MEASURE-THEORETIC PROBLEMS IN TOPOLOGICAL DYNAMICS

By

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In memory of my sister

Abstract. This paper, motivated by a conjecture raised by Choksi in 1984 about homogeneous spaces, investigates the topological connexions between transformation groups and product spaces; our approach, based on the Furstenberg structure theorem, provides a unified treatment for (Baire) measures on any minimal distal flow and for measures on a product of compact metric spaces of the same topological weight.

0. Introduction

The problem of 'point realizations of σ-homomorphisms' goes back to the well known publication [26] of von Neumann about measure algebras on Polish spaces. His investigations were continued by Maharam [21] and completed in the work [4] of Choksi, who considered arbitrary Baire measures on any product of Polish spaces. In subsequence papers, Maharam [22], Choksi and Fremlin [8], Graf [15], and particularly Fremlin [13] discussed various aspects of the problem; for further information see [7].

We shall here study especially the problem for measures on a minimal distal flow [1], [10]; motivations for this kind of topological measure theory are numerous: most of them have their roots in Choksi [5], Choksi and Simha [9], as well as Choksi [6], three references of particular interest.

The first important theorem on the structure of distal flows is the Furstenberg structure theorem [14]. Developments on the same topic will be found in Veech [27], Ellis, Glasner and Shapiro [12], McMahon and Wu [24], McMahon [23] and later on Ellis [11]. A detailed account of topological dynamics is given in [28]; see also [29].

The present paper deals with the structure of topological transformation groups and emphasizes the topological connexions between minimal distal flows and product spaces. By combining the Furstenberg theorem and ideas in [9] with some classical facts about compact transformation groups, it is shown, in Theorem 3.7, that (the phase space of) any minimal distal flow is Baire isomorphic to a product of compact metric spaces; the isomorphism takes some invariant, completion regular probability measure to a direct product measure. This simple fact, which is one of the main results, combined with arguments due to Maharam [20], Choksi [4] and

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Choksi and Fremlin [8], allows the rapid resolution of several measure-theoretic problems; in particular, it provides a rather simple approach to the 'point realization of homorphisms', in the setting of transformation groups, only using topology, and extends previous studies in the field (cf. [9], [16], [17]).

The paper is organized in three parts. Section 1 deals with topological tools about compact transformation groups. These tools are used in Section 2 to study measure-theoretic problems, while Section 3 investigates the situation for minimal distal flows.

1. Some facts about compact transformation groups

Theorem 2.3 in [17] states that the (normalized) Haar measure on any compact group is Baire isomorphic to a direct product measure on some product of compact metric spaces. The natural question, based on a problem posed by Choksi [6, p. 89] about measures on homogeneous spaces, arises from that result: is there a corresponding fact for homogeneous spaces?

In Sections 1 and 2 it is shown that this is exactly the case. In passing, we also prove some auxiliary results, which may be of independent interest. We start by introducing some notation and terminology.

Definitions Let X be a compact (Hausdorff) topological space and G a locally compact group. The pair (G,X) is a (left) transformation group (t.g.), or flow, if there is a continuous action ([3], [25]) $G \times X \to X : (g,x) \to gx$ of G on X (one defines a right t.g. in the obvious way). In this case X is called the phase space of the t.g. and the symbol X/G denotes the quotient, or orbit space, i.e. the space of G-orbits with the quotient topology (note that the canonical projection $X \to X/G$ is an open mapping). For a couple (C,D) of closed subgroups of G, with $C \subset D$, $p_{C,D}$ denotes the (canonical) projection of X/C to X/D.

Say that (G,X) is compact (resp free, or G acts freely) if G is compact (resp $g \neq \mathrm{id}_G$, then $gx \neq x$, $x \in X$).

By a compact homogeneous space, we mean a topological space of the form G/H, where G is a compact group and H is a closed subgroup of G, under the action of G.

Next, let K, L be compact spaces. A map $f : K \to L$ is called Baire measurable iff $f^{-1}B$ is Baire in K, for all Baire sets B in L. A Baire measurable bijection f is said to be a Baire isomorphism iff f^{-1} is also Baire measurable.

For a product $A_1 \times A_2$, we denote by p_i $(i = 1,2)$ the i-projection of $A_1 \times A_2$ to A_i.

In the sequel we shall identify a cardinal with its initial ordinal.

The main result of this section is somewhat technical, but nevertheless, has consequences which are useful in the following sections.

Theorem 1.1 *Let (G,X) be a free, compact t.g. and F a closed subgroup of G. There are an ordinal α, a product of compact metric spaces $Y = \Pi_{\beta<\alpha} Y_\beta$, each Y_β with at least two points, and a Baire isomorphism q from $X/G \times Y$ onto X/F such that $p_{F,G} \circ q = p_1$.*

[*Note.* The conclusion of 1.1 holds for $X = G$. It follows that every compact homogeneous space is Baire isomorphic to a product of compact metric spaces. Thus, in view of Choksi's work [4] on products of Polish spaces, 1.1 just stated provides an alternative and totally different proof of Theorem 1 in [9], in the compact case.]

Before embarking on the proof of 1.1, we need a generalization of Lemma 2.1 in [17].

Lemma 1.2 *Let (H,Y) be a (left) free compact t.g. and N a closed subgroup of H. If H is Lie, then there exists a Baire isomorphism q from $Y/H \times H/N$ onto Y/N such that $p \circ q = p_1$ (with $p = p_{N,H}$, the projection).*

Proof By a theorem of Gleason (see e.g. Theorem 1 in Sec. 5.4 of [25], also [3, Ch. II, §4, Remarks p. 86]), for each $y \in Y$ there exist a compact neighborhood $Q_1 = Q_1(y)$ of y (in Y) and a compact $F_1 = F_1(y) \subset Q_1$ such that (i) $Q_1 = F_1 . H$, (ii) $p_H^{-1}\{[x]_H\} \cap F_1$ is a single point, say $r_y([x]_H)$, for $[x]_H \in p_H Q_1$ (where, for $D \subset H$, p_D is the projection of Y to Y/D and $[x]_D = Dx$ the D-orbit of $x \in Y$).

Then for every $y \in Y$ the mapping

$$\begin{aligned}\theta = \theta(y) : p_H F_1 \times H/N &\to p^{-1}(p_H Q_1) \quad (= p^{-1}(p_H F_1)) \\ : ([x]_H, [g]_N) &\to [g.r_y([x]_H)]_N, \quad x \in F_1, \quad g \in H,\end{aligned}$$

is a homeomorphism such that $p' \circ \theta = p'_1$ (where p' and p'_1 are the restrictions $p/_{p^{-1}(p_H F_1)}$ and $p_1/_{p_1 F_1 \times H/N}$ resp).

Let $Q = Q(y) \subset p^{-1}(p_H Q_1)$ be an open Baire neighborhood of $[y]_N$ in Y/N. Then, clearly, the restriction $\theta' = \theta'(y)$ of θ to $pQ \times H/N$ is a homeomorphism onto $p^{-1}Q$.

If now $\{P_1 = pQ(y_1),\ P_2 = pQ(y_2), \ldots, P_L = pQ(y_L)\}$ is any finite cover of Y/N, then we can easily construct (see the proof of Lemma 2.1 in [17]) a cover $\{S^1, S^2, \ldots, S^M\}$ of Baire, pairwise disjoint subsets of Y/H such that

each S^m $(1 \leqq m \leqq M)$ is a subset of some P_k $(1 \leqq k \leqq L)$.

Set $i_m = \min\{k : S^m \subset P_k\}$. Then for every $m = 1, 2, \ldots, M$ the restriction θ_m of $\theta'(y_{i_m})$ to $S^m \times H/N$ is a homeomorphism onto some Baire subset of Y/N. (Note that θ_m has the form $\theta_m([x]_H, [g]_N) = [g.r_m([x]_H)]_N$, with $r_m = r_{y_{i_m}}$. Note also that for different values of m, the images of the maps θ_m are disjoint and partition Y/N.)

To complete the proof of the lemma, we define

$$q : Y/H \times H/N \to Y/N$$
$$: q(w) = \theta_m(w), \quad w \in S^m \times H/N.$$

Then, by the definition, it follows that q satisfies the required conditions (see also proof of Lemma 2.1 in [17]). □

Discussion 1.3 Let X, G, F be as in the statement of 1.1, $e = \mathrm{id}_G$ and let $\mathcal{R}_G$ denote the set of closed normal subgroups of G, directed under the relations: $H_1 < H_2$ if $H_2 \subset H_1$.

If $(P_j)_{j\in J}$ is any decreasing directed family of groups in $\mathcal{R}_G$, with the G/P_j Lie, such that $\bigcap_j P_j = \{e\}$, then X can be identified, in a natural way, with $\mathrm{proj}_j \lim X/P_j$. Similarly, $\{X/FP_j,\ p_{FP_j,FP_i}, i < j\}$ is a projective system of compact spaces and $X/F \cong \mathrm{proj}_j \lim X/FP_j$. Note that

1. If $\pi_j : X/FP_j \to X/G$ is the canonical projection, then the fibres (inverse images of points) are homeomorphic to G/FP_j.

2. In some cases the projection map $X/FP_i \underset{j<i}{\to} X/FP_j$ may be the identity.

It follows that there are $w(G/F)$ many (distinct) G/FP_j, so

$$X/F \cong \mathrm{proj}_{j\in J_0} \lim X/FP_j \quad \text{with } |J_0| = w(G/F). \tag{1.4}$$

Let $P = \bigcap_{j\in J_0} P_j$. It is easily seen that $X/G \cong (X/P)/(G/P)$, $X/F \cong (X/P)/(F/P)$, that is X/P is a free group extension of X/F. Thus, replacing (G,X) and (F,X) by $(G/P,X/P)$ and $(F/P,X/P)$ respectively (i.e. replacing G by G/P), we may, if we wish, regard G as a compact group of topological weight $w(G/F)$, i.e.

(1.5) we may assume without loss of generality(w.l.o.g.) that $w(G) = w(G/F)$.

We turn now to the proof of 1.1. Basically, it is a rehash of the proof of Theorem 2.3 in [17] with modifications due to the fact that F is now assumed to be any closed (not necessarily normal) subgroup of G.

Proof of Theorem 1.1. By (1.5), we may suppose that $w(G) = w(G/F)$. There is a directed set $\Gamma = \{F_j,\ j \in J\}$ of groups in $\mathcal{R}_G$, with the G/F_j Lie, of cardinal $\alpha = w(G)$, such that $\bigcap_{j\in J} F_j = \{e\}$ [25]. Enumerating Γ as $\Gamma = \{F_\xi,\ \xi < \alpha\}$ and taking $H_0 = G$, we set $H_\gamma = H_\delta \cap F_\gamma$ if $\gamma = \delta + 1$ for some $\delta < \alpha$ and $H_\gamma = \bigcap_{\delta<\gamma} H_\delta$

otherwise. Then (see Lemma 2.2 in [17]), $(H_\gamma)_{\gamma<\alpha}$ is a family of groups in $\mathcal{R}_G$, with $\bigcap_{\gamma<\alpha} H_\gamma = \{e\}$, such that

$(i_1) H_\gamma \subset H_\delta$ for $\delta \leqq \gamma \leqq \alpha$,

(i_2) each $H_\gamma/H_{\gamma+1}$ is Lie,

$(i_3) H_\gamma = \bigcap_{\beta<\gamma} H_\beta$ if γ is a limit ordinal.

Then clearly,

$$X \cong \mathrm{proj}_{\gamma<\alpha} \lim X/H_\gamma, \qquad X/F \cong \mathrm{proj}_{\gamma<\alpha} \lim X/FH_\gamma. \tag{1.6}$$

Case 1 First we consider the case $\alpha \leq \omega$ (we assume: $\alpha = \omega$, the case $\alpha < \omega$ being trivial).

Set $H^\gamma = FH_\gamma/H_{\gamma+1}$, $Y^\gamma = X/H_{\gamma+1}$, $N^\gamma = FH_{\gamma+1}/H_{\gamma+1}$, $\gamma < \omega$. Evidently N^γ is (isomorphic to) a compact subgroup of H^γ. On the other hand, since H^γ acts freely on Y^γ, by Lemma 1.2, there is a Baire isomorphism q^γ between $X/(FH_\gamma \times FH_\gamma/FH_{\gamma+1})$ and $X/FH_{\gamma+1}$ such that

$$p_\gamma \circ q^\gamma = p_1^\gamma, \tag{1.7}$$

where $p_\gamma : X/FH_{\gamma+1} \to X/FH_\gamma$ is the canonical projection and p_1^γ the projection of $(Y^\gamma/H^\gamma) \times (H^\gamma/N^\gamma)$ to Y^γ/H^γ.

If we set $M_\beta = FH_\beta/FH_{\beta+1}$ ($\cong H^\beta/N^\beta$), $\beta < \alpha$ and $Y = \Pi_{\beta<\alpha} M_\beta$, then, by (1.6), there exists a mapping q of $X/G \times Y$ to X/F. A standard argument involving projective limits of compact spaces shows that Y and q, as defined, are as required.

Case 2 The second step is, essentially, the reduction to Case 1. In detail: suppose that $\alpha = w(G)$ is an arbitrary cardinal.

Let $G^\gamma = FH_\gamma/H_{\gamma+1}$, $F^\gamma = FH_{\gamma+1}/H_{\gamma+1}$ and $Z^\gamma = X/H_{\gamma+1}$, $\gamma < \alpha$. First notice that each G^γ acts freely on Z^γ and

$$Z^\gamma/G^\gamma \cong X/FH_\gamma, \qquad Z^\gamma/F^\gamma \cong X/FH_{\gamma+1}. \tag{1.8}$$

Since $FH_\gamma/FH_{\gamma+1} \cong H_\gamma/(H_\gamma \cap FH_{\gamma+1})$ is compact and metrizable, by (1.5), we can take in (1.8) the G^γ to be metrizable (with a denumerable base).

Now the proof proceeds as that in [17, Theorem 2.3]. Set $Y_0 = G/H_1$ and $Y_\gamma = FH_\gamma/FH_{\gamma+1}$ for $\gamma > 0$. By (1.6) and Case 1, for every $\gamma < \alpha$ we find a Baire isomorphism q_γ from $K_\gamma = \Pi_{\delta<\gamma} Y_\delta$ onto X/FH_γ such that $q_\varepsilon \circ r_{\zeta,\varepsilon} = p_{\zeta,\varepsilon}$ for $\varepsilon < \zeta < \alpha$, where $p_{\zeta,\varepsilon}$ (resp $r_{\zeta,\varepsilon}$) is the projection of X/FH_ζ to X/FH_ε (resp K_ζ to K_ε). [If γ is a successor ordinal, say $\gamma = \delta+1$, then (1.8) yields a Baire isomorphism $q^\gamma : X/(FH_\delta \times Y_\delta) \to X/FH_\gamma$. We take $q_\gamma = q^\gamma \circ (q_\delta \times I_{Y_\delta}) : K_\gamma \to X/FH_\gamma$, where $q_\delta \times I_{Y_\delta} : K_\delta \times Y_\delta \to (X/FH_\delta) \times Y_\delta : (x, y) \to (q_\delta(x), y)$. If γ is a limit ordinal, then we take q_γ to be the (unique) mapping satisfying: $q_\delta \circ r_{\gamma,\delta} = p_{\gamma,\delta} \circ q_\gamma$, $\delta < \gamma < \alpha$.]

From the family $\{q_\gamma\}$, in view of (1.6), we obtain, in a natural way, a Baire isomorphism $q : X/G \times (\Pi_{\beta<\alpha} Y_\beta) \to X/F$ satisfying the required conditions. This completes the proof of the theorem. □

The last theorem gives the following

Corollary 1.9 *If (G, X) is a free compact t.g., then there exists a product of compact metric spaces $Y = \Pi_{\gamma<w(G)} Y_\gamma$, each Y_γ with at least two points, such that X and $X/G \times Y$ are Baire isomorphic; in particular, the phase space of any free compact t.g., with metrizable orbit space, is Baire isomorphic to some product of compact metric spaces.*

Remark 1.10 [In the language of 1.1] If $w(X/F)$ is uncountable, then we can assume: $\alpha + w(X/G) = w(X/F)$. [If $w(X/G) < w(X/F)$, then we may take $\alpha = w(X/F)$. If $w(X/G) = w(X/F)$, then we can choose α to be $\leqq w(X/F)$.]

2. Measure-theoretic facts

This section makes use of the topological techniques we developed to study measure-theoretic problems.

All measures are assumed to be (positive) Radon measures on compact (Hausdorff) topological spaces. For a compact space Z, $\mathcal{B}_Z^0$ denotes its Baire σ-algebra. If (L, Σ, μ) is any measure space and $g : L \to Z$ a mapping with $g^{-1}C \in \Sigma$ for every $C \in \mathcal{B}_Z^0$, then $g[\mu]$ will denote the (Baire) measure on Z defined by: $g[\mu](D) = \mu(g^{-1}D)$, $D \in \mathcal{B}_Z^0$ (in the sequel, when no ambiguity arises, we shall identify a Radon measure with its Baire contraction, i.e. its restriction to the Baire σ-algebra).

Next, let K, L be compact spaces and μ, ν Baire measures on K, resp L. A map $f : K \to L$ is called completion Baire measurable iff $f^{-1}B$ is μ-measurable for all ν-measurable sets B. A completion Baire measurable bijection f is said to be a completion Baire isomorphism iff f^{-1} is also completion Baire measurable. If such a bijection exists, then the measure spaces $(K, \mu), (L, \nu)$ (or just μ, ν) are said to be completion Baire isomorphic.

For any family $\{\mu_i\}$ of Radon probability measures, $\bigotimes_i \mu_i$ will denote the Radon product of the μ_i; in this case, $\times_i \mu_i$ will be its Baire contraction.

Consider now some compact t.g. (G, X). If β_G is the (normalized) Haar measure on G and μ a Radon probability measure on $Y = X/G$, then the G-Haar (or simply, the Haar) lift $\lambda = \lambda[\mu, G, X]$ of μ is defined as follows ([2], [19], [30, pp. 42–45]):

$$\lambda(f) = \int_Y \Big(\int_G f(tx) d\beta_G(t) \Big) d\mu(y), \quad f \in C(X). \tag{2.1}$$

Remark 2.2 If F is a closed subgroup of G and $p_F : X \to X/F$ the canonical projection, then $\lambda[\mu, G, X] = \lambda[p_F[\lambda], F, X]$; in particular, if E is a compact homogeneous space acted upon by a compact group G, then the G-Haar lift of the quasi-invariant measure on E is (equivalent to) β_G ([2], [30]).

The following result will be used in proving the main Theorem 2.5 of this section.

Lemma 2.3 *Let H, Y, N, p, q be as in Lemma 1.2 and let μ be a Radon probability measure on Y/H. If λ is the H-Haar lift of μ and $p_H : Y \to Y/H$, $p_N : Y \to Y/N$ the projections, then there is a Radon measure σ on H/N such that $q[\mu \times \sigma] = p_N[\lambda]$.*

Proof Put $\sigma = \psi[\beta_H]$ and $\nu = p_N[\lambda]$ (where $\psi : H \to H/N$ is the canonical projection). It suffices to show that given $A \subset Y/H$ Baire and $W \subset H/N$ open,

$$\nu(q(A \times W)) = \mu(A) \cdot \sigma(W). \tag{2.4}$$

Let θ_m, S^m and r_m be as in the proof of Lemma 1.2. If ϕ denotes a characteristic function, $D_m = r_m(A \cap S^m) \cdot \psi^{-1}(W) \cdot N$ and $\dot{y} = Hy$ (the H-orbit of $y \in Y$), then

$$\begin{aligned}
\nu(q(A \times W)) &= \nu\Big(p_N \bigcup_m (r_m(A \cap S^m) \cdot \psi^{-1}(W)) \overset{\prime\prime}{=} \sum_{m=1}^{M} \lambda(r_m(A \cap S^m)) \cdot \psi^{-1}(W) \cdot N\Big) \\
&= \sum_m \int_{Y/H} \Big\{ \int_H \phi_{D_m}(yt) d\beta_H(t) \Big\} d\mu(\dot{y}) \quad \text{by (2.1)} \\
&= \sum_m \int_{A \cap S^m} d\mu(\dot{y}) \int_H \phi_{D_m}(yt) d\beta_H(t) \\
&= \sum_m \int_{A \cap S^m} d\mu(\dot{y}) \int_{\psi^{-1}(W) \cdot N} d\beta_H(t) \\
&= \sum_m \beta_H(\psi^{-1}(W) \cdot N) \cdot \mu(A \cap S^m) \\
&= \mu(A) \cdot \sigma(W),
\end{aligned}$$

which proves (2.4). □

[*Note.* The measure σ, as defined, is the quasi-invariant measure on the homogeneous space H/N.]

Now let $G, X, F, Y = \Pi_{\beta<\alpha} Y_\beta, q$ be as in Theorem 1.1, μ a Radon probability measure on X/G and $\lambda = \lambda[\mu, G, X]$. The main result of this section is

Theorem 2.5 *There exists a family $(\mu_\beta)_{\beta<\alpha}$ of Radon probability measures, each μ_β on Y_β, such that $q[\mu \times (\times_{\beta<\alpha} \mu_\beta)] = p_F[\lambda]$.*

Proof [As in the proof of 1.1, we may assume w.l.o.g. that $\alpha = w(G) = w(G/F)$.] The proof is almost identical with that of Theorem 2.3 in [17]. One sets up the projective system of compact spaces $\{X/FH_\gamma,\ \gamma < \alpha\}$ of (1.6), one needs Lemma 2.3 to define, in an obvious way, inductively the family $(\mu_\gamma)_{\gamma<\alpha}$ and concludes the proof of the theorem. □

Remark 2.6 (i) For the case of compact groups (i.e. for $X = G$ and $F = \{e\}$), Theorem 2.5 is Theorem 2.3 in [17].

(ii) Evidently, by taking supp $\mu = X/G$, we may also take the μ_γ to have full support.

Recall that a measure algebra (Ω, μ)—or simply μ—of finite magnitude is called homogeneous if every non-zero principal ideal has a minimal σ-basis of the same cardinal; this cardinal is called the Maharam type of the measure algebra ([20], [8], [13]).

Corollary 2.7 *Let (G,X) be a free compact t.g. If X/G is metrizable, then (1) X is Baire isomorphic with some product of $w(G)$ many compact metric spaces; the isomorphism takes any G-invariant probability measure on X to a direct product measure; (2) if $w(G)$ is uncountable, then every G-invariant probability measure on X is homogeneous, of Maharam type $w(G)$; (3) every automorphism of any measure algebra on X is induced by an invertible completion Baire point transformation of X; (4) if μ, ν are probability measures on X, with (measure-preserving) isomorphic measure algebras, then μ, ν are completion Baire isomorphic.*

Proof Combine the following: (i) Theorems 1.1 and 2.5 just proved; (ii) [26]; (iii) [20], [4] and [8]. □

The next result generalizes Theorem 2.3 in [17].

Theorem 2.8 *Every compact homogeneous space is Baire isomorphic to some product of compact metric spaces of the same topological weight; the isomorphism takes the (normalized) quasi-invariant measure to a direct product measure.*

Proof Immediate from Theorem 2.5 (with $X = G$).

Corollary 2.9 *The quasi-invariant measure on any compact homogeneous space of uncountable topological weight is homogeneous of Maharam type equal to its weight.*

As will appear from the arguments below, the last theorem is the first step in a more general structure theorem for minimal distal flows.

3. Consequences of Furstenberg's structure theorem

We are now ready to begin the main work of this paper.

Let Z be a compact space. Recall that a (Baire) measure μ on Z is called completion regular if every Borel set in Z is μ-measurable. The following result, which is inspired by ideas of Choksi and Fremlin [8, Theorem 3], will be the initial point for the discussion of distal flows.

Proposition 3.1 *Let (G,X) be a free compact t.g. and μ a Radon probability measure on X/G. The following assertions are equivalent.*

(1) *μ is completion regular (as a measure on $Y = X/G$),*

(2) *the Haar lift $\lambda = \lambda[\mu, G, X]$ of μ is completion regular (as a measure on X).*

Proof

(2) $\Rightarrow$ (1): Immediate from [18, Lemma 1.6, p. 275].

(1) $\Rightarrow$ (2): It suffices to prove that every open set in X has a Baire cover of the same measure.

(a) First let G be a compact Lie group. By [25] (see also proof of Lemma 1.2), for each $x \in X$ there are a compact neighborhood Q_1^x of x and a compact $F_1^x \subset Q_1^x$, with $Q_1^x = G \cdot F_1^x$, such that

$$\text{(3.2)} \qquad \begin{array}{l} \text{each } q \in Q_1^x \text{ has a unique representation} \\ \text{of the form } q = g.d,\ (g,d) \in G \times F_1^x. \end{array}$$

Let $Q_2^x \subset Q_1^x$ be an open Baire set in X. Since the projection $p : X \to Y = X/G$ is surjective and open, pQ_2^x is a Baire subset of Y (see e.g. [18, Lemma 1.6, p. 275]). Thus, the open set $Q^x = p^{-1}pQ_2^x = G \cdot Q_2^x$ must be Baire in X.

Set $F^x = \{z \in F_1^x : p(z) \in pQ_2^x\}$. Then, clearly, F^x is Baire in X and $Q^x = G \cdot F^x$. Now let $\{R_1 = Q^x1,\ R_2 = Q^x2, \dots, R_N = Q^xN\}$ be a cover of $X, \mathcal{B}$ a countable base for the topology of G and $\mathcal{B}_k$ a base for the topology of (the space) $P_k = F^xk$ $(1 \leqq k \leqq N)$. Since the mapping $G \times P_k \to R_k : (g,z) \to g \cdot z$ is a homeomorphism, the sets W_k of the form $W_k = V \cdot B_k,\ V \in \mathcal{B},\ B_k \in \mathcal{B}_k$ form a base for the topology of R_k. On the other hand, in view of (3.2), $W_k = G \cdot B_k \cap V \cdot P_k$ (note that $G \cdot B_k$ is open and G-invariant, $V \cdot P_k$ is Baire and $p(G \cdot B_k \cap V \cdot P_k) = p(G \cdot B_k) = p(R_k)$). Also, the family $\mathcal{B}_1 = \{V \cdot P_k,\ V \in \mathcal{B},\ 1 \leqq k \leqq N\}$ is countable. It follows that every open set W in X is expressible in the form

$$W = \bigcup_{k=1}^{N} U_k, \text{ where each } U_k \text{ is a countable union of the form :}$$

$$\bigcup_{n \in \mathbb{N}} (A^n \cap D^n) \text{ with the } A^n \text{ open } G - \text{invariant and } D^n \in \mathcal{B}_1.$$

In conclusion, it suffices to show that every open G-invariant set in X has a Baire cover of the same λ-measure. In other words, every open set in X/G has a Baire cover of the same μ-measure, which is true because of the completion regularity of μ.

Let $\alpha = w(G)$. By the proof of Theorem 1.1 (with $F = \{e\}$), there exist two families $(F_j)_{j\in J}, (H_\gamma)_{\gamma<\alpha}$ of groups in $\mathcal{R}_G$, as in the proof of Theorem 1.1.

(b) Suppose that $\alpha \leqq \omega$ (we assume: $\alpha = \omega$). Since $X \cong \operatorname{proj}_\gamma \lim X/H_\gamma$, every open set U is of the form $U = \bigcup_{\gamma<\alpha} U_\gamma$, where each U_γ is open and H_γ-invariant. By (a), there is a Baire, H_γ-invariant set C_γ containing U_γ, such that $\lambda(C_\gamma) = \lambda(U_\gamma)$. Now the set $C = \bigcup_{\gamma<\omega} C_\gamma$ is a Baire cover of U with $\lambda(C) = \lambda(U)$.

(c) The general case ($\alpha = w(G)$ is an arbitrary cardinal). The final step is to see that given $W \subset X$ open, there exists a Baire set W_1 such that

$$W \subset W_1, \quad \lambda(W) = \lambda(W_1). \tag{3.3}$$

Let W_0 be any open Baire subset of W with $\lambda(W) = \lambda(W_0)$. Since $X \cong \operatorname{proj}_{j\in J} \lim X/F_j$, we can take the set $A = X \setminus W_0$ to be R-invariant, where $R = \bigcap_i F_i$, for some countable subfamily $\{F_i\}$ of $\{F_j,\ j \in J\}$.

Now, if $p_R : X \to X/R : x \to \dot{x}$ is the projection, $\nu = p_R[\lambda]$ and $\lambda_{\dot{x}}$ the measure on X defined by: $\lambda_{\dot{x}}(h) = \int_R h(tx)d\beta_R(t)$, $h \in C(X)$ ($\dot{x} \in X/R$), then (see e.g. [2]), $\lambda = \int_{X/R} \lambda_{\dot{x}} d\nu(\dot{x})$, i.e. $\lambda(B) = \int_{X/R} \lambda_{\dot{x}}(B)d\nu(\dot{x})$ for every Borel set B in X. (Note that the R-Haar lift of ν is λ.) Because W is open, $\lambda_{\dot{x}}(W) > 0$ is equivalent to $W \cap Rx \neq \emptyset$ and this is equivalent to $p_R^{-1}p_R(W) \cap Rx \neq \emptyset$, i.e. $\lambda_{\dot{x}}(p_R^{-1}p_R W) > 0$. Since A is R-invariant, using the formula: $\lambda(A \cap W) = \int_{X/R} \lambda_{\dot{x}}(A \cap W)d\nu(\dot{x})$, one gets: $\lambda(A \cap p_R^{-1}p_R W) = 0$ iff $\lambda(A \cap W) = 0$. On the other hand, since G/R is metrizable, (b) yields a Baire set W_2 in X/R such that $p_R(W) \subset W_2$ and $p_R[\lambda](W) = p_R[\lambda](W_2)$. It follows that the set $W_1 = p_R^{-1}W_2$ satisfies (3.3), completing the proof of the proposition. □

We now turn to a discussion of certain concepts which are of interest in the context of topological dynamics. More terminology is needed.

Definitions (1) Let (G,X) be a compact t.g. and T a locally compact group. The triple (G,X,T) is a bitransformation group (bt.g.) if (i) (G,X) and (X,T) are (left and right resp) t.gs; (ii) $(gx)t = g(xt)$ ($g \in G,\ x \in X,\ t \in T$).

(2) Let (X,T) be a (right) t.g. Say that (X,T) is a distal (resp. minimal) flow if whenever x,y are distinct points of X, there is no net $\{t_\eta\} \subset T$ such that $\lim_\eta xt_\eta = \lim_\eta yt_\eta$ (resp for each $x \in X$ the orbit xT is dense in X).

(3) Let $(X,T),(Y,T)$ be t.gs. Say (X,T) is an almost-periodic (a.p.) extension of (Y,T) if there is a bt.g. (G,Z,T) and a closed subgroup H of G such that $(Z/G,T) \cong (Y,T)$ and $(Z/H,T) \cong (X,T)$.

Let now (X,T) be a minimal distal flow. The next classical theorem plays a fundamental role in topological dynamics. We have chosen its statement from [19], whose formulation seems more directly applicable to our purpose; see also [11], [14], [28].

Theorem 3.F [Furstenberg] *There is an ordinal α and a family of flows $\{(X_\gamma, T),\ \gamma \leqq \alpha\}$ such that:* (i$_1$) *X_0 contains just one point and $(X_\alpha, T) = (X, T)$;* (i$_2$) *each $(X_{\gamma+1}, T)$ is an a.p. extension of (X_γ, T);* (i$_3$) *if β is a limit ordinal, then (X_β, T) is an inverse limit of $\{(X_\delta, T),\ \delta < \beta\}$.*

Notation 3.4 Let (X,T) be a minimal distal flow and let $\{X_\gamma, T),\ \gamma \leqq \alpha\}$ be as in Theorem 3.F. For $\gamma \leqq \alpha$ let (G_γ, Z_γ, T) be a bt.g. and $H_\gamma \subset G_\gamma$ a closed subgroup with: (i) $(Z_\gamma/G_\gamma, T) = (X_\gamma, T)$; (ii) $(Z_\gamma/H_\gamma, T) \cong (X_{\gamma+1}, T)$. If $\delta \leq \beta \leq \alpha$, there is a homomorphism (of flows) $\pi_{\beta\delta} : (X_\beta, T) \to (X_\delta, T)$. Write π_γ for the homomorphism taking (X,T) to (X_γ, T) and $\eta_\gamma : (Z_\gamma, T) \to (Z_\gamma/H_\gamma, T) \cong (X_{\gamma+1}, T)$. If μ is a Radon measure on X, let $\mu_\gamma = \pi_\gamma[\mu]$.

Corollary 3.5 *There is a T-invariant, completion regular, probability measure $\mu = \mu[X,T]$ on X, such that:* (i) $\pi_{\beta\delta}[\mu_\beta] = \mu_\delta,\ \delta \leqq \beta \leqq \alpha$; (ii) $\mu_{\gamma+1} = \eta_\gamma[\lambda[\mu_\gamma, G_\gamma, Z_\gamma]],\ \gamma \leqq \alpha$.

Proof The construction of the measure μ, which is due to Johnson [19, Prop. 3.6], is an easy application of Theorem 3.F and transfinite induction.

To see that μ is completion regular, note first that μ_0 is completion regular and $\mu_\alpha = \mu$. Fix some $\theta \leqq \alpha$ and take any θ-invariant subset A of X (for convenience, a subset S of X is called γ-invariant if $\gamma \leqq \alpha$ and $S = \pi_\gamma^{-1}\pi_\gamma S$).

Claim 1 If for every couple (γ, W') with $\gamma < \theta$ and W' open in X, $\lambda(A \cap W') = 0$ implies $\lambda(A \cap \pi_\gamma^{-1}\pi_\gamma W') = 0$, then for every open set W in X, $\lambda(A \cap W) = 0$ implies $\lambda(A \cap \pi_\theta^{-1}\pi_\theta W) = 0$.

We may assume w.l.o.g. that $\theta < \alpha$.

1. If θ is a successor ordinal, say $\theta = \delta + 1$, then the claim is a simple application of the arguments used in the proof of Proposition 3.1 (note that $\mu_\theta = \eta_\delta[\lambda[\mu_\delta, G_\delta, Z_\delta]]$).

2. If θ is a limit ordinal, then every W as in the claim has the form $W = \bigcup_{\rho<\theta} W_\rho$, where W_ρ is open and ρ-invariant. Then $0 = \mu(A \cap W) = \mu(A \cap (\bigcup_{\rho<\theta} W_\rho)) \Rightarrow \forall \rho < \theta,\ \mu(A \cap W_\rho) = 0 \Rightarrow \forall \rho < \theta,\ \mu(A \cap \pi_\theta^{-1}\pi_\theta W_\rho) = 0 \Rightarrow \mu(A \cap (\bigcup_{\rho<\theta} \pi_\theta^{-1}\pi_\theta W_\rho)) = \mu(A \cap \pi_\theta^{-1}\pi_\theta W) = 0$. This ends the proof of Claim 1.

Combining now Claim 1 with an induction argument, one easily gets

(3.6) for every triple (θ, A, W), where $A \subset X$, θ–invariant and $W \subset X$ open, $\mu(A \cap W) = 0$ implies $\mu(A \cap \pi_\theta^{-1}\pi_\theta W) = 0$.

Claim 2 Let $\theta \leqq \alpha$. If μ_γ is completion regular, for all $\gamma < \theta$, then μ_θ is again completion regular.

Observe that if θ is a successor ordinal, then Claim 2 is immediate from Proposition 3.1.

Suppose that θ is a limit ordinal. In this case, it suffices to prove that

$(*)$ every open set U in X_θ has a Baire cover U_1 of the same μ_θ – measure.

Let $U_0 \subset U$ open, Baire in X_θ, with $\mu_\theta(U_0) = \mu_\theta(U)$. Since $X_\theta = \text{proj}_{\gamma<\theta} \lim X_\gamma$, there is some countable $I \subset \{\gamma : \gamma < \theta\}$ with $\beta = \sup I$, such that the set $\pi_\theta^{!}\pi_\theta U_0$ is β-invariant.

(a) $\beta < \theta$. Let $B = X_\theta \setminus U_0$, $A = \pi_\theta^{-1}B$, $W = \pi_\theta^{-1}U$. Then $0 = \mu_\theta(B \cap U) = \mu(A \cap W) \Rightarrow$ (by (3.6)) $\mu(A \cap \pi_\theta^{-1}\pi_\theta W) = 0$. On the other hand, because of the completion regularity of μ_β, we can find a Baire set W_0 in X_θ, with $\pi_\beta(W) \subset W_0$, such that $\mu_\beta(\pi_\beta(W)) = \mu_\beta(W_0)$. Thus, the Baire set $U_1 = \pi_{\theta\beta}^{-1}(W_0)$ satisfies $(*)$.

(b) $\beta = \theta$. In this case θ is the sup of the countable set I, and thus the measure space (X_θ, μ_θ) is completion regular, as an inverse limit of a sequence of completion regular measure spaces. This ends the proof of Claim 2.

Finally, by a transfinite induction argument (using Claim 2 repeatedly), we easily see that μ is completion regular. □

Theorem 3.7 *Let (X,T) be a minimal distal flow. There exist a family of Radon probability measures $(\nu_\gamma)_{\gamma<w(X)}$, each ν_γ supported on some compact metric space Q_γ with at least two points and a Baire isomorphism $q_{(X,T)}$ from $Q_{(X,T)} = \Pi_{\gamma<w(X)}Q_\gamma$ onto X such that $q_{(X,T)}[\bigotimes_{\gamma<w(X)}\nu_\gamma] = \mu[X,T]$, hence $q_{(X,T)}$ carries, in a natural way, the T-action on X to a T-action on $Q_{(X,T)}$.*

Proof By Theorem 3.F and Corollary 3.5, there are an ordinal α, a family of flows $\{(X_\gamma, T),\ \gamma \leqq \alpha\}$ as in Theorem 3.F and a Radon measure $\mu = \mu[X,T]$ on X as in Corollary 3.5. First note that Theorem 3.F shows that each (X_γ, T) is either an a.p. extension of $(X_{\gamma-1}, T)$, or an inverse limit of $\{(X_\delta, T),\ \delta < \gamma\}$.

The rest of the proof is a natural application of Theorem 2.5, Corollary 3.5 and transfinite induction. Summary steps are sketched as follows:

An easy induction argument yields, for each $\gamma \leqq \alpha$, a family $(\nu_i)_{i\in I_\gamma}$ of Radon probability measures, each ν_i on a compact metric space Q_i and a Baire isomorphism f_γ from $\Pi_{i\in I_\gamma}Q_i$ onto X_γ, taking $\bigotimes_{i\in I_\gamma}\nu_i$ to μ_γ, such that $I_\delta \subset I_\beta$, $\pi_{\beta\delta} \circ f_\beta = f_\delta \circ p_{\beta\delta}$, where $p_{\beta\delta} : \Pi_{i\in I_\beta}Q_i \to \Pi_{j\in I_\delta}Q_j$ is the canonical projection ($\delta \leqq \beta$). [Note that, according to Remark 1.10, we can assume: $\text{card}(I_\gamma) = w(X_\gamma)$.] The proof then follows, by taking $q_{(X,T)} = q_\alpha$. □

Corollary 3.8 *If $w(X)$ is uncountable, then $\mu[X,T]$ is homogeneous, of Maharam type $w(X)$.*

Take now a minimal distal flow (X,T) to be fixed throughout. Theorem 3.7, combined with classical results of Maharam [20], Choski [4] and Choksi and Fremlin [8], leads to several measure-theoretic consequences. Most of them have been discussed in great detail in [16], [17], for the case of compact groups; there is no need to reproduce this discussion here. We shall only present three results of a specific nature.

Theorem 3.9 *If ν is any finite Baire measure on X, then every automorphism of the measure algebra of ν is induced by an invertible completion Baire point transformation of X.*

Proof Immediate from [4] and Theorem 3.7. □

Similarly, from [8, Theorem 1] and Theorem 3.7, one gets

Theorem 3.10 *If λ, ν are Radon probability measures on X and if there exists a measure-preserving isomorphism of their measure algebras, then λ, ν are completion Baire isomorphic.*

Combining now Maharam's famous theorem [20] and Theorem 3.10, we obtain

Theorem 3.11 *If $w(X)$ is uncountable, then every homogeneous probability measure on X, of Maharam type $w(X)$, is completion Baire isomorphic with some T-invariant, completion regular probability measure on X.*

Note added in proof. After this paper was accepted the author learned that the first part of Theorem 2.8 was obtained by L. B. Shapiro, *On Baire isomorphisms of spaces of uncountable weight*, Soviet Math. Dokl. **32** (1985), 113–117. Russian original: Dokl. Akad. Nauk SSSR **283** (1985), 321–325.

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