QUASICONFORMAL MAPPINGS AND PERIODIC SPECTRAL PROBLEMS IN DIMENSION TWO

By

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Abstract. We study spectral properties of second-order elliptic operators with periodic coefficients in dimension two. These operators act in periodic simply-connected waveguides, with either Dirichlet, or Neumann, or the third boundary condition. The main result is the absolute continuity of the spectra of such operators. The cornerstone of the proof is an isothermal change of variables, reducing the metric to a flat one and the waveguide to a straight strip. The main technical tool is the quasiconformal variant of the Riemann mapping theorem.

1 Introduction

According to a common belief, second-order elliptic differential operators with periodic coefficients should not have degenerate bands in their spectra or, in other words, their spectra should be purely absolutely continuous (see [8], [19], [32]). The first rigorous proof of this fact was given by L. Thomas in [33] for the Schrödinger operator $-\Delta + V$ with a periodic real-valued potential V. Further developments in this area were driven by attempts to consider operators with ever increasing "strength" of the periodic perturbation, i.e., to pass from zero order (as in [33]) to first- and second-order perturbations, and ultimately to tackle the absolute continuity of the elliptic operator

$$H = \sum_{j,l=1}^{d} (D_j - a_j) g_{jl} (D_l - a_l) + V, \quad D_j = -i\partial_j, \tag{1.1}$$

with a periodic variable metric $\{g_{jl}\} = \mathbf{G}$ and a magnetic potential $\mathbf{a} = \{a_l\}$, for arbitrary $d \geq 2$.

The case of first-order perturbations, i.e., that of constant $\mathbf{G}$'s and variable $\mathbf{a}$'s, was handled in [15] (small $\mathbf{a}$'s), [6], [7] ($d = 2$) and [30] (arbitrary $d \geq 2$). If the metric is conformal, i.e., the matrix $\mathbf{G}$ is given by a scalar multiple of the identity

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matrix, then the problem can be easily reduced to the case of a constant metric. This situation is discussed in [8]. The most difficult case, that of a general variable $\mathbf{G}$, remained unaccessible until the paper [25], where it was resolved for infinitely differentiable $\mathbf{G}$'s, $\mathbf{a}$'s and $d = 2$. An important breakthrough was made in recent work [11], where the absolute continuity was proved in all dimensions $d \geq 2$, but with the additional requirement of the reflectional symmetry of the operator. Without the symmetry assumption, the question is still open. At present, it is known only that without smoothness assumptions on the coefficients the absolute continuity may break down. An appropriate example with $V = 0$, $\mathbf{a} = 0$ and "non-smooth" $\mathbf{G}$ was constructed in [10].

Most of the progress was achieved in the two-dimensional case, which we shall discuss in more detail. The paper [25], as well as all earlier papers on absolute continuity, relied on the approach suggested by L. Thomas in [33]. Later it was observed in [19] that the general periodic metric can be reduced to a conformal (or scalar) periodic one by a suitable isothermal change of variables. This allows one to reproduce the results of [25] under much weaker assumptions, by reducing the problem to the one considered in [6], [7], [8]. This approach was exploited in [4] for an operatoı of the form (1.1) with a delta-like periodic potential supported by a periodic system of curves. Even more general perturbations are studied in [27], [28].

The main aim of the present paper is to prove absolute continuity for an operator of the form (1.1) with non-constant periodic coefficients defined in a periodic domain $\Omega \subset \mathbb{R}^2$ (usually referred to as a waveguide) with the Dirichlet or "natural" boundary conditions. In these two cases, we use the notation H_D or H_N for the operator at hand. By natural boundary conditions, we mean either Neumann or the third boundary condition. As in [4], we also include in the operator a delta-like periodic perturbation supported by smooth curves, which allows us to study the cases of the Neumann condition and the third boundary condition simultaneously. This problem with Dirichlet and Neumann conditions was considered in [31] in somewhat restricted generality. In the present paper, the smoothness conditions on the coefficients are substantially relaxed; and the case of the third boundary-value problem is also treated. This progress is a result of a different approach to the problem: instead of Morame's techniques, we now use the isothermal coordinate change. A parallel approach is used in [29], which focuses on this problem in a slightly different setting. Similar spectral questions for a periodic Helmholz-type operator were studied in [9]. More general elliptic operators with constant coefficients and a periodic potential perturbation are considered in [18], Theorem 5.4.9. We refer to the book [18], Section 5.4 for further relevant references.

The cornerstone of our method is an isothermal change of coordinates, which reduces the metric to a conformal one. The new coordinates are given by functions satisfying the Beltrami equation (see Section 6 below) with a dilatation coefficient q determined by the matrix G (see (6.2)), i.e., they define a q-quasiconformal mapping. We use a q-quasiconformal change of variables which maps the waveguide Ω homeomorphically onto a straight strip. The existence of such a transformation follows from the "quasiconformal" version of Riemann's mapping theorem (see, e.g., [17], Ch. 1). Here a key point is that the uniqueness part of Riemann's mapping theorem guarantees a certain natural "periodicity" property of the above homeomorphism (see (6.4) below). This ensures that the transformed operators H_D and H_N have periodic coefficients.

To be precise, the quasiconformal Riemann mapping theorem alone is not sufficient for our needs. It is also necessary to study the boundary behaviour of the quasiconformal homeomorphism. Moreover, we allow the boundary of the waveguide Ω to have corners and inward peaks, which give rise to singularities of the map. For conformal maps, this circle of questions is exhaustively studied in the relevant literature (see, e.g., [26]) and is usually associated with the names of P. Koebe, O. D. Kellogg and S. E. Warschawski. The extension to the quasiconformal case is either well-known or evident to experts in complex analysis. Nevertheless, we have decided to provide precise statements and complete proofs, since we have been unable to find them in the literature in the form readily suitable for our purposes. These results along with the quasiconformal analogue of Riemann's mapping theorem are collected in Section 7.

As mentioned earlier, periodic isothermal coordinates were used in [19], [4] to establish the absolute continuity of the operator H acting on $L^2(\mathbb{R}^2)$, which we denote below by H_F. They were constructed in [19] in the following way. Using results from [3], [35], one can define an analytic structure on the two-dimensional torus with the help of local q-quasiconformal coordinates. It then follows from the theory of Riemann surfaces that integration of the analytic differential on the torus leads to the desired isothermal coordinates in $\mathbb{R}^2$. Note that the above analytic differential exists and is unique up to a constant factor, since the torus is a surface of genus one. The secondary aim of our paper is to give another, more direct, proof of the existence of such coordinates. Instead of geometrical considerations of [19], we rely on the well-known fact (see [1], [2], [3], [35]) that there exists a unique q-quasiconformal homeomorphism of $\mathbb{C}$ which preserves the points 0, 2π and ∞. Similarly to the waveguide case, a crucial observation here is that uniqueness combined with the periodicity of G automatically implies the required periodicity of the homeomorphism at hand (see Theorem 6.1). Using the isothermal coordinates as in [19] and [4], we obtain the absolute continuity of the operator H_F. Moreover, applying a stronger regularity result for the coordinate change, we are able to relax the smoothness restrictions on $\mathbf{G}$ in comparison with [19], [4]: for instance, under the assumption $\det \mathbf{G} = 1$, the matrix $\mathbf{G}$ does not need to be Hölder-continuous, but only bounded.

After reduction to a conformal metric, the absolute continuity of H_F results immediately from the earlier papers [7], [8]. As to the operators H_D and H_N, the isothermal change of variables reduces them to operators of the same type acting on a straight strip in $\mathbb{R}^2$, with a scalar constant $\mathbf{G}$. From this point on, we follow the strategy suggested in [31]. It consists in further reduction to an auxiliary operator H_P with *periodic* conditions on the boundary of the strip of the double width. Then a reference to [4] secures the required absolute continuity.

The paper is organised as follows. Section 2 contains some preliminary material and the precise statements of all main results of the paper (Theorems 2.6 and 2.9). In Section 3, we make first reductions simplifying the problem. In particular, it is shown that it suffices to prove the main results for the case $\det \mathbf{G} = 1$. The important Section 4 is devoted to a detailed description of the isothermal change of variables. Two central theorems of this section (Theorems 4.1 and 4.2) are proved in Section 6 after having been translated into the language of the quasiconformal maps. The proof of the main results is completed in Section 5. The necessary information on the quasiconformal maps and their boundary properties is collected in Section 7.

2 Main results

2.1 Notation. Lattices and domains. Let $\mathbf{e}_1, \mathbf{e}_2$ be the canonical basis in $\mathbb{R}^2$. Along with the standard two-dimensional square lattice $\mathbf{\Gamma} = (2\pi\mathbb{Z})^2$, introduce two "one-dimensional" lattices:

$$\gamma_1 = (2\pi\mathbb{Z}) \times \{0\} = \{2\pi n\mathbf{e}_1, n \in \mathbb{Z}\},$$
$$\gamma_2 = \{0\} \times (2\pi\mathbb{Z}) = \{2\pi n\mathbf{e}_2, n \in \mathbb{Z}\}.$$

We say that a function f is γ_j-periodic (resp., $\mathbf{\Gamma}$-periodic), if $f(\mathbf{x} + 2\pi n\mathbf{e}_j) = f(\mathbf{x})$ a.a. $\mathbf{x}$ and all $n \in \mathbb{Z}$ (resp., $f(\mathbf{x} + \boldsymbol{\xi}) = f(\mathbf{x})$ a.a. $\mathbf{x}$ and all $\boldsymbol{\xi} \in \mathbf{\Gamma}$).

For any set $\mathcal{F} \subset \mathbb{R}^2$, define its translates as follows:

$$\mathcal{F}^{(\mathbf{n})} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} - 2\pi\mathbf{n} \in \mathcal{F}\}, \quad \mathbf{n} \in \mathbb{Z}^2,$$
$$\mathcal{F}^{(n)} = \mathcal{F}^{(\mathbf{n})}, \quad \text{with } \mathbf{n} = (n, 0), \quad n \in \mathbb{Z}.$$

We say that the set $\mathcal{F}$ is $\boldsymbol{\Gamma}$*-periodic* if $\mathcal{F} = \mathcal{F}^{(\mathbf{n})}$ for all $\mathbf{n} \in \mathbb{Z}^2$ and that $\mathcal{F}$ is γ_1*-periodic* if $\mathcal{F} = \mathcal{F}^{(n)}$ for all $n \in \mathbb{Z}$. When it does not lead to confusion, instead of "γ_1-" or "$\boldsymbol{\Gamma}$-periodicity" we use the term "periodicity". Similarly, we define the periodicity of sets $\mathcal{F}$ on the cylinder

$$\mathcal{C} = \mathbb{R}^2/\gamma_2.$$

Precisely, $\mathcal{F} \subset \mathcal{C}$ is said to be periodic (or γ_1-periodic) if $\mathcal{F}^{(n)} = \mathcal{F}$ for all $n \in \mathbb{Z}$.

We are going to study periodic operators in two-dimensional domains of three types: on the entire plane $\mathbb{R}^2$, on the cylinder $\mathcal{C}$, or on a periodic domain $\Omega \subset \mathbb{R}^2$. Our model periodic domain will be the straight strip

$$\mathcal{S} = \mathcal{S}_{\mathrm{d}} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < \pi\,\mathrm{d}\}, \quad \mathrm{d} > 0. \tag{2.1}$$

Obviously, the cylinder $\mathcal{C}$ can be viewed as the closed strip $\overline{\mathcal{S}_2}$ with identified lower and upper boundaries. In general, we assume that Ω is as described below.

Definition 2.1. We say that a domain $\Omega \subset \mathbb{R}^2$ is *admissible* if there exists a finite collection of bounded domains $\mathcal{E}_j,\ j = 1, 2, \dots, N$ with Lipschitz boundaries such that the set

$$\mathcal{E}_0 = \bigcup_{j=1}^{N} \mathcal{E}_j \tag{2.2}$$

is connected, and $\Omega = \bigcup_{n\in\mathbb{Z}} \mathcal{E}_0^{(n)}$.

Note that the domain Ω is automatically γ_1-periodic and bounded in the direction $\mathbf{e}_2$. We also point out that Ω satisfies the interior cone condition (see, e.g., [24], Section 1.3.3), since the domains $\mathcal{E}_j, j = 1, 2, \dots, N$ do so. Without loss of generality, we **always assume that**

$$\mathcal{E}_0 \supset \{\mathbf{x} \in \Omega : -1 \le x_1 \le 2\pi + 1\}. \tag{2.3}$$

Certainly, the choice of the domains $\mathcal{E}_j$ for a given admissible Ω is not unique.

As a rule, we try to treat all three cases simultaneously, and therefore we use the notation Λ either for $\mathbb{R}^2$ or $\mathcal{C}$ or Ω. Identifying the points that differ by a vector of the lattice, we define

$$\Upsilon = \begin{cases} \Omega/\gamma_1, & \text{if } \Lambda = \Omega; \\ \mathbb{T}^2 = \mathbb{R}^2/\boldsymbol{\Gamma}, & \text{if } \Lambda = \mathbb{R}^2 \text{ or } \mathcal{C}. \end{cases} \tag{2.4}$$

We also introduce the *fundamental domains*

$$\mathcal{O} = \begin{cases} (0,2\pi) \times (0,2\pi), & \text{if } \Lambda = \mathbb{R}^2 \text{ or } \mathcal{C}, \\ \{\mathbf{x} \in \Omega : 0 < x_1 < 2\pi\}, & \text{if } \Lambda = \Omega. \end{cases}$$

In the case $\Lambda = \Omega$, the set $\mathcal{O}$ may be nonconnected.

Similarly to Ω, for our purposes it will be necessary to view $\mathbb{R}^2$ as being covered by bounded domains. More precisely, we assume that there are finitely many bounded domains $\mathcal{E}_j, j = 1, 2, \dots, N$, with Lipschitz boundaries such that

$$\overline{\mathcal{O}} \subset \mathcal{E}_0 = \bigcup_{j=1}^{N} \mathcal{E}_j.$$

System of curves. We also need to introduce a system of curves $\mathbf{\Sigma}$ in Λ associated with the covering of Λ by $\mathcal{E}_j$'s and their translates. Below, by a "$\mathsf{C}^{m+\alpha}$-arc", $m \in \mathbb{N}$, $0 \le \alpha < 1$, we mean a Jordan arc in $\mathbb{R}^2$ which is parametrised by a $\mathsf{C}^{m+\alpha}$-smooth function $\psi : [0,1] \to \mathbb{R}^2$ such that $|\psi'(t)| > 0$, $t \in [0,1]$.

Definition 2.2. Let $\Lambda = \mathbb{R}^2$ or Ω, and let the parameter $\mathbf{n}$ vary over the set $\mathbb{Z}^2$ (for $\Lambda = \mathbb{R}^2$) or $\mathbb{Z} \times \{0\}$ (for $\Lambda = \Omega$). Let $\ell_j \subset \overline{\mathcal{E}_j}$, $j = 1, 2, \dots N$, be a finite set of C^1-arcs such that

$$\ell_j^{(\mathbf{n})} \cap \ell_j = \emptyset, \quad \forall \mathbf{n} \neq 0.$$

Then *a (periodic) system of curves* $\mathbf{\Sigma}$ *in* Λ is a family $\{\Sigma_j\}$ of the closed sets $\Sigma_j \subset \overline{\Lambda}$ of the form

$$\Sigma_j = \bigcup_{\mathbf{n}} \ell_j^{(\mathbf{n})}. \tag{2.5}$$

Let $\Lambda = \mathcal{C}$. Then the set $\mathbf{\Sigma}$ is a *system of curves in* $\mathcal{C}$ if there exists a system of curves $\mathbf{M}$ in $\mathbb{R}^2$ such that $\mathbf{\Sigma} = \mathbf{M}/\gamma_2$, the latter being defined as the family $\{\mathrm{M}_j/\gamma_2\}$, $j = 1, 2, \dots, N$.

Similarly, one defines

$$\Xi = \begin{cases} \mathbf{\Sigma}/\gamma_1, & \text{if } \Lambda = \Omega \text{ or } \mathcal{C}, \\ \mathbf{\Sigma}/\mathbf{\Gamma}, & \text{if } \Lambda = \mathbb{R}^2. \end{cases} \tag{2.6}$$

Remark 2.3. (i) In the above definition, any two curves ℓ_j, ℓ_l or their parts can coincide. This means, in particular, that $\mathbf{\Sigma}$ may contain several "copies" of the same curve. The curves are also allowed to meet at zero angle.

(ii) Definition 2.2 prescribes exactly one curve ℓ_j for each of the domains $\mathcal{E}_j$. The seemingly more general situation in which there are finitely many curves in $\overline{\mathcal{E}_j}$ is easily reduced to the original one by adding a suitable number of copies of $\mathcal{E}_j$ in (2.2).

For any bounded set $\mathcal{E} \subset \overline{\Lambda}$, we denote

$$\Sigma_{\mathcal{E}} = \{\Sigma_{j,\mathcal{E}}\}, \quad \Sigma_{j,\mathcal{E}} = \bigcup_{\mathbf{n}} \ell_j^{(\mathbf{n})}, \tag{2.7}$$

where $\mathbf{n}$ are such that $\ell_j^{(\mathbf{n})} \cap \mathcal{E} \neq \emptyset$. Obviously, each $\Sigma_{j,\mathcal{E}}$ contains finitely many translates of ℓ_j.

Function spaces. Apart from the standard notational conventions for classes of differentiable functions, we use the notation $\mathsf{W}^{s,p}(\mathcal{C})$, $s = 0, 1, p \geq 1$, for the space of all γ_2-periodic functions from $\mathsf{W}^{s,p}_{\mathrm{loc}}(\mathbb{R}^2)$ equipped with the norm $\|\cdot\|_{\mathsf{W}^{s,p}(\mathcal{S}_2)}$. Similarly, $\mathsf{W}^{s,p}(\Upsilon)$, $s = 0, 1$, $p \geq 1$, is the space of all γ_1-periodic (for $\Lambda = \Omega$) or $\mathbf{\Gamma}$-periodic (for $\Lambda = \mathbb{R}^2$ or $\Lambda = \mathcal{C}$) functions $u \in \mathsf{W}^{s,p}_{\mathrm{loc}}(\Lambda)$ with the norm $\|\cdot\|_{\mathsf{W}^{s,p}(\mathcal{O})}$. In the case $p = 2$, we use the standard notation $\mathsf{H}^s \equiv \mathsf{W}^{s,2}$. The same convention applies to Hölder spaces $\mathsf{C}^{m+\alpha}$. For instance, $\mathsf{C}^{m+\alpha}(\Upsilon)$, $m \in \mathbb{N} \cup \{0\}$, $\alpha \in (0, 1)$ (resp., $\mathsf{C}^{m+\alpha}(\overline{\Upsilon})$) denotes the space of all γ_1-periodic (for $\Lambda = \Omega$) or $\mathbf{\Gamma}$-periodic (for $\Lambda = \mathbb{R}^2$ or $\Lambda = \mathcal{C}$) functions $u \in \mathsf{C}^{m+\alpha}(\Lambda)$ (resp., $\mathsf{C}^{m+\alpha}(\overline{\Lambda})$). Certainly, in the cases $\Lambda = \mathbb{R}^2$ and $\Lambda = \mathcal{C}$ the space $\mathsf{C}^{m+\alpha}(\Upsilon)$ coincides with $\mathsf{C}^{m+\alpha}(\overline{\Upsilon})$.

We use boldface letters to denote spaces of vector-valued functions, e.g., $\mathbf{L}^p(\Lambda)$. The symbol $\mathfrak{D}\mathbf{f}$ stands for the Jacobian matrix of the function $\mathbf{f}$. Slightly abusing notation, we do not always distinguish between $\mathsf{L}^2(\mathcal{C})$ and $\mathsf{L}^2(\mathcal{S}_2)$ ($\mathsf{L}^2(\Upsilon)$ and $\mathsf{L}^2(\mathcal{O})$).

Function spaces on $\mathbf{\Sigma}$ are defined in a natural way. Namely, by definition the space $\mathbf{L}^p(\mathbf{\Sigma})$ is the set of functions $\boldsymbol{\sigma} = \{\sigma_j\}$ such that $\sigma_j \in \mathsf{L}^p(\Sigma_j)$, $j = 1, 2, \dots, N$. One defines $\mathbf{L}^p(\Xi)$ similarly. We say that $\boldsymbol{\sigma}$ is real-valued if all the components σ_j are real-valued.

The definition of traces on $\mathbf{\Sigma}$ of functions on Λ requires special comment. Suppose that $f \in \mathsf{H}^1(\Lambda)$. The trace $f|_{\ell_j}$ is defined to be the trace of the function $f|_{\mathcal{E}_j} \in \mathsf{H}^1(\mathcal{E}_j)$. Sometimes we write f instead of the trace $f|_{\ell_j}$ when it does not cause confusion. By the embedding theorems and multiplicative inequality for traces (see [23], Corollary 1.4.7/2), $f \in \mathsf{L}^r(\ell_j)$ with any $r < \infty$ and

$$\|f\|_{\mathsf{L}^r(\ell_j)} \leq \epsilon \|\nabla u\|_{\mathsf{L}^2(\mathcal{E}_j)} + C_j(\epsilon)\|u\|_{\mathsf{L}^2(\mathcal{E}_j)} \tag{2.8}$$

for all $\epsilon > 0$. Similarly, one defines the traces on the translates $\ell_j^{(\mathbf{n})}$, which leads in a natural way to the collection of traces $f|_{\mathbf{\Sigma}} = \{f|_{\Sigma_j}\}$. Note that if an arc $\ell = \ell_j = \ell_m$ belongs to $\mathcal{E}_j \cap \mathcal{E}_m$, then $f|_{\ell_j} = f|_{\ell_m}$. On the other hand, if $\ell \subset \partial\mathcal{E}_j \cap \partial\mathcal{E}_m$ for two distinct j and m, then $f|_{\ell_j}$ may be different from $f|_{\ell_m}$. The latter situation can occur in the case when $\Lambda = \Omega$ and ℓ is a part of the boundary of Ω such that Ω lies on both sides of ℓ.

Weights and coefficients. We work in the weighted space $L^2(\mu, \Lambda)$ with the norm

$$\|u\|_\mu = \left[\int_\Lambda |u|^2 \mu d\mathbf{x}\right]^{1/2},$$

where μ is a real-valued periodic function, satisfying the conditions

$$\begin{cases} \text{mes}\{\mathbf{x} \in \Lambda : \mu(\mathbf{x}) \le 0\} = 0, \\ \mu \in L^t(\Upsilon),\ t > 1. \end{cases} \tag{2.9}$$

When it does not cause any confusion, the subscript μ will be omitted from the norm. We are interested in the properties of the Schrödinger-type periodic operators in $L^2(\mu, \Lambda)$, defined by the formal expression

$$H = \frac{1}{\mu}\langle(\mathbf{D} - \mathbf{a}), \omega^2 \mathbf{G}(\mathbf{D} - \mathbf{a})\rangle + \frac{1}{\mu}V + \sigma\delta_\Sigma, \quad \mathbf{D} = -i\nabla,$$

where $V, \omega, \mathbf{a}, \mathbf{G}$ are real-valued periodic (vector/matrix) functions defined on Λ and σ is a real-valued function defined on a system of curves Σ. Let us now give a precise definition of the operators in question. The *electric potential* V, the *magnetic vector-potential* $\mathbf{a}$ and the function σ are assumed to satisfy the conditions

$$\mathbf{a} \in \mathbf{L}^s(\Upsilon),\ s > 2; \quad V \in L^p(\Upsilon),\ p > 1, \tag{2.10}$$

$$\sigma \in L^r(\Xi), \quad r > 1. \tag{2.11}$$

The coefficient $\mathbf{G} = \{g_{jl}(\mathbf{x})\}, j, l = 1, 2$ is a symmetric matrix-valued function on Υ with real-valued entries $g_{jl}(\mathbf{x})$ that satisfy

$$\begin{cases} c|\boldsymbol{\xi}|^2 \le \langle \mathbf{G}(\mathbf{x})\boldsymbol{\xi}, \boldsymbol{\xi}\rangle \le C|\boldsymbol{\xi}|^2, \\ \det \mathbf{G}(\mathbf{x}) = C', \end{cases} \quad \forall \boldsymbol{\xi} \in \mathbb{R}^2,\ \text{a.a. } \mathbf{x} \in \Upsilon. \tag{2.12}$$

Here and below we denote by C and c with or without indices various positive constants whose precise value is unimportant. As a rule, we assume that $\det \mathbf{G} = 1$, but sometimes it is convenient not to have this restriction.

As to ω, it is a real-valued function on Υ, such that

$$c \le \omega(\mathbf{x}) \le C, \quad \text{a.a. } \mathbf{x} \in \Upsilon. \tag{2.13}$$

We are interested in four different realisations of the operator H. Namely, we study

- H as an operator on $\mathsf{L}^2(\mu, \mathbb{R}^2)$, which will be later referred to as the "full" operator H_F;
- H as an operator on $\mathsf{L}^2(\mu, \mathcal{C})$, or which is the same, as an operator on $\mathsf{L}^2(\mu, \mathcal{S}_2)$ with periodic conditions on the boundary of $\mathcal{S}_2$. In this case, we write H_P;
- H acting on $\mathsf{L}^2(\mu, \Omega)$ with the Dirichlet or natural boundary condition. In these cases we write H_D or H_N, respectively.

When we need to treat all four situations simultaneously, we use the notation $H_\aleph$, where $\aleph$ can mean any of the four letters F, P, D, N.

For each of these four problems the operator will be defined via its quadratic form. To give this definition it suffices to assume (2.9), (2.10), (2.11), (2.12) and (2.13), although later on we shall need more restrictive conditions on $\mathbf{G}$ and ω. Consider the quadratic form

$$h[u] = \int_\Lambda \omega^2 \langle \mathbf{G}(\mathbf{D} - \mathbf{a})u, \overline{(\mathbf{D} - \mathbf{a})u} \rangle d\mathbf{x} + \int_\Lambda V|u|^2 d\mathbf{x} + \int_{\boldsymbol{\Sigma}} \sigma |u|^2 dS, \tag{2.14}$$

where

$$\int_{\boldsymbol{\Sigma}} \sigma |u|^2 dS = \sum_{j=1}^N \int_{\Sigma_j} \sigma_j \left| u|_{\Sigma_j} \right|^2 dS \tag{2.15}$$

defined either on $\mathcal{D}_F = \mathsf{H}^1(\mathbb{R}^2)$ (for the full problem), or $\mathcal{D}_P = \mathsf{H}^1(\mathcal{C})$ (for the γ_2-periodic problem), or $\mathcal{D}_D = \mathsf{H}^1_0(\Omega)$ or $\mathcal{D}_N = \mathsf{H}^1(\Omega)$. Depending on the domain, we denote the form h by h_F, h_P, h_D or h_N, respectively. Sometimes, in order to distinguish operators defined on different domains and/or with different coefficients and weights, we use the full notation $H_\aleph(\omega, \mathbf{G}, \mathbf{a}, V, \mu, \sigma; \Lambda)$ or such short variants as $H_\aleph(\omega, \mathbf{G})$. A similar convention applies to the notation for the quadratic forms $h_\aleph$.

Let us check that these forms are closed in $\mathsf{L}^2(\mu, \Lambda)$. To this end, split the form (2.14) into the *unperturbed* form and the *perturbation* form:

$$\begin{aligned}
h_\aleph[u] &= h_\aleph^{(0)}[u] + w_\aleph[u]; \\
h_\aleph^{(0)}[u] &= \int_\Lambda \omega^2 \langle \mathbf{G}\mathbf{D}u, \overline{\mathbf{D}u} \rangle d\mathbf{x}, \\
w_\aleph[u] &= \int_\Lambda \left[-\omega^2 \langle \mathbf{G}\mathbf{D}u, \mathbf{a} \rangle \overline{u} - \omega^2 u \langle \mathbf{G}\mathbf{a}, \overline{\mathbf{D}u} \rangle + \omega^2 \langle \mathbf{G}\mathbf{a}, \mathbf{a} \rangle |u|^2 \right] d\mathbf{x} \\
&\qquad + \int_\Lambda V|u|^2 d\mathbf{x} + \int_{\boldsymbol{\Sigma}} \sigma |u|^2 dS.
\end{aligned}$$

The necessary properties of these forms are contained in

Proposition 2.4. *Let μ satisfy the condition* (2.9) *and let* **a**, V *and* σ *satisfy* (2.10), (2.11). *Then*

(i) *the standard* H^1*-norm is equivalent to the norm induced by the form* $h_\aleph^{(0)}$, *i.e.,*

$$C^{-1}(\|\nabla u\|_1^2+\|u\|_1^2) \le h_\aleph^{(0)}[u]+\|u\|_\mu^2 \le C(\|\nabla u\|_1^2+\|u\|_1^2), \quad \forall u \in \mathcal{D}_\aleph, \tag{2.16}$$

with some constant C depending on $\omega, \mathbf{G}, \mu$;

(ii) *the forms* $h_\aleph^{(0)}$ *with the domains* $\mathcal{D}_\aleph$ *are closed in* $\mathsf{L}^2(\mu, \Lambda)$;

(iii) *for any* $\epsilon > 0$, *there exists a constant* C_ϵ *such that*

$$|w_\aleph[u]| \le \epsilon h_\aleph^{(0)}[u] + C_\epsilon\|u\|_\mu^2, \quad \forall u \in \mathcal{D}_\aleph, \tag{2.17}$$

so that the perturbed forms $h_\aleph$ *are also closed.*

The plan of the proof is to establish the required estimates on the domain $\mathcal{E}_0$ using the embedding theorems for the domains with the interior cone condition, and the estimate (2.8) for the traces. The bounds for the entire Λ are obtained by using the periodicity of Λ and an appropriate partition of unity. The details are fairly standard and are omitted.

By Proposition 2.4, all four forms $h_\aleph$ are closed; and therefore they uniquely define four self-adjoint operators on $\mathsf{L}^2(\mu, \Lambda)$, which we denote by H_F, H_P, H_D and H_N. We do not need to know the domains of these operators, although they can be specified under supplementary regularity conditions on the coefficients, the boundary of Ω and the curves from Σ. For instance, if the system of curves contains only the boundary $\partial\Omega$, then H_N is the operator of the third boundary value problem with the condition

$$\omega^2\langle \mathbf{G}(\nabla - i\mathbf{a})u, \mathbf{n}\rangle + \sigma u = 0, \quad \mathbf{x} \in \partial\Omega,$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the exterior unit normal to the boundary at $\mathbf{x} \in \partial\Omega$. If Σ contains a curve which has an arc ℓ strictly inside Ω and separated from other components of Σ, then the integral over Σ in (2.14) induces the condition

$$\big[\omega^2\langle \mathbf{G}\nabla u, \mathbf{n}\rangle\big] + \sigma u = 0, \quad \mathbf{x} \in \ell,$$

on the jump $[\ldots]$ of the conormal derivative across the curve ℓ.

An important role will be played by the general observation that the singular continuous spectra of the operators $H_\aleph$ are empty, which we state separately for later reference.

Proposition 2.5. *Let Λ be either $\mathbb{R}^2$ or C or Ω. Suppose that the conditions* (2.9), (2.10), (2.12), (2.13) *and* (2.11) *are fulfilled. Then the singular continuous spectra of $H_\aleph$ are empty.*

The proof of this property is based on the standard direct integral representation for $H_\aleph$, known as *the Floquet decomposition.* The crucial fact is that the resolvents of the fibers of $H_\aleph$ in this representation are compact operator-functions, analytic in the quasi-momentum. We are not going to provide all the details of this argument, but refer to the comprehensive exposition of this issue in [18], and also to [19], [15] and [36]. The Floquet decomposition for H_F can be found, for example, in [4] and [8]. For the operators H_D and H_N, it is clearly explained in [5].

2.2 Results. Our main goal is to go further and prove that the spectrum of $H_\aleph$ is absolutely continuous. Here $\aleph$ takes the values F, D or N. The operator H_P plays an auxiliary role: we prove its absolute continuity only in the case of a diagonal constant matrix G. This result will be decisive in the proofs for the cases D and N.

Now the conditions (2.12), (2.13) are insufficient. Suppose, in addition, that

$$\omega \in \mathsf{W}^{1,q}(\Upsilon) \quad \text{and} \quad \mathbf{G}\nabla\omega \in \mathbf{W}^{1,q}(\Upsilon), \quad q > 1. \tag{2.18}$$

As far as $\mathbf{G}$ is concerned, a number of results will be obtained under the additional restriction

$$\mathbf{G} \in \mathbf{C}^{\alpha}(\overline{\Upsilon}), \quad \alpha \in (0,1). \tag{2.19}$$

Clearly, for a uniformly Lipschitz matrix $\mathbf{G}$, the condition (2.18) is equivalent to $\omega \in \mathsf{W}^{2,q}, q > 1$. One is tempted to say that due to the presence of the function ω in (2.14), the condition $\det \mathbf{G} = \text{const}$ in (2.12) does not restrict generality. We emphasise, however, that the smoothness conditions for the functions $\mathbf{G}$ and ω are different.

The main result for the operator H_F is contained in the next theorem.

Theorem 2.6. *Let $\Lambda = \mathbb{R}^2$ and $\sigma = 0$. Suppose that the conditions* (2.9), (2.10), (2.12), (2.13), *and* (2.18) *are fulfilled. Then the spectrum of H_F is absolutely continuous.*

Remark 2.7. For a variable $\mathbf{G}$ satisfying (2.19), Theorem 2.6 was proved in [4] even with $\sigma \neq 0$. R. Shterenberg [28] recently proved the absolute continuity of H_F under the reduced smoothness assumption $\omega \in \mathsf{W}^{1,q}$, $q > 2$. For our proof, we need an earlier result from the paper [7], where Theorem 2.6 was established for constant matrices $\mathbf{G}$ and $\omega = 1$.

To prove the absolute continuity of the operators H_D and H_N, we need to impose some extra conditions on the domain Ω. We assume that Ω is γ_1-periodic and that there exists a homeomorphism of $\overline{\Omega}$ onto $\overline{S_1}$. This means, in particular, that the boundary of Ω consists of two disjoint γ_1-periodic Jordan curves, which we denote by ℓ_-, ℓ_+. We impose the following condition on these curves.

Condition 2.8. (1) *Locally, each curve ℓ_-, ℓ_+, is a piecewise $\mathrm{C}^{1+\alpha}$-smooth Jordan arc with $\alpha \in (0,1)$.*

(2) *The domain Ω does not have any outward peaks, i.e., the interior angle between any two smooth components of the boundary at each point of non-smoothness is strictly greater than zero.*

Note that Condition 2.8 does not exclude *inward* peaks, and that the cone condition is satisfied. Using this fact and recalling that every bounded domain with the cone condition can be represented as a union of finitely many domains with the Lipschitz boundaries (see, e.g., [23]), one can easily show that the domain Ω is admissible in the sense of Definition 2.1.

Without loss of generality, the parameter α above can be chosen to be the same as in the condition (2.19). Below, we denote by $\mathbf{n}(\mathbf{x})$, a.a. $\mathbf{x} \in \partial\Omega$, the exterior unit normal to the boundary $\partial\Omega$ and by $Z \subset \partial\Omega$ the discrete set where the smoothness of the boundary breaks down. Clearly, Z has no finite accumulation points.

Theorem 2.9. *Let $\Lambda = \Omega$. Suppose that the domain Ω satisfies Condition 2.8, the set $\boldsymbol{\Sigma}$ is a system of curves in Ω in the sense of Definition 2.2, and that conditions* (2.9), (2.10), (2.12), (2.13), (2.11), (2.18), *and* (2.19) *are fulfilled. Then the spectra of H_D and H_N are absolutely continuous.*

For later convenience, we make a couple of simplifying assumptions which do not restrict generality.

First, we include the boundary $\partial\Omega$ in the system of curves $\boldsymbol{\Sigma}$, even if $\boldsymbol{\Sigma}$ already contains either pieces of $\partial\Omega$ or the entire boundary. The notation for the components of $\boldsymbol{\Sigma}$ will be as follows. By Condition 2.8, there are finitely many $\mathrm{C}^{1+\alpha}$-arcs $\ell_j \subset \partial\Omega$, $j = 1, 2, \dots, M < \infty$, such that

$$\partial\Omega = \bigcup_{j=1}^{M} \Sigma_j,$$

where Σ_j are defined by (2.5). Since the boundary $\partial\Omega$ is a Jordan curve, we can assume that no pair of arcs ℓ_j, ℓ_s with $s, j = 1, 2, \dots, M$ has common interior points. Moreover, since outward peaks are absent, there are M bounded domains

$\mathcal{E}_j \subset \Omega$ with Lipschitz boundaries such that $\ell_j \subset \overline{\mathcal{E}_j}$. This allows us to include $\Sigma_j, j = 1, 2, \ldots, M$, in the initial system of curves in Ω. From now on, we consider the sets $\Sigma_j, j = 1, 2, \ldots, M$, to be the first M components of $\mathbf{\Sigma}$, and the original components of $\mathbf{\Sigma}$ will be relabelled to have numbers from $M+1$ to N. Also we set $\sigma_j = 0, j = 1, 2, \ldots, M$. Obviously, this procedure does not change the operators H_D and H_N, since the quadratic form (2.14) remains unchanged.

Secondly, we assume that if a curve ℓ_j, $j = 1, 2, \ldots, N$, contains a point $\mathbf{z} \in Z$, then $\mathbf{z}$ is either the start or the end point of ℓ_j, i.e., given a parametrisation $\psi_j : [0,1] \to \ell_j$, we have $\psi_j(0) = \mathbf{z}$ or $\psi_j(1) = \mathbf{z}$. This can be done by breaking, if necessary, every ℓ_j containing a $\mathbf{z} \in Z$ into subarcs, and using Remark 2.3(ii).

3 Preliminary conclusions

3.1 Reduction to $\mu = \omega = 1$. Proposition 2.5 allows one to show that it suffices to prove Theorems 2.6 and 2.9 for $\omega = \mu = 1$. The following lemma is a variant of a well-known result (see e.g., [8]) and is a crucial ingredient in our argument.

Lemma 3.1. *In addition to the conditions of Proposition 2.5, assume also that* (2.18) *is satisfied. In the case of the operator H_N, assume also that Condition 2.8 is fulfilled. Then*

$$\omega^{-1} H_{\aleph}(\omega, V, \sigma)\omega^{-1} = H_{\aleph}(1, \tilde{V}, \tilde{\sigma}),$$
$$\tilde{V} = \omega^{-2} V + \omega^{-1}\langle \nabla, \mathbf{G}\nabla\rangle\omega,$$
$$\tilde{\sigma}_j = \begin{cases} -\omega^{-1}\langle \mathbf{G}\nabla\omega, \mathbf{n}\rangle, & \text{if } j = 1, 2, \ldots, M, \\ \omega^{-2}\sigma_j, & \text{if } j = M+1, \ldots, N. \end{cases} \tag{3.1}$$

Proof. We prove the lemma only for the case $\aleph = N$. To avoid cumbersome calculations, assume that $\mathbf{a} = 0, V = 0, \sigma = 0$. The general case requires only obvious modifications. The second condition in (2.18) and (2.12) imply that $\nabla\omega \in \mathsf{L}^h(\Upsilon)$ with some $h > 2$. Therefore, $\omega \in \mathsf{W}^{1,h}(\Upsilon) \subset \mathsf{C}(\Upsilon)$; and, using (2.13), one can show that the functions ω, ω^{-1} are multipliers in $\mathsf{H}^1(\Omega)$. This implies that the quadratic forms of the operators in the r.h.s. and l.h.s. of (3.1) are both closed on $\mathsf{H}^1(\Omega)$. Thus it suffices to prove that the corresponding bilinear forms coincide. Let us consider the form of the operator in the l.h.s. for $u, v \in \mathsf{H}^1(\Omega)$ (all integrals below are over Ω unless indicated otherwise):

$$
\begin{aligned}
h_N^{(0)}(\omega)[\omega^{-1}u, \omega^{-1}v] &= \int \omega^2 \langle \mathbf{G}\nabla(\omega^{-1}u), \overline{\nabla(\omega^{-1}v)}\rangle d\mathbf{x} \\
&= \int \omega^2 \langle \mathbf{G}(\omega^{-1}\nabla u - \omega^{-2}u\nabla\omega), \omega^{-1}\overline{\nabla v} - \omega^{-2}\bar{v}\nabla\omega\rangle d\mathbf{x} \\
&= \int \langle \mathbf{G}\nabla u, \overline{\nabla v}\rangle d\mathbf{x} + \int \omega^{-2}\langle \mathbf{G}\nabla\omega, \nabla\omega\rangle u\bar{v} d\mathbf{x} \\
&\quad - \int \omega^{-1}u\langle \mathbf{G}\nabla\omega, \overline{\nabla v}\rangle d\mathbf{x} - \int \omega^{-1}\langle \mathbf{G}\nabla u, \nabla\omega\rangle \bar{v} d\mathbf{x}.
\end{aligned}
$$

Integrate the last integral by parts, recalling that $\mathbf{G}\nabla\omega \in \mathbf{W}^{1,q}$, $q > 1$; and using the notation $\mathbf{n}(\mathbf{x})$ for the exterior unit normal to the boundary $\partial\Omega$ at the point $\mathbf{x}$:

$$
\begin{aligned}
- \int \omega^{-1}\langle \mathbf{G}\nabla u, \nabla\omega\rangle \bar{v} d\mathbf{x} &= - \int_{\partial\Omega} \omega^{-1}\langle \mathbf{G}\nabla\omega, \mathbf{n}\rangle u\bar{v} dS \\
&+ \int \omega^{-1}u\langle \mathbf{G}\nabla\omega, \overline{\nabla v}\rangle d\mathbf{x} + \int \omega^{-1}u\bar{v}\langle \nabla, \mathbf{G}\nabla\rangle\omega d\mathbf{x} \\
&- \int \omega^{-2}\langle \mathbf{G}\nabla\omega, \nabla\omega\rangle u\bar{v} d\mathbf{x}.
\end{aligned}
$$

Substituting this in the initial formula for the bilinear form, we arrive at the relation

$$
\begin{aligned}
h_N^{(0)}(\omega)[\omega^{-1}u, \omega^{-1}v] &= h_N^{(0)}(1)[u, v] + \int \tilde{V} u\bar{v} d\mathbf{x} - \int_{\partial\Omega} \omega^{-1}\langle \mathbf{G}\nabla\omega, \mathbf{n}\rangle u\bar{v} dS \\
&= h_N(1, \tilde{V}, \tilde{\sigma})[u, v].
\end{aligned}
$$

It remains to notice that

$$
\int_{\partial\Omega} \omega^{-1}\langle \mathbf{G}\nabla\omega, \mathbf{n}\rangle u\bar{v} dS = \sum_{j=1}^{M} \int_{\Sigma_j} \omega^{-1}\langle \mathbf{G}\nabla\omega, \mathbf{n}\rangle u\bar{v} dS,
$$

which completes the proof. □

Corollary 3.2. *It suffices to prove Theorems 2.6 and 2.9 for $\omega = 1$ and $\mu = 1$.*

Proof. To be definite, consider the case of Theorem 2.9 only. Suppose that it holds for $\omega = \mu = 1$. First we show that the operator $H_\aleph(1, V, \mu)$, $\aleph = D, N$, is absolutely continuous if μ satisfies (2.9). According to Proposition 2.5, it suffices to check that it has no point spectrum. Aiming for a contradiction, let us suppose that λ is an eigenvalue of $H_\aleph$ with an eigenfunction u. Recall that by Proposition

2.4 the quadratic form $h_\aleph$ is closed on the domain $\mathcal{D}_\aleph$ *independent* of μ. Therefore, the equality

$$h_\aleph(1, V, \mu, \sigma)[u, v] - \lambda \int_\Omega u\bar{v}\mu dx = 0, \quad \forall v \in \mathcal{D}_\aleph,$$

implies that the point $\tilde{\lambda} = 0$ is an eigenvalue of the operator

$$\tilde{H}_\aleph = H_\aleph(1, V - \lambda\mu, 1, \sigma).$$

On the other hand, the potential $V - \lambda\mu$ satisfies (2.10), so that by Theorem 2.9 with $\omega = \mu = 1$, the spectrum of $\tilde{H}_\aleph$ is absolutely continuous. This contradiction proves the claim.

Let us now remove the condition $\omega = 1$. Again, in view of Proposition 2.5, it suffices to show that the operator $H_\aleph(\omega, V, \mu, \sigma)$ has no point spectrum. Assuming the contrary, we obtain from (3.1) that for any eigenfunction u of $H_\aleph(\omega, V, \mu, \sigma)$ associated with an eigenvalue λ, the function ωu will be an eigenfunction of the operator $\tilde{H}_\aleph = H_\aleph(1, \tilde{V} - \lambda\omega^{-2}, \mu, \tilde{\sigma})$ associated with the eigenvalue $\tilde{\lambda} = 0$. This contradicts the absolute continuity of $\tilde{H}_\aleph$, which follows from Theorem 2.9 with arbitrary μ and $\omega = 1$ in the same way as in the first part of the proof. □

Referring to this Corollary, from now on **we always assume that** $\omega = \mu = 1$.

3.2 Absolute continuity. The proof of Theorem 2.9 will be based on the further reduction to the operator $H_P(1, \mathbf{B}, \mathbf{a}, V, 1, \sigma; \mathcal{C})$ with a constant *diagonal* matrix $\mathbf{B}$.

Proposition 3.3. *Let* $\omega = \mu = 1$. *Suppose that the conditions* (2.10), (2.11) *are fulfilled and that* $\mathbf{G} = \mathbf{B}$ *is a constant* ***diagonal*** *matrix with positive entries. Then the operator* $H_P(1, \mathbf{B}, \mathbf{a}, V, 1, \sigma; \mathcal{C})$ *is absolutely continuous.*

Although this result is not contained in [4], it follows immediately from the estimates obtained in [4]; we do not comment on the details.

4 Isothermal coordinates. Proof of Theorem 2.6

As was already explained, the proofs of Theorems 2.6 and 2.9 are based on a reduction of the operator $H_\aleph(\mathbf{G})$ to the canonical form, i.e., to the operator $H_\aleph(\mathbf{A})$ with a constant positive-definite matrix $\mathbf{A}$. (Recall again that we may assume without loss of generality that $\omega = \mu = 1$.) This reduction is done using isothermal coordinates. The required properties of this coordinate change are stated in Theorems 4.1 and 4.2. Their proof is postponed until Section 6.

We consider two mappings. One is a homeomorphism of the entire plane onto itself, and the other is a homeomorphism of the periodic domain Ω onto the straight strip $\mathcal{S}_d$.

We always denote $\mathbf{F} = \sqrt{\mathbf{G}}$ and assume that

$$\det \mathbf{G} = 1, \tag{4.1}$$

so that $\det \mathbf{F} = 1$ as well.

4.1 Change of variables.

Let

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The first theorem below describes a suitable coordinate change in $\mathbb{R}^2$.

Theorem 4.1. *Let* $\mathbf{G}$ *satisfy* (2.12) *and* (4.1). *Then there exists a unique homeomorphism* $\mathbf{f} = \mathbf{f}_{\mathbb{R}^2} = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{f} \in \mathsf{H}^1_{\mathrm{loc}}(\mathbb{R}^2)$ *such that*

(i) $\mathbf{f}(0) = 0$, $\mathbf{f}(2\pi\mathbf{e}_1) = 2\pi\mathbf{e}_1$, *and* $|\mathbf{f}(\mathbf{x})| \to \infty$ *as* $|\mathbf{x}| \to \infty$; *and*

(ii) *the components* f_1, f_2 *satisfy the equation*

$$\nabla f_2 = \mathbf{J}\mathbf{G}\nabla f_1, \quad \text{a.a. } \mathbf{x} \in \mathbb{R}^2. \tag{4.2}$$

Moreover, the map $\mathbf{f}$ *possesses the following properties.*

(iii) *The Jacobian* $J_{\mathbf{f}}(\mathbf{x}) = \det(\mathfrak{D}\mathbf{f}(\mathbf{x}))$ *is positive a.a.* $\mathbf{x} \in \mathbb{R}^2$. *The function* $\mathbf{f}$ *and its inverse* $\mathbf{f}^{-1}$ *both belong to* $\mathsf{W}^{1,\tau}_{\mathrm{loc}}(\mathbb{R}^2)$ *with some* $\tau > 2$.

(iv) *For any* $u \in \mathsf{H}^1_{\mathrm{loc}}$, *the usual chain rule holds:*

$$\nabla(u \circ \mathbf{f}) = [\mathfrak{D}\mathbf{f}]^T(\nabla u \circ \mathbf{f}), \quad \text{a.a. } \mathbf{x} \in \mathbb{R}^2.$$

Moreover, for any $h \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}^2)$, *the function* $h \circ \mathbf{f}$ *belongs to* $\mathsf{L}^1_{\mathrm{loc}}(J_{\mathbf{f}}, \mathbb{R}^2)$ *and*

$$\int_{\mathbf{f}(\Omega_1)} h(\mathbf{y})d\mathbf{y} = \int_{\Omega_1} (h \circ \mathbf{f})(\mathbf{x})J_{\mathbf{f}}(\mathbf{x})d\mathbf{x}$$

for any open bounded $\Omega_1 \subset \mathbb{R}^2$.

(v) *For* $\mathbf{h}_1 = 2\pi\mathbf{e}_1$ *and some linearly independent vector* $\mathbf{h}_2$, *one has*

$$\mathbf{f}(\mathbf{x} + 2\pi\mathbf{n}) = \mathbf{f}(\mathbf{x}) + n_1\mathbf{h}_1 + n_2\mathbf{h}_2$$

for all $\mathbf{x} \in \mathbb{R}^2$ *and all* $\mathbf{n} \in \mathbb{Z}^2$.

(vi) *If* (2.19) *is fulfilled, then* $\mathbf{f} \in \mathbf{C}^{1+\alpha}(\mathbb{R}^2)$ *and the Jacobian satisfies the estimate* $J_{\mathbf{f}}(\mathbf{x}) \geq c$ *for all* $\mathbf{x} \in \mathbb{R}^2$.

Note that the properties stated in Theorem 4.1(iv) are certainly standard for smooth, or even Lipschitz maps $\mathbf{f}$. For homeomorphisms $\mathbf{f}$ of class $\mathbf{H}^1_{\mathrm{loc}}$, such "standard" results as the chain rule or the change of variables under the integral are not obvious. They require in addition that the functions $\mathbf{f}$ and $\mathbf{f}^{-1}$ should map sets of measure zero into sets of measure zero. This property follows from Theorem 4.1(iii) according to [13], Ch. 5, §3.

Let us now state the appropriate result for an admissible domain Ω satisfying Condition 2.8. Recall that the boundary $\partial\Omega$ consists of two disjoint Jordan curves ℓ_+ and ℓ_-. Without loss of generality, we assume that $\mathbf{0} \in \ell_-$. As defined in Section 2, $Z \subset \partial\Omega$ is the set of the boundary points where the smoothness of ℓ_+, ℓ_- breaks down.

Theorem 4.2. *Suppose that an admissible domain* Ω *satisfies Condition 2.8. Let* $\mathbf{G}$ *satisfy* (2.12), (4.1), *and* (2.19). *Then there exists a unique homeomorphism* $\mathbf{f} = \mathbf{f}_\Omega = (f_1, f_2) : \Omega \to \mathcal{S} = \mathcal{S}_1$, $\mathbf{f} \in \mathbf{H}^1_{\mathrm{loc}}(\Omega)$ *such that*

(i) $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $f_1(\mathbf{x}) \to +\infty$ *as* $x_1 \to +\infty$, *and* $f_1(\mathbf{x}) \to -\infty$ *as* $x_1 \to -\infty$; *and*

(ii) *the components* f_1, f_2 *satisfy the equation* (4.2) *for a.a.* $\mathbf{x} \in \Omega$.

The map $\mathbf{f}$ *satisfies the following properties.*

(iii) *For some number* $h > 0$, *one has*

$$\mathbf{f}(\mathbf{x} + 2\pi n\mathbf{e}_1) = \mathbf{f}(\mathbf{x}) + 2\pi nh\mathbf{e}_1$$

for all $\mathbf{x} \in \Omega$ *and all* $n \in \mathbb{Z}$.

(iv) $\mathbf{f} \in \mathbf{C}^{1+\alpha}(\overline{\Omega} \setminus Z)$, $\mathbf{f}^{-1} \in \mathbf{C}^{1+\alpha}(\overline{\mathcal{S}} \setminus \mathbf{f}(Z))$, *and the Jacobian* $J_{\mathbf{f}}(\mathbf{x})$ *is positive everywhere in* Ω. *Moreover, for each* $\mathbf{x}_0 \in Z$, *there exist a number* $\nu \in (0, 2]$ *and four non-degenerate Hölder-continuous matrix-functions* $\mathbf{M}, \mathbf{T}$ *and* $\mathbf{\Phi}$, $\mathbf{\Psi}$ *with real-valued entries such that in the vicinity of* $\mathbf{x}_0$ *and* $\mathbf{z}_0 = \mathbf{f}(\mathbf{x}_0) \in \mathbf{f}(Z)$, *one has the representations*

$$\mathfrak{D}\mathbf{f}(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_0|^{1/\nu - 1}\mathbf{\Phi}\left(\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}\right)\mathbf{M}(\mathbf{x}),$$

(4.3)

$$\mathfrak{D}\mathbf{f}^{-1}(\mathbf{z}) = |\mathbf{z} - \mathbf{z}_0|^{\nu - 1}\mathbf{\Psi}\left(\frac{\mathbf{z} - \mathbf{z}_0}{|\mathbf{z} - \mathbf{z}_0|}\right)\mathbf{T}(\mathbf{z}).$$

The next lemma establishes some further properties of $\mathbf{f}$ that follow from Theorems 4.1 and 4.2.

Lemma 4.3. *Let the matrix* $\mathbf{G}$ *and the map* $\mathbf{f}$ *be as in Theorem 4.1 or 4.2. Then the following identities hold:*

$$\nabla f_1 = -\mathbf{J}\mathbf{G}\nabla f_2, \tag{4.4}$$

$$\langle \mathbf{F}\nabla f_1, \mathbf{F}\nabla f_2\rangle = 0, \tag{4.5}$$

$$J_{\mathbf{f}} = |\mathbf{F}\nabla f_1|^2 = |\mathbf{F}\nabla f_2|^2, \tag{4.6}$$

$$J_{\mathbf{f}}^{-1}\mathfrak{D}\mathbf{f}\ \mathbf{G}\ \mathfrak{D}\mathbf{f}^T = \mathbf{I}. \tag{4.7}$$

Proof. As $\det \mathbf{F} = \det \mathbf{G} = 1$, a direct calculation shows that

$$\mathbf{J} = \mathbf{G}\mathbf{J}\mathbf{G}, \quad \mathbf{J} = \mathbf{F}\mathbf{J}\mathbf{F}.$$

Noticing also that $\mathbf{J}^2 = -\mathbf{I}$, from (4.2) we obtain (4.4) and the relation

$$\mathbf{F}\nabla f_2 = \mathbf{J}\mathbf{F}\nabla f_1.$$

This implies the orthogonality (4.5) in view of the obvious equality $\langle \mathbf{J}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle = 0$, for all $\boldsymbol{\xi} \in \mathbb{R}^2$. It also yields the equality $|\mathbf{F}\nabla f_1|^2 = |\mathbf{F}\nabla f_2|^2$. The equality (4.7) is a direct consequence of (4.5) and (4.6).

To prove (4.6), compute the Jacobian, using (4.4):

$$\begin{aligned} J_{\mathbf{f}}(\mathbf{x}) = \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} &= -\langle \nabla f_1, \mathbf{J}\nabla f_2\rangle \\ &= -\langle \nabla f_1, \mathbf{G}\mathbf{J}\mathbf{G}\nabla f_2\rangle = \langle \nabla f_1, \mathbf{G}\nabla f_1\rangle \\ &= |\mathbf{F}\nabla f_1|^2 = |\mathbf{F}\nabla f_2|^2. \end{aligned}$$

□

Remark 4.4. By (4.6), the norm $|\mathfrak{D}\mathbf{f}|$ can be estimated as follows: $|\mathfrak{D}\mathbf{f}|^2 \le K J_{\mathbf{f}}$, where the positive constant K depends on the matrix $\mathbf{G}$. Recall that this inequality serves as a definition of K-quasiconformal maps (see [12], Section 12.1).

Similarly, $|(\mathfrak{D}\mathbf{f})^{-1}|^2 = \big(|\mathfrak{D}\mathbf{f}| J_{\mathbf{f}}^{-1}\big)^2 \le K J_{\mathbf{f}}^{-1}$.

4.2 Unitary transformation. Notice that the mapping $\mathbf{f} = \mathbf{f}_{\mathbb{R}^2}$ constructed in Theorem 4.1 transforms the lattice $\Gamma = (2\pi\mathbb{Z})^2$ into the lattice generated by the vectors $\mathbf{h}_1, \mathbf{h}_2$. It is slightly more convenient to reduce this lattice back to Γ by applying the non-degenerate linear transformation $\mathbf{R} = \mathbf{R}_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the relations

$$\mathbf{R}\mathbf{h}_1 = 2\pi\mathbf{e}_1, \quad \mathbf{R}\mathbf{h}_2 = 2\pi\mathbf{e}_2.$$

By Theorem 4.1(v), the composite mapping $\mathbf{g} = \mathbf{R} \circ \mathbf{f}$ satisfies

$$\mathbf{g}(\mathbf{x} + 2\pi \mathbf{n}) = \mathbf{g}(\mathbf{x}) + 2\pi n_1 \mathbf{e}_1 + 2\pi n_2 \mathbf{e}_2, \quad \mathbf{x} \in \mathbb{R}^2, \mathbf{n} \in \mathbb{Z}^2. \tag{4.8}$$

Note that in view of (4.7), we have

$$\mathbf{A} := J_{\mathbf{g}}^{-1} \mathfrak{D}\mathbf{g}\, \mathbf{G}\, \mathfrak{D}\mathbf{g}^T = (\det \mathbf{R})^{-1} \mathbf{R}\mathbf{R}^T. \tag{4.9}$$

Similarly, in Theorem 4.2, the one-dimensional lattice γ_1 is transformed into the lattice $\gamma_1^h = \{2\pi n h \mathbf{e}_1\}, n \in \mathbb{Z}$. We rescale γ_1^h back to γ_1 by applying the transformation $\mathbf{R} = \mathbf{R}_\Omega : \mathcal{S} \to \mathcal{S}$ defined by

$$\mathbf{R}\mathbf{e}_2 = \mathbf{e}_2, \quad \mathbf{R}\mathbf{e}_1 = h^{-1}\mathbf{e}_1,$$

Then, clearly, the mapping $\mathbf{g} = \mathbf{R} \circ \mathbf{f}$ satisfies the relation

$$\mathbf{g}(\mathbf{x} + 2\pi n \mathbf{e}_1) = \mathbf{g}(\mathbf{x}) + 2\pi n \mathbf{e}_1, \quad \mathbf{x} \in \Omega, n \in \mathbb{Z}. \tag{4.10}$$

Note also that the matrix $\mathbf{A}$ in (4.9) is diagonal in this case. Depending on the context, below we write either $\mathbf{g}, \mathbf{A}$ or $\mathbf{g}_\Lambda, \mathbf{A}_\Lambda$ where Λ is either $\mathbb{R}^2$ or Ω.

Denote

$$\tilde{\Lambda} = \begin{cases} \Lambda, & \text{if } \Lambda = \mathbb{R}^2 \text{ or } \mathcal{C}, \\ \mathcal{S}, & \text{if } \Lambda = \Omega. \end{cases}$$

In the case $\Lambda = \Omega$, under the conditions of Theorem 4.2, one can easily show that the set $\tilde{\Sigma} = \mathbf{g}(\Sigma)$ is again a system of curves. More precisely, if $\ell_j, j = 1, 2, \dots, N$, are the C^1-arcs from Definition 2.2 (see also the end of Section 2), then each $\mathbf{g}(\ell_j)$ is again a C^1-arc satisfying all the required properties. In the case of curves ℓ_j ending or starting at the points of Z, this is done by a suitable re-parametrisation, using (4.3). Recall that $\mathbf{g}(\Sigma)$ contains the boundary of the strip $\mathcal{S}$.

Introduce also the sets $\tilde{\Upsilon}$, $\tilde{\Xi}$, defined similarly to (2.4) and (2.6) with $\tilde{\Lambda}$ and $\tilde{\Sigma}$ instead of Λ and Σ. Note that according to Theorem 4.2, the following is fulfilled for the mappings $\mathbf{g}, \mathbf{g}^{-1}$ in the case $\Lambda = \Omega$:

$$\mathbf{g} \in \mathbf{W}^{1,\tau}(\Omega'), \quad \mathbf{g}^{-1} \in \mathbf{W}^{1,\tau}(\Omega''), \quad \mathfrak{D}\mathbf{g}^{-1}|_{\tilde{\Xi}} \in \mathsf{L}^{\tau/2}(\tilde{\Xi}) \tag{4.11}$$

for any bounded domains $\Omega' \subset \Omega$, $\Omega'' \subset \mathcal{S}$ and some $\tau > 2$.

Using Theorem 4.1(iii), (iv) and Theorem 4.2 (iv), it is easy to show that the operator

$$(Su)(\mathbf{x}) = u\big(\mathbf{g}^{-1}(\mathbf{x})\big), \quad u \in \mathsf{L}^2(\Lambda),$$

is unitary from $\mathsf{L}^2(\Lambda)$ onto the space $\mathsf{L}^2(\tilde{\mu}, \tilde{\Lambda})$ with the weight

$$\tilde{\mu}(\mathbf{x}) = \Big(J_{\mathbf{g}}\big(\mathbf{g}^{-1}(\mathbf{x})\big)\Big)^{-1} = J_{\mathbf{g}^{-1}}(\mathbf{x}).$$

Let

$$\begin{cases} \tilde{\mathbf{a}}(\mathbf{x}) = \left(((\mathfrak{D}\mathbf{g}^T)^{-1}\mathbf{a}) \circ \mathbf{g}^{-1}\right)(\mathbf{x}), \\ \tilde{V}(\mathbf{x}) = \left((J_{\mathbf{g}}^{-1}V) \circ \mathbf{g}^{-1}\right)(\mathbf{x}), \end{cases} \quad \mathbf{x} \in \tilde{\Lambda},$$

and, in the case $\Lambda = \Omega$, write

$$\tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_N),$$

$$\tilde{\sigma}_j(\mathbf{x}) = |(\mathfrak{D}\mathbf{g}^{-1})(\mathbf{x})\mathbf{t}_j(\mathbf{x})|(\sigma_j \circ \mathbf{g}^{-1})(\mathbf{x}), \quad \mathbf{x} \in \tilde{\Sigma}_j, \ j = 1, 2, \ldots, N,$$

where $\mathbf{t}_j(\mathbf{x})$ is the unit tangent vector to $\tilde{\Sigma}_j$ at the point $\mathbf{x} \in \tilde{\Sigma}_j$.

In the next theorem, among other properties of the map S we show that the coefficients $\tilde{V}, \tilde{\mu}$, $\tilde{\mathbf{a}}$ and $\tilde{\sigma}$ satisfy the conditions of Theorems 2.6 or 2.9.

Theorem 4.5. *Let S be as defined above.*

(i) *Under the conditions of Theorems 2.6 or 2.9, one has $\tilde{\mu} \in \mathsf{L}^{\tau/2}(\tilde{\Upsilon})$,*

$$\tilde{V} \in \mathsf{L}^{\tilde{p}}(\tilde{\Upsilon}), \quad \tilde{p} = \frac{p\tau}{2(p-1)+\tau},$$
$$\tilde{\mathbf{a}} \in \mathbf{L}^{\tilde{s}}(\tilde{\Upsilon}), \quad \tilde{s} = \frac{s\tau}{s-2+\tau},$$
$$\tilde{\sigma} \in \mathsf{L}^{\tilde{r}}(\tilde{\Xi}), \quad \tilde{r} = \frac{r\tau}{2(r-1)+\tau},$$

where $\tau > 2$ is as in Theorem 4.1(iii) or (4.11). The exponents $\tilde{p}$ and $\tilde{s}$ satisfy the inequalities $\tilde{p} > 1$, $\tilde{s} > 2$, $\tilde{r} > 1$.

(ii) *The map S (resp., S^{-1}) is bounded as an operator from $\mathsf{H}^1(\Lambda)$ to $\mathsf{H}^1(\tilde{\Lambda})$ (resp., $\mathsf{H}^1(\tilde{\Lambda})$ to $\mathsf{H}^1(\Lambda)$) and from $\mathsf{H}_0^1(\Omega)$ to $\mathsf{H}_0^1(S)$ (resp., $\mathsf{H}_0^1(S)$ to $\mathsf{H}_0^1(\Omega)$).*

(iii) *Let $\mathbf{A}$ be as defined in (4.9). Then one has the unitary equivalence*

$$SH_{\aleph}(\mathbf{G}, \mathbf{a}, V, 1, \sigma; \Lambda)S^* = H_{\aleph}(\mathbf{A}, \tilde{\mathbf{a}}, \tilde{V}, \tilde{\mu}, \tilde{\sigma}; \tilde{\Lambda}).$$

Proof. (i) The inequalities $\tilde{p} > 1$, $\tilde{s} > 2$ and $\tilde{r} > 1$ are immediate from the conditions $p > 1$, $s > 2$, $r > 1$, $\tau > 2$.

It follows from Theorem 4.1(iii) and (v) or (4.11) that $\tilde{\mu} \in \mathsf{L}^{\tau/2}(\tilde{\Upsilon})$. Let us prove that $\tilde{V} \in L^{\tilde{p}}(\tilde{\Upsilon})$. By Hölder's inequality, for any bounded domain $\Omega_1 \subset \mathbb{R}^2$ we have

$$\int_{\Omega_1} |J_{\mathbf{g}}^{-1}V \circ \mathbf{g}^{-1}(\mathbf{x})|^{\tilde{p}} d\mathbf{x}$$
$$\leq \left[\int_{\Omega_1} |V \circ \mathbf{g}^{-1}(\mathbf{x})|^{\tilde{p}\beta^{-1}} J_{\mathbf{g}^{-1}}(\mathbf{x}) d\mathbf{x}\right]^{\beta} \left[\int_{\Omega_1} \left(J_{\mathbf{g}^{-1}}(\mathbf{x})\right)^{(\tilde{p}-\beta)(1-\beta)^{-1}} d\mathbf{x}\right]^{1-\beta}$$

with $\beta = \tilde{p}p^{-1}$. Noticing that $(\tilde{p}-\beta)(1-\beta)^{-1} = \tau/2$, and using Theorem 4.1(iii), (iv), or (4.11), we conclude that the r.h.s. of the last inequality does not exceed

$$C\left[\int_{\mathbf{g}^{-1}(\Omega_1)} |V(\mathbf{x})|^p d\mathbf{x}\right]^\beta \left[\int_{\Omega_1} \left(J_{\mathbf{g}^{-1}}(\mathbf{x})\right)^{\tau/2} d\mathbf{x}\right]^{1-\beta} < \infty.$$

To prove that $\tilde{\mathbf{a}} \in \mathsf{L}^{\tilde{s}}(\tilde{\Upsilon})$, note that in view of Remark 4.4,

$$\int_{\Omega_1} |\tilde{\mathbf{a}}(\mathbf{x})|^{\tilde{s}} d\mathbf{x} \le C \int_{\Omega_1} \left[J_{\mathbf{g}^{-1}}(\mathbf{x}) \left|(\mathbf{a}\circ\mathbf{g}^{-1})(\mathbf{x})\right|^2\right]^{\tilde{s}/2} d\mathbf{x}.$$

Now, repeating the argument used in the first part of the proof, we arrive at the required property.

A similar calculation can be done for the function σ. Precisely, for any C^1-arc $\ell \subset \tilde{\Sigma}_j$, we have

$$\begin{aligned}
&\int_\ell \left[|(\mathfrak{D}\mathbf{g}^{-1})(\mathbf{x})\mathbf{t}_j(\mathbf{x})| \; |(\sigma_j\circ\mathbf{g}^{-1})(\mathbf{x})|\right]^{\tilde{r}} dS \\
&\qquad \le \left[\int_\ell \left(|(\sigma_j\circ\mathbf{g}^{-1})(\mathbf{x})|\right)^{\tilde{r}\gamma^{-1}} |(\mathfrak{D}\mathbf{g}^{-1})(\mathbf{x})\mathbf{t}_j(\mathbf{x})| dS\right]^\gamma \\
&\qquad\qquad \left[\int_\ell \left(|(\mathfrak{D}\mathbf{g}^{-1})(\mathbf{x})\mathbf{t}_j(\mathbf{x})|\right)^{(\tilde{r}-\gamma)(1-\gamma)^{-1}} dS\right]^{1-\gamma}
\end{aligned}$$

with $\gamma = \tilde{r}r^{-1}$. Noticing that $(\tilde{r}-\gamma)(1-\gamma)^{-1} = \tau/2$, we conclude that the r.h.s. of the last inequality does not exceed

$$C\left[\int_{\mathbf{g}^{-1}(\ell)} |\sigma_j(\mathbf{x})|^r dS\right]^\gamma \left[\int_\ell |\mathfrak{D}\mathbf{g}^{-1}(\mathbf{x})|^{\tau/2} dS\right]^{1-\gamma}$$

and is therefore bounded in view of (4.11).

(ii) Let $u \in \mathsf{H}^1(\Lambda)$ and $v = Su$. Let us first prove that $\nabla v \in \mathsf{L}^2(\tilde{\Lambda})$. Since the matrix $\mathbf{A}$ defined in (4.9) is positive-definite, changing variables we have

$$\begin{aligned}
c\|\nabla v\|^2 &\le \int_{\tilde{\Lambda}} \langle \mathbf{A}\nabla v, \overline{\nabla v}\rangle d\mathbf{x} = \int_{\tilde{\Lambda}} \langle \mathbf{G}\nabla u, \overline{\nabla u}\rangle \circ \mathbf{g}^{-1} J_{\mathbf{g}^{-1}} d\mathbf{x} \\
&= \int_\Lambda \langle \mathbf{G}\nabla u, \overline{\nabla u}\rangle d\mathbf{x} \le C\|\nabla u\|^2.
\end{aligned}$$

In the last inequality, we have used (2.12); to secure the change of variables in the case $\Lambda = \mathbb{R}^2$, we refer to Theorem 4.1(iv).

Since $\tilde{\mu} = J_{\mathbf{g}^{-1}} \in \mathsf{L}^{\tau/2}(\tilde{\Upsilon})$ with $\tau > 2$, in order to prove the boundedness of the operator $S : \mathsf{H}^1(\Lambda) \to \mathsf{H}^1(\tilde{\Lambda})$, it remains to use the unitarity of $S : \mathsf{L}^2(\Lambda) \to \mathsf{L}^2(\tilde{\mu}, \tilde{\Lambda})$ and Proposition 2.4. Similarly for S^{-1}.

To show that S maps $\mathsf{H}_0^1(\Omega)$ into $\mathsf{H}_0^1(\mathcal{S})$, it suffices to notice that for any $u \in \mathsf{C}_0^1(\Omega)$, we have $u \circ \mathbf{g}^{-1} \in \mathsf{C}_0^1(\mathcal{S})$.

(iii) This result follows by a straightforward calculation. □

Proof of Theorem 2.6. As was explained in Remark 2.7, Theorem 2.6 was established in [7] for constant matrices $\mathbf{G}$ and $\omega = \mu = 1$. In view of the unitary equivalence established in Theorem 4.5(iii) and by Corollary 3.2, Theorem 2.6 for general $\mathbf{G}$ and ω, μ follows immediately from [7]. □

As far as Theorem 2.9 is concerned, by Theorem 4.5(iii) and Corollary 3.2, it follows from

Theorem 4.6. *Suppose that $\Omega = \mathcal{S}_1$, $\omega = \mu = 1$. Let the conditions* (2.10), (2.11) *be satisfied, and let $\mathbf{G} = \mathbf{B}$ be a constant diagonal matrix. Then the spectra of the operators $H_D(\mathbf{B}, \mathbf{a}, V, \sigma; \Omega)$, $H_N(\mathbf{B}, \mathbf{a}, V, \sigma; \Omega)$ are absolutely continuous.*

5 Proof of Theorems 4.6 and 2.9

From now on, we assume that the conditions of Theorem 4.6 are fulfilled.

5.1 Extension operators. The proof is based on a reduction of H_D and H_N to a periodic operator. To construct the appropriate operator, we reflect the strip $\mathcal{S}_1$ in the line $x_2 = 0$ and extend the functions $\mathbf{a}, V$ and σ into the lower half of the domain obtained. More precisely, denote

$$\mathcal{S}_u = \mathcal{S}_1, \quad \mathcal{S}_l = \{\mathbf{x} \in \mathbb{R}^2 : (x_1, -x_2) \in \mathcal{S}_u\},$$
$$\mathcal{S}_0 = \mathcal{S}_u \cup \mathcal{S}_l \cup \{\mathbf{x} : x_2 = 0\},$$

and define subspaces

$$\mathsf{L}^2_{\pm}(\mathcal{C}) = \{u \in \mathsf{L}^2(\mathcal{C}) : u(x_1, x_2) = \pm u(x_1, -x_2), \quad \text{a.a. } \mathbf{x} \in \mathcal{S}_0\}$$

of all even (L^2_+) and odd (L^2_-) functions from $\mathsf{L}^2(\mathcal{C})$. One easily concludes by inspection that the projections onto these subspaces are given by the formula

$$\mathcal{P}_{\pm} u = \frac{1}{2}\big(u(x_1, x_2) \pm u(x_1, -x_2)\big). \tag{5.1}$$

We also need *the extension operators* $W_{\pm} : \mathsf{L}^2(\mathcal{S}_u) \to \mathsf{L}^2_{\pm}(\mathcal{C})$. For $\mathbf{x} \in \mathcal{S}_0$, they are defined by

$$(W_{\pm}u)(x_1, x_2) = \begin{cases} u(x_1, x_2)/\sqrt{2}, & \mathbf{x} \in \mathcal{S}_u, \\ \pm u(x_1, -x_2)/\sqrt{2}, & \mathbf{x} \in \mathcal{S}_l, \end{cases}$$

and extended to $\mathbb{R}^2$ as γ_2-periodic functions. Using the formula

$$(W_{\pm}^{-1}u)(\mathbf{x}) = \sqrt{2}\, u(\mathbf{x}), \quad \mathbf{x} \in \mathcal{S}_u, \quad u \in \mathsf{L}^2_{\pm}(\mathcal{C}), \tag{5.2}$$

one readily proves that $W_{\pm}$ is unitary on $\mathsf{L}^2(\mathcal{S}_u)$. To find out how the functions from the Sobolev spaces $\mathsf{H}^1(\mathcal{S}_u)$, $\mathsf{H}^1_0(\mathcal{S}_u)$ transform under the extensions $W_{\pm}$, denote

$$\mathsf{H}^1_{\pm}(\mathcal{C}) = \mathcal{P}_{\pm}\mathsf{H}^1(\mathcal{C}).$$

Given the explicit form (5.1) of the operator $\mathcal{P}_{\pm}$, it is easy to see that

$$\mathsf{H}^1_{\pm}(\mathcal{C}) = \mathsf{H}^1(\mathcal{C}) \cap \mathsf{L}^2_{\pm}(\mathcal{C}). \tag{5.3}$$

Now, observing that

$$u(x_1, -\pi) = u(x_1, 0) = u(x_1, \pi) = 0, \quad \text{a.a. } x_1 \in \mathbb{R}, \tag{5.4}$$

for all $u \in \mathsf{H}^1_-(\mathcal{C})$, one sees that

$$W_+\mathsf{H}^1(\mathcal{S}_u) = \mathsf{H}^1_+(\mathcal{C}), \quad W_-\mathsf{H}^1_0(\mathcal{S}_u) = \mathsf{H}^1_-(\mathcal{C}). \tag{5.5}$$

5.2 Reduction. Now we can describe the periodic operator associated with the operators H_D and H_N. Assume that the conditions of Theorem 4.6 are satisfied. We begin with the definition of the corresponding system of curves. Define $\mathbf{\Sigma}_u = (\Sigma_{1,u}, \Sigma_{2,u}, \dots, \Sigma_{N,u})$, where $\Sigma_{j,u} = \Sigma_j, j = 1, 2, \dots, N$, and

$$\mathbf{\Sigma}_l = (\Sigma_{1,l}, \Sigma_{2,l}, \dots, \Sigma_{N,l}),$$
$$\Sigma_{j,l} = \{\mathbf{x} \in \overline{\mathcal{S}_l} : (x_1, -x_2) \in \Sigma_{j,u}\},$$
$$\mathbf{\Sigma}_0 = (\mathbf{\Sigma}_u, \mathbf{\Sigma}_l).$$

By Definition 2.2, $\mathbf{\Sigma}_0$ is a system of curves in $\mathcal{C}$. We emphasise that this system contains two copies of the upper and lower boundaries of the strip $\mathcal{S}_1$.

Define the γ_2-periodic functions $\mathbf{b}, Q$ by applying the extension operators $W_{\pm}$ to $\mathbf{a}, V$ in the following way:

$$Q = \sqrt{2}\, W_+V, \quad b_1 = \sqrt{2}\, W_+a_1, \quad b_2 = \sqrt{2}\, W_-a_2.$$

It is clear that the functions Q, b_1 are even, and b_2 is odd. Clearly, the new coefficients $\mathbf{b}, Q$ satisfy (2.10) with $\Upsilon = \mathbb{T}^2$. In a similar way, we extend the function σ to $\mathbf{\Sigma}_0$. Specifically, let ρ be the function on $\mathbf{\Sigma}_0$ defined by

$$\rho(\mathbf{x}) = \begin{cases} \sigma(\mathbf{x}), & \mathbf{x} \in \mathbf{\Sigma}_u \\ \sigma(x_1, -x_2), & \mathbf{x} \in \mathbf{\Sigma}_l \end{cases}$$

and extended γ_2-periodically. Clearly, ρ satisfies (2.11) with $\Xi = \Sigma_0/\gamma_1$.

Now define the reference periodic operator $H = H_P(1, \mathbf{B}, \mathbf{b}, Q, 1, \rho; \mathcal{C})$. Using the symmetry properties of $\mathbf{b}, Q, \rho$, we decompose the operator H in the orthogonal sum associated with the subspaces $\mathsf{L}^2_\pm = \mathsf{L}^2_\pm(\mathcal{C})$.

Lemma 5.1. *The subspaces $\mathsf{L}^2_\pm$ are invariant subspaces of the operator H_P.*

Proof. We need to check that

$$\mathcal{P}_\pm D[h] = \mathsf{L}^2_\pm \cap D[h], \tag{5.6}$$

and that for any $u, v \in D[h]$

$$h[u, v] = h[\mathcal{P}_+u, \mathcal{P}_+v] + h[\mathcal{P}_-u, \mathcal{P}_-v]. \tag{5.7}$$

The equality (5.6) follows from (5.3). To prove (5.7), it suffices to verify that

$$h[\mathcal{P}_+u, \mathcal{P}_-v] = 0, \quad \forall u, v \in \mathsf{H}^1(\mathcal{C}). \tag{5.8}$$

Write out the l.h.s. using the notation $w_\pm = \mathcal{P}_\pm w$:

$$\sum_{l=1}^{2}\big(b_{ll}(D_l - b_l)u_+, (D_l - b_l)v_-\big) + (Qu_+, v_-) + \int_{\Sigma_0} \rho u_+ \overline{v_-} dS.$$

The last two terms vanish since Q and ρ are even functions. In particular, the integrals over ∂S equal zero in view of (5.4). To handle the first term, make the following table using the properties of the coefficients:

$(D_1 - b_1)u_+$ even,	$(D_1 - b_1)v_-$ odd,
$(D_2 - b_2)u_+$ odd,	$(D_2 - b_2)v_-$ even.

Now it is easy to see that the first term also vanishes, which implies (5.8). $\square$

Denote the parts of the operator H in this orthogonal decomposition by H_+ and H_-. It can be shown that $H_\pm$ are unique self-adjoint operators associated with the closed quadratic forms $h_\pm[\,\cdot\,]$ obtained from $h[\,\cdot\,]$ by restricting the domain $D[h]$ to $D[h_\pm] = \mathsf{H}^1_\pm(\mathcal{C})$. The final step of the reduction of H_D and H_N to H is implemented in the following lemma.

Lemma 5.2. *Let the conditions of Theorem 4.6 be satisfied. Then*

$$W_- H_D W_-^* = H_-, \quad W_+ H_N W_+^* = H_+. \tag{5.9}$$

Proof. By (5.5), $W_+\mathcal{D}_N = D[h_+]$ and $W_-\mathcal{D}_D = D[h_-]$; hence it suffices to show that the bilinear forms of the operators in (5.9) coincide on the domains $D[h_-] = \mathsf{H}^1_-(\mathcal{C})$ and $D[h_+] = \mathsf{H}^1_+(\mathcal{C})$, respectively. Let $u_\pm, v_\pm \in \mathsf{H}^1_\pm(\mathcal{C})$. Using (5.2) and referring again to the symmetry properties of the coefficients, one can write

$$\begin{aligned}
&\left(\mathbf{B}(\mathbf{D}-\mathbf{a})W_\pm^* u_\pm, (\mathbf{D}-\mathbf{a})W_\pm^* v_\pm\right) \\
&\qquad + \left(VW_\pm^* u_\pm, W_\pm^* v_\pm\right) + \int_\Sigma \sigma W_\pm^* u_\pm \overline{W_\pm^* v_\pm}\, dS \\
&= 2\int_{\mathcal{S}_u}\left[\sum_{l=1}^2 b_{ll}(D_l - a_l)u_\pm \overline{(D_l - a_l)v_\pm} + V u_\pm \overline{v_\pm}\right] d\mathbf{x} + 2\int_\Sigma \sigma u_\pm \overline{v_\pm}\, dS \\
&= \int_{\mathcal{S}_0}\left[\sum_{l=1}^2 b_{ll}(D_l - b_l)u_\pm \overline{(D_l - b_l)v_\pm} + Q u_\pm \overline{v_\pm}\right] d\mathbf{x} + \int_{\Sigma_0} \rho u_\pm \overline{v_\pm}\, dS.
\end{aligned}$$

This form coincides with the bilinear form of the operator $H_\pm$. □

Proof of Theorem 4.6. The coefficients $\mathbf{b}, Q$ satisfy the conditions of Proposition 3.3. Therefore, the periodic operator H is absolutely continuous; and so are the orthogonal parts H_+ and H_-. By virtue of Lemma 5.2, the operators H_N and H_D are unitarily equivalent to H_+, H_-; and thus they are also absolutely continuous, as required. □

Theorem 4.6 combined with Corollary 3.2 and Theorem 4.5 leads to Theorem 2.9.

6 Quasiconformal maps

In this section, we prove Theorems 4.1 and 4.2. We are using the standard approach to second-order elliptic equations in dimension two, which consists in passing to the complex variable and using the theory of quasi-analytic functions (see, e.g., [3], [35]). Let us define $z = x_1 + ix_2$, $f = f_1 + if_2$.

To begin with, notice that the equation (4.2) for f_1, f_2 is equivalent to the *Beltrami* equation (see [35])

$$\partial_{\bar z} f = q\partial_z f, \tag{6.1}$$

where

$$\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \quad \partial_{\bar z} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$$

with the complex-valued function

$$q = \frac{-g_{12} + i(1 - g_{22})}{g_{12} - i(g_{22} + 1)}. \tag{6.2}$$

Note that $\|q\|_{\mathsf{L}^\infty}$ is strictly less than 1 and that the Jacobian of $\mathbf{f}$ satisfies the relation

$$J_{\mathbf{f}} = |\partial_z f|^2 - |\partial_{\bar z} f|^2 = (1 - |q|^2)|\partial_z f|^2.$$

We say that a continuous function f is q-quasiconformal (or quasiconformal) if $\partial_z f \in \mathsf{L}^2_{\mathrm{loc}}$ and f satisfies the equation (6.1) for a.a. z.

6.1 Mappings of $\mathbb{R}^2$. The following theorem is a one-to-one "translation" of Theorem 4.1 into the language of quasiconformal mappings, except for the additional item (v).

Theorem 6.1. *Let $q \in \mathsf{L}^\infty(\mathbb{C})$ be a function such that $\|q\|_{\mathsf{L}^\infty} < 1$. Suppose that q is periodic, that is,*

$$q(z) = q(z + 2\pi) = q(z + 2\pi i), \quad a.a.\ z \in \mathbb{C}.$$

Then there exists a unique q-quasiconformal homeomorphism f of the complex plane onto itself such that $f(0) = 0$, $f(2\pi) = 2\pi$, and $f(\infty) = \infty$.

Moreover,

(i) *$\partial_z f \neq 0$ almost everywhere;*

(ii) *there exists a number $\tau > 2$ such that $f, f^{-1} \in \mathsf{W}^{1,\tau}_{\mathrm{loc}}(\mathbb{C})$;*

(iii) *for any $u \in \mathsf{H}^1_{\mathrm{loc}}(\mathbb{C})$, the derivatives $\partial_z(u \circ f)$ and $\partial_{\bar z}(u \circ f)$ are found by the standard chain rule. Also, for any $h \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{C})$, the function $h \circ \mathbf{f}$ belongs to $\mathsf{L}^1_{\mathrm{loc}}(J_{\mathbf{f}}, \mathbb{C})$ and*

$$\int_{f(\Omega_1)} h(\mathbf{x})d\mathbf{x} = \int_{\Omega_1} (h \circ f)(\mathbf{x}) J_{\mathbf{f}}(\mathbf{x}) d\mathbf{x}$$

for any open bounded $\Omega_1 \subset \mathbb{C}$;

(iv) *the mapping f has the periodicity property*

$$f(z + 2\pi n + 2i\pi m) = f(z) + 2\pi n + \varkappa m, \quad z \in \mathbb{C}, \quad m, n, \in \mathbb{Z}, \tag{6.3}$$

with some $\varkappa$ which has a non-zero imaginary part;

(v) *if $q(\bar z) = \overline{q(z)}$ almost everywhere, then $\varkappa$ in (6.3) is purely imaginary: $\varkappa = 2i\,\mathrm{Im} f(\pi i)$;*

(vi) *if $q \in \mathsf{C}^\alpha(\mathbb{C})$, $0 < \alpha < 1$, then $f \in \mathsf{C}^{1+\alpha}(\mathbb{C})$ and $|\partial_z f| > 0$ for all $z \in \mathbb{C}$*

Proof. By Proposition 7.1, the mapping with the fixed three points $0, 2\pi, \infty$ exists, is unique, and satisfies properties (i), (ii), (iii) and (vi). We need only prove (6.3) and (v). Let ζ be either 2π or $2\pi i$. Along with f, the function

$$\tilde{f}(z) = 2\pi \frac{f(z+\zeta) - f(\zeta)}{f(2\pi+\zeta) - f(\zeta)}$$

is also a solution of the equation (6.1), due to the periodicity of q. Note that the denominator here is not zero as the function f is a homeomorphism, and for the same reason $f(\zeta) \neq 0$ as well. Notice also that $\tilde{f}(0) = 0$, $\tilde{f}(2\pi) = 2\pi$ and $\tilde{f}(\infty) = \infty$. By the uniqueness of such a solution, $\tilde{f} = f$, or, what is the same,

$$f(z+\zeta) = c_1 f(z) + c_2$$

with

$$c_1 = \frac{f(2\pi+\zeta) - f(\zeta)}{2\pi}, \quad c_2 = f(\zeta).$$

Let us consider separately four possibilities: $|c_1| < 1$, $|c_1| > 1$, $c_1 = e^{i\theta}, \theta \in (0, 2\pi)$ and $c_1 = 1$. We shall eliminate the first three of them, thus proving that $c_1 = 1$.

Case 1: $|c_1| < 1$. For any integer $n \geq 1$, we have

$$f(n\zeta) = c_2 \sum_{k=0}^{n-1} c_1^k = c_2 \frac{1 - c_1^n}{1 - c_1};$$

hence $f(n\zeta) \to c_2(1-c_1)^{-1}$ as $n \to \infty$. This contradicts the fact that $f(\infty) = \infty$.

Case 2: $|c_1| > 1$. The sought contradiction follows from Case 1 by rewriting

$$f(z-\zeta) = \frac{1}{c_1} f(z) - \frac{c_2}{c_1},$$

and noting that $|c_1^{-1}| < 1$.

Case 3: $c_1 = e^{i\theta}$, $\theta \in (0, 2\pi)$. Again, as in Case 1, we have

$$f(n\zeta) = c_2 \frac{1 - c_1^n}{1 - c_1}.$$

The r.h.s. remains bounded as $n \to \infty$, which contradicts the requirement that $f(\infty) = \infty$.

Consequently, the only possible option is $c_1 = 1$.

Note that $f(2\pi) = 2\pi$ by definition of f, so that (6.3) with $m = 0$ is proved. Let us now prove that the imaginary part of $\varkappa = f(2\pi i)$ is non-zero. Suppose, on the contrary, that $\operatorname{Im} \varkappa = 0$. Then, for any integer m, one can find another integer $n = n(m)$ such that $|2\pi n + \varkappa m| \leq 2\pi$, so that the r.h.s. of the equality

$$f(2\pi n + 2\pi m i) = 2\pi n + \varkappa m$$

remains bounded as $m \to \infty$. On the other hand, $|2\pi n + 2\pi m i| \to \infty$ as $m \to \infty$. Again we get the same contradiction, which proves that $\operatorname{Im} \varkappa \neq 0$.

It is left to prove (v). Let $\varphi(z) := \overline{f(\overline{z})}$. Then $\varphi(0) = 0$, $\varphi(2\pi) = 2\pi$ and $\varphi(\infty) = \infty$. Further, it is easy to see (see, e.g., [1], Ch. I, Section C, (1)) that

$$\partial_z(f(\overline{z})) = (\partial_{\overline{z}} f)(\overline{z}) = (q\partial_z f)(\overline{z}) = q(\overline{z})(\partial_z f)(\overline{z}) = q(\overline{z})\partial_{\overline{z}}(f(\overline{z})).$$

Taking the complex conjugates of both sides and using the equality $q(\overline{z}) = \overline{q(z)}$, we obtain $\partial_{\overline{z}}\varphi(z) = q(z)\partial_z\varphi(z)$. By uniqueness, we then conclude that $\varphi \equiv f$, i.e., $\overline{f(\overline{z})} = f(z)$, for all $z \in \mathbb{C}$; in particular, $f(-\pi i) = \overline{f(\pi i)}$. On the other hand, (6.3) implies

$$f(\pi i) = f(-\pi i + 2\pi i) = f(-\pi i) + \varkappa = \overline{f(\pi i)} + \varkappa.$$

Hence $\varkappa = 2i \operatorname{Im} f(\pi i)$. □

Theorem 4.1 follows immediately.

We emphasise again that the crucial periodicity property (6.3) of the quasiconformal map f is a direct consequence of the uniqueness in Proposition 7.1. Besides, we have included in Theorem 6.1 statement (v), which also follows from the uniqueness. Using (6.2), one easily sees that the condition $q(\overline{z}) = \overline{q(z)}$ in part (v) is equivalent to the following symmetry conditions on the matrix $\mathbf{G}$:

$$g_{jj}(x_1, x_2) = g_{jj}(x_1, -x_2), \quad j = 1, 2,$$
$$g_{jl}(x_1, x_2) = -g_{jl}(x_1, -x_2), \quad j \neq l.$$

Then $\operatorname{Re} \varkappa = 0$ means that the isothermal change of variables transforms the initial square lattice $(2\pi\mathbb{Z})^2$ into another *orthogonal* lattice. Although this observation is not needed in this paper, we consider it worth mentioning.

6.2 Mapping of the domain Ω. The next theorem restates Theorem 4.2 in the language of quasiconformal mappings.

Theorem 6.2. *Let Ω be a domain as in Theorem 4.2; in particular, Condition 2.8 is fulfilled and $0 \in \ell_-$. Let $q \in C^\alpha(\overline{\Omega})$ be a periodic function (i.e., $q(z) = q(z+2\pi)$ for all $z \in \Omega$) such that $\|q\|_{L^\infty} < 1$. Then there exists a unique q-quasiconformal homeomorphism f of the domain Ω onto the strip S_1 such that $f(0) = 0$, $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$. The map f has the following properties.*

(i) *For all $z \in \Omega$,*

(6.4) $$f(z + 2\pi n) = f(z) + \varkappa n, \quad \textit{for all } n \in \mathbb{Z},$$

with some $\varkappa > 0$.

(ii) *$f \in C^{1+\alpha}(\overline{\Omega} \setminus Z)$, $f^{-1} \in C^{1+\alpha}(\overline{S} \setminus f(Z))$, and $|\partial_z f| > 0$ everywhere in Ω. Moreover, for each $z_0 \in Z$, there exist a number $\nu \in (0, 2]$ and four Hölder-continuous functions M, T and Φ, Ψ, separated from zero, such that in the vicinity of z_0 and $\zeta_0 = f(z_0) \in f(Z)$ one has the representations*

$$\partial_z f(z) = |z - z_0|^{1/\nu - 1} \Phi\big(\arg(z - z_0)\big) M(z),$$
$$\partial_\zeta f^{-1}(\zeta) = |\zeta - \zeta_0|^{\nu - 1} \Psi\big(\arg(\zeta - \zeta_0)\big) T(\zeta).$$

Proof. Pick the following accessible boundary points (see, e.g., [14], Ch. II, §3 for definition) of $\partial\Omega$: $-\infty, +\infty, 0$. Then by Theorem 7.2(i) combined with Remark 7.3, there exists a uniquely defined quasiconformal homeomorphism $f : \Omega \to \mathcal{S}_1$ which preserves these boundary points. Moreover, the smoothness properties required in (ii) follow from Theorem 7.2(ii) and (iii). It remains to prove (6.4).

In view of the periodicity of the domain Ω, the mapping $\tilde{f}(z) = f(z + 2\pi)$ is also a quasiconformal homeomorphism of Ω onto the straight strip $\mathcal{S}_1$. Note that $\tilde{f}$ sends $-\infty$ and ∞ into themselves and the point 0 into $z_0 = f(2\pi)$. Also, z_0 is real, since it lies on the lower portion of the boundary of $\mathcal{S}_1$, i.e., on the horizontal line $\operatorname{Im} z = 0$. Consequently, the composition function

$$h(z) = \tilde{f}\big(f^{-1}(z)\big)$$

defined on $\mathcal{S}_1$ is a conformal homeomorphism of $\mathcal{S}_1$ onto itself (see [3], Part II, Section 6.2), acting in such a way that $-\infty$, $+\infty$ are preserved and $h(0) = z_0$. It is easy to see that the function $\tilde{h}(z) = z + z_0$ satisfies the same conditions. On the other hand, such a conformal mapping is unique (see, e.g., [14], Ch. II, §3, Theorem 6). Therefore, $h(z) = \tilde{h}(z)$; and hence

$$\tilde{f}(z) = f(z) + \varkappa, \quad \varkappa = \overline{\varkappa} = z_0.$$

It is clear that $\varkappa \neq 0$, for otherwise the function f would remain bounded as $|z| \to \infty$, which contradicts the assumption that f preserves $\pm\infty$. For the same reason, $\varkappa > 0$ since otherwise $+\infty$ and $-\infty$ would exchange places under the mapping f. □

For conformal mappings, the above argument can be found in [14], Ch. V, §1. Theorem 4.2 now follows from Theorem 6.2.

6.3 Bilipschitz mappings of the domain Ω. In this subsection, we address a question which has no direct effect on the results of the paper but is nevertheless natural and important. If the domain Ω has no corners or peaks, then according to Theorem 6.2, the homeomorphism $f : \Omega \to \mathcal{S}_1$ is $\mathrm{C}^{1+\alpha}$-smooth. This is guaranteed by the initially assumed $\mathrm{C}^{1+\alpha}$-smoothness of the boundary $\partial\Omega$ (see Condition 2.8) and the C^{α}-smoothness of the matrix G (see (2.19)). In the presence of corners or peaks, one or both of the derivatives $\partial_z f$, $\partial_z f^{-1}$ are unbounded. In particular, the homeomorphism f may fail to be Lipschitz even if the boundary $\partial\Omega$ is Lipschitz. Therefore, one may ask whether a domain Ω with Lipschitz boundary $\partial\Omega$ admits a periodic bilipschitz map $\Omega \to \mathcal{S}_1$.

To state the question in a precise form, recall that a mapping F from a metric space (X_1, d_1) into a metric space (X_2, d_2) is called *bilipschitz* if there exists a constant $M > 0$ such that

$$d_1(x,y)/M \le d_2(F(x), F(y)) \le M d_1(x,y), \quad x, y \in X_1.$$

We say that a curve $\ell \in \mathbb{C}$ is Lipschitz if it is a bilipschitz image of $\mathbb{R}$. Note in passing that it is easy to give an intrinsic characterisation of a Lipschitz curve. First of all, it is clear that a Lipschitz curve is Jordan and locally rectifiable. Conversely, let $\ell : \mathbb{R} \to \mathbb{C}$ be a Jordan locally rectifiable curve such that $|\ell(t)| \to \infty$ as $t \to \pm\infty$. Using the arclength parametrisation, one can easily show that ℓ is a Lipschitz curve if and only if it is an chord-arc curve, i.e., if there exists a constant $K \ge 1$ such that the length of the subarc of ℓ joining any two points is bounded by K times the distance between them.

Assume that the boundary of a periodic domain Ω consists of two disjoint Lipschitz curves ℓ_+, ℓ_-. Our objective is to find a bilipschitz mapping F of $\mathcal{S}_1$ onto Ω such that

$$F(z + 2\pi) = F(z) + 2\pi, \quad z \in \mathcal{S}_1. \tag{6.5}$$

It is evident that the bilipschitz regularity of the curves $\ell_\pm$ is necessary for the existence of such a mapping. This condition is also sufficient.

Theorem 6.3. *Let Ω be a simply connected periodic domain with a boundary consisting of two disjoint Lipschitz curves ℓ_+ and ℓ_-. Then there exists a bilipschitz mapping F of $\mathcal{S}_1$ onto Ω satisfying* (6.5).

Proof. Let $\varphi : \mathcal{S}_1 \to \Omega$ be a conformal homeomorphism, mapping $\pm\infty$ to $\pm\infty$, 0 to a given point of ℓ_- and such that $\varphi(\zeta + \varkappa) = \varphi(\zeta) + 2\pi$, for all $\zeta \in \mathcal{S}_1$ where $\varkappa > 0$. The existence of such a map follows from [14], Ch. V, §1. Since $\ell_\pm$ are locally rectifiable Jordan curves, it follows that φ can be extended to a homeomorphism of the closure of $\mathcal{S}_1$ onto the closure of Ω (see [14], Ch. II, §3, Theorem 4); that, for almost all $x_0 \in \mathbb{R}$, the finite limits

$$\lim_{\tau \to 0+} \varphi'(x_0 + i\tau) \ne 0, \quad \lim_{\tau \to 0+} \varphi'(x_0 + i\pi - i\tau) \ne 0$$

exist; and that φ is conformal at x_0 and $x_0 + i\pi$ (see [14], Ch. X, §1, Theorems 1 and 3). This latter implies that the curves $\ell_\pm$ meet the arcs $\ell_0 = \varphi([x_0, x_0 + i\pi])$ and $\ell_1 = \varphi([x_0 + \varkappa, x_0 + \varkappa + i\pi])$ at the right angle. Now let us define a closed "fundamental domain" of the strip $\mathcal{S}_1$:

$$\Delta := \{x + iy : \ x_0 \leq x \leq x_0 + \varkappa, \ 0 \leq y \leq \pi\}.$$

The boundary of Δ consists of four segments

$$\gamma_- = [x_0, x_0 + \varkappa], \quad \gamma_+ = [x_0 + i\pi, x_0 + \varkappa + i\pi],$$
$$\gamma_0 = [x_0, x_0 + i\pi], \quad \gamma_1 = [x_0 + \varkappa, x_0 + \varkappa + i\pi].$$

The next step is to define a bilipschitz map F_0 from $\partial\Delta$ onto $\varphi(\partial\Delta) = \partial\varphi(\Delta)$. Let $\tilde{\ell}_\pm := \varphi(\gamma_\pm) \subset \ell_\pm$. Since $\ell_\pm$ are Lipschitz curves, there exists a bilipschitz homeomorphism $F_0 : \gamma_\pm \to \tilde{\ell}_\pm$. Further, set $F_0(z) = \varphi(z)$ if $z \in \gamma_0$ or γ_1. In view of the periodicity of φ, we have

$$F_0(z + \varkappa) = F_0(z) + 2\pi, \quad z \in \gamma_0. \tag{6.6}$$

Since the arcs $\tilde{\ell}_+$, $\tilde{\ell}_-$ and ℓ_0, ℓ_1 meet at the right angle, it is easy to see that F_0 is a bilipschitz mapping of $\partial\Delta$ onto $\partial\varphi(\Delta) = \varphi(\partial\Delta)$. Then it follows from [34] (see also [26], Theorem 7.10 and [22]) that F_0 can be extended to a bilipschitz homeomorphism of $\mathbb{C}$ onto itself. This defines a bilipschitz homeomorphism $F_0 : \Delta \to \varphi(\Delta)$. By virtue of (6.6), it is now straightforward to see that the extension F_1 of F_0, defined by

$$F_1(\zeta + n\varkappa) := F_0(\zeta) + 2\pi n, \quad \zeta \in \Delta, \ n \in \mathbb{Z},$$

is a bilipschitz mapping of $\mathcal{S}_1$ onto Ω such that $F_1(\zeta + \varkappa) = F_1(\zeta) + 2\pi$, $\zeta \in \mathcal{S}_1$. It remains to define F by

$$F(x + iy) := F_1\left(\frac{\varkappa}{2\pi} x + iy\right). \qquad \square$$

7 General properties of quasiconformal mappings

We begin with describing a quasiconformal homeomorphism of the complex plane onto itself. The following general result can be found in [1], [2], [3] Part II, Ch. 6, or [35], Ch. 2.

Proposition 7.1. *Let $q \in \mathsf{L}^\infty(\mathbb{R}^2)$ be a function such that $\|q\|_{\mathsf{L}^\infty} < 1$. Then there exists a unique q-quasiconformal homeomorphism f of the complex plane such that $f(0) = 0$, $f(2\pi) = 2\pi$ and $f(\infty) = \infty$. Moreover,*

(i) *the conclusions* (i), (ii), (iii) *of Theorem 6.1 hold;*

(ii) *if* $q \in C^\alpha$, $0 < \alpha < 1$, *in a neighbourhood of some point* z_0, *then* $f \in C^{1+\alpha}$ *and* $|\partial_z f| > 0$ *in this neighbourhood.*

The next theorem is a "quasiconformal" version of Riemann's mapping theorem and the results on the boundary behaviour of conformal mappings. The results collected in this theorem probably are not new, but we have not been able to find them stated in the form convenient for us.

Theorem 7.2. *Let* $\Omega \subset \mathbb{C}$ *be a simply connected open domain with more than one boundary point such that all points of* $\partial\Omega$ *are accessible. Suppose* $q \in L^\infty(\Omega)$ *and* $\|q\|_{L^\infty} < 1$.

(i) *There exists a unique q-quasiconformal map w of* Ω *onto the unit disk* $\mathcal{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ *which maps three given points* $z_1, z_2, z_3 \in \partial\Omega$ *indexed in order of their occurrence as one proceeds in the positive direction along* $\partial\Omega$ *(see [14], Ch. II, §3) onto three given points* $\zeta_1, \zeta_2, \zeta_3 \in \partial\mathcal{D}$ *similarly indexed. This map and its inverse* f^{-1} *belong to* $W^{1,\tau}_{loc}(\Omega)$ *for some* $\tau > 2$. *The map f defines a homeomorphism of the compactification* $\widehat{\Omega}$ *of* Ω *by the prime ends onto the closed unit disk.*

(ii) *Let* $m \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$, $z_0 \in \widehat{\Omega}$, *and let q be* $C^{m+\alpha}$*-smooth in a neighbourhood of* z_0. *Suppose also that if* $z_0 \notin \Omega$, *then the intersection of* $\partial\Omega$ *with a neighbourhood of* z_0 *is a* $C^{m+1+\alpha}$*-smooth Jordan curve. Then f is* $C^{m+1+\alpha}$*-smooth in a neighbourhood of* z_0 *and* $|\partial_z f(z_0)| > 0$.

(iii) *Let* $z_0 \in \partial\Omega$ *and suppose that the intersection of* $\partial\Omega$ *with a neighbourhood of* z_0 *is a piecewise* $C^{1+\alpha}$*-smooth Jordan curve,* $0 < \alpha < 1$, *with the only angular point at* z_0, *which is not an outward cusp, i.e., the interior angle at* z_0 *with respect to* Ω *is nonzero. Let* $\gamma : [t_1, t_2] \to \mathbb{C}$, $\gamma(t_0) = z_0$ *be a parametrisation of this curve in the positive direction. Suppose also that q is* C^α*-Hölder continuous in a neighbourhood of* z_0. *Then f and its inverse* $g := f^{-1}$ *satisfy*

$$\partial_z f(z) = ((z - z_0) + q(z_0)(\bar{z} - \bar{z}_0))^{1/\nu - 1} F(z), \tag{7.1}$$

$$\partial_\zeta g(\zeta) = (\zeta - \zeta_0)^{\nu-1} G(\zeta), \quad \partial_{\bar\zeta} g(\zeta) = -q(g(\zeta))\overline{\partial_\zeta g(\zeta)}, \tag{7.2}$$

where $\zeta_0 = f(z_0)$,

$$\nu = \frac{1}{\pi} \arg\left\{ -\frac{\gamma'(t_0 - 0) + q(z_0)\overline{\gamma'(t_0 - 0)}}{\gamma'(t_0 + 0) + q(z_0)\overline{\gamma'(t_0 + 0)}} \right\} \in (0, 2], \tag{7.3}$$

and F, G are Hölder continuous and nowhere zero in neighbourhoods of z_0 and ζ_0, respectively.

Remark 7.3. The theorem above uses the disk $\mathcal{D}$ as the target domain. This choice was made only for definiteness and convenience in the proof. One can easily restate Theorem 7.2 choosing other simply connected domains as targets. In particular, making obvious modifications, $\mathcal{D}$ can be replaced by the strip $\mathcal{S}_1$. This can be done by mapping $\mathcal{D}$ onto $\mathcal{S}_1$ using a standard conformal map and noticing that composition of a conformal and a q-quasiconformal mappings is again q-quasiconformal (see [3], Part II, §6.2).

Proof of Theorem 7.2. Step I. Take an arbitrary extension $q_0 \in \mathsf{L}^\infty(\mathbb{C})$ of q such that $\|q_0\|_{\mathsf{L}^\infty} < 1$. There exists a q_0-quasiconformal homeomorphism $w : \mathbb{C} \to \mathbb{C}$ which belongs, together with its inverse, to $\mathsf{W}^{1,\tau}_{\mathrm{loc}}(\mathbb{C})$ with $\tau > 2$ (see Proposition 7.1). Let $\Omega_0 := w(\Omega)$, $z'_k := w(z_k)$, $k = 1,2,3$. It is clear that Ω_0 is a simply connected domain and all points of $\partial\Omega_0$ are accessible. Hence there exists a unique conformal map ψ of Ω_0 onto $\mathcal{D}$ which maps z'_1, z'_2, z'_3 onto $\zeta_1, \zeta_2, \zeta_3$ (see, e.g., [14], Ch. II, §3, Theorem 6). It is not difficult to see that $f := \psi \circ w : \Omega \to \mathcal{D}$ is a q-quasiconformal map (see [3], Part II, §6.2) having all the properties announced in (i).

Step II. Let $f_1 : \Omega \to \mathcal{D}$ be an arbitrary q-quasiconformal homeomorphism mapping z_1, z_2, z_3 onto $\zeta_1, \zeta_2, \zeta_3$. Then $f_1 \circ f^{-1} : \mathcal{D} \to \mathcal{D}$ is an analytic (see [3], Part II, §6.2) homeomorphism and hence a conformal automorphism (see [14], Ch. II, §1) with fixed points $\zeta_1, \zeta_2, \zeta_3$. By the uniqueness result for conformal maps, we have $f_1 \circ f^{-1}(z) \equiv z$, i.e., $f_1 = f$. This proves uniqueness and shows that the map f constructed above **does not depend on the choice of an extension q_0**.

Step III. Under the conditions of (ii), there exists an extension q_0 which is $\mathsf{C}^{m+\alpha}$-smooth in a neighbourhood of z_0. It then follows from [35], Theorem 2.9 and the proof of Theorem 2.12 that w from Step I is $\mathsf{C}^{m+1+\alpha}$-smooth in a neighbourhood of z_0 and

$$|\partial_z w(z_0)|^2 - |\partial_{\bar z} w(z_0)|^2 > 0. \tag{7.4}$$

So (ii) follows if we can prove that the conformal map ψ from Step I is $\mathsf{C}^{m+1+\alpha}$-smooth in a neighbourhood of $w(z_0)$ and $\psi'(w(z_0)) \neq 0$. We need to do this only if $w(z_0) \in \partial\Omega_0$, i.e., $z_0 \in \partial\Omega$.

Step IV. It follows from Step III that, under the conditions of Part (ii), the intersection of $\partial\Omega_0$ with a neighbourhood of $w(z_0)$ is a $\mathsf{C}^{m+1+\alpha}$-smooth Jordan curve. Let us take a simply connected subdomain $\Omega_1 \subset \Omega_0$ with a $\mathsf{C}^{m+1+\alpha}$-smooth boundary such that $\Gamma := \partial\Omega_1 \bigcap \partial\Omega_0$ is a $\mathsf{C}^{m+1+\alpha}$-smooth subarc of the above

Jordan curve such that $w(z_0)$ belongs to Γ and is different from its endpoints. By the uniqueness result for conformal maps, we can choose conformal maps $\psi_1 : \Omega_1 \to \mathcal{D}$ and $\varphi : \mathcal{D} \to \psi(\Omega_1)$ so that

$$\psi = \varphi \circ \psi_1 \quad \text{in } \Omega_1.$$

The map ψ_1 is $C^{m+1+\alpha}$-smooth in the closure of Ω_1, and $\psi_1'(w(z_0)) \neq 0$ (see [26], Theorems 3.5, 3.6)

The mapping φ maps an open circular arc containing $\psi_1(w(z_0))$ onto a circular arc. Hence φ is holomorphic in a neighbourhood of $\psi_1(w(z_0))$, and $\varphi'(\psi_1(w(z_0))) \neq 0$ (see [14], Ch. II, §3, Theorem 5). Therefore, ψ is $C^{m+1+\alpha}$-smooth in a neighbourhood of $w(z_0)$; and $\psi'(w(z_0)) \neq 0$. We note in passing that this is a variant of Kellogg's theorem (cf., e.g., [21], Section 29 or [20], Ch. II, §1, Theorem 1).

The proof of (ii) is now completed.

Step V. The proof of (iii) is similar to that of (ii). Let q_0 be an extension of q which is C^α-Hölder continuous in a neighbourhood of z_0. Then as in Step III, w from Step I is $C^{1+\alpha}$-smooth in a neighbourhood of z_0 and satisfies (7.4). Therefore, the intersection of $\partial\Omega_0$ with a neighbourhood of $w(z_0)$ is a piecewise $C^{1+\alpha}$-smooth Jordan curve whose only singular point is at $w(z_0)$. This curve is parameterized by $[t_1, t_2] \ni t \mapsto w(\gamma(t)) \in \mathbb{C}$. The equality $\partial_{\bar z} w(z) = q(z)\partial_z w(z)$ implies

$$\begin{aligned}\frac{dw(\gamma(t_0 \pm 0))}{dt} &= \partial_z w(z_0)\gamma'(t_0 \pm 0) + \partial_{\bar z} w(z_0)\overline{\gamma'(t_0 \pm 0)} \\ &= (\gamma'(t_0 \pm 0) + q(z_0)\overline{\gamma'(t_0 \pm 0)})\partial_z w(z_0).\end{aligned}$$

Therefore, the interior angle with respect to $\Omega_0 = w(\Omega)$ at $w(z_0)$ equals $\pi\nu$, where ν is given by (7.3).

Step VI. Now we need to investigate the properties of the conformal map $\psi : \Omega_0 \to \mathcal{D}$. Using the argument from Step IV, we can reduce this to the study of a conformal map $\psi_1 : \Omega_1 \to \mathcal{D}$, where Ω_1 is a simply connected domain with a piecewise $C^{1+\alpha}$-smooth boundary whose only singular point is $w(z_0)$, where the interior angle with respect to Ω_1 equals $\pi\nu > 0$. Applying Warschawski's theorem (see, e.g., [35], Theorem 1.9, and [16], Ch. 3, §3), we obtain that ψ and its inverse $\eta := \psi^{-1}$ satisfy the conditions

$$\psi'(w) = (w - w(z_0))^{1/\nu - 1}\Psi(w), \quad \eta'(\zeta) = (\zeta - \zeta_0)^{\nu-1}E(\zeta), \quad \zeta_0 = \psi(w(z_0)),$$

where Ψ and E are Hölder continuous and nowhere zero in neighbourhoods of $w(z_0)$ and ζ_0, respectively. Now (iii) follows from the formulae $f = \psi \circ w$, $g = w^{-1} \circ \eta$

and

$$\begin{aligned} w(z) - w(z_0) &= (\partial_z w(z_0)(z - z_0) + \partial_{\bar z} w(z_0)(\bar z - \bar z_0))\, \omega(z, z_0) \\ &= ((z - z_0) + q(z_0)(\bar z - \bar z_0))\, \partial_z w(z_0)\, \omega(z, z_0), \end{aligned}$$

where $\omega(\cdot, z_0)$ is Hölder continuous in a neighbourhood of z_0 and $\omega(z_0, z_0) = 1$. The second equality in (7.2) follows from [3], Part II, Ch. 6, Appendix, Theorem 3(iv) (see also [1], Ch. I, Section C). □

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