DECAY AND REGULARITY FOR DISPERSIVE EQUATIONS WITH CONSTANT COEFFICIENTS

By

TOSHIHIKO HOSHIRO*

Dedicated to Professor Norio Shimakura

Abstract. This article discusses some smoothing estimates of the initial value problem for dispersive equations with constant coefficients. In particular, it is shown that a certain condition for the principal part of the symbol (see the assumption (1.3) below, which is equivalent to the one "of principal type" in the paper by Ben-Artzi and Devinatz [2]) is necessary and sufficient for the maximal smoothing in space-time.

1 Introduction and results

The purpose of this paper is to investigate local and global smoothing properties of dispersive equations with constant coefficients. In particular, we shall give a necessary and sufficient condition for the maximal smoothing in space-time. We consider the initial value problem of the form

$$(1.1) \qquad \begin{cases} i\dfrac{\partial u}{\partial t} + P(D)u = f(t,x), & (t,x) \in \mathbf{R}^{n+1}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \end{cases}$$

where $u = u(t,x)$, $D = i^{-1}(\partial/\partial x_1, \dots, \partial/\partial x_n)$, and the symbol $P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha$ is a real polynomial of degree $m > 1$.

Heuristically, if the equation is dispersive, a particle (in the sense of semi-classical approximation) can have any velocity, and the velocity depends on its frequency. As a result, some kind of homogenization occurs in space-time. So it is a reasonable expectation that the solution $u(t,x)$ might be a bit smoother than the initial datum $u_0(x)$. For more than a decade, this phenomenon has attracted a lot of attention and has been studied by many authors. In studies of the convergence of

*The author was supported in part by Grant-in-Aid for Scientific Research, Ministry of Education, Culture, Sports, Science and Technology, Japan (No. 13640187).

the Schrödinger semi-group, Sjölin [18] and Vega [19] proved that $u_0(x) \in L^2(\mathbf{R}^n)$ implies $\langle D\rangle^{1/2}u(t,x) \in L^2_{loc}(\mathbf{R}^{n+1})$ if $P(D) = \Delta$ and $f \equiv 0$. Constantin and Saut [5] generalized this local result for general higher order equations. They proved that if there exist positive constants c and R such that

$$\left|\frac{\partial P}{\partial \xi_j}(\xi)\right| \geq c(1+|\xi|)^{m-1} \cdot \frac{|\xi_j|}{|\xi|}, \tag{1.2}$$

for all $\xi \in \mathbf{R}^n$, $|\xi| \geq R$, $j = 1, \dots, n$, then $u_0(x) \in L^2(\mathbf{R}^n)$ implies $\langle D\rangle^{(m-1)/2}u(t,x) \in L^2_{loc}(\mathbf{R}^{n+1})$ for the solutions of the initial value problem (1.1) in the homogeneous case $f \equiv 0$. The first interest of this paper is devoted to the relationship between the above maximal smoothing and a dispersiveness assumption like (1.2).

The global decay and the smoothness estimates were obtained by Kato and Yajima [13] and Ben-Artzi and Klainerman [3] in the case of the Schrödinger equation. There are some delicate aspects related the estimates for the low frequency part of the solution, and they affect some involved assumptions for the exponent and the dimension. However, from their proofs, it is easy to see that for $s > 1/2$ and $T > 0$, there exists a constant C such that

$$\int_0^T \int_{\mathbf{R}^n} (1+|x|^2)^{-s} \left|\langle D\rangle^{1/2} e^{it\Delta} u_0(x)\right|^2 dxdt \leq C \|u_0\|^2_{L^2}.$$

The second interest of this paper is to generalize this kind of global result to the initial value problem (1.1).

Let us denote by $P_m(\xi)$ the principal part of the symbol, namely

$$P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha.$$

Also, instead of (1.2), we introduce the following dispersiveness condition:

$$|\nabla P_m(\xi)| \neq 0, \quad \text{for } \xi \in S^{n-1}, \tag{1.3}$$

where S^{n-1} is the unit sphere $\{\xi \in \mathbf{R}^n : |\xi| = 1\}$.

Our first result is

Theorem 1.1. *Let $\chi \in C_0^\infty(\mathbf{R}^n)$ be a function satisfying $\chi(x) \equiv 1$ for $x \in U$, where U is a non-void bounded open set. Suppose that there exist positive constants C and T such that, for the solutions of the initial value problem* (1.1) *in the homogeneous case $f \equiv 0$,*

$$\int_0^T \|\chi(\cdot)\, u(t,\cdot)\|^2_{H^{(m-1)/2}}\, dt \leq C \|u_0\|^2_{L^2}. \tag{1.4}$$

Then we have (1.3).

This theorem indicates a necessary condition for the local maximal smoothing. The converse is also true. We have the following global result.

Theorem 1.2. *Let $u(t,x)$ be a solution of the initial value problem* (1.1). *Assume that the principal part of symbol satisfies the dispersiveness condition* (1.3). *Then, for any $s > 1/2$ and $T > 0$, there exists a constant C such that*

$$\begin{aligned}(1.5)\qquad &\int_0^T \|\langle x\rangle^{-s}\langle D\rangle^{(m-1)/2}u(t,\cdot)\|_{L^2}^2 dt \\ &\le C\left(\|u_0\|_{L^2}^2 + \int_0^T \|\langle x\rangle^{s}\langle D\rangle^{-(m-1)/2}f(t,\cdot)\|_{L^2}^2 dt\right).\end{aligned}$$

Notice that the terms $\langle x\rangle^{-s}\langle D\rangle^{(m-1)/2}$ and $\langle x\rangle^{s}\langle D\rangle^{-(m-1)/2}$ in (1.5) can be replaced, respectively, by $\langle D\rangle^{(m-1)/2}\langle x\rangle^{-s}$ and $\langle D\rangle^{-(m-1)/2}\langle x\rangle^{s}$. This is because $\langle D\rangle^{(m-1)/2}\langle x\rangle^{-s}\langle D\rangle^{-(m-1)/2}\langle x\rangle^{s}$ and $\langle D\rangle^{-(m-1)/2}\langle x\rangle^{s}\langle D\rangle^{(m-1)/2}\langle x\rangle^{-s}$ are bounded in $L^2(\mathbf{R}^n)$.

We now compare the dispersiveness conditions (1.2) and (1.3). The assumption (1.2) is not invariant with respect to linear change of variables. For second order equations, (1.2) is equivalent to (1.3), if we admit a linear change of variables. However, if $m \ge 3$, (1.2) is more restrictive than (1.3). For example, $P(\xi) = \xi_1^3 + \xi_1^2\xi_2 - \xi_1\xi_2^2 + \frac{1}{6}\xi_2^3$ satisfies (1.3), but does not satisfy (1.2), even after some linear change of variables. To see this, let us observe that the assumption (1.2) allows at most two zeros of $\partial P/\partial\xi_j$ $(j = 1, 2)$ in the unit circle S^1, but $c_1\partial P/\partial\xi_1 + c_2\partial P/\partial\xi_2$ has four zeros, for any real constants c_1 and c_2. Moreover, from the point of the view of microlocal analysis, the condition (1.3) looks natural.

Remark 1.3. In the homogeneous initial problem (i.e., the case $f \equiv 0$ in (1.1)), the estimate similar to (1.5) under the assumption (1.3) was first obtained by Ben-Artzi and Devinatz [2]. They called the assumption equivalent to (1.3) "of principal type" (cf. (3.1) below and (3.2) in [2]). They also improved their result to the non-maximal smoothing in space-time. We refer the reader to [2] for details.

The global result of Theorem 1.2 can be improved to an optimal estimate of some Besov spaces, which were introduced by Agmon and Hörmander [1]. Let $D_k = \{x \in \mathbf{R}^n : 2^{k-1} \le |x| \le 2^k\}$ for $k \ge 1$, and $D_0 = \{x \in \mathbf{R}^n : |x| \le 1\}$. Denote by χ_{D_k} the characteristic function of D_k. We introduce the function spaces as follows:

$$B_T = \left\{u \in L^2_{loc}([0,T]\times\mathbf{R}^n) \,:\, \|u\|_{B_T} = \sum_{k=0}^{\infty} 2^{k/2}\|\chi_{D_k}u\|_{L^2([0,T]\times\mathbf{R}^n)} < \infty\right\},$$

and

$$B_T^* = \left\{ u \in L^2_{loc}([0,T] \times \mathbf{R}^n) : \|u\|_{B_T^*} = \sup_{k \geq 0} 2^{-k/2} \|\chi_{D_k} u\|_{L^2([0,T]\times\mathbf{R}^n)} < \infty \right\}.$$

Then it is easy to see that for $s > 1/2$, $\langle x \rangle^{-s} L^2([0,T] \times \mathbf{R}^n) \subset B_T$ and $B_T^* \subset \langle x \rangle^{s} L^2([0,T] \times \mathbf{R}^n)$. The refined result is

Theorem 1.4. *Let $u(t,x)$ be a solution of the initial value problem* (1.1). *Assume that the principal part of the symbol satisfies* (1.3). *Then for any positive number T, there exists a constant C such that*

$$\|\langle D \rangle^{(m-1)/2} u\|_{B_T^*} \leq C \left(\|u_0\|_{L^2} + \|\langle D \rangle^{-(m-1)/2} f\|_{B_T} \right). \tag{1.6}$$

We now explain our method of proof. Theorem 1.1 is shown by constructing one-parameter family of solutions, which contradicts the estimate (1.4). The argument of the proof is similar to those in Doi [8] and Ichinose [11], and it can be regarded as a version of the energy method initiated by Mizohata [15]. The proofs of Theorems 1.2 and 1.4 are based on Mourre's commutator method, which was also an essential tool in our proof of the results in our previous paper [10]. This method seems useful for deriving estimates for operators with variable coefficients. An improvement will be elaborated in a subsequent paper.

The paper is organized as follows. Section 2 contains an argument of the energy method and the proof of Theorem 1.1. In Section 3, we discuss Mourre's commutator method to derive a smoothing estimate related to an inhomogeneous initial value problem. This is the essential part of the proof of Theorem 1.2. Section 4 is devoted to the completion of the proof of Theorem 1.2. In Section 5, we extend the arguments in Section 3 in order to prove the improved estimate (1.6).

Remark 1.5. Since this paper was written, the author learned that Chihara [4] has showed the estimate (1.5) for a class of the equations with variable coefficients.

Acknowledgements. The author is grateful to the referee for useful comments and criticisms.

2 Energy method and necessity

We prove Theorem 1.1 by contradiction. More precisely, we are going to show that the estimate (1.4) cannot hold if $|\nabla P_m|$ has zeros in the unit sphere.

First take a point x_0 from the open set U. Also let ω_0 be a point of the unit sphere S^{n-1} and δ a positive constant to be chosen sufficiently small in the last part of this section. Take a real-valued smooth function $h(\xi) \in C_0^\infty(\mathbf{R}^n)$ satisfying

$h(\xi) = 1$ for $|\xi| \le 10^{-10}$, $h(\xi) = 0$ for $|\xi| \ge 2 \cdot 10^{-10}$, and let the initial datum with a positive large parameter λ be

$$u_0(x) = h\left(\delta^{-1}\left(\frac{D}{\lambda} - \omega_0\right)\right)\delta(x - x_0),$$

where $\delta(x)$ is the Dirac delta function, and $h\left(\delta^{-1}\left(\frac{D}{\lambda} - \omega_0\right)\right)$ is a pseudodifferential operator (more precisely, Fourier multiplier) defined in the usual way. Notice that the energy of $u_0(x)$ is concentrated in a neighborhood of $(x, \xi) = (x_0, \lambda\omega_0)$. We consider the solution $u = u(t, x)$ of the initial value problem (1.1) in the homogeneous case $f(t, x) \equiv 0$. Roughly speaking, the energy of $u(t, x)$ is concentrated in a neighborhood of the point $(x_0 - t\nabla P_m(\lambda\omega_0), \lambda\omega_0)$ for small t and large λ. Our first task of the present section is to describe this vague fact precisely.

Let $d = \mathrm{dist}(x_0, \partial U)$ and let $h_1(x) \in C^\infty(\mathbf{R}^n)$ be a real-valued function satisfying $h_1(x) = 1$ for $|x| \le d \cdot 10^{-10}$ and $h_1(x) = 0$ for $|x| \ge 2d \cdot 10^{-10}$. To localize the energy near the point $x_0 - t\nabla P_m(\lambda\omega_0)$, we introduce the following notation:

$$\phi = \phi(t, x, \xi) = h_1(x - x_0 + t\nabla P_m(\xi)),$$
$$\phi^{(\alpha)} = \phi^{(\alpha)}(t, x, \xi) = (D_x^\alpha h_1)(x - x_0 + t\nabla P_m(\xi)),$$

and

$$\Phi(\alpha)u = \phi^{(\alpha)}(t, x, D) \cdot h\left((2\delta)^{-1}\left(\frac{D}{\lambda} - \omega_0\right)\right)u,$$

where the symbol of the operator $h\left((2\delta)^{-1}\left(\frac{D}{\lambda} - \omega_0\right)\right)$ is

$$h_{2\delta} = h\left((2\delta)^{-1}\left(\frac{\xi}{\lambda} - \omega_0\right)\right).$$

The microlocal energy of the solution $u = u(t, x)$ is

$$\sigma_\lambda(u)(t) = \sum_{|\alpha| \le N} \lambda^{-|\alpha|/4} \|\Phi^*(\alpha)\, u(t, \cdot)\|^2,$$

where $\lambda > 0$, $\|\ \| = \|\ \|_{L^2(\mathbf{R}^n)}$, $\Phi^*(\alpha)$ is the adjoint operator of $\Phi(\alpha)$, and N is a positive integer chosen large below.

We have

Proposition 2.1. *Take N so that $n + m - N/4 \le 0$. Then there exists a positive constant C such that*

$$\sigma_\lambda(u)(t) \ge \exp\left(-C \cdot \lambda^{m-7/4} \cdot t\right)\{\sigma_\lambda(u)(0) - C\} \tag{2.1}$$

for λ large and $0 \le t \le \lambda^{-m+7/4}$.

Remark 2.2. Notice that $\|u_0\| = c\,(\delta\cdot\lambda)^{n/2}$ and $C^{-1}(\delta\cdot\lambda)^n \leq \sigma_\lambda(0) \leq C\,(\delta\cdot\lambda)^n$ as $\lambda \to \infty$. Thus, for some positive constant c,

$$\sigma_\lambda(u)(t) \geq c\,(\delta\cdot\lambda)^n, \quad \text{for } \lambda \text{ large and } 0 \leq t \leq \lambda^{-m+7/4}, \tag{2.2}$$

which means that a large part of the energy of the solution is concentrated near the point $(x_0 - t\nabla P_m(\lambda\omega_0), \lambda\omega_0)$.

Proof of Proposition 2.1. As in the proof of the Lax–Mizohata theorem [15], we prove (2.1) by deriving a differential inequality. First, observe that

$$\frac{\partial}{\partial t}\,\Phi(\alpha) = \Phi(\alpha)\,\frac{\partial}{\partial t} + i\sum_{j=1}^{n}\Phi(\alpha+j)\,P_m^{(j)}(D),$$

where $\alpha + j = (\alpha_1, \ldots, \alpha_j + 1, \ldots, \alpha_n)$ and $P_m^{(j)}(D)$ is an operator with symbol $P_m^{(j)}(\xi) = (\partial_{\xi_j})P_m(\xi)$. Hence, by taking the adjoint of the both sides of the equation, we have

$$\begin{aligned}\frac{d}{dt}\|\Phi^*(\alpha)u\|^2 &= 2\mathrm{Re}\left(\frac{\partial}{\partial t}(\Phi^*(\alpha)u), \Phi^*(\alpha)u\right)\\ &= 2\mathrm{Re}\,\{-i\sum_{j=1}^{n}(P_m^{(j)}(D)\,\Phi^*(\alpha+j)u,\ \Phi^*(\alpha)u)\\ &\quad + (\Phi^*(\alpha)\frac{\partial u}{\partial t},\ \Phi^*(\alpha)u)\}\\ &= 2\mathrm{Re}\,\{-i\sum_{j=1}^{n}(P_m^{(j)}(D)\,\Phi^*(\alpha+j)u,\ \Phi^*(\alpha)u)\\ &\quad + i(\Phi^*(\alpha)\,P(D)u,\ \Phi^*(\alpha)u)\}.\end{aligned} \tag{2.3}$$

On the other hand,

$$P(D)\,\Phi(\alpha) = \Phi(\alpha)\,P(D) + \sum_{j=1}^{n}\Phi(\alpha+j)\,P^{(j)}(D) + \sum_{2\leq|\alpha'|\leq m}\frac{1}{(\alpha')!}\,\Phi(\alpha+\alpha')\,P^{(\alpha')}(D);$$

hence

$$\begin{aligned}\Phi^*(\alpha)\,P(D)u &= P(D)\,\Phi^*(\alpha)u + \sum_{j=1}^{n}P^{(j)}(D)\,\Phi^*(\alpha+j)u\\ &\quad + \sum_{2\leq|\alpha'|\leq m}\frac{1}{(\alpha')!}\,P^{(\alpha')}(D)\,\Phi^*(\alpha+\alpha')u,\end{aligned} \tag{2.4}$$

where $P^{(j)}(D)$ and $P^{(\alpha)}(D)$ are operators with symbols $P^{(j)}(\xi) = (\partial_{\xi_j}P)(\xi)$ and $P^{(\alpha)}(\xi) = (\partial_\xi^\alpha P)(\xi)$, respectively.

Substitute (2.4) into the last term of the right side of (2.3). Observe that the value of $(P(D)\,\Phi^*(\alpha)u,\ \Phi^*(\alpha)u)$ is real, the operator $P^{(j)}(D) - P_m^{(j)}(D)$ is of order $m-2$, and the energy of the solution is concentrated in a neighborhood of $\xi = \lambda\,\omega_0$. Hence, for large λ,

$$\frac{d}{dt}\|\Phi^*(\alpha)\,u\|^2 \geq -C\cdot\Bigg[\lambda^{m-2}\Bigg(\sum_{j=1}^{n}\|\Phi^*(\alpha+j)\,u\|^2 + \|\Phi^*(\alpha)\,u\|^2\Bigg) - \sum_{2\leq|\alpha'|\leq m}\lambda^{m-|\alpha'|}(\|\Phi^*(\alpha+\alpha')\,u\|^2 + \|\Phi^*(\alpha)\,u\|^2) - 1\Bigg].$$

Multiplying the weight $\lambda^{-|\alpha|/4}$ and taking the summation with respect to α, we have for large λ

$$\frac{d}{dt}\sigma_\lambda(u)(t) \geq -C\lambda^{m-7/4}\sigma_\lambda(u)(t) - C\lambda^m\sum_{N\leq|\alpha|\leq N+m}\lambda^{-|\alpha|/4}\|\Phi^*(\alpha)\,u\|^2 - C, \tag{2.5}$$

since $m-|\alpha'|+|\alpha'|/4 \leq m-7/4$ for α' satisfying $|\alpha'|\geq 2$.

Notice now that

$$\|\Phi^*(\alpha)\,u\| = O(\lambda^{n/2})$$

for $0\leq t\leq \lambda^{-m+7/4}$ and $\lambda\geq\delta^{-1}$. This follows from the facts that $\|u\| = c\,(\delta\cdot\lambda)^{n/2}$ and that the operator $\Phi^*(\alpha)$ is uniformly bounded in $L^2(\mathbf{R}^n)$ for $0\leq t\leq\lambda^{-m+7/4}$ and $\lambda\geq\delta^{-1}$. Hence

$$\lambda^m\sum_{N\leq|\alpha|\leq N+m}\lambda^{-|\alpha|/4}\|\Phi^*(\alpha)\,u\|^2 = O(\lambda^{n+m-N/4}).$$

Finally, take N so that $n+m-N/4\leq 0$ and integrate both sides of the equation (2.5) to obtain the desired estimate (2.1). □

Now we are in position to start the proof of Theorem 1.1. First, observe that the operator $h\left((2\delta)^{-1}\left(\frac{D}{\lambda}-\omega_0\right)\right)\cdot\Phi^*(\alpha)$ is uniformly bounded in $L^2(\mathbf{R}^n)$ for $0\leq t\leq\lambda^{-m+7/4}$ and $\lambda\geq\delta^{-1}$. Hence there exists a positive constant c such that

$$\|\langle D\rangle^{(m-1)/2}\chi(x)u\| \geq c\left\|h\left((2\delta)^{-1}\left(\frac{D}{\lambda}-\omega_0\right)\right)\Phi^*(\alpha)\langle D\rangle^{(m-1)/2}\chi(x)u\right\|. \tag{2.6}$$

We now show that the right hand term of the inequality (2.6) is not small, until the time when the point $x_0 - t\nabla P_m(\lambda\omega_0)$ reaches a neighborhood of the boundary ∂U.

Set

$$T(\lambda) = \min\left(\lambda^{-m+7/4},\ \frac{d}{2\lambda^{m-1}\cdot\sup_{|\omega-\omega_0|\leq 4\delta}|\nabla P_m(\omega)|}\right). \tag{2.7}$$

It is clear that $0 \le t \le T(\lambda)$ and $\xi \in \operatorname{supp} h_{2\delta}$ imply $|t\nabla P_m(\xi)| \le d/2$. Also $(x,\xi) \in \operatorname{supp} \phi^{(\alpha)}$ implies $|x - x_0 + t\nabla P_m(\xi)| \le 2d \cdot 10^{-10}$. Thus $0 \le t \le T(\lambda)$, $\xi \in \operatorname{supp} h_{2\delta}$ and $(x,\xi) \in \operatorname{supp} \phi^{(\alpha)}$ imply

$$\begin{aligned} |x - x_0| &\le |x - x_0 + t\nabla P_m(\xi)| + |t\nabla P_m(\xi)| \\ &\le 2d \cdot 10^{-10} + d/2 \\ &< d, \end{aligned}$$

and hence $\chi(x) = 1$.

On the other hand, the inequality (2.2) shows that there exists α such that $\|\Phi^*(\alpha)\, u\| \ge c\,(\delta \cdot \lambda)^{n/2}$ for $0 \le t \le \lambda^{-m+7/4}$. Thus we have for some positive constant c_1

$$\begin{aligned} (2.8) \qquad & \left\| h\left((2\delta)^{-1}\left(\frac{D}{\lambda} - \omega_0\right)\right) \Phi^*(\alpha)\langle D\rangle^{(m-1)/2}\chi(x)u \right\| \\ &\ge \left\| \langle D\rangle^{(m-1)/2} h\left((2\delta)^{-1}\left(\frac{D}{\lambda} - \omega_0\right)\right) \chi(x)\, \Phi^*(\alpha)\, u \right\| - O(\lambda^{(n+m-3)/2}) \\ &\ge c_1 \lambda^{(m-1)/2} \|\Phi^*(\alpha)\, u\| - O(\lambda^{(n+m-3)/2}) \\ &\ge \frac{c_1}{2} \cdot c_2 \cdot \delta^{n/2} \cdot \lambda^{(n+m-1)/2}, \end{aligned}$$

for $0 \le t \le T(\lambda)$ and λ large.

Now we are in position to finish the proof of Theorem 1.1. Observe that the estimates (1.4), (2.6) and (2.8) imply that

$$\begin{aligned} c_0\,(\delta \cdot \lambda)^n &= C\|u_0\|^2 \\ &\ge \int_0^{T(\lambda)} \|\langle D\rangle^{(m-1)/2}\chi(x)\, u\|^2\, dt \\ &\ge (c \cdot c_1 \cdot c_2/2)^2 \cdot \delta^n \cdot T(\lambda) \cdot \lambda^{n+m-1}. \end{aligned}$$

Consequently,

$$T(\lambda) \le C \cdot \lambda^{-m+1},$$

for λ large. The constant C in the last inequality can be taken independent of $\omega_0 \in S^{n-1}$ and $\delta > 0$, even though the range of the parameter λ may depend on them. This is impossible if $|\nabla P_m|$ has zeros on the unit sphere. Indeed, by taking ω_0 so that $|\nabla P_m(\omega_0)| = 0$ and $\delta > 0$ sufficiently small, the definition (2.7) implies that, with any given positive number ϵ, the inequality

$$T(\lambda) \ge \epsilon^{-1} \cdot \lambda^{-m+1},$$

holds for λ large. This completes the proof of Theorem 1.1.

3 Mourre's commutator method

As noted in our previous paper [10], the global smoothing estimate (1.5) is closely related to the limiting absorption principle in spectral theory. It can be regarded as an estimate for the Fourier multiplier with singular symbol $(-\tau + P(\xi))^{-1}$ and depends heavily on the level set $\{x \in \mathbf{R}^n | \tau = P(\xi)\}$ for $\tau \in \mathbf{R}$. Notice that the dispersiveness assumption (1.3) implies that this level set is a hypersurface if $|\tau|$ is large, and its shape becomes similar to the one for the principal part $P_m(\xi)$ as $|\tau| \to \infty$. In the present section, we derive an estimate for the high frequency part (see Proposition 3.1 below) from the dispersiveness assumption (1.3).

We begin by introducing some notation. Let Ω_j ($j = \pm 1, \dots, \pm n$) be a covering of the unit sphere S^{n-1} (i.e., each Ω_j is open and $S^{n-1} \subset \cup_j \Omega_j$) such that $\partial_{\xi_j} P_m(\omega) > 0$ ($\omega \in \Omega_j$) and $\partial_{\xi_j} P_m(\omega) < 0$ ($\omega \in \Omega_{-j}$) for $j = 1, 2, \dots, n$. Take a partition of unity $\varphi_j(\omega)$ ($j = \pm 1, \dots, \pm n$) subordinate to this covering (i.e., for each j, $\varphi_j \in C_0^\infty(\Omega_j)$, $0 \le \varphi_j \le 1$ and $\sum_j \varphi_j(\omega) = 1$ for $\omega \in S^{n-1}$). Choosing $R > 0$ sufficiently large so that, for some positive constant c,

$$
|\partial_{\xi_j} P(\xi)| \ge c|\xi|^{m-1}, \quad \text{if } |\xi| \ge R, \text{ and } \frac{\xi}{|\xi|} \in \Omega_j \cup \Omega_{-j} \tag{3.1}
$$

for all j (notice that the hypothesis (1.3) allows us such a choice), we denote by φ_0 a function of class $C^\infty(\mathbf{R}_+)$ such that $0 \le \varphi_0 \le 1$, $\varphi_0(\lambda) = 1$ for $\lambda \ge 2R$, and $\varphi_0(\lambda) = 0$ for $0 < \lambda \le R$. Set

$$
\psi_j(\xi) = \varphi_0(|\xi|)\, \varphi_j\left(\frac{\xi}{|\xi|}\right),
$$

$$
Q_{j,(\epsilon,\epsilon')}(\xi) = i\epsilon' + \left\{\sum_{k=1}^{m} \frac{(i\epsilon)^k}{k!} \left(\partial_{\xi_{|j|}}^k P\right)(\xi)\right\} \cdot \operatorname{sgn} j,
$$

$$
G_{j,(\epsilon,\epsilon')}(D) = \psi_j(D)\,(i\partial_t + P(D) + Q_{j,(\epsilon,\epsilon')}(D))^{-1},
$$

for $j = \pm 1, \dots, \pm n$, and

$$
G = \sum_j G_{j,(\epsilon,\epsilon')}(D).
$$

Observe that $\operatorname{Im} Q_{j,(\epsilon,\epsilon')}(\xi) > 0$ for $\xi \in \operatorname{supp} \psi_j$; hence the operator G is well-defined as a Fourier multiplier, with small positive parameters ϵ and ϵ'.

The purpose of the present section is to prove

Proposition 3.1. *Assume* (3.1). *Then, for $s > 1/2$, the operator norm*

$$
\|\langle x\rangle^{-s} \langle D\rangle^{m-1} G \langle x\rangle^{-s}\|_{\mathcal{L}(L^2(\mathbf{R}^{n+1}))}
$$

has a bound which is independent of the small positive parameters ϵ and ϵ'.

Since it suffices to prove the assertion for each $G_{j,(\epsilon,\epsilon')}(D)$, we abbreviate below $Q_{j,(\epsilon,\epsilon')}(D)$ by $Q(D)$, $G_{j,(\epsilon,\epsilon')}(D)$ by $G(D)$, ψ_j by ψ, and $\| \ \|_{\mathcal{L}(L^2(\mathbf{R}^{n+1}))}$ by $\| \ \|$. Also, without loss of generality, we may assume $j > 0$. First we show the assertion in the special case $s = 1$. Then we give the proof of the general case $1/2 < s \leq 1$.

Denote

$$G_0 = (i\partial_t + P(D) + Q(D))^{-1}.$$

Observe that G_0 might not be well-defined, but that the operators $\psi(D)\,G_0$, $(\partial_{\xi_j}\psi)(D)\,G_0$ and $\psi^{1/2}(D)\,G_0$ are well-defined nevertheless. Let

$$F = \langle x\rangle^{-1}\langle D\rangle^{m-1}\psi(D)G_0\langle x\rangle^{-1}.$$

Then a direct calculation shows

$$\begin{aligned}\frac{dF}{d\epsilon} &= -\langle x\rangle^{-1}\langle D\rangle^{m-1}\psi(D)[x_j, G_0]\langle x\rangle^{-1}\\ &= -\langle x\rangle^{-1}x_j\langle D\rangle^{m-1}\psi(D)G_0\langle x\rangle^{-1} - \langle x\rangle^{-1}[\langle D\rangle^{m-1}\psi(D), x_j]G_0\langle x\rangle^{-1}\\ &\quad + \langle x\rangle^{-1}\langle D\rangle^{m-1}\psi(D)G_0\,x_j\langle x\rangle^{-1}.\end{aligned}$$

Observe now that $0 \leq \psi(\xi) \leq 1$ and the smoothness of ψ imply $\psi(\xi) \leq \psi^{1/2}(\xi)$ and $|\nabla\psi(\xi)| \leq C\,\psi^{1/2}(\xi)$ for some positive constant C. Hence we arrive at the following inequality, which plays an important role in our procedure.

$$(3.2)\qquad \left\|\frac{dF}{d\epsilon}\right\| \leq C\left(\|\langle D\rangle^{m-1}\psi^{1/2}(D)G_0\langle x\rangle^{-1}\| + \|\langle x\rangle^{-1}G_0\langle D\rangle^{m-1}\psi^{1/2}(D)\|\right).$$

The following lemma uses mainly the ideas of Mourre [16].

Lemma 3.1. *There exist positive constants ϵ_0 and C such that for $0 < \epsilon < \epsilon_0$ and $\epsilon' > 0$,*

$$(3.3)\qquad \|F\| \leq C\cdot\epsilon^{-1},$$

$$(3.4)\qquad \begin{aligned}&\|\langle D\rangle^{m-1}\psi^{1/2}(D)G_0\langle x\rangle^{-1}\| + \|\langle x\rangle^{-1}G_0\langle D\rangle^{m-1}\psi^{1/2}(D)\|\\ &\leq C\cdot\epsilon^{-1/2}\cdot\|F\|^{1/2}.\end{aligned}$$

Proof of Lemma 3.2. Both inequalities follow from the hypothesis (3.1). Indeed, the inequality (3.3) is a consequence of the fact that there exist positive constants ϵ_0 and C such that

$$\begin{aligned}|\langle\xi\rangle^{m-1}\psi(\xi)G_0(\xi)| &\leq \langle\xi\rangle^{m-1}\psi(\xi)\cdot|\mathrm{Im}\,Q(\xi)|^{-1}\\ &\leq C\cdot\epsilon^{-1}\end{aligned}$$

for $0 < \epsilon < \epsilon_0$.

To obtain (3.4), observe that

$$\|\langle D\rangle^{m-1}\psi^{1/2}(D)G_0\langle x\rangle^{-1}f\|^2 = (\langle x\rangle^{-1}G_0^*\langle D\rangle^{2m-2}\psi(D)G_0\langle x\rangle^{-1}f, f)$$

and

$$G_0^* - G_0 = 2G_0^*\left(i\epsilon' + \sum_{k=0}^{[(m-1)/2]} \frac{(i\epsilon)^{2k+1}}{(2k+1)!}\cdot(\partial_{\xi_j}^{2k+1}P)(D)\right) G_0.$$

Thus

$$\begin{aligned}
&\|\langle D\rangle^{m-1}\psi^{1/2}(D)G_0\langle x\rangle^{-1}f\|^2\\
&\le \frac{C}{\epsilon}\left|(\langle x\rangle^{-1}G_0^*\langle D\rangle^{m-1}\psi(D)\langle x\rangle^{-1}f, f) - (\langle x\rangle^{-1}\langle D\rangle^{m-1}\psi(D)G_0\langle x\rangle^{-1}f, f)\right|\\
&\le \frac{2C}{\epsilon}\|Ff\|\cdot\|f\|,
\end{aligned}$$

which implies the estimate for $\langle D\rangle^{m-1}\psi^{1/2}(D)G_0\langle x\rangle^{-1}$.

The evaluation of the operator norm of $\langle x\rangle^{-1}G_0\langle D\rangle^{m-1}\psi^{1/2}(D)$ is similar, after taking its adjoint. □

Now we are in position to finish the proof in the case $s = 1$. The argument is quite routine. First, (3.2), (3.3) and (3.4) yield

$$\left\|\frac{dF}{d\epsilon}\right\| \le C\cdot\epsilon^{-1}.$$

After integrating this inequality with respect to ϵ, we apply (3.2) and (3.4) again. Then

$$\left\|\frac{dF}{d\epsilon}\right\| \le C\cdot\epsilon^{-1/2}|\log\epsilon|^{1/2},$$

so that

$$\|F\| \le C,$$

which shows that the operator norm of $\langle x\rangle^{-1}\langle D\rangle^{m-1}G(D)\langle x\rangle^{-1}$ has a bound independent of ϵ and ϵ'. This completes the proof in the case $s = 1$.

Proof of Proposition 3.1 for the general case $1/2 < s \le 1$. Here we shall make some modifications in the above arguments. First put

$$G_1 = (\partial_t + P(D) + Q(\epsilon+\eta, \epsilon'))^{-1},$$

where $0 < \eta \ll 1$ and $Q(\epsilon+\eta,\epsilon') = Q_{j,(\epsilon+\eta,\epsilon')}(D)$. As in the case $s = 1$, the operators $\psi(D)\,G_1$, $(\partial_{\xi_j}\psi)(D)\,G_1$ and $\psi^{1/2}(D)\,G_1$ are well-defined, even though G_1 might not be well-defined. Set

$$W = \langle x\rangle^{-s}\,(\eta\,|x| + 1),$$

and

$$F_1 = W \langle D\rangle^{m-1}\psi(D)\, G_1 W.$$

Then

$$\frac{dF_1}{d\eta} = \frac{dW}{d\eta}\cdot \langle D\rangle^{m-1}\psi(D)\, G_1 W + W \langle D\rangle^{m-1}\psi(D)\cdot \frac{dG_1}{d\eta}\cdot W + W\langle D\rangle^{m-1}\psi(D)\, G_1 \cdot \frac{dW}{d\eta}. \tag{3.5}$$

The evaluation of the second term on the right side of (3.5) is similar to that in the case $s = 1$. In fact,

$$\|W\cdot x_j\| \le C\cdot \eta^{s-1}$$

and

$$\begin{aligned} W \langle D\rangle^{m-1}\psi(D)\, \frac{dG_1}{d\eta} W &= -W \langle D\rangle^{m-1}\psi(D)\,[x_j, G_1]\, W \\ &= -W\, x_j\, \langle D\rangle^{m-1}\psi(D)\, G_1\, W - W[\langle D\rangle^{m-1}\psi(D),\, x_j] G_1\, W \\ &\quad + W \langle D\rangle^{m-1}\psi(D)\, G_1\, x_j W \end{aligned}$$

imply

$$\begin{aligned} &\left\| W \langle D\rangle^{m-1}\psi(D)\, \frac{dG_1}{d\eta} W \right\| \\ &\le C\cdot \eta^{s-1}\left(\|\langle D\rangle^{m-1}\psi^{1/2}(D) G_1 W\| + \|W\, G_1 \langle D\rangle^{m-1}\psi^{1/2}(D)\|\right). \end{aligned}$$

On the other hand, concerning the other terms, let us notice that

$$\frac{dW}{d\eta} = (s-1)\cdot \langle x\rangle^{-s}\cdot |x|\cdot (\eta|x| + 1)^{s-2}$$

implies

$$\left\|\frac{dW}{d\eta}\right\| \le C\cdot \eta^{s-1}.$$

Thus, applying the argument of the proof of (3.4), we arrive at the following inequality, which plays an important role in our procedure:

$$\begin{aligned} \left\|\frac{dF_1}{d\eta}\right\| &\le C\cdot \eta^{s-1}\left(\|\langle D\rangle^{m-1}\psi^{1/2}(D) G_1 W\| + \|W\, G_1 \langle D\rangle^{m-1}\psi^{1/2}(D)\|\right) \\ &\le C'\cdot \eta^{s-3/2}\|F_1\|^{1/2}. \end{aligned} \tag{3.6}$$

Now we are in position to finish the proof for the general case. As in the case $s = 1$, the argument is quite routine. At first, let us assume that

$$\|F_1\| \le C\cdot \eta^{-\gamma}$$

for some positive constant γ. Substituting this inequality into (3.6) yields

$$\left\| \frac{dF_1}{d\eta} \right\| \leq C \cdot \eta^{s-3/2-\gamma/2},$$

so that

$$\|F_1\| \leq C \cdot \eta^{s-1/2-\gamma/2}.$$

Observe now that if $s > 1/2$,

$$s - 1/2 - \gamma/2 - (-\gamma) > s - 1/2 > 0,$$

which shows that the exponent increases a fixed amount after such a procedure. We conclude that, starting from the estimate $\|F_1\| \leq C \cdot \eta^{-1}$, after a finite number of steps the above procedure yields

$$\|F_1\| \leq C,$$

which immediately implies that the operator norm of $\langle x\rangle^{-s}\langle D\rangle^{m-1}\, G_{j,(\epsilon,\epsilon')}(D)\, \langle x\rangle^{-s}$ has a bound independent of ϵ and ϵ'. This completes the proof.

4 Global smoothing estimate

In this section, we discuss the reduction of Theorem 1.2 to Proposition 3.1. We derive the estimate (1.5) by reducing it to special cases as follows.

Case (i). The estimate of the high frequency part in the case $u_0(x) \equiv 0$

First, we explain the relationship between Proposition 3.1 and the initial value problem (1.1). The argument here was already described in our previous paper [10]. Denote

$$u_{\epsilon'}(t,x) = (i\partial_t + P(D) + i\epsilon')^{-1} f(t,x),$$

and

$$u(t,x) = \lim_{\epsilon' \downarrow 0} u_{\epsilon'}(t,x).$$

Here the operator $(i\partial_t + P(D) + i\epsilon')^{-1}$ is defined as a Fourier multiplier, and $u(t,x)$ has a sense (at least) when the right hand side has a limit in the space of distributions. Then $u(t,x)$ is a solution of (1.1) with $u_0(x) \equiv 0$ if $f(t,x) \equiv 0$ for $t < 0$. In fact, Cauchy's integral formula shows that

$$\begin{aligned} u(0,x) &= \lim_{\epsilon' \downarrow 0} \frac{1}{2\pi} \iint e^{-i\tau s}(-\tau + P(D) + i\epsilon')^{-1} f(s,x)\, ds d\tau \\ &= -i \int_{-\infty}^{0} f(s,x)\, ds. \end{aligned}$$

Thus, by taking the limit $\epsilon \to 0$ in Proposition 3.1, we arrive at the estimate of the high frequency part as follows. Let $u = u(t,x)$ be a solution of (1.1) with $u_0(x) \equiv 0$. Then, for $s > 1/2$, there exists a constant C_s such that

$$
\begin{aligned}
(4.1)\qquad & \int_0^\infty \|\langle x\rangle^{-s}\langle D\rangle^{(m-1)/2}\varphi_0(|D|)\,u(t,\cdot)\|_{L^2}^2\,dt \\
&\le C_s\int_0^\infty \|\langle x\rangle^{s}\langle D\rangle^{-(m-1)/2} f(t,\cdot)\|_{L^2}^2\,dt.
\end{aligned}
$$

Case (ii). The estimate (1.5) *in the case* $u_0 \equiv 0$

The estimate (4.1) does not include any information about the low frequency part of the solution $u(t,x)$. So it remains to prove that part of the estimate (1.5) This is a consequence of standard facts concerning the initial value problem (1.1) Indeed, the representation

$$
(4.2)\qquad \hat{u}(t,\xi) = -i\int_0^t e^{i(t-s)\,P(\xi)}\hat{f}(s,\xi)ds
$$

implies that

$$
(4.3)\qquad \sup_{0\le t\le T}\|u(t,\cdot)\|_{L^2} \le C_T\int_0^T \|f(t,\cdot)\|_{L^2}\,dt,
$$

for the solution of (1.1) with $u_0(x) \equiv 0$. Observe also that

$$
\int_0^T \|\langle x\rangle^{-s}\langle D\rangle^{(m-1)/2}(1-\varphi_0(|D|))\,u(t,\cdot)\|_{L^2}^2 dt \le C\cdot \sup_{0\le t\le T}\|\langle D\rangle^{-(m-1)/2}u(t,\cdot)\|_{L^2}^2
$$

and

$$
\begin{aligned}
\left(\int_0^T \|\langle D\rangle^{-(m-1)/2} f(t,\cdot)\|_{L^2}\,dt\right)^2 &\le T\int_0^T \|\langle D\rangle^{-(m-1)/2} f(t,\cdot)\|_{L^2}^2\,dt \\
&\le T\int_0^T \|\langle x\rangle^{s}\langle D\rangle^{-(m-1)/2} f(t,\cdot)\|_{L^2}^2\,dt.
\end{aligned}
$$

Thus we have for the low frequency part

$$
\begin{aligned}
(4.4)\qquad & \int_0^T \|\langle x\rangle^{-s}\langle D\rangle^{(m-1)/2}(1-\varphi_0(|D|))\,u(t,\cdot)\|_{L^2}^2\,dt \\
&\le C\int_0^T \|\langle x\rangle^{s}\langle D\rangle^{-(m-1)/2} f(t,\cdot)\|_{L^2}^2\,dt.
\end{aligned}
$$

Now (4.1) and (4.4) immediately imply the estimate (1.5) in the case $u_0(x) \equiv 0$.

Remark 4.1. We comment here on the relationship between the smoothing estimate (4.1) and the dispersiveness assumption (1.3). In general, smoothing estimates are closely related to the level set $\{\xi \in \mathbf{R}^n : P(\xi) = \tau\}$ for $\tau \in \mathbf{R}$. The assumption (1.3) implies that the level set is a hypersurface if $|\tau|$ is sufficiently large. Moreover, it becomes asymptotically similar as $|\tau| \to \infty$; hence the estimate (4.1) holds. However, the assumption (1.3) does not affect the level set for small $|\tau|$. It might be a point, or a point plus a hypersurface for some value of τ, so we can only get the trivial estimate (4.3). It seems rather difficult to discuss significant estimates for the low frequency part, since there are some involved features even in the case of the Schrödinger equation (cf. [3] and [13]).

Case (iii). The estimate (1.5) *in the case* $f(t,x) \equiv 0$

The result for the homogenous initial value problem comes from the above estimate for the inhomogenous case. The procedure below (which is called the T^*T method) was described already in our previous paper [10]. First let

$$u(t,x) = i \lim_{\epsilon \downarrow 0} \left\{ (i\partial_t + P(D) + i\epsilon)^{-1} - (i\partial_t + P(D) - i\epsilon)^{-1} \right\} f(t,x).$$

Then

$$\hat{u}(t,\xi) = \int_0^T e^{i(t-s)\,P(\xi)} \hat{f}(s,\xi) ds$$

if one extends the function f so that $f(t,x) \equiv 0$ for $t \notin [0, T]$. Thus the arguments up to this point imply the following estimate for the correspondence $f \mapsto u$ of the functions in $[0, T] \times \mathbf{R}^n$:

$$\int_0^T \|\langle x\rangle^{-s} \langle D\rangle^{(m-1)/2} u(t,\cdot)\|_{L^2}^2 \, dt \le C \int_0^T \|\langle x\rangle^{s} \langle D\rangle^{-(m-1)/2} f(t,\cdot)\|_{L^2}^2 \, dt.$$

On the other hand, it is clear that the function $u(t,x)$ satisfies the homogeneous equation $(i\partial_t + P(D))u \equiv 0$, and

$$u(0,x) = \int_0^T e^{-is\,P(D)} f(s,x)\, ds.$$

Denote the correspondence

$$f = f(t,x) \mapsto u_0 = u(0,x)$$

by U. Then its adjoint U^* becomes

$$u_0 \mapsto u = e^{it\,P(D)} u_0,$$

and the above correspondence $f \mapsto u$ becomes U^*U. Therefore, we have

$$\begin{aligned}
&\|Uf\|_{L^2}^2 \\
&= \left|(Uf, Uf)_{L^2(\mathbf{R}^n)}\right| \\
&= \left|(u, f)_{L^2([0,T]\times\mathbf{R}^n)}\right| \\
&\le \left(\int_0^T \|\langle x\rangle^{-s}\langle D\rangle^{(m-1)/2} u(t,\cdot)\|_{L^2}^2\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^T \|\langle x\rangle^{s}\langle D\rangle^{-(m-1)/2} f(t,\cdot)\|_{L^2}^2\,dt\right)^{\frac{1}{2}} \\
&\le C\int_0^T \|\langle x\rangle^{s}\langle D\rangle^{-(m-1)/2} f(t,\cdot)\|_{L^2}^2\,dt,
\end{aligned}$$

so that

$$\left|(U^*u_0, f)_{L^2([0,T]\times\mathbf{R}^n)}\right| \le C\|u_0\|_{L^2}\cdot\left(\int_0^T \|\langle x\rangle^{s}\langle D\rangle^{-(m-1)/2} f(t,\cdot)\|_{L^2}^2\,dt\right)^{1/2}.$$

Finally, observe that the last inequality is equivalent to

$$\int_0^T \|\langle x\rangle^{-s}\langle D\rangle^{(m-1)/2} U^*u_0\|_{L^2}^2\,dt \le C\,\|u_0\|_{L^2}^2,$$

which completes the proof of the estimate (1.5) in the case of the homogeneous initial value problem.

5 Besov space estimate

In this section, we prove the improved estimate (1.6). As preparation for the proof, we recall the arguments and the procedures mentioned in Section 3 and Section 4. First, the estimate for the homogeneous initial value problem comes from the estimate for the inhomogeneous one. This argument was discussed in the case (iii) of the previous section. Second, the estimate for the low frequency part immediately comes from the representation (4.2) and the fact that $B_T \subset L^2 \subset B_T^*$. The similar argument to see this was discussed in the case (ii) of the previous section. It remains to prove the estimate for the high frequency part as (4.1), and this is the essential part of the proof. We follow below the procedure of Jensen and Perry [12], where the arguments of the proof are based on those in Mourre [17].

Let

$$B_k = \{x \in \mathbf{R}^n : |x| \le 2^k\},$$

and denote by χ_{B_k} the characteristic function of the ball B_k. We use the notation introduced in Section 3. The required estimate is a consequence of the following bound for the operator G defined in Section 3.

Proposition 5.1. *Assume the dispersiveness assumption* (1.3). *Then*

$$\|\chi_{B_k}\langle D\rangle^{m-1} G\,\chi_{B_{k'}}\|_{\mathcal{L}(L^2(\mathbf{R}^{n+1}))} \le C\cdot 2^{(k+k')/2},$$

where C is a constant independent of ϵ, ϵ', k and k'.

Admitting this proposition for the moment, let us proceed to the proof of the estimate for the high frequency part. It is clear that

$$\begin{aligned}
&2^{-k/2}\|\chi_{D_k}\langle D\rangle^{m-1}G\,f\|_{L^2([0,T]\times\mathbf{R}^n)}\\
\le\,&2^{-k/2}\sum_{k'}\|\chi_{D_k}\langle D\rangle^{m-1}G\,\chi_{D_{k'}}\|_{\mathcal{L}(L^2(\mathbf{R}^{n+1}))}\cdot\|\chi_{D_{k'}}f\|_{L^2([0,T]\times\mathbf{R}^n)}\\
\le\,&2^{-k/2}\sum_{k'}\|\chi_{B_k}\langle D\rangle^{m-1}G\,\chi_{B_{k'}}\|_{\mathcal{L}(L^2(\mathbf{R}^{n+1}))}\cdot\|\chi_{D_{k'}}f\|_{L^2([0,T]\times\mathbf{R}^n)}\\
\le\,&2^{-k/2}\sum_{k'}C\cdot 2^{(k+k')/2}\cdot\|\chi_{D_{k'}}f\|_{L^2([0,T]\times\mathbf{R}^n)}\\
\le\,&C\sum_{k'}2^{k'/2}\cdot\|\chi_{D_{k'}}f\|_{L^2([0,T]\times\mathbf{R}^n)},
\end{aligned}$$

provided $f(t,x)\equiv 0$ for $t\notin[0,T]$. Hence we have

$$\left\|\langle D\rangle^{(m-1)/2}G\,f\right\|_{B_T^*}\le C\left\|\langle D\rangle^{-(m-1)/2}f\right\|_{B_T}.$$

Taking the limit of the last inequality as $\epsilon\to 0$ and $\epsilon'\to 0$, we arrive at the required estimate for the high frequency part

$$\left\|\langle D\rangle^{(m-1)/2}\varphi_0(|D|)\,u\right\|_{B_T^*}\le C\left\|\langle D\rangle^{-(m-1)/2}f\right\|_{B_T}.$$

This completes the proof of Theorem 1.4.

Proof of Proposition 5.1. Since $G=\sum_j G_j$, it suffices to show the estimate for each j. We adopt below the abbreviations in Proposition 3.1. For example, we denote G_j by G as in the proof of that proposition. The proof of Proposition 5.1 is accomplished by decomposing the balls B_k and $B_{k'}$ into slabs.

First let χ_ℓ ($\ell=0,\pm1,\pm2,\dots$) be the characteristic function of the slab $\{x\in\mathbf{R}^n:\ell\le x_j<\ell+1\}$, i.e., $\chi_\ell(x_j)=1$ for $\ell\le x_j<\ell+1$ and $\chi_\ell(x_j)=0$ otherwise. Moreover, let us denote $\chi_{<\ell}=\sum_{m=-\infty}^{\ell-1}\chi_m$ and $\chi_{\ge\ell}=\sum_{m=\ell}^{\infty}\chi_m$.

Lemma 5.1. *Assume the dispersiveness assumption* (1.3). *Then*

$$\|\chi_\ell\langle D\rangle^{m-1}G\,\chi_m\|_{\mathcal{L}(L^2(\mathbf{R}^{n+1}))}$$

has a bound independent of ϵ, ϵ', ℓ and m.

Proof. The following arguments are due to Mourre [17]. The point of the proof is to derive the estimate when $|\ell - m|$ is large. First observe that

$$
\begin{aligned}
(5.1)\qquad \chi_\ell \langle D\rangle^{m-1} G \chi_m &= \chi_\ell \langle D\rangle^{m-1} \psi(D) G_0 \chi_m \\
&= \chi_\ell \chi_{<m} \langle D\rangle^{m-1} \psi(D) G_0 \chi_m \\
&\quad + \chi_\ell \chi_{\geq m} \langle D\rangle^{m-1} \psi(D) G_0^* \chi_m \\
&\quad + \chi_\ell \chi_{\geq m} \langle D\rangle^{m-1} \psi(D) G_0^* (Q^* - Q) G_0 \chi_m .
\end{aligned}
$$

For the third term of the right side of (5.1), the argument in the proof of the estimate (3.4) implies that

$$
\begin{aligned}
&\|\chi_\ell \chi_{\geq m} \langle D\rangle^{m-1} \psi(D) G_0^* (Q^* - Q) G_0 \chi_m \| \\
\leq\ & \|\chi_\ell G_0^* \epsilon^{1/2} \langle D\rangle^{m-1} \psi^{1/2}(D)\| \cdot \|\epsilon^{-1/2} \psi^{1/2}(D) (Q^* - Q) G_0 \chi_m \| \\
\leq\ & C \|\chi_\ell G_0^* \epsilon^{1/2} \langle D\rangle^{m-1} \psi^{1/2}(D)\| \cdot \|\epsilon^{1/2} \psi^{1/2}(D) \langle D\rangle^{m-1} G_0 \chi_m \| \\
\leq\ & C' \|\chi_\ell \langle D\rangle^{m-1} \psi(D) (G_0^* - G_0) \chi_\ell \|^{1/2} \cdot \|\chi_m \langle D\rangle^{m-1} \psi(D) (G_0^* - G_0) \chi_m \|^{1/2}.
\end{aligned}
$$

Hence one sees that it has a bound independent of ℓ and m. Concerning the other terms we have

Lemma 5.2. *Assume the dispersiveness assumption* (1.3). *Then*

$$\|\chi_{<m} \langle D\rangle^{m-1} \psi(D) G_0 \chi_m \|_{\mathcal{L}(L^2(\mathbf{R}^{n+1}))}$$

and

$$\|\chi_{\geq m} \langle D\rangle^{m-1} \psi(D) G_0^* \chi_m \|_{\mathcal{L}(L^2(\mathbf{R}^{n+1}))}$$

have bounds independent of ϵ, ϵ' *and* m.

Proof. As in Proposition 3.1, the proof is based on a differential inequality. Here we consider the first operator norm. The proof of the second one is quite similar. Let G_1 be as in the proof of Proposition 3.1 for the general case $1/2 < s \leq 1$. For small positive η, denote

$$F_2 = \chi_{<m} e^{\eta (x_j - m)} \langle D\rangle^{m-1} \psi(D) G_1 \chi_m .$$

Then

$$
\begin{aligned}
\frac{dF_2}{d\eta} &= \chi_{<m} e^{\eta (x_j - m)} \cdot (x_j - m) \cdot \langle D\rangle^{m-1} \psi(D) G_1 \chi_m \\
&\qquad - \chi_{<m} e^{\eta (x_j - m)} \langle D\rangle^{m-1} \psi(D) [x_j, G_1] \chi_m \\
&= \chi_{<m} e^{\eta (x_j - m)} [x_j, \langle D\rangle^{m-1} \psi(D)] G_1 \chi_m \\
&\qquad + \chi_{<m} e^{\eta (x_j - m)} \langle D\rangle^{m-1} \psi(D) G_1 \cdot (x_j - m) \cdot \chi_m .
\end{aligned}
$$

Concerning the first term on the right side of the last inequality, the argument in the proof of the estimate (3.4) shows that

$$\begin{aligned}
&\|\chi_{<m} e^{\eta(x_j-m)}[x_j, \langle D\rangle^{m-1}\psi(D)]\, G_1\, \chi_m\| \\
&\le C\,\|\langle D\rangle^{m-1}\psi^{1/2}(D)\, G_1\, \chi_m\| \\
&\le C'\cdot\eta^{-1/2}\cdot\|\chi_m\langle D\rangle^{m-1}\psi(D)\, G_1\, \chi_m\|^{1/2} \\
&\le C''\cdot\eta^{-1/2}.
\end{aligned}$$

On the other hand, concerning the second term, we have

$$\begin{aligned}
&\|\chi_{<m} e^{\eta(x_j-m)}\langle D\rangle^{m-1}\psi(D)\, G_1\cdot(x_j-m)\cdot\chi_m\| \\
\le\ &\|\chi_{<m} e^{\eta(x_j-m)}\langle D\rangle^{m-1}\psi(D)\, G_1\chi_m\| \\
\le\ &\|F_2\|.
\end{aligned}$$

Also, it is clear that $\|F_2\| \le C\cdot\eta^{-1}$. Thus, from these observations, one can easily conclude that $\|F_2\| \le C$, which implies the desired bound.

End of the proof of Proposition 5.1. Now we are in position to finish the proof of Proposition 5.1. It is quite straightfoward. Indeed, denoting

$$\begin{aligned}
F_{\ell,m} &= \chi_\ell\,\chi_{B_k}\langle D\rangle^{m-1} G\,\chi_{B_{k'}}\,\chi_m \quad (-2^k\le\ell\le 2^k,\ -2^{k'}\le\ell\le 2^{k'}), \\
g_\ell &= \chi_\ell\,\chi_{B_k}\langle D\rangle^{m-1} G\,\chi_{B_{k'}}\, f, \qquad f_m = \chi_m\chi_{B_{k'}}\, f,
\end{aligned}$$

we obtain

$$\begin{aligned}
\|g\|^2_{L^2(B_k\times\mathbf{R})} &= \sum_{-2^k\le\ell\le 2^k}\|g_\ell\|^2 \\
&= \sum_{-2^k\le\ell\le 2^k}\Big\|\sum_{-2^{k'}\le m\le 2^{k'}} F_{\ell,m}\, f_m\Big\|^2 \\
&\le \sum_{-2^k\le\ell\le 2^k}\ \sum_{-2^{k'}\le m\le 2^{k'}}\|F_{\ell,m}\|^2\cdot\sum_{-2^{k'}\le m\le 2^{k'}}\|f_m\|^2 \\
&\le C\cdot 2^{k+k'}\cdot\|f\|^2_{L^2(B_{k'}\times\mathbf{R})}.
\end{aligned}$$

This completes the proof of Proposition 5.1.

REFERENCES

[1] S. Agmon and L. Hörmander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Analyse Math. **30** (1976), 1–38.

[2] M. Ben-Artzi and A. Devinatz, *Local smoothing and convergence properties of Schrödinger type equations*, J. Funct. Anal. **101** (1991), 231–254.

[3] M. Ben-Artzi and S. Klainerman, *Decay and regularity for the Schrödinger equation*, J. Analyse Math. **58** (1992), 25–37.

[4] H. Chihara, *Smoothing effects of dispersive pseudodifferential equations*, Comm. Partial Differential Equations **27** (2002), 1953–2005.

[5] P. Constantin and J.C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. **1** (1988), 413–439.

[6] W. Craig, T. Kappeler and W. Strauss, *Microlocal dispersive smoothing for the Schrödinger equation*, Comm. Pure Appl. Math. **48** (1995), 769-860.

[7] H. Cycon and P. Perry, *Local time-decay of high energy scattering states for the Schrödinger equation*, Math. Z. **188** (1984), 125–142.

[8] S. Doi, *Smoothing effects of Schrödinger evolution groups on Riemannian manifolds*, Duke Math. J. **82** (1996), 679–706.

[9] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive equations*, Comm. Math. Phys. **144** (1992), 163–188.

[10] T. Hoshiro, *Mourre's method and smoothing properties of dispersive equations*, Comm. Math. Phys. **202** (1999), 255–265.

[11] W. Ichinose, *Some remarks on the Cauchy problem for Schrödinger type equations*, Osaka J. Math. **21** (1984), 565–581.

[12] A. Jensen and P. Perry, *Commutator method and Besov space estiamtes for Schrödinger operators*, J. Operator Theory **14** (1985), 181–188.

[13] T. Kato and K. Yajima, *Some examples of smooth operators and the associated smoothing effect*, Rev. Math. Phys. **1** (1989), 481–496.

[14] C. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Koteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620.

[15] S. Mizohata, *Some remarks on the Cauchy problem*, J. Math. Kyoto Univ. **1** (1961), 109–127.

[16] E. Mourre, *Absence of singular continuous spectrum for certain selfadjoint operators*, Comm. Math. Phys. **78** (1978), 391–408.

[17] E. Mourre, *Operateur conjugués et propriétés de propagation*, Comm. Math. Phys. **91** (1983), 279–300.

[18] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55** (1987), 699–715.

[19] L. Vega, *The Schrödinger equation: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. **102** (1988), 874–878.

T. Hoshiro
DEPARTMENT OF MATHEMATICS
HIMEJI INSTITUTE OF TECHNOLOGY
SHOSHA 2167, HIMEJI, HYOGO 671-2201, JAPAN
email: hoshiro@sci.himeji-tech.ac.jp

(Received April 9, 2002 and in revised form July 22, 2002)