_i = v_i + w_i + \partial^* f_i. \tag{10.15}$$

Then also

$$\alpha_i(e) = v_i(e) \qquad (e \neq e_R, \alpha^*(e) > 0). \tag{10.16}$$

If $\beta^*(e_1) = 0$, and if $\alpha^*(e_R) > 0$, then (10.16) *holds for $e = e_R$.*

Proof. Let $\alpha = \alpha_i - v_i$, $\beta = -w_i$, and $f = f_i$ in (10.7). If $e \neq e_R$, and if $\alpha^*(e) > 0$, the proof of Lemma 10.11 shows f_i vanishes on ∂e. Therefore (10.16) is true. If $\beta^*(e_1) = 0$, and if $\alpha^*(e_R) > 0$, then $\partial^* f_i(e_R) = 0$, and (10.16) holds for $e = e_R$.

10.17. Lemma. *Keep the same notations and assumptions as in Lemma* 10.14. *If* $\beta^*(e_1) = 0$, *then* $v_i = 0$.

Proof. If e is an edge such that $v_i(e) \neq 0$, then because $v_i \in V$, we have also $\alpha^*(e) > 0$. Lemma 10.14 now implies $v_i \geqq 0$, i.e., $v_i \in \overline{P_h^1}$. Recall the assumption (3) of the first paragraph of this section: α_i is an extremal of the cone $\overline{P_h^1}$. Since $\alpha_i - v_i \in \overline{P_h^1}$ and $v_i \in \overline{P_h^1}$, it follows that there exists $t \geqq 0$ such that $v_i = t\alpha_i$. Since $i \leqq q$, and since A is by assumption a linearly independent set, it cannot be that $\alpha_i \in V$. Therefore, $t = 0$, and $v_i = 0$. The lemma is proved.

If $\beta^*(e_1) = 0$, Lemma 10.17 reduces the study of the equation (10.15) to the equation

$$\alpha_i = w_i + \partial^* f_i. \tag{10.18}$$

Lemmas 10.6 and 10.9 imply the relation

$$\alpha_i(U) = 0 \qquad (\beta^*(U) = 0). \tag{10.19}$$

In preparation for the next lemma, define $L(\beta^*)$ to be the union of the closures, $\bar{U}$, of all components $U \subseteq \Lambda^c$ such that $\beta^*(U) = 0$. Similarly, define $L(\alpha^*)$. We observe that (10.5) implies $L(\alpha^*) \cup L(\beta^*) = X$.

10.20. Lemma. *If the hypothesis of Lemma* 10.14 *is true for each* i, $1 \leqq i \leqq q$, *then* $\beta^*(e_1) > 0$.

Proof. If $\beta^*(e_1) = 0$, then Lemma 10.17 and the discussion which precedes this lemma imply (10.19) is true for $1 \leqq i \leqq q$. Let e be a horizontal edge in $L(\beta^*)$. By (10.19) and the definition of $L(\beta^*)$ it must be that $\alpha_i(e) = 0$, $1 \leqq i \leqq q$. Assumption (2) of the first paragraph of this section now implies $\alpha^*(e) > 0$. Now (10.5) implies $\beta^*(U) = 0$ for every component of Λ^c which has e on its boundary. Therefore, $e \not\subseteq \partial L(\beta^*)$. Since e is an arbitrary horizontal edge in the closed set $L(\beta^*)$, we have $\partial L(\beta^*) \subseteq \mathscr{C}_1^v$. Lemma 4.8 implies $\partial L(\beta^*) = 0$, and therefore $L(\beta^*) = X$ or $L(\beta^*) = \varnothing$. If $L(\beta^*) = X$, then $\beta^* = 0$, contradicting the assumption $p \geqq 1$. If $L(\beta^*) = \varnothing$, then $L(\alpha^*) = X$, and $\alpha^* = 0$. This contradicts the assumption $q < N$. The lemma is proved.

The lemma which follows is the first form of our basic lemma.

10.21. Lemma. *Let notations and assumptions be as in the first two paragraphs of this section. If* $N = \dim \mathscr{X}_h^1$, *and if* $\beta^*(e_1) = 0$, *then* $p < q$.

Proof. Let $\bar{V}$ and $\bar{W}$ be the images of V and W in $H^1_{\mathbf{R}}(X, S)$ under the canonical projection. Proposition 4.16 implies $\dim \bar{V} = \dim V = N - q$ and $\dim \bar{W} = \dim W = p$. Since $\beta^*(e_1) = 0$, Lemma 10.11 implies $\bar{V}$ and $\bar{W}$ are linearly independent in $H^1_{\mathbf{R}}$. As $N = \dim H^1_{\mathbf{R}}(X, S)$ by assumption (and Proposition 4.16), it follows $p + (N - q) \leqq N$ or $p \leqq q$. If $p = q$, then $\bar{V} + \bar{W} = H^1_{\mathbf{R}}(X, S)$, and it follows the hypothesis of Lemma 10.14 is true for $1 \leqq i \leqq q$. But $\beta^*(e_1) = 0$, and we have reached a contradiction to Lemma 10.20. The lemma is proved.

If (10.10) is not true, the conclusion of Lemma 10.21 may also not be true. The lemmas which follow will lead to a second form of the basic lemma. Recall from Lemma 10.12 that $V \cap (W + \partial^* \mathscr{C}_0(S)^\perp) \subseteq \mathbf{R}e_R$.

10.22. Lemma. *If $V \cap (W + \partial^* \mathscr{C}_0(S)^\perp) = \mathbf{R}e_R$, then*
(a) *e_R is the left-hand edge on the top of U_1, and*
(b) *if $e \neq e_R$ is a horizontal edge on ∂U_1, then $\alpha^*(e) = 0$.*

Proof. The hypothesis implies there exist $\beta \in W, f \in \mathscr{C}_0(S)^\perp$ such that $e_R = \beta + \partial^* f$. Since $\alpha^*(e_R) > 0$, the proof of Lemma 10.6 implies $f(x') = 0$, where $\partial e_R = y_1 - x'$. Because $\alpha^*(U_1) \geqq \alpha^*(e_R) > 0$, (10.5) implies $\beta^*(U_1) = 0$, and therefore $f(y_1) = f(x_1)$. Since $\partial^* f(e_R) = 1, f(y_1) = f(x_1) = 1 \neq 0$. Choose the edge e on top of U_1 which satisfies both
(i) $\alpha^*(e) > 0$ and
(ii) the right endpoint, x'', of e is closest to x_1, subject to (i).
If $e \neq e_R$, Lemma 10.6 implies $f(x'') = 0$, and (i)–(ii) imply $f(x_1) = f(x'')$. This is impossible, and therefore $e = e_R$. We claim x', the left endpoint of e_R, is interior to the base of U_1. For otherwise e_R is the entire base of U_1, and consideration of $\Sigma_{i=1}^N \alpha_i \in P^1_h$ implies e_R is the entire top of U_1. But then U_1 is a cylinder of closed leaves for $\mathscr{F}_v(\xi)$. This contradicts the assumption $\xi \in \Omega_0^+(S)$ is in general position. Finally, since x' is interior to the base of U_1, x' is the terminus of an outgoing vertical end. This implies x' is the left-hand endpoint of the top of U_1, and the lemma is proved.

10.23. Lemma. *Suppose $V \cap (W + \partial^* \mathscr{C}_0(S)^\perp) = \mathbf{R}e_R$. There exist i_0, $q < i_0 \leqq N$, and $t > 0$ such that $\alpha_{i_0} = te_R$. Moreover, if $1 \leqq i \leqq N$ and $i \neq i_0$, then $\alpha_i(e_R) = 0$.*

Proof. Let $1 \leqq i_0 \leqq N$ be such that $\alpha_{i_0}(e_R) > 0$. The hypothesis of the lemma implies $e_R \in \overline{P^1_h}$, and the choice of i_0 implies there exists $\varepsilon > 0$ such that $\alpha_{i_0} - \varepsilon e_R \in \overline{P^1_h}$. Arguing as in the proof of Lemma 10.17, there exists $\delta \geqq 0$ such that $\varepsilon e_R = \delta \alpha_{i_0}$. This time, $\varepsilon > 0$ implies $\delta > 0$, and therefore $\alpha_{i_0} = (\varepsilon/\delta)e_R$. The facts $i_0 > q$ and i_0 is unique follow from $\alpha^*(e_R) > 0$ and the assumed linear independence of $A = \{\alpha_1, \ldots, \alpha_N\}$.

In the event that $V \cap (W + \partial^* \mathscr{C}_0(S)^\perp) = \mathbf{R}e_R$, Lemma 10.23 implies there is no loss of generality in assuming $\alpha_{q+1} = e_R$ and $\alpha_i(e_R) = 0$, $1 \leqq i \leqq N$, $i \neq i_0$. In this situation let V_1 be the linear span of $\alpha_{q+2}, \ldots, \alpha_N$. Replace q by $q + 1$ in the second paragraph of the introduction. Because $V_1 \cap \mathbf{R}e_R = \{0\}$, Lemma 10.12 yields

10.24. Lemma. *Let the assumptions be as in Lemma* 10.23. *Then* $\bar{V}_1$ *and* $\bar{W}$ *are linearly independent subspaces of* $H^1_{\mathbf{R}}(X, S)$.

We continue to assume the hypothesis of Lemma 10.23. It is possible that $q + 1 = N$, but in this case it is automatic that $p \leqq q$. In what follows we suppose $q + 1 < N$, and we use $\hat{\alpha} = \Sigma_{i=q+2}^{N} \alpha_i$ to correspond to α^*. Replace V by V_1 and α^* by $\hat{\alpha}$ in the statement of Lemma 10.14. Because $\hat{\alpha}(e_R) = 0$, the conclusion of Lemma 10.17 is valid, even though necessarily $\beta^*(e_1) > 0$.

Lemma 10.23 implies $p \leqq q + 1$. If $p = q + 1$, then $\bar{V}_1 + \bar{W} = H^1_{\mathbf{R}}(X, S)$. The hypothesis of Lemma 10.14 is therefore true for $1 \leqq i \leqq q$, with V replaced by V_1. Now Lemma 10.6 (but not 10.9) applies as in (10.19), to yield

$$\alpha_i(U) = 0 \qquad (1 \leqq i \leqq q, \beta^*(U) = 0, U \neq U_1). \tag{10.25}$$

Now define $L_0(\beta^*) = \overline{L(\beta^*) - \bar{U}_1}$. (Observe that $\beta^*(U_1) = 0$ because $\alpha^*(U_1) \geqq \alpha^*(e_R) > 0$.) Because e_R is interior to $\bar{U}_1$, $e_R \not\subseteq L_0(\beta^*)$. If e is a horizontal edge which lies in $L_0(\beta^*)$, the argument used in Lemma 10.20 implies $e \not\subseteq \partial L_0(\beta^*)$. It follows as before that $\partial L_0(\beta^*) = 0$, and since $L_0(\beta^*) \neq X$, it must be that $L_0(\beta^*) = \varnothing$. That is, $L(\beta^*) = \bar{U}_1$. As $L(\alpha^*) \cup L(\beta^*) = X$, it is the case that $L(\alpha^*) = \overline{(\bar{U}_1)^c}$. Lemma 10.22 implies $q = N - 1$ (and $V = \mathbf{R}e_R$), contradicting the assumption $q + 1 < N$. We have

10.26. Lemma. (Basic Lemma — Part I). *Let the notations and assumptions be as in the first two paragraphs of this section. Then* $p \leqq q$, *and* $p = q$ *only if* $\beta^*(e_1) > 0$.

11. Basic Lemma — Part II

In this section we shall establish a counterpart to Lemma 10.26 for certain weavings associated to $\Omega^+(S, \tau)$ (see (8.5)). Some preparations are necessary.

11.1. Definition. Let $\xi \in \Omega^+(S, \tau)$ be in general position. A weaving Λ of ξ shall be called *τ-special* if

(a) Λ is a special weaving, and
(b) in the notation of 9.1a, b, $\tau\gamma \cap e_1 = \varnothing$ (i.e., $\tau\gamma$ contains no point interior to e_1).

11.2. Lemma. *Let* $\xi = \Omega^+(S, \tau)$. *If* ξ *is in general position, there exists a* τ*-special weaving of* ξ.

Proof. Let s_0, γ_0, s_1, δ_0, and x_1 be as in the proof of Proposition 9.4. Define $x_1^* \in \tau\delta_0 \cap \gamma_0$ to be the last point of the intersection before $\tau\delta_0$ reaches τs_1. Let a_1 be the arc length along δ_0 from x_1 to s_1, and let a_2 be the arc length along $\tau\delta_0$ from x_1^* to τs_1. If $a_1 \leqq a_2$, continue with the proof of Proposition 9.4. If $a_2 < a_1$, replace δ_0 by $\tau\delta_0$, x_1 by x_1^*, and reletter. Again continue with the proof of Proposition 9.4. Since $\tau\gamma \cap e_1 \neq \varnothing$ if and only if $\gamma \cap \tau e_1 \neq \varnothing$, the construction above implies $\tau\gamma \cap e_1 = \varnothing$. The lemma is proved.

If Λ is a τ-special weaving, define Λ_τ by

$$\Lambda_\tau = \gamma \cup \tau\gamma \cup (\Lambda \cap \tau\Lambda). \tag{11.3}$$

We prove

11.4. Lemma. *Let Λ be a τ-special weaving, and let Λ_τ be defined by* (11.3). *Λ_τ satisfies*

(i) *Λ_τ is a weaving,*
(ii) $\tau\Lambda_\tau = \Lambda_\tau$,
(iii) *Λ_τ contains e_1 and τe_1 as edges.*

Proof. Since $\tau S = S$, Λ_τ is a finite union of ends. Fix $s \in S$, and let δ be a vertical separatrix at s. Because Λ is (τ-) special, Λ and $\tau\Lambda$ contain ends, e' and e'' respectively, of δ, and $\Lambda_\tau \cap \delta = e' \cap e''$. If x is the terminus of $e' \cap e''$, then x is the terminus of one of e' or e'', and therefore x is interior to one of γ or $\tau\gamma$. Since s and δ are arbitrary, Λ_τ satisfies the weaving property at each vertical terminus. Since Λ_τ contains an end of *every* vertical separatrix, Λ_τ satisfies the weaving property at every $s \in S$. Finally let $e'' \subseteq \tau\Lambda$ be the maximal vertical end at s_1 such that $e'' \cap (e_{y_1} + e_1)$ is nontrivial. Because Λ is τ-special, e'' contains e_1. Also, because $\tau\gamma \cap \gamma = \varnothing$ (τ interchanges notions of "incoming" and "outgoing"), e'' contains e_1 properly. Therefore, x_1 is interior to the vertical end $e'' \cap (e_{y_1} + e_1) \subseteq \Lambda_\tau$. The weaving property is established for x_1 and, by the symmetry of (11.3), for τx_1. Property (i) is proved. Property (ii) is obvious from (11.3), and Property (iii) has already been noted. The lemma is proved.

In all that follows Λ is a fixed τ-special weaving associated to an element $\xi \in \Omega^+(S, \tau)$ which is in general position. Λ_τ is defined by (11.3). Denote by U_1^* the component of Λ_τ^c which has x_1 in its upper-right hand corner. In the notation used for the proof of the previous lemma, the right-hand side of U_1^* is an edge, $e^* = e'' \cap e_{y_1}$, which contains no element of S. e^* and $\tau e^* \subseteq \partial\tau U_1^*$ are the only sides of components of Λ_τ^c which do not contain elements of S. We have

11.5. Lemma. *Let notations be as above, and suppose U is a component of Λ_τ^c. If $U \neq U_1^*$, τU_1^*, both vertical sides of U contain an element of S. If $U = U_1^*$,* *the left side of U contains an element of S; if $U = \tau U_1^*$, the right side of U contains an element of S.*

Proof. τe^* is the left side of τU_1^*. If $\tau U_1^* = U_1^*$, then U_1^* contains a fixed point of τ (at its center with respect to a ξ-compatible chart), contradicting the assumption $\mathrm{Fix}(\tau) \subseteq S$. Therefore, $\tau U_1^* \neq U_1^*$, and τe^* is not the left side of U_1^*. The lemma now follows from the preceding discussion.

Let $N = \dim \mathscr{X}^1_{h,\tau} = \mathscr{X}^1_{h,\tau}(\Lambda_\tau)$. We recall that N is the dimension of the -1 eigenspaces for τ^* on $H^1_{\mathbf{R}}(X, S)$ and for τ on $\mathscr{X}^1_m(\Lambda_\tau)$, $m = h$ or v.

Let there now be given linearly independent sets (bases), $A = \{\alpha_1, \dots, \alpha_N\} \subseteq \mathscr{X}^1_{h,\tau}$, and $B = \{\beta_1, \dots, \beta_N\} \subseteq \mathscr{X}^1_{v,\tau}$. We assume in addition the properties

(1) $A \subseteq \overline{P^1_{h,t}}$ and $B \subseteq \overline{P^1_{v,\tau}}$, and

(2) for each i, $1 \leqq i \leqq N$, α_i is an extremal of $\overline{P^1_{h,\tau}}$.

With notations as above, assume integers p and q are such that $1 \leqq p, q < N$ and

$$[\alpha_i, \beta_j] = 0 \qquad (q < i \leqq N,\ 1 \leqq j \leqq p). \tag{11.6}$$

V and W denote the linear spans of $\{\alpha_{q+1}, \dots, \alpha_N\}$ and $\{\beta_1, \dots, \beta_p\}$, respectively. Also, define $\alpha^* = \Sigma_{i=q+1}^N \alpha_i$ and $\beta^* = \Sigma_{j=1}^p \beta_j$.

As in Lemma 10.3 the assumption (1) above implies (11.6) is equivalent to each of the conditions

$$[\alpha^*, \beta^*] = 0 \tag{11.7}$$

and

$$\alpha^*(U)\beta^*(U) = 0 \qquad (\text{all components } U \subseteq \Lambda_\tau^c). \tag{11.8}$$

Lemma 10.6 is concerned with the equation

$$\alpha + \beta = \partial^* f \tag{11.9}$$

in which $\alpha \in \mathscr{X}^1_h$, $\beta \in W$, and $f \in \mathscr{C}_0(S)^\perp$. If $U \neq U_1^*, \tau U_1^*$, the conclusion (10.8) has the same proof, word for word, and we have

11.10. Lemma. *With notations as above*

$$\alpha(U) = 0 \qquad (U \neq U_1^*, \tau U_1^*, \beta^*(U) = 0). \tag{11.11}$$

Moreover, f vanishes at the four corners of U. If $U = U_1^*$ (resp. $U = \tau U_1^*$), f vanishes at the left-hand (resp. right-hand) corners of U, and f is constant along the right (resp. left) side of U. If $\beta^*(e_1) = 0$, (11.11) holds also for $U = U_1^*$, τU_1^*, assuming $\beta^*(U_1^*) = 0$.

The next result has the same proof, word for word, as Lemma 10.11:

11.12. Lemma. *If* $\beta^*(e_1) = 0$, *then* V *and* W *are linearly independent modulo* $\partial^* \mathscr{C}_0(S)^\perp$.

Denote by e_R^* *the edge on the bottom of* U_1^* *whose right-hand endpoint is the lower right-hand corner of* U_1^*. *We have by analogy with Lemma* 10.14

11.13. Lemma. *Suppose* $1 \leqq i \leqq q$ *is such that there exists a solution,* $(v_i, w_i, f_i) \in V \times W \times \partial^* \mathscr{C}_0(S)^\perp$, *to*

$$\alpha_i = v_i + w_i + \partial^* f_i. \tag{11.14}$$

Then also

$$\alpha_i(e) = v_i(e) \qquad (e \neq e_R^*, \tau e_R^*, \alpha^*(e) > 0). \tag{11.15}$$

If $\beta^*(e_1) = 0$, *and if* $\alpha^*(e_R^*) > 0$, *then* (11.15) *holds also for* $e = e_R$.

Proof. The proof uses Lemma 11.10 exactly the same way the proof of Lemma 10.14 uses Lemma 10.6 and (the proof of) Lemma 10.11.

If we assume $\beta^*(e_1) = 0$ (which is equivalent to $\beta^*(\tau e_1) = 0$), the argument used in Lemma 10.17 implies $v_i = 0$ in (11.14). (From (11.15) and the fact $v_i \in V$ it follows that $v_i \in V \cap \overline{P_h^1} \subseteq \overline{P_{h,\tau}^1}$, and then $\alpha_i = v_i + (\alpha_i - v_i)$ with v_i, $\alpha_i - v_i \in \overline{P_{h,\tau}^1}$.) Just as in (10.19) the conclusion is

11.16. Lemma. *Let notations and assumptions be as in Lemma* 11.13. *If* $\beta^*(e_1) = 0$, *then*

$$\alpha_i(U) = 0 \qquad (\beta^*(U) = 0). \tag{11.17}$$

Because $A = \{\alpha_i, \ldots, \alpha_N\} \subseteq \overline{P_{h,\tau}^1}$ is a basis for $\mathscr{L}_{h,\tau}^1$, it must be that $\Sigma_{i+1}^N \alpha_i = \alpha_0 \in P_{h,\tau}^1$, i.e., $\alpha_0(e) > 0$ for every horizontal edge, $e \subseteq \Lambda_\tau$. The proof of Lemma 10.20 may be repeated, essentially word-for-word, and from this we obtain

11.18. Lemma. *Let notations and assumptions be as in* (11.6) *and the paragraphs which precede* (11.6). *If* $\beta^*(e_1) = 0$, *then* $p < q$.

Proof. Denote by $\bar{V}$ and $\bar{W}$ the images of V and W in $H_{\mathbf{R}}^1(X, S)$ under the canonical projection. $\bar{V}$ and $\bar{W}$ are linearly independent subspaces of the (N-dimensional) -1 eigenspace for τ^*, and therefore

$$\begin{aligned}(N - q) + p &= \dim V + \dim W \\ &= \dim \bar{V} + \dim \bar{W} \\ &= \leqq N.\end{aligned}$$

Thus, $p \leqq q$, and if equality holds, (11.14) admits a solution for $1 \leqq i \leqq q$. The analog of Lemma 10.20, mentioned above, implies $\beta^*(e_1) > 0$, contrary to hypothesis. therefore, $p < q$, and the lemma is proved.

Suppose now $\alpha \in V$, $\beta \in W$, $f \in \mathscr{C}_0(S)^\perp$ are such that (11.9) holds, $\alpha + \beta = \partial^* f$. Since $\alpha^*(e) = 0$ implies $\alpha(e) = 0$, Lemma 11.10 applies to tell us $\alpha = c_1 e_R^* + c_2 \tau e_R^*$. In order that $\tau\alpha = -\alpha$, it is necessary that $c_1 = -c_2$, $\alpha = t(e_R^* - \tau e_R^*)$ for some $t \in \mathbf{R}$.

It is possible $e_R^* - \tau e_R^* \notin \mathscr{X}_h^1$. Indeed, because $\tau e_R^* \not\subseteq \partial U_1^*$ (else $\tau U_1^* = U_1^*$, see the proof of Lemma 11.5), $\partial^*(e_R^* - \tau e_R^*) = 0$ if and only if e_R^* lies on the bottom and top of U_1^*. In particular, it is necessary that $e_{y_1} + e_1 \subseteq e''$ in the notation of the proof of Lemma 11.4. Observe this condition is tantamount to the requirement

$$U_1^* = U_1, \qquad \tau U_1^* = \tau U_1. \tag{11.19}$$

11.20. *Let the notations and assumptions be as in Lemma* 11.18, *except that possibly* $\beta^*(e_1) > 0$. *If* (11.19) *is false, then* $p \leqq q$.

In what follows we assume (11.19) is true, and we also assume $e_R^*(= e_R)$ lies on both the top and bottom of $U_1^* = U_1$. Define $\alpha_0 = e_R^* - \tau e_R^*$. If $1 \leqq i \leqq N$, and if $\alpha_i(e_R^*) > 0$, then there exists $t > 0$ such that $\alpha_i - t\alpha_0 \in \overline{P_{h,\tau}^1}$. As $\alpha_0 \in \overline{P_{h,\tau}^1}$, it follows that $\alpha_0 = t'\alpha_i$ for some $t' > 0$, because α_i is extremal. Because A is by assumption a linearly independent set, the value of i above is unique. We have

11.21. Lemma. *If* $V \cap (W + \partial^* \mathscr{C}_0(S)^\perp) \neq \{0\}$, *there exist* i_0, $q < i_0 \leqq N$ *and* $t > 0$ *such that* $\alpha_{i_0} = t\alpha_0$. *Moreover*, $\alpha_i(e_R^*) = 0$ *for* $i \neq i_0$.

If the hypothesis of Lemma 11.20 is in force, it is no loss of generality to suppose $\alpha_{q+1} = \alpha_0$. Of course,

$$V \cap (W + \partial^* \mathscr{C}_0(S)^\perp) = \mathbf{R}\alpha_0. \tag{11.22}$$

Define V_1 to be the linear span of $\{\alpha_{q+2}, \ldots, \alpha_N\}$. The discussion above implies $\bar{V}_1$ and $\bar{W}$ are linearly independent subspaces of the -1 eigenspace of τ^*, and it follows that $p \leqq q + 1$. Should $p = q + 1$, then (11.14) admits a solution for $1 \leqq i \leqq q$. Because $\hat{\alpha}(e_R^*) = 0$, where $\hat{\alpha} = \Sigma_{i=q+2}^N \alpha_i$, (11.15) holds for all edges e such that $\hat{\alpha}(e) > 0$, and then $v_i = 0$, $1 \leqq i \leqq q$. Next, we find $\alpha_i(U) = 0$ for $1 \leqq i \leqq q$ and all components $U \subseteq \Lambda_\tau^c$ such that $U \neq U_1^*$, τU_1^* and $\beta^*(U) = 0$. The argument preceding Lemma 10.26 implies $q = N - 1$, $\alpha_N = \alpha_0$, and $p \leqq N - 1$. We have

11.23. Lemma. (Basic Lemma — Part II). *Let the notations and assumptions be as in* (11.6) *and the paragraph which precedes* (11.6). *Then* $p \leqq q$, *and* $p = q$ *only if* $\beta^*(e_1) > 0$.

11.24. Remark. If Λ is a τ-special weaving, it can be shown there is a version of Lemma 11.23 for the weaving $\Lambda \cup \tau\Lambda$. We believe there are versions of the Basic Lemma(s) for a large class of weavings, perhaps all weavings.

12. Operations on weavings

Let $\xi_0 \in \Omega_0^+(S)$, and let Λ be a weaving of ξ_0. If $\lambda \in P^1(\Lambda)$ is in general position, it is possible to construct from the pair $(\Lambda, \xi(\lambda))$ an infinite set of weavings of $\xi(\lambda)$. The construction will be described in this section. In Section 13 the construction will be used to define finite covers of moduli spaces by "nice" open sets.

12.1. Lemma. *Assume $\xi_0 \in \Omega_0^+(S)$ is in general position. If Λ is a weaving of ξ_0, Λ contains a horizontal edge e such that $\partial e = \pm(x-y)$, where x is a horizontal terminus and $y \notin S$.*

Proof. Λ must contain a vertical end δ. If y_1 is the terminus of δ, y_1 is interior to a horizontal end $\gamma \subseteq \Lambda$. Let x be the terminus of γ. Between y_1 and x on γ there is a vertex y, $y \neq x$, which is nearest to x. Let e be the edge such that $\partial e = \pm(x-y)$. By construction $y \notin S$, and the lemma is proved.

In what follows there is given a weaving Λ of $\xi_0 \in \Omega_0^+(S)$, ξ_0 in general position. Let e be an edge which satisfies the conclusion of Lemma 12.1. We construct a new weaving Λ_1 according to the dichotomy (A) $\Lambda_1 - \{e\}$ is a weaving or (B) $\Lambda_1 - \{e\}$ is not a weaving.

Procedure A. ($\Lambda_1 - \{e\}$ is a weaving.) In this case we remark that if Λ has v horizontal or vertical edges, then Λ_1 has $v-1$ horizontal or vertical edges (recall Lemma 4.5). If $m = h$ or v, and if $\rho: \mathscr{L}_m^1(\Lambda) \to \mathscr{L}_m^1(\Lambda_1)$ is the canonical map (Proposition 4.16), then

$$\rho P^1(\Lambda) \subseteq P^1(\Lambda_1). \tag{12.2}$$

Indeed, if $\lambda = (\alpha, \beta) \in P^1(\Lambda)$, $\rho\alpha$ is the restriction of α to the horizontal edges other than e, while $\rho\beta(\tilde{e}) = \beta(\tilde{e})$ for each vertical edge $\tilde{e}$ which is also an edge of Λ, and $\rho\beta(\tilde{e}) = \beta(e_1) + \beta(e_2)$ for the unique edge $\tilde{e} = e_1 \cup e_2$, e_1, e_2 distinct edges of Λ.

Procedure B. ($\Lambda - \{e\}$ is not a weaving.) In this case the construction of Λ_1 will depend upon a given $\lambda \in P^1(\Lambda)$. In preparation of the construction let U^+ and U^- be the components of Λ^c which have e on their bottom and top sides, respectively. Because $\partial e = \pm(x-y)$, where $y \notin S$ and x is a horizontal terminus, one of $U^\pm$ has e for an entire side. We claim this implies $U^+ \neq U^-$. For if $U^+ = U^-$, e is (say) the bottom of U^+ and a portion of the top of U^+. Consideration of elements $\alpha \in P_h^1(\Lambda)$ implies e is the entire top of U^+. It follows then that $\mathscr{F}_v(\xi_0)$ has closed leaves, contrary to the assumption ξ_0 is in general position. Therefore $U^+ \neq U^-$.

We now describe the construction. Let Λ, e be as above, and let $\lambda \in P^1(\Lambda)$. If $\partial e = \pm(x-y)$, also as above, let δ be the vertical end which contains y. Since $\Lambda - \{e\}$ is not a weaving, y must be the terminus of δ. Assume for concreteness δ is an incoming end so that the left hand side of U^+ is contained in δ or contains δ. Consideration of $R(U^-, \lambda)$ (Section 5) shows that δ can be extended as a $\xi(\lambda)$-geodesic until it reaches Λ again for the first time. Moreover, if δ' is the extended end, the terminus of δ' lies in $\Lambda - \{e\}$ (e is not on the bottom of U^-). Now define $\Lambda_1 = (\Lambda - \{e\}) \cup \delta'$. Λ_1 is a weaving of $\xi(\lambda)$, provided the terminus of δ' is not already a (vertical) terminus of Λ or an element of S. For example, if $\lambda \in P^1(\Lambda)$ is chosen so that $\xi(\lambda)$ is in general position, Λ_1 is a weaving in the sense of Section 3. It should be emphasized that Λ_1 can be a weaving even when $\xi(\lambda)$ is not in general position. Note that Λ_1 depends upon the choice of λ.

In both cases A and B above there are canonical isomorphisms, $\rho: \mathscr{X}^1_m(\Lambda) \to \mathscr{X}^1_m(\Lambda_1)$, for $m = h$ or v. However, (12.2) cannot be expected to hold always in the case of B.

12.3. Remark. Let $\nu = \nu(\Lambda)$ be the number of horizontal or vertical edges of Λ. We have

$$\nu(\Lambda_1) = \begin{cases} \nu(\Lambda) - 1 & \text{Procedure A,} \\ = & \\ \nu(\Lambda) & \text{Procedure B.} \end{cases} \tag{12.4}$$

The dichotomy (12.4) depends upon the choice of e with $\partial e = \pm (x - y)$, $y \notin S$. If $y \in S$, it is possible to construct Λ_1; however, it may be necessary to add *two* edges to effect the weaving property, and (12.4) would not be true.

12.5. Notation. When Procedure B is used, e' will denote the new vertical edge which is added, that is, $\delta' = \delta \cup e'$.

12.6. Lemma. *With notations as above suppose* $\lambda_1 \in P^1(\Lambda)$ *is such that* $\rho\lambda_1 \in P^1(\Lambda_1)$. *There exists* $\phi \in H_0(X, S)$ *such that*

$$\xi(\lambda_1) = \xi(\rho\lambda_1)\phi. \tag{12.7}$$

Proof. We shall give the proof only when Procedure B is followed in the construction of Λ_1. It is necessary first to recall the construction of the map(s) $\xi(\cdot)$ in (12.7). Let U be a component of Λ^c, and let $\lambda_0 = \hat{\xi}_0$, $\lambda \in P^1(\Lambda)$. Then the $\xi(\lambda)$-chart function on U is

$$F^U_\lambda = \phi^U_{\lambda_0\lambda} \circ f^U \tag{12.8}$$

where (U, f^U) is an ξ_0-chart (see (5.4)). The map $\phi^U_{\lambda_0\lambda}$ is canonically determined by objects $r(U)$ and $r(U, \lambda)$ ($r(U) = r(U, \lambda_0)$), and the construction implies for all $\lambda_1 \in P^1(\Lambda)$

$$\phi^U_{\lambda_0\lambda_1} = \phi^U_{\lambda\lambda_1} \circ \phi^U_{\lambda_0\lambda}. \tag{12.9}$$

Here $\phi^U_{\lambda\lambda_1}$ is canonically determined by $r(U, \lambda)$ and $r(U, \lambda_1)$.

In what follows we set $U = U^- \cup e \cup U^+$. While U is not a component of Λ^c, the discussion in Procedure B implies there is an ξ_0-compatible chart, $(U, f^U) = (U, f)$, such that $fU^- = R(U^-)$ and $fU^+ = R(U^+) + c$ for some constant $c \neq 0$.

The definitions of $\phi_{\lambda\lambda_1}^{U\pm}$, etc., can be modified to determine continuous maps which satisfy (12.9) on fU.

Now f_λ^U, $U = U^+ \cup e \cup U^-$, is defined by (12.8), and F_λ^U is a chart function for both $\xi(\lambda)$ and $\xi(\rho\lambda)$. Of course, (12.9) implies

$$F_{\lambda_1}^U = \phi_{\lambda\lambda_1}^U \circ F_\lambda^U. \tag{12.10}$$

Let V^+ and V^- be the components of Λ_1^c which have e' on their left and right sides, respectively.

F_λ^U is a chart function for $\xi(\rho\lambda)$ on U because Λ_1 is defined in terms of $\xi(\lambda)$. Let $\phi_{\rho\lambda\rho\lambda_1}^U$ be defined, as above, on $F_\lambda^U U = F_\lambda^U(V^+ \cup e' \cup V^-)$, so that

$$F_{\rho\lambda_1}^U = \phi_{\rho\lambda\rho\lambda_1}^U \circ F_{\rho\lambda}^U \tag{12.11}$$

is a chart function on U for $\xi(\rho\lambda_1)$. Now we define ϕ on X by

$$\phi(x) = \begin{cases} x & (x \notin \bar{U}), \\ (F_{\rho\lambda_1}^U)^{-1} \circ F_{\lambda_1}^U(x) & (x \in \bar{U}). \end{cases} \tag{12.12}$$

It is clear that $\phi \in H_0(X, S)$. Because $F_{\lambda_1}^V = F_{\rho\lambda_1}^V$ for every component $V \subseteq \Lambda_1^c$ such that $V \cap U = \varnothing$, (12.12) implies $\xi(\lambda) = \xi(\rho\lambda_1)\phi$. The lemma is proved.

If $\lambda \in P^1(\Lambda)$, $l = (\alpha, \beta)$, introduce norms

$$\begin{aligned} |\alpha| &= \sum_{e \subseteq \Lambda} \alpha(e), \\ |\beta| &= \sum_{e \subseteq \Lambda} \beta(e). \end{aligned} \tag{12.13}$$

Observe that $|\alpha|$ (resp. $|\beta|$) is the sum of the $\xi(\lambda)$-lengths of the horizontal (resp. vertical) ends which are contained in Λ.

If Procedure B is followed in the construction of Λ_1, and if $\lambda \in P^1(\Lambda)$, $\rho\lambda \in P^1(\Lambda_1)$, then

$$\begin{aligned} |\rho\alpha| &= |\alpha| - \alpha(e), \\ |\rho\beta| &= |\beta| + \rho\beta(e'). \end{aligned} \tag{12.14}$$

It is clear that $\rho\beta(e')$ is at least as large as the smallest value of $\beta(e'')$, $e'' \subseteq \Lambda$ a vertical edge, and so (12.14) implies

$$\begin{aligned} |\rho\beta| &\geqq |\beta| + \min_{\substack{e'' \subseteq \Lambda \\ e'' \text{ vertical}}} \beta(e''), \\ \rho\beta(e') &\geqq \min_{\substack{e'' \subseteq \Lambda \\ e'' \text{ vertical}}} \beta(e''). \end{aligned} \tag{12.15}$$

Assume now that $\lambda \in P^1(\Lambda)$ is in general position. Select e by Lemma 12.1, and let Λ_1 and $\lambda_1 = \rho\lambda \in P^1(\Lambda_1)$ be constructed by the applicable procedure, A or B. Select $e_1 \subseteq \Lambda_1$ by Lemma 12.1, and repeat the above. There results a sequence $(\Lambda_n, \lambda_n, e_n)$, $n \geqq 1$, such that $J\lambda = J\lambda_1 = J\lambda_2 \cdots$ (e.g., if $\lambda_1 = \lambda$ in (12.12), then $\phi = \mathrm{Id}$). Also, $\lambda_{n+1} = \rho\lambda_n$; and by (12.4) $\nu(\lambda_n)$ is a nonincreasing sequence of integers. Moreover, if n is sufficiently large, (12.4) implies Λ_{n+1} arises from $(\Lambda_n, \lambda_n, e_n)$ by Procedure B. By (12.15)

$$\lim_{n\to\infty} |\beta_n| = +\infty. \tag{12.16}$$

Because $\xi(\lambda)$ is in general position, each separatrix of $\mathscr{F}_v\xi(\lambda)$ is dense in X. It follows from this that

$$\lim_{n\to\infty} \left(\max_{e \subseteq \Lambda_n} \alpha_n(e) \right) = 0. \tag{12.17}$$

Again because $\nu(\Lambda_n)$ is bounded, (12.17) implies

$$\lim_{n\to\infty} |\alpha_n| = 0. \tag{12.18}$$

It is possible to apply the procedure above with the roles of "horizontal" and "vertical" reversed. Indeed, when Procedure B is used to produce Λ_1 and the vertical edge e' from Λ and the horizontal edge e, for given $\lambda \in P^1(\Lambda)$, the reverse procedure produces Λ, e, and $\lambda = \rho^{-1}\rho\lambda$ from Λ_1, e', and $\rho\lambda$.

12.19. Example. Let Λ be a special weaving, Definition 9.1 being amended as in Remark 9.5. In this case there is a unique choice for e, $e = e_T$ in the notation of 9.2(ii). If $\lambda \in P^1(\Lambda)$, only Procedure B applies to (Λ, λ, e_T), and since e_T is unique, we define

$$\begin{aligned} \mathscr{U}(\Lambda, \lambda) &= (\Lambda_1, \rho\lambda) \\ &= (\Lambda_1, \lambda_1). \end{aligned} \tag{12.20}$$

It has been remarked in the description of Procedure B that Λ_1 can fail to be a weaving. In the present setting Λ_1 fails to be a weaving if and only if e_T is the base of U^+ and the top of U^-.

To invert $\mathscr{U}$ we observe that while Λ has many vertical ends, the edge which satisfies the conclusion of the ("vertical version" of) Lemma 12.1 is unique; it is the edge e_0 defined in 9.2(iii). Apply Procedure B to Λ, λ and e_0, with "horizontal" and "vertical" interchanged. If the new horizontal end, the extension of γ as an end for $\mathscr{F}_h(\xi(\lambda))$ to its next encounter with Λ, is not a saddle connection, there will result a weaving Λ_{-1} and $\lambda_{-1} \in P^1(\Lambda_{-1})$ such that

$$\mathscr{U}(\Lambda_{-1}, \lambda_{-1}) = (\Lambda, \lambda). \tag{12.21}$$

We set $(\Lambda_{-1}, \lambda_{-1}) = \mathscr{U}^{-1}(\Lambda, \lambda)$.

12.22. Proposition. *If Λ is a special weaving, and if $\lambda \in P^1(\Lambda)$ is such that $\xi(\lambda)$ is in general position, then $\mathscr{U}^n(\Lambda, \lambda) = (\Lambda^{(n)}, \lambda^{(n)})$ is defined for all $n \in Z$. If $e_T^{(n)}$ is the unique terminal horizontal edge of Λ_n, there exists a unique value of n such that*

$$(12.23) \qquad 1 \leqq |\alpha_n| < 1 + \alpha_n(e_T^{(n)}).$$

Proof. Apply (12.16) with horizontal and vertical interchanged to conclude

$$\lim_{n \to \infty} |\alpha_{-n}| = +\infty.$$

Combine this with (12.18) to conclude there is a value of n such that

$$|\alpha_{n+1}| < 1 \leqq |\alpha_n|.$$

Now (12.14) implies $|\alpha_{n+1}| = |\alpha_n| - \alpha_n(e_T^{(n)})$, and the proposition follows.

12.24. Example. Let notations be as in Section 11. If $\xi_0 \in \Omega^+(S, \tau)$ is in general position, and if Λ is a τ-special weaving of ξ_0, define Λ_τ as in (11.3). Λ_τ admits two terminal horizontal edges. One, which we denote by e_T^τ, lies on γ, while the other, τe_T^τ, lies on $\tau\gamma$. Let $U_\tau^\pm \subseteq \Lambda_\tau^c$ be associated to e_T^τ, and form $U_\tau = U_\tau^+ \cup e_T^\tau \cup U_\tau^-$. We claim $\tau U_\tau \cap U_\tau = \varnothing$. For otherwise since $\tau U_\tau^+ \neq U_\tau^+$, we have $\tau U_\tau^+ = U_\tau^-$, and e_T^τ is the bottom of U_τ^+ and the top of U_τ^-. This implies ξ_0 has a saddle connection and contradicts the assumption ξ_0 is in general position.

If $\lambda \in P_\lambda^1(\Lambda_\tau)$, apply Procedure B to Λ_τ, λ and e_T^τ, τe_T^τ (simultaneously or one after the other). Call the result $\mathscr{U}_\tau(\Lambda_\tau, \lambda) = (\Lambda^*, \lambda^*)$.

12.25. Lemma. *With notations as above suppose $\xi(\lambda)$* is in general position and Λ^* *is a weaving of $\xi(\lambda)$ and $\xi(\lambda^*)$. There exists a τ-special weaving, Λ^0, of $\xi(\lambda)$ such that $\Lambda^* = (\Lambda^0)_\tau$.*

Proof. Let $\gamma_0 = \gamma \cap \Lambda^*$ (γ as in (11.3)). If δ is a vertical end in Λ^*, the terminus of δ is interior to γ_0 or to $\tau\gamma_0$. In the latter case, extend δ until its terminus is interior to γ. Let Λ^0 be the union of S, γ_0, and the ends just described. Λ^0 is a special weaving, and clearly $\Lambda^* = \gamma_0 \cup \tau\gamma_0 \cup (\Lambda^0 \cap \tau\Lambda^0)$. Let e_1^0 be such that $\partial e_1 = \pm(s_1^0 - x^0)$, where $s_1^0 \in S$ and x^0 is the terminus of γ_0. Since e_1^0 is an edge of Λ^*, it cannot be that $\tau\gamma_0 \cap e_1^0 \neq \varnothing$. That is, Λ^0 is τ special. The lemma is proved.

12.26. Proposition. *Let Λ, Λ_τ be as above, and define $\mathscr{U}(\Lambda, \lambda) = (\Lambda^*, \lambda^*)$. If $\xi(\lambda) \in \Omega^+(S, \tau)$ is in general position, there exists a unique n such that if $\mathscr{U}^n\lambda) = (\alpha_n, \beta_n) \in P_\tau^1(\Lambda_\tau^{(n)})$, then*

(12.27) $$1 \leqq |\alpha_n| < 1 + 2\alpha_n(e^{\tau}_{T,n}).$$

Proof. $e^{\tau}_{T,n}$ is to $\Lambda^{(n)}_{\tau}$ as e^{τ}_T is to Λ_τ. The proof of the lemma is similar to that of Proposition 12.22 and will be omitted.

13. Moduli spaces

The moduli spaces $\mathcal{M}_0^+(S)/\Gamma(X, S)$ and $\mathcal{M}^+(S, \tau)/\Gamma(\tau)$ can be studied with the help of special and τ-special weavings. Let $\mathcal{W}^*(S)$ be the set of special weavings (Definition 9.1), and let $\mathcal{W}^*(S, \tau)$ be the set of weavings Λ_τ such that Λ is τ-special (see (11.3)). Each of the "moduli spaces" $\mathcal{W}^*(S)/H(X, S)$ and $\mathcal{W}^*(S, \tau)/Z(\tau)$ is finite. We denote these moduli spaces by $\mathcal{W}_0^*(S)$ and $\mathcal{W}_0^*(S, \tau)$, respectively.

With notations as above, select a complete set of representatives $\{\Lambda_1, \ldots, \Lambda_N\}$ for $\mathcal{W}_0^*(S)$ (resp. $\mathcal{W}_0^*(S, \tau)$), and let $O_j = (JP^1(\Lambda_j))\Gamma(X, S)$ (resp. $O_j = (JP^1_\tau(\Lambda_j))\Gamma(\tau)$, $1 \leqq j \leqq N$. The open sets $\{O_1, \ldots, O_N\}$ cover the moduli space in question, but for a set of real codimension 2 ([V86], Proposition 2.12). In fact, if we define

$$Q^1(\Lambda_j) = \{\lambda = (\alpha, \beta) \in P^1(\Lambda_j) \mid 1 < |\alpha| < 1 + \alpha(e_T)\}$$

and

$$Q^1_\tau(\Lambda_j) = \{\lambda \in P^1_\tau(\Lambda_j) \mid 1 < |\alpha| < 1 + 2\alpha(e^{\tau}_T)\},$$

as $\Lambda_j \in \mathcal{W}^*(S)$ or $\Lambda_j \in \mathcal{W}^*(S, \tau)$, the sets $\tilde{O}_j = (JQ^1(\Lambda_j))\Gamma(X, S)$ (resp. $\tilde{O}_j = JQ^1_\tau(\Lambda_j)\Gamma(\tau)$), $1 \leqq j \leqq N$, also cover the moduli space(s) modulo a (more complicated) set of real codimension 1. This statement follows from Propositions 12.22 and 12.26.

The analysis in Section 14 will be directed toward proving the constant volume hypersurfaces $\tilde{O}_j \cap V^{-1}1$, V the volume function, have finite volume with respect to the contraction of the measure(s) μ from Theorems 7.17 and (the remark following) Theorem 8.10. The remainder of the present section is devoted to the observation that the moduli spaces $V(\pi)/\Gamma(g, n)$ are not connected in general.

In what follows we suppose $n = 0$ and $\pi = (0, \nu, +1)$. Because $3g - 3 + n > 0$, by assumption, $g > 1$, and the sum $N(\pi) = \Sigma_{l=1}^{\infty} \nu(l)$ is positive. The space $W(\pi) \cong \tilde{V}(\pi)$ from Section 1 is identified with a subspace, $Y(\pi)$, of $\mathcal{M}_0^+(X, S)$, $X = M_g$, $S = S_{N(\pi)}$. Strictly speaking, the canonical map $Y(\pi) \to W(\pi)$ is two-to-one because elements of $Y(\pi)$ are "oriented."

Let $Y_0(\pi) = \mathcal{M}_0^+ \cap Y(\pi)$, where $\mathcal{M}_0^+$ is defined by (7.3). As noted earlier, Proposition 2.12 of [V86] implies $(Y_0(\pi))^c$ has real codimension at least 2 in $Y(\pi)$. As $Y_0(\pi)$ is open, $Y_0(\pi)$ is connected if $Y(\pi)$ is connected. In what follows we will describe nonempty pairwise disjoint open sets $U_1(\pi), \ldots, U_\nu(\pi)$, $\nu = \nu(\pi)$, which exhaust $Y_0(\pi)$. In certain instances we can show that $\nu > 1$, and therefore $Y(\pi)$ is not connected. These open sets will also satisfy $U_j(\pi)\Gamma(X, S) = U_j(\pi)$, all j, and therefore the moduli space $Y(\pi)/\Gamma(X, S)$ is also not connected. In the notation of Section 1 the moduli space $\tilde{V}(\pi)/\Gamma(g, N(\pi))$ is not connected. The constructions will imply the components of $\tilde{V}(\pi)/\Gamma(g, N(\pi))$ are invariant under the (finite) group of the cover $\tilde{V}(\pi)/\Gamma(g, N(\pi)) \to V(\pi)/\Gamma(g, n)$ (Section 1), and therefore *the moduli space* $V(\pi)/\Gamma(g, n)$, $\pi = (n, \nu, \varepsilon)$ *is not, in general connected*. (Recall we are assuming $n = 0$ and $\varepsilon = +1$.)

In what follows $\mathscr{G}_0(m)$, $m > 0$ denotes the set of irreducible permutations $\alpha : k \to \alpha k$, $1 \leqq k \leqq m$. We recall the Rauzy operations ([R79]) $\alpha \to c\alpha$, where $c = a$ or b is defined by

$$a\alpha(j) = \begin{cases} \alpha j, & j \leqq \alpha^{-1}m, \\ \alpha m, & j \leqq \alpha^{-1}m + 1, \\ \alpha(j-1), & \text{other } j, \end{cases} \qquad b\alpha(j) = \begin{cases} \alpha j, & j \leqq \alpha m, \\ \alpha j + 1, & \alpha m \leqq \alpha j < m, \\ \alpha m + 1, & \alpha j = m. \end{cases}$$

Also, define $R\alpha = \sigma_m \alpha \sigma_m$, where $\sigma_m j = m - j + 1$, $1 \leqq j \leqq m$. The operations on a, b, R generate a group of transformations of $\mathscr{G}_m^0$. If $\alpha \in \mathscr{G}_m^0$, we use $O(\alpha)$ to denote the orbit of α under this group. $O(\alpha)$ is an *extended Rauzy class*. (Rauzy classes arise from a, b alone.)

Let $[\xi] \in Y_0(\pi)$ be such that ξ is in general position; by definition the form ω_ξ, $\omega_\xi^2 = \sigma_\xi$ is given along with ξ. Let Λ be a special weaving of ξ with horizontal end γ (Definition 9.1). Identify γ with the interval $[0, |\gamma|_\xi] \subseteq \mathbf{R}^+$, and let the components of Λ^c be represented as rectangles with bases on $[0, |\gamma|_\xi]$. The map which sends the base of one of these rectangles to the subinterval of $[0, |\gamma|_\xi]$ to which the top is identified is an (η, α)-interval exchange for some m, $\alpha \in \mathscr{G}_m^0$ and $\eta \in (\mathbf{R}^+)^m$, $|\eta| \overset{\text{def}}{=} \Sigma\, \eta_j = |\gamma|_\xi$. The procedure "erase e_T" and elongate the vertical end which abuts the left endpoint of e_T corresponds to one of the Rauzy operations $(\eta, \alpha) \to (\eta', c\alpha)$, $c = a$ or b, as extended to "zippered rectangles" in [V82]. (Note: $|\eta'| = |\gamma|_\xi - |e_T|_\xi$.) Proposition 9.1 of [V82] asserts that the Rauzy class of α depends only upon ξ and the separatrix upon which γ lies. On the other hand every $\xi(\lambda)$, $\lambda \in P^1(\Lambda)$, is assigned the same Rauzy class.

If we extend the definition of "special" weaving in the obvious way to allow the single horizontal end $\gamma \subseteq \Lambda$ to be incoming, the edge which corresponds to e_T is an initial edge e_I, i.e., the left-most edge of Λ on γ. Erasure of e_I and extension of the vertical end of Λ which abuts the *right* endpoint of e_I has the following counterpart for the (η, α) exchange: Let $t = \min(\eta_1, \eta_{\alpha^{-1}1})$, and let (η', α') be the interval exchange which is equivalent to the exchange obtained from the first return map on $[t, |\eta|)$. It is clear that there is a choice $c = a$ or b such that $\alpha' = \sigma_m[c(\sigma_m \alpha \sigma_m)]\sigma_m$, and therefore $\alpha' \in O(\alpha)$.

Having fixed Λ as above with γ, say, outgoing, suppose δ is a distinct horizontal separatrix. Truncate δ when it first meets a vertical edge of Λ, and let γ_1 be the horizontal end which results. There is a unique weaving Λ_1 of ξ which has γ_1 for its horizontal end. Notice that δ is allowed to be incoming or outgoing. Let $\gamma_1^* \subseteq \gamma$ be the projection of γ_1 "down" to γ. (Λ_1, ξ) induces an interval exchange on γ_1 and γ_1^*, and this interval exchange can be obtained from (Λ, ξ) as follows: Iterate the operation $\mathscr{U}$ from (12.20) until the right endpoint of γ_1^* is the right endpoint of the horizontal end of Λ_n, $\mathscr{U}^n(\Lambda, \hat{\xi}) = (\Lambda_n, \lambda_n)$. This corresponds to the application of a series of operations $c = a$ or b on the permutation α associated to Λ. Next, apply erasure operations from the left as in the preceding paragraph, stopping when the left endpoint of γ_1^* is reached. This corresponds to a series of operations $\alpha \to \sigma_m c(\sigma_m \alpha \sigma_m)\sigma_m$, $c = a$ or b. The rectangle between γ_1 and γ_1^* is now partitioned into m rectangles. If the bases of these rectangles are now erased, the weaving Λ_1 is obtained. It follows that if α and α_1 are the permutations associated to Λ and Λ_1, then $O(\alpha) = O(\alpha_1)$. It makes sense to define $O([\xi]) = O(\alpha)$, initially for $[\xi] \in Y_0(\pi)$ such that ξ is in general position. If $[\xi] \in Y_0(\pi)$ is such that ξ is not in general position, there exists Λ as above which is a weaving of ξ. If Λ_1 is a second such weaving, then $[\xi] \in E(\Lambda) \cap E(\Lambda_1)$, and therefore there exists $[\xi'] \in E(\Lambda) \cap E(\Lambda_1)$ with ξ' in general position. It follows from the first part of the discussion that $O([\xi]) = O(\alpha)$, α the permutation determined by Λ, does not depend upon the choice of Λ.

A moment's reflection shows that if $[\xi'] \in Y_0(\pi)$ corresponds to the form $-\omega_\xi$, then $O([\xi']) = \sigma_m O([\xi])^{-1}\sigma_m = O([\xi])^{-1}$.

Example. When $m = 2p$ and $\alpha = \sigma_m$, the class $O(\sigma_m)$ is just the Rauzy class of σ_m. $O(\sigma_m)$ has $2^{m-1} - 1$ elements ([R79]). Moreover, $O(\sigma_m) = O(\sigma_m)^{-1}$. In this instance the corresponding symbol π is $\pi = (0, \nu, +1)$ with $\nu(l) = \delta_{l, 2p-2}$, δ_{ij} the Kronecker delta. (That is, X is of genus p, $S = \{s\}$ is a singleton, and ω_ξ has a single zero of order $2p - 2$ if $[\xi] \in Y(\pi)$.) However, when $p = 3$, there is a second Rauzy class which gives rise to the same symbol π. This class has 134 elements and is the class of the permutation $(\alpha_0 1, \dots, \alpha_0 6) = (4, 6, 5, 3, 2, 1)$. It follows that $Y(\pi)$ is not connected.

Let $Y_1(\pi)$ and $Y_2(\pi)$ be the open sets identified in the previous paragraph. That is, $Y_1(\pi) = \{[\xi] \mid O([\xi]) = O(\sigma_6\}$ and $Y_2(\pi) = \{[\xi] \mid O([\xi]) = O(\alpha_0)\}$. Each of Y_1, Y_2 is invariant under the Teichmüller geodesic flow, and when this flow is restricted to the level surfaces $V^{-1}1 \cap Y_j$, it is ergodic [V86]. In particular, $V^{-1}1 \cap Y_j$ is the closure of an embedded line. It follows that Y_1 and Y_2 are connected sets. Finally, one finds by inspection that if $\alpha \in \mathscr{G}_6^0$ gives rise to an element of $Y(\pi)$ (by the zippered rectangle construction [V82]), then $\alpha \in O(\sigma_6 \cup O(\alpha_0)$.

13.1. Theorem. *Let* $\pi = (0, \nu, +1)$ *with* $\nu(l) = \delta_{l4}$. *The moduli space* $Y(\pi)/\Gamma(3)$ *has two components, one corresponding to* $O(\sigma_6)$ *and one corresponding to* $O(\alpha_0)$.

Remark. If $[\xi] \in Y_1(\pi)$ above, it can be seen that ξ endows X with the structure of a (genus 3) hyperelliptic surface. The elements of $Y_2(\pi)$ correspond to (genus 3) surfaces which, while *not* hyperelliptic, support holomorphic 1-forms with a single zero (of order $2p-2=4$). It has been pointed out to us by R. Bryant that the Fermat quartic ($x^4+y^4=z^4$) is an example of such a surface.

Remark. $Y(\pi)/\Gamma(g, n)$ is always finitely connected. It would be interesting to express the connectivity in terms of π. In this regard we mention that a computer calculation by P. Arnoux (private communication) implies that in genus 4, $\pi=(0, \nu, +1)$, $\nu(l)=\delta_{l6}$, $l \geqq 1$, $Y(\pi)/\Gamma(4)$ has three components, corresponding to the permutations σ_8 and $\sigma(1, 2, \ldots, 8)=(6, 8, 7, 5, 4, 3, 2, 1)$ and $(6, 5, 8, 7, 4, 3, 2, 1)$. In each case the extended Rauzy class is the same as the Rauzy class; Arnoux's calculation shows the classes have, respectively, $127(=2^7-1)$, 2327, and 5209 elements. (In effect, our proof that $Y(\pi)/\Gamma(4)$ has finite volume consists of proving each of the 7,663 sets in its "natural" cover has finite volume.)

14. Finiteness of integrals over the moduli spaces

We begin by introducing notation which enables us to give a unified treatment of "finite integral" statements.

In this section Λ denotes either (a) a special weaving of a $\xi \in \Omega_0^+(S)$ in general position, or (b) the weaving Λ_τ associated to a τ-special weaving of $\xi \in \Omega^+(S, \tau)$ in general position. The notations $\mathscr{X}_h^1$, P_h^1, etc. have the usual meanings when Λ is as in (a) above; however, when Λ is as in (b), $\mathscr{X}_h^1$, P_h^1 etc. shall be understood to denote the (-1 eigen-) spaces $\mathscr{X}_{h,\tau}^1$, $P_{h,\tau}^1$, etc. In all that follows N denotes

$$N=\dim \mathscr{X}_h^1.$$

N is the complex dimension of $\mathscr{M}_0^+(S)$ or $\mathscr{M}^+(\tau)$, depending on (a), (b).

14.1. Remark. Let $H(\Lambda)$ be the set of horizontal edges $e \subseteq \Lambda$ such that ∂e contains a horizontal terminus. We have $|H(\Lambda)|=1$ or 2 as Λ falls under (a) or (b) above. In the case of (b) $\alpha(\cdot)$ is constant on $H(\Lambda)$ for each $\alpha \in \mathscr{X}_h^1$. Similarly, if e_1 is as in Section 11, $\beta(e_1)=\beta(-\tau e_1)$ for all $\beta \in \mathscr{X}_v^1$.

Select bases $A=\{\alpha_1, \ldots, \alpha_N\}$ and $B=\{\beta_1, \ldots, \beta_N\}$ for $\mathscr{X}_h^1$ and $\mathscr{X}_v^1$, respectively. A and B will be further restricted below; for now we require only the additional property that each be an integral basis for the respective integral lattice. Both A and B project to bases, either for $H_{\mathbf{Z}}^1(X, S)$ or for the -1 eigenspace of τ^* in $H_{\mathbf{Z}}^1(X, S)$, under the canonical projection of Section 4.

Superscripts will be used to denote coordinate functions, i.e., $\alpha=\Sigma\alpha^j\alpha_j$, $\beta=\Sigma\beta^j\beta_j$. Introduce volume forms, $\Sigma(\alpha)$ and $\Sigma(\beta)$, where

$$\Sigma(\alpha) = d\alpha^1 \wedge \cdots \wedge d\alpha^N \quad \text{and} \quad \Sigma(\beta) = d\beta^1 \wedge \cdots \wedge d\beta^N.$$

Also, define $\sigma(x)$ on $\mathbf{R}^N$ by

$$\sigma(x) = \sum_{i=1}^{N} (-1)^i x_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^N$$

where ^ denotes omission.

Define $F_1 : \mathbf{R}^N \times \mathbf{R}^+ \to \mathbf{R}^N$ by $F_1(a, \theta) = \theta a$. We have

$$F_1^*\Sigma(\alpha) = (-1)^N \theta^{N-1} \sigma(a) \wedge d\theta + \theta^N \Sigma(a). \tag{14.2}$$

Next, define $F_2 : \mathbf{R}^N \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^N$ by $F_2(b, \theta, \zeta) = \zeta b/\theta$. This time we compute

$$F_2^*\Sigma(\beta) = (-1)^N \frac{\zeta^{N-1}}{\theta^N} \sigma(b) \wedge d\zeta + E_1 \Sigma(b) + E_2 \wedge d\theta \tag{14.3}$$

where E_1 is a scalar and E_2 an $N-1$ form. Define $F : \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^N \times \mathbf{R}^N$ by $F(a, b, \theta, \zeta) = (F_1(a, \theta), F_2(b, \theta, \zeta))$. Introduce the form

$$\omega(a, b, \theta) = \sigma(a) \wedge d\theta/\theta \wedge \sigma(b),$$

and compute from (14.2)–(14.3)

$$F^*(\Sigma(\alpha) \wedge \Sigma(\beta)) = \zeta^{N-1} \omega(a, b, \theta) \wedge d\zeta + E_3 \wedge \Sigma(a) + E_4 \wedge \Sigma(b) \tag{14.4}$$

where E_3 and E_4 are N-forms.

Introduce on $\mathscr{X}_h^1 \times \mathscr{X}_v^1$ two functions. The first is $\theta(\alpha, \beta) = \Sigma_e \alpha(e) = \theta(\alpha)$, the sum extending over all (horizontal) edges of Λ. The second is the intersection pairing, $\zeta(\alpha, \beta) = [\alpha, \beta]$. Define $E \subseteq \mathscr{X}_h^1 \times \mathscr{X}_v^1 \simeq \mathbf{R}^N \times \mathbf{R}^N$ by

$$E = \{(a, b) \mid \theta(a, b) = 1 = \zeta(a, b)\}.$$

Let F_E be the restriction of F to $E \times \mathbf{R}^+ \times \mathbf{R}^+$. F_E is a diffeomorphism onto its image, and

$$F_E^*\Sigma(\alpha) \wedge \Sigma(\beta) = i^*F^*\Sigma(\alpha) \wedge \Sigma(\beta)$$

where $E \times \mathbf{R}^+ \times \mathbf{R}^+ \xrightarrow{i} (\mathbf{R}^N \times \mathbf{R}^N) \times \mathbf{R}^+ \times \mathbf{R}^+$ is the inclusion map. The definition of E combines with (14.4) to imply

$$F_E^*\Sigma(\alpha) \wedge \Sigma(\beta) = \zeta^{N-1} i^*\omega(a, b, \theta) \wedge d\zeta. \tag{14.5}$$

Observe from the definitions that P^1 is in the image of F_E. Indeed if $E^+ = E \cap P^1$, then

$$P^1 = F_E(E^+ \times \mathbf{R}^+ \times \mathbf{R}^+). \tag{14.6}$$

This being so, use (14.5) to construct forms ω_Λ and σ_Λ on P^1,

$$\omega_\Lambda = (F_E^{-1})^* \circ i^*\omega(a, b, \theta), \tag{14.7}$$
$$\sigma_\Lambda = (F_E^{-1})^* \circ i^*\zeta^{N-1}d\zeta.$$

By construction

$$\omega_\Lambda \wedge \sigma_\Lambda = \Sigma(\alpha) \wedge \Sigma(\beta). \tag{14.8}$$

Let there now be given a pair Λ_1, Λ_2 as above, and suppose T is a diffeomorphism from an open set $O_1 \subseteq P^1(\Lambda_2)$ onto an open set $O_2 \subseteq P^1(\Lambda_2)$. Let η_Λ be the form on the right in (14.8), and assume of T that

$$T^*\eta_{\Lambda_2} = \eta_{\Lambda_1}, \tag{14.9}$$
$$\zeta_2 \circ T = \zeta_1$$

($\zeta_i(\lambda) = [\alpha, \beta]$, $\lambda = (\alpha, \beta) \in O_j$). The second line of (14.9) implies

$$T^*\sigma_{\Lambda_2} = \sigma_{\Lambda_1}. \tag{14.10}$$

Also, there exists a scalar function, ϕ, and a $2N-1$ form, $\tilde{\omega}_\Lambda$ on O_1, such that (a) $\tilde{\omega}_{\Lambda_1} \wedge \sigma_{\Lambda_1} = 0$, and (b) $T^*\omega_{\Lambda_2} = \phi\omega_{\Lambda_1} + \tilde{\omega}_{\Lambda_1}$. From (14.10), (14.8) and the first line of (14.9) it follows that $\phi \equiv 1$. We have

14.11. Lemma. *With all notations as above we have*

$$T^*\omega_{\Lambda_2} = \omega_{\Lambda_1} + \tilde{\omega}_{\Lambda_1}, \tag{14.12}$$
$$\tilde{\omega}_{\Lambda_1} \wedge \sigma_{\Lambda_1} = 0.$$

Consider now the manifolds $\mathscr{M}_0^+(S)$ and $\mathscr{M}^+(S, \tau)$. Each of these manifolds admits an atlas of the form $\{(A^{-1}(\phi)E(\Lambda), F_\Lambda \circ A(\phi))\}$ ($E(\Lambda) = E(\Lambda, \tau)$ in the case of $\mathscr{M}^+(S, \tau)$). If T is a coordinate transition for this atlas, T satisfies (14.9)–(14.10), and therefore $(F_\Lambda \circ A(\phi))^*\omega_\Lambda$ is a $2N-1$ form modulo the kernel of $(F_\Lambda \circ A(\phi))^*\sigma_\Lambda$ under wedge product. With notations as in Sections 7–8, it follows ω_Λ restricts to a volume form on each "constant energy" surface $V^{-1}t$, $t > 0$. Moreover, if ω_Λ^t is the restriction of ω_Λ, we have

$$\gamma^*\omega_\Lambda^t = \omega_\Lambda^t$$

for each element $\gamma \in G = \mathrm{SL}(2, \mathbf{R})$.

We shall now consider the forms

$$|\omega(a, b, \theta)| = \left|\sigma(a) \wedge \frac{d\theta}{\theta} \wedge \sigma(b)\right| \tag{14.13}$$

which by the above discussion determine a (real analytic) invariant measure for the action of G on the level sets of $V(\cdot)$ in the manifolds $\mathscr{M}_0^+(S)$ and $\mathscr{M}^+(S, \tau)$.

Let $\psi(a)$ be the value $a(\cdot)$ assumes on $H(\Lambda)$ (see Remark 14.1). If $E_0 = \{(a, b, \theta) \mid 1 \leqq \theta < 1 + \psi(a)\}$, the first three paragraphs of Section 13 lead us to consider the integrals

$$\begin{aligned} \int_{E_0} |\omega(a, b, \theta)| &= \int_E \log(1 + \psi(a)) |\sigma(a) \wedge \sigma(b)| \\ &\leqq \int_E \psi(a) |\sigma(a) \wedge \sigma(b)|. \end{aligned} \tag{14.14}$$

The object of the discussion which follows is a proof of the finiteness of the right side of (14.14).

14.15. Remark. $P^1(\Lambda) = P_h^1 \times P_v^1$ is a product of cones which are defined by finite sets of rational linear inequalities. It follows that P_h^1 and P_v^1 are each a finite union of cones with simplex bases, each cone being spanned by extremals of $\overline{P_h^1}$ or $\overline{P_v^1}$. In what follows we shall suppose given sets A and B as at the beginning of this section, with the additional properties (a) $A \subseteq \overline{P_h^1}$ and $B \subseteq \overline{P_v^1}$, (b) $|\alpha_i| = 1$, $1 \leqq i \leqq N$, and (c) each element α_i or β_j is an extremal.

Let $\Delta = \operatorname{co} A$, and let $\mathscr{C}(\Delta)$ be the cone with base Δ, $\mathscr{C}(\Delta) = \mathbf{R}^+\Delta$.

If $a \in \Delta$, denote by $E_1(a)$ the simplex

$$E_1(a) = \operatorname{co}\left\{\frac{\beta_1}{[a, \beta_1]}, \ldots, \frac{\beta_N}{[a, \beta_N]}\right\}. \tag{14.16}$$

Also, define $E(\Delta) \subseteq E$ by

$$E(\Delta) = \bigcup_{a \in \Delta} \{a\} \times E_1(a). \tag{14.17}$$

If $a \in \Delta$, $E(\Delta)$ contains the set of (a, b) such that

$$(a, b) \in E \cap (\mathbf{R}^+\Delta \times \mathbf{R}^+ \operatorname{co}\{\beta_1, \ldots, \beta_N\}).$$

Let $E_2(a) = \{tb \mid 0 < t \leqq 1, b \in E_1(a)\}$. Replace a by b and θ by t in (14.2) to find

$$\int_{E_2(a)} |\Sigma(b)| = \frac{1}{N} \int_{E_1(a)} |\sigma(b)|. \tag{14.18}$$

The value of the integral on the left side of (14.18) differs by a constant factor, $C = C(N, \beta_1, \ldots, \beta_N)$ from the determinant of the diagonal matrix with ith diagonal entry $[a, \beta_i]^{-1}$. From (14.18) and (14.17) we conclude

(14.19) $$\int_{E(\Delta)} \psi(a)\,|\sigma(a)\wedge\sigma(b)| = NC \int_{\Delta} \frac{\psi(a)}{\prod_{i=1}^{N}[\alpha,\beta_i]}\,|\sigma(a)|.$$

On Δ the form $|\sigma(a)|$ is a dimensional constant times the euclidean measure on Δ.

In order to analyze the right hand side of (14.19) set up on $\mathscr{C} = \mathscr{C}(\Delta)$ the function, ϕ, defined by

(14.20) $$\phi(a) = \psi(a) \prod_{j=1}^{N} [a,\beta_j]^{-1}.$$

Let F be a face of $\mathscr{C}$ such that $1 \leqq \dim F = q < N$. There exist $1 \leqq i_1 < \cdots < i_q \leqq N$ such that F is spanned by $\{\alpha_{i_1}, \ldots, \alpha_{i_q}\}$. Call this latter set $s(F)$.

Define $M(F) \subseteq F = \{\beta_1, \ldots, \beta_N\}$ by

(14.21) $$M(F) = \{\beta_j \mid [\alpha_i, \beta_j] = 0, \alpha_i \not\subseteq s(F)\}.$$

We observe that $F_1 \subseteq F_2$ implies $M(F_1) \subseteq M(F_2)$. Also, define $M^+(F) \subseteq M(F)$ by

(14.22) $$M^+(F) = \{\beta_j \in M(F) \mid \beta_j(e_1) > 0\}$$

(see Remark 14.1). It is possible that $M(F) = \varnothing$ or that $M(F) \neq \varnothing$ and $M^+(F) = 0$. In present notation the Basic Lemma, Parts I and II, from Sections 10, and 11, yields

14.23. Lemma. *If F is a face of $\mathscr{C}$ such that $1 \leqq \dim F < N$, then*

(14.24) $$\operatorname{Card} M(F) \leqq \dim F.$$

If equality holds in (14.24), *then*

(14.25) $$M^+(F) \neq \varnothing \qquad (\operatorname{Card} M(F) = \dim F).$$

Associate to F a function ψ_F, on $\mathscr{C}$, where

(14.26) $$\psi_F(a) = \begin{cases} 1, & M^+(F) = \varnothing, \\ \min\limits_{\beta_j \in M^+(F)} [a,\beta_j], & M^+(F) \neq \varnothing. \end{cases}$$

Because $\mathscr{C}$ has a simplex base, each $a \in \mathscr{C}$ has a unique representation $a = a' + a''$, where $a' \in F$ and $a'' \in F' = \operatorname{span}\{\alpha_i \mid \alpha_i \notin s(F)\}$. The definition (14.26) implies

(14.27) $$\psi_F(a) = \psi_F(a') \qquad (a = a' + a'', a' \in F, a'' \in F').$$

Associate to the face F a function ϕ_F on $\mathscr{C}$, defined by

$$\phi_F(a)=\psi_F(a)\sum_{\beta_j\in M(F)}[a,\beta_j]^{-1}. \tag{14.28}$$

If $a=a'+a''$, $a'\in F$, $a''\in F'$, as above, the definition of $M(F)$ implies $[a'',\beta_j]=0$, $\beta_j\in M(F)$, and using (14.27) we find

$$\phi_F(a)=\phi_F(a') \qquad (a=a'+a'', a'\in F, a''\in F'). \tag{14.29}$$

Finally, associate to the face F a function $B(F,a)$ on $\mathscr{C}$, defined by

$$B(F,a)=\frac{\phi(a)}{\phi_F(a)}. \tag{14.30}$$

By construction

$$\phi(a)=B(F,a)\phi_F(a). \tag{14.31}$$

14.32. Lemma. *With notations as above, ϕ_F is homogeneous of degree* $-\nu_F$, where

$$\nu_F<\dim F. \tag{14.33}$$

Proof. Immediate from the definitions (14.28) and (14.26) and Lemma 14.23.

Let $F_1\subseteq F_2$ be faces of $\mathscr{C}$. Define $s(F_1,F_2)=s(F_1)^c\cap s(F_2)$. If $a\in F_2$, express a as $a=a'+a''$, where $a'\in F_1$ and a'' is in the span of $s(F_1,F_2)$. Define for $\varepsilon>0$

$$Q(F_1,F_2,\varepsilon)=\left\{a\in F_2 \,\middle|\, a=a'+a'', a'\in F_1, a''=\sum_{\alpha_i\in s(F_1,F_2)}\gamma_i\alpha_i, \gamma_i\geqq\varepsilon, \text{all } i\right\}.$$

14.34. Lemma. *Let $F_1\subseteq F_2$ be faces, and let $\varepsilon>0$. There exists $R=R(\varepsilon)<\infty$ such that*

$$\sup_{a\in Q(F_1,F_2,\varepsilon)}\frac{B(F_1,a)}{B(F_2,a)}<R. \tag{14.35}$$

Proof. By definition the quotient in (14.35) has the form

$$\begin{aligned}\frac{B(F_1,a)}{B(F_2,a)}&=\frac{\phi_{F_2}(a)}{\phi_{F_1}(a)}\\&=\frac{\psi_{F_2}(a)}{\psi_{F_1}(a)}\prod_{\substack{\beta_j\in M(F_2)\\ \beta_j\notin M(F_1)}}[a,\beta_j]^{-1}.\end{aligned} \tag{14.36}$$

To say $\beta_j\in M(F_2)\cap M(F_1)^c$ is to say in particular there exists $\alpha_i\in s(F_1,F_2)$ such that $[\alpha_i,\beta_j]>0$. It follows the product on the right in (14.36) is uniformly bounded on $Q(F_1, F_2, \varepsilon)$. If $M^+(F_1) \neq \varnothing$, or if $M^+(F_2) = 0$, $\psi F_2(a)/\psi F_1(a) \leqq 1$. In the case that $M^+(F_1) = \varnothing$ but $M^+(F_2) \neq 0$, we have

$$\frac{\psi_{F_2}(a)}{\psi_{F_1}(a)} = \min_{\beta_j \in M^+(F_2)} [a, \beta_j]. \tag{14.37}$$

Since $M^+(F_2) \subseteq M(F_2) \cap M(F_1)^c$, it is true for each $a \in Q(F_1, F_2, \varepsilon)$ that (14.37) is dominated by some $[a, \beta_j]$ which appears in the denominator in (14.36). The lemma is proved.

14.38. Proposition. *Let ϕ be a positive function on $\mathscr{C}$ such that ϕ is homogeneous of degree $-v$, $v < \dim \mathscr{C}$. Assume for each face F there exists a representation $\phi = \phi_F B(F, \cdot)$ such that ϕ_F depends only upon F. ($\phi_F(a' + a'') = \phi_F(a')$, as above.) If ϕ_F satisfies the conclusion of Lemma* 14.32, *and if $B(F, \cdot)$ satisfies the conclusion of Lemma* 14.34, *then*

$$\int_\Delta \phi(a)\,|\sigma(a)| < \infty. \tag{14.39}$$

Proof. This is the content of Proposition 13.2 of [V82].

14.40. Theorem. *Let $t > 0$ and let μ^t be the natural measure (above) on the surface $V^{-1}t \subseteq \mathscr{M}_0^+(S)$ or $\mathscr{M}^+(S, \tau)$. Then*

$$\mu^t(V^{-1}t/\Gamma) < \infty \tag{14.41}$$

where $\Gamma = \Gamma(X, S)$ or $\Gamma(\tau)$.

We now will establish Theorem 0.2. Let $\xi \in \mathscr{M}_0^+(S)$ be in general position, and let δ be an ξ-geodesic with endpoints in S. If Λ is any weaving of ξ, the natural projection of δ on the horizontal end in Λ will have length at least $\min a(e)$, $\xi = (a, b) \in P^1(\Lambda)$, as e ranges over the horizontal edges of Λ. (This minimum may be less than the minimum width of a component of Λ^c.) Proposition 14.38 and the proof of Theorem 14.40 imply the function which is the reciprocal of this minimum belongs to L^p on $V^{-1}1 \cap \mathscr{M}_0^+(S)/\Gamma(X, S)$ for $0 < p < 1$. This is valid for $\mathscr{M}_0^+(S)$ or $\mathscr{M}^+(S, \tau)$. We have

14.42. Theorem. *Define $\phi(x)$, $x \in V(\pi)$ to be the minimum length of a $|q|$ goedesic δ with endpoints in the singular set of a quadratic differential q which represents x. Then $\phi^{-1} \in L^p(V_1(\pi)/\Gamma(g, n))$, $0 < p < 1$, relative to the "Liouville measure" μ_π.*

15. Degenerate F-structures

Let $\xi_0 \in \Omega_0^+(S)$ be in general position, and let Λ be a fixed weaving of ξ_0. Set $\lambda_0 = \hat{\xi}_0 \in P^1(\Lambda)$. Recall from Section 5 that we associate to ξ_0 and each component $U \subseteq \Lambda^c$ an element $r(U) = r(U, \lambda_0) \in \mathscr{R}$. For each U, $r(U) = (R(U), \Sigma)$, where $R(U) \subseteq \mathbf{C}$ is a rectangle in the first quadrant having $z = 0$ for one vertex, and $\Sigma \subseteq \partial R(U)$ is a finite set containing the vertices. There is a ξ_0-compatible chart (U, f^U) such that $f^U \bar{U} = \overline{R(U)}$ and $(f^U)^{-1}\Sigma$ is the set of vertices of Λ on ∂U.

If $\lambda = (\alpha, \beta) \in P^1(\Lambda)$, λ is used to associate to each U as above an element $r(U, \lambda) \in \mathscr{R}$, $r(U, \lambda) \sim r(U)$ in the sense of Section 5, and $\phi^U_{\lambda_0\lambda} : R(U) \to R(U, \lambda)$ is the canonical map. A partial atlas for $\xi(\lambda) \in \Omega_0^+(S)$ is defined by (5.4),

$$F_\lambda^U(x) = \phi^U_{\lambda_0\lambda} \circ f^U(x) \qquad (x \in \bar{U}). \tag{15.1}$$

If $\lambda \in \overline{P^1(\Lambda)}$, there is a natural definition of $r(U, \lambda)$. If e is an edge on ∂U such that $\alpha(e) + \beta(e) = 0$, the corresponding edge on $r(U)$ is assigned length 0, and two vertices of Λ on ∂U are identified. This means $r(U, \lambda)$ does not in general satisfy $r(U) \sim r(U, \lambda)$. Indeed, $R(U, \lambda)$ may degenerate to a line segment or even a point; and even when $R(U, \lambda)$ is a rectangle, distinct vertices on ∂U may correspond to the same point on $\partial R(U, \lambda)$.

If $\lambda \in \overline{P^1(\Lambda)}$ as above, there is still a canonical map, $\phi^U_{\lambda_0\lambda} \to r(U, \lambda)$. This map sends the center of $R(U)$ to the center of $R(U, \lambda)$, elements of $\Sigma \subseteq \partial R(U)$ to corresponding points on $\partial R(U)$, albeit in a many to one fashion, and is extended to $R(U)$ to be PL. Now define F_λ^U as in (15.1).

If λ is fixed as above, define $xE_\lambda^0 y$, $x, y \in \bar{U}$, to be the relation $F_\lambda(x) = F_\lambda(y)$. Define $xE_\lambda y$ to be the closed equivalence relation on X generated by E_λ^0. Let $X(\lambda) = X/E_\lambda$ be the space of equivalence classes with the quotient topology, and let $X \xrightarrow{\pi_\lambda} X(\lambda)$ be the canonical projection.

If U is a component of Λ^c, define $U(\lambda) = \pi_\lambda U$. There is a natural homeomorphism $\psi_\lambda^{U(\lambda)} : U \to R(U, \lambda)$ such that the diagram

$$\begin{array}{ccc} U & & \\ \downarrow \pi_\lambda & \searrow F_\lambda^U & \\ U(\lambda) & \xrightarrow{\psi_\lambda^{U(\lambda)}} & R(U, \lambda) \end{array} \tag{15.2}$$

is commutative.

Let $X^0(\lambda)$ be the "two dimensional interior" of $X(\lambda)$. $X^0(\lambda)$ is the union of (a) the sets $U(\lambda)$ such that $R(U, \lambda)$ is a rectangle and (b) edges e such that (i) $\pi_\lambda e$ is not degenerate and (ii) $e \subseteq \partial U$ implies $U(\lambda) \subseteq X^0(\lambda)$.

With notations as above, define $\mathscr{U}^0(\lambda)$ to be the set of charts $(U(\lambda), \psi_\lambda^{U(\lambda)})$ such that $U(\lambda) \subseteq X^0(\lambda)$. It follows from (15.1)–(15.2) that transitions associated to $\mathscr{U}^0(\lambda)$ on sets $\overline{U_1(\lambda)} \cap \overline{U_2(\lambda)} \cap X^0(\lambda)$ are locally given by $z \to z + c$, and $\mathscr{U}^0(\lambda)$ extends to an atlas $\mathscr{U}(\lambda)$ such that $\xi(\lambda) = (X^0(\lambda), \mathscr{U}(\lambda))$ is an F-structure. It is possible that $X^0(\lambda) = \varnothing$, but this will not occur in the cases which are of interest to us.

The F-structure $\xi(\lambda)$ will not in general be admissible. For while $X^0(\lambda)$ is totally bounded relative to $d_{\xi(\lambda)}$, the completion of $X^0(\lambda)$ will in general involve adding a one-dimensional set. In the case that every set $U(\lambda)$ is two-dimensional, $X(\lambda)$ will be a surface of genus $g = g(X)$, and π_λ is a homotopy equivalence. However, π_λ *cannot* be one-to-one if $\lambda \in \partial P^1(\Lambda)$.

In what follows we shall consider the case of $\lambda = (\alpha, \beta)$ when $\beta \in P_v^1(\Lambda)$ and $\alpha \in \overline{P_h^1(\Lambda)}$. In this situation $U(\lambda) \subseteq X^0(\lambda)$ as soon as ∂U contains one horizontal edge e such that $\alpha(e) > 0$.

15.3. Proposition. *Suppose* $\lambda = (\alpha, \beta)$ *satisfies* (i) $\lambda \in \overline{P_h^1} \times P_v^1$ *and* (ii) α *is a rational vector. Then* $\mathscr{F}_v(\xi(\lambda))$ *consists entirely of closed leaves and saddle connections.*

Proof. The remark which precedes the proposition implies $X^0(\lambda)$ is a surface with boundary consisting of images, $\pi_\lambda e$, of vertical edges $e \subseteq \Lambda$. It is no loss of generality to suppose α is an integral vector. Define $\Lambda^0 = \pi_\lambda \Lambda \cap X^0(\lambda)$, and let Λ_h^0 be the "horizontal part" of Λ^0. If $x \in \Lambda_h^0$, x lies on an edge $e^0 = \pi_\lambda e$ of integer length. Define $\phi(x)$ to be the distance from x to the left endpoint of e^0, relative to $\xi(\lambda)$, but measured modulo 1. Also, if $0 < \phi(x) < 1$, define $Tx \in \Lambda_h^0$ to be the next point of intersection between Λ_h^0 and the vertical geodesic through x, oriented in the "upward direction." The condition $x \in \Lambda_h^0$ implies there is a component $U(\lambda)$ such that x lies on the base of $U(\lambda)$, and Tx is the point on the top of $U(\lambda) \equiv R(U, \lambda)$ which lies above x. If $0 < \phi(x) < 1$, then $Tx \in \Lambda_h^0$ and $\phi(x) = \phi(Tx)$. Since $\{x' \mid \phi(x') = \phi(x)\}$ is a finite set, and since T is one-to-one, $T^n x = x$ for some $n > 0$, and the vertical leaf through x is closed. The conclusion of the proposition follows readily.

Let $\lambda = (\alpha, \beta)$ be as in Proposition 15.3. α descends to an element α^0 of $P_h^1(\Lambda^0)$. Moreover, if $\alpha' \in \overline{P_h^1(\Lambda^0)}$, there exists $\alpha'' \in \overline{P_h^1(\Lambda)}$ which descends to α'. Indeed, if $e \subseteq \Lambda$ is a horizontal edge such that $\pi_\lambda e \subseteq X^0(\lambda)$ is not degenerate, define $a''(e) = \alpha'(\pi_\lambda e)$. The condition $\partial^* \alpha' = 0$ and the obvious identity.

$$\langle U, \partial^* \alpha'' \rangle = \langle U(\lambda), \partial^* \alpha' \rangle$$

imply $\partial^* \alpha''(U) = 0$ for all U such that $U(\lambda)$ is not degenerate. Since $\partial^* \alpha''(U) = 0$ trivially when $U(\lambda)$ is degenerate, the claim is established.

Continuing with $\lambda = (\alpha, \beta)$ as above, define $Y(\lambda) \subseteq X^0(\lambda)$ to be the complement of the critical set for $\mathscr{F}_v(\xi(\lambda))$. Let $Y_1(\lambda)$ be a component of $Y(\lambda)$, and let $Y_2(\lambda) = Y_1^c(\lambda)$. Write $\alpha^0 = \alpha_1^0 + \alpha_2^0$, where α_j^0 is the restriction of α^0 to $\Lambda_h^0 \cap Y_j(\lambda)$. We can state

15.4. Proposition. *Let $\lambda = (\alpha, \beta)$ be as above, and suppose α is an extremal element of $\overline{P_h^1}(\Lambda^0)$. Then* (a) *$\alpha$ is a multiple of a rational vector, and* (b) *$Y(\lambda)$ is connected.*

Proof. $\overline{P_p^1(\Lambda^0)}$ is a cone defined by a finite set of rational linear equalities and inequalities. This implies (a). As for (b), let α descend to α^0, and suppose $\alpha^0 = \alpha_1^0 + \alpha_2^0$ as in the discussion preceding the proposition. Then α_j^0 lifts to $\alpha_j \in \overline{P_h^1}$, and $\alpha = \alpha_1 + \alpha_2$. Since α is an extremal, $\alpha_j = t_j\alpha$ and then $\alpha_j^0 = t_j\alpha^0$. If, say, $\alpha_1^0 \neq 0$, it must be that $t_1 \neq 0$, and then that $t_2 = 0$. That is, $\alpha_2^0 = 0$ and $Y_1(\lambda)^c = \varnothing$. The proposition is proved.

At this point it is worthwhile remarking on the situation when $\xi_0 \in \Omega^+(S, \tau)$, $\tau\Lambda = \Lambda$, and $\lambda = (\alpha, \beta) \in \overline{P_{h,\tau}^1} \times P_{v,\tau}^1$. All constructions are the same. If α is an extremal of $\overline{P_{h,\tau}^1}$, α is a multiple of a rational vector. If $Y_1(\lambda)$ is a component of $Y(\lambda)$, define $Y_1^\tau(\lambda) = Y_1(\lambda) \cup \tau Y_1(\lambda)$, and let $Y_2^\tau = (Y_1^\tau)^c$. If α_j^0 is the restriction of α^0 to Y_j^τ, then $\alpha_j^0 \in \overline{P_{h,\tau}^1(\Lambda_h^0)}$, and so $\alpha_j \in \overline{P_{h,\tau}^1(\Lambda_h)}$. The argument used in Proposition 15.4 implies $\alpha_j^0 = 0$ for some j, and we have

15.5. Proposition. *Let $\lambda = (\alpha, \beta) \in \overline{P_{h,\tau}^1} \times P_{v,\tau}^1$, and suppose α is an extremal. Then* (a) *α is a multiple of a rational vector, and* (b) *$Y(\lambda)$ has one or two components. In the latter case the components are permuted by τ.*

With notations as above, let $Y_1(\lambda)$ be a component of the complement of the critical set. While $X^0(\lambda)$ is not a closed surface, Theorem 2.3 applies, and $Y_1(\lambda)$ is holomorphically equivalent to an annulus, $A = \{z \mid r < |z| < s\}$, in such a way that the biholomorphism, $Y_1(\lambda) \xrightarrow{\phi} A$, satisfies

$$\phi^* \frac{dz}{z} = \omega_{\xi(\lambda)}\big|_{Y_1(\xi)}. \tag{15.6}$$

This is because the vertical foliation of $(dz)/z$ is the foliation of A by circles centered at 0.

Because $Y_1(\lambda)$ has finite volume relative to $\xi(\lambda)$, the numbers r and s above satisfy $0 < r < s < \infty$. We normalize and assume $r = 1$, $s = s/r = R$. In the case of $\alpha \in P_{h,\tau}^1$ ($\tau\Lambda = \Lambda$), if $Y_1(\lambda) \neq \tau Y_1(\lambda)$, each component is assigned the same annulus A and form $(dz)/z$. (The annuli are the same because they are biholomorphic and have inner radii 1.) In either case the fact $\tau^*\omega_{\xi(\lambda)} = -\omega_{\xi(\lambda)}$ implies τ induces $A \xrightarrow{\tau_0} A$ such that

$$\tau_0^* \frac{dz}{z} = -\frac{dz}{z}.$$

This implies $\tau_0 z = R/z$, up to a scalar factor of absolute value one.

16. Primitive Jenkins–Strebel differentials

Let X be a closed Riemann surface of genus $g \geqq 0$, and let $S \subseteq X$ be a nonempty finite set. If $\sigma \in M(X)$ is such that $S^*(\sigma) \subseteq S$, we follow the convention that each point s of S is a zero of σ of order $l(s) \geqq 0$. Let $\xi(\sigma)$ be the associated special F-structure.

If $\varepsilon(\sigma) = +1$, define $\mathscr{F}_v(\sigma) = \mathscr{F}_v(\xi(\sigma))$. If $\varepsilon(\sigma) = -1$, let $\xi'(\sigma) \xrightarrow{q} \xi(\sigma)$ be the canonical positive extension with sheet interchange τ. Because $\tau\mathscr{F}_v(\xi'(\sigma)) = \mathscr{F}_v(\xi'(\sigma))$, $\mathscr{F}_v(\xi'(\sigma))$ projects under q to a foliation, $\mathscr{F}_v(\xi(\sigma))$, with singularities in S. In this case $\mathscr{F}_v(\xi(\sigma))$ is a nonorientable foliation.

Let $C(\sigma)$ be the union of S and the set of separatrices of $\mathscr{F}_v(\xi'(\sigma))$ in the case $\varepsilon(\sigma) = -1$. $C(\sigma)$ contains $C(\pi/2)$ from Section 2 when $\varepsilon(\sigma) = +1$. $C(\sigma)$ is the *critical set* of σ.

16.1. Definition. With notations as above, σ is a *Jenkins–Strebel differential* if $C(\sigma)$ is compact.

By construction $C(\sigma) \neq \varnothing$ (i.e., even when $g = 1$). *Theorem* 2.3 *implies* $C(\sigma)^c$ decomposes into a finite union of cylinders (annuli) of closed leaves.

16.2. Definition. A Jenkins-Strebel differential σ shall be said to be *primitive* if $C(\sigma)^c$ is connected.

In what follows we shall describe a procedure for constructing primitive Jenkins–Strebel differentials.

Notations are as in the second paragraph of Section 14. In what follows we suppose Λ_1 and Λ_2 are weavings of ξ and that (Λ_j, ξ), $j = 1, 2$, satisfy (a) or (b) of that paragraph. Define $\Lambda = \Lambda_1 \cup \Lambda_2$, observing that $\tau\Lambda = \Lambda$ when (v) applies. There is a commutative diagram of canonical maps,

$$(16.3) \qquad \begin{array}{ccc} & P^1(\Lambda) & \\ {\scriptstyle \rho_2}\swarrow & & \searrow{\scriptstyle \rho_1} \\ P^1(\Lambda_2) & \xrightarrow{\rho_{21}} & P^1(\Lambda_1) \end{array}$$

However, ρ_{21} is defined only on $\rho_2 P^1(\Lambda)$. Since $\xi \in P^1(\Lambda)$, the domain of ρ_{21} is a nonempty open subcone of $P^1(\Lambda_2)$.

Because $P^1(\Lambda)$ is a product space, the domain of ρ_{21} has the form $U \times V$. We shall be interested in the case that $U = P^1(\Lambda_2)$, i.e., that

$$\rho_2 P_h^1(\Lambda) = p_h^1(\Lambda_2). \tag{16.4}$$

It is useful to observe that if Λ and Λ_1 are as in Section 12, and if in (16.4) we replace Λ by $\Lambda \cup \Lambda_1$ and Λ_2 by Λ_1, then (16.4) is true.

Let us now assume in addition to (16.4)

$$\overline{\rho_{21} P_h^1(\Lambda_2)} \subseteq P_h^1(\Lambda_1) \cup \{0\}. \tag{16.5}$$

Suppose now $\lambda_2 = (\alpha_2, \beta_2) = \rho_2(\alpha, \beta) = \rho_2 \lambda$, $\overline{\alpha \in P_h^1(\Lambda)}$ and $\beta \in P_v^1(\Lambda)$. From (16.3) we have that $\rho_1 \lambda = \rho_{21} \lambda_2$, and (16.5) implies $\rho_1 \lambda \in P^1(\Lambda_1)$. If U_1 is a component of Λ_1^c, then U_1 is a union of (edges and) components of Λ^c. Moreover, because $\rho_1 \lambda \in P^1(\Lambda_1)$, one of these components, call it U, has the property that $U(\lambda) \subseteq X^0(\lambda)$. If $U_1^\lambda = \pi_\lambda U_1$, then U_1^λ is a union of (edges and) components of $(\Lambda^0)^c$, and there is determined a natural map $U_1^\lambda \to R(U_1, \rho_1 \lambda)$ which is an isometry. It follows that $X^0(\lambda)$ is a closed Riemann surface and there is an equivalence $\xi(\lambda) \xrightarrow{\phi} \xi(\rho_1 \lambda)$.

Suppose now $\rho_2 \alpha$ is an extremal of $\overline{P_h^1(\Lambda_2)}$. Propositions 15.4–15.5 imply the complement of the critical set of $\mathscr{F}_v(\xi(\lambda))$ is one or two components, each biholomorphic to an annulus, and that $\mathscr{F}_v(\xi(\lambda))$ corresponds to the foliation of the annulus (annuli) by concentric circles. It follows that $\mathscr{F}_v(\xi(\rho, \lambda))$ has the same property. If we are in Case (a), $\sigma_{\xi(\rho, \lambda)}$ is a primitive Jenkins–Strebel differential. If we are in Case (b), then $\sigma_{\xi(\rho, \lambda)}$ projects on X/τ to a primitive Jenkins–Strebel differential. Indeed, if there are two annuli in X, these project to a single annulus in X/τ.

16.6. Proposition. *Let notations and assumptions be as above. If $\lambda = (\alpha, \beta) \in P^1(\Lambda_2)$ is such that α is an extremal of $\overline{\alpha \in P_h^1(\Lambda_2)}$, then $\xi(\rho_2, \lambda) = \xi$ satisfies either* (1) *(in Case* (a)*) σ_ξ is a primitive Jenkins–Strebel differential or* (2) *(in Case* (b)*) σ_ξ projects under τ to a primitive Jenkins–Strebel differential on X/τ, relative to the quotient $T(\xi)$ complex structure.*

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