SCATTERING IN THE ENERGY SPACE AND BELOW FOR 3D NLS

By

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0 Introduction

The purpose of this note is to clarify a uniform boundedness issue arising in the context of scattering in the energy space $H^1(\mathbb{R}^3)$ for defocusing 3D NLS

$$\begin{cases} iu_t + \Delta u - u|u|^{p-2} = 0, \quad p < 6, \\ u(0) = \varphi \in H^1. \end{cases}$$

More precisely, it is shown that the solution satisfies

$$\|u\|_{L_t^q L_x^p} < C(\|\varphi\|_{H^1}) \tag{0.1}$$

for all admissible pairs (p, q), an issue left open in the paper [G-V]. If moreover $\varphi \in H^s(\mathbb{R}^3)$, $s \geq 1$, then the corresponding solution u^* satisfies a uniform estimate

$$\|u(t)\|_{H^s} \leq C(\|\varphi\|_{H^s}) \tag{0.2}$$

and

$$\|u(t) - e^{it\Delta}(\Omega_+(\varphi))\|_{H^s} \overset{t\to\infty}{\longrightarrow} 0 \tag{0.3}$$

(only (0.2) is significantly new).

The main ingredient is essentially a refinement of the method used in [L-S] and [G-V] to get some initial decay property, based on Morawetz' inequality.

Next we develop the argument from [B_2] to the 3D-situation, considering the equation

$$iu_t + \Delta u - u|u|^2 = 0 \tag{0.4}$$

*We restrict ourselves to the case $p = 4$ because of smoothness of nonlinearity.

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as an example, establishing global wellposedness results and scattering for $\phi \in H^s, s < 1$. Without further restriction, it is shown that (0.4) is globally wellposed for general data $\phi \in H^s$, $s > \frac{11}{13}$ and $u(t) - e^{it\Delta}\phi \in H^1$ for all time. If ϕ is radially symmetric, then the result holds for $s > \frac{5}{7}$ and there is moreover scattering in H^s-space.

1 3D NLS

We consider

$$\begin{array}{c} iu_t - \Delta u + u|u|^{p-2} = 0 \\ u(0) = \phi \in H^1 \end{array} \qquad \Big(\frac{10}{3} < p < 6\Big).$$

2 Morawetz inequality

We repeat the proof of the Morawetz inequality for NLS, following [L-S].

$$0 = \mathrm{Re}\,\Big[\big(i\dot{u} - \Delta u + |u|^{p-2}u\big)\Big(\overline{u}_r + \frac{1}{r}\overline{u}\Big)\Big] = \frac{\partial X}{\partial t} + \nabla.Y + Z,$$

$$X = -w\Big(v_r + \frac{1}{r}v\Big),$$

$$Y = \frac{x}{r}v_t w - \nabla v\Big(v_r + \frac{v}{r}\Big) - (\nabla w)\Big(w_r + \frac{w}{r}\Big) + \frac{x}{2r}|\nabla u|^2 + \frac{x}{pr}|u|^p - \frac{x}{2r^3}|u|^2,$$

$$Z = \frac{1}{r}(|\nabla u|^2 - |u_r|^2) + \Big(1 - \frac{2}{p}\Big)\frac{|u|^p}{r},$$

$$u = v + iw,\ r = |x|,$$

$$0 = \int [X(T) - X(0)]dx + 2\pi \int_0^T |u(0,t)|^2 dt + \int_0^T \int \frac{1}{r}\{[|\nabla u|^2 - |u_r|^2] + \tfrac{1}{2}|u|^p\} dx dt.$$

Hence

$$\text{(2.1)} \qquad \int_0^T |u(0,t)|^2 dt + \int_0^T \int \frac{|\nabla u|^2 - u_r^2}{r} dx dt + \int_0^T \int \frac{|u|^p}{r} dx dt < C \sup_{0 \le t \le T} \|u(t)\|^2_{H^{1/2}}$$

and, in particular,

$$\text{(2.2)} \qquad \int_0^\infty \int \frac{|u(x,t)|^p}{|x|} dx dt < C\big(\|\phi\|_2^2 + H(\phi)\big).$$

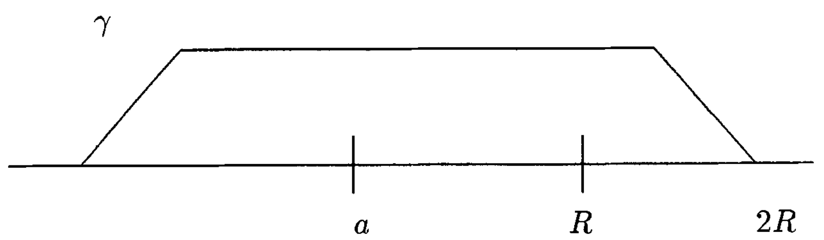
By translation, also

$$\int_0^\infty \int \frac{|u(x,t)|^p}{|x-a|} dxdt < C \qquad \text{for all } a \in \mathbb{R}^3. \tag{2.3}$$

3 Dispersion of L^2-norm

$$\int_{|x-a|<2R} |u(t_2)|^2 dx \geq \int_{|x-a|<R} |u(t_1)|^2 dx - C\,\frac{t_2 - t_1}{R} \text{ for } t_1 < t_2. \tag{3.1}$$

Proof. Consider the localizing function γ



where

$$\gamma(x) = 1 \text{ for } |x-a| < R,$$
$$\gamma(x) = 0 \text{ for } |x-a| > 2R,$$
$$|\gamma| < \frac{1}{R}.$$

Define

$$I(t) = \int |u|^2 \gamma(x) dx.$$

Then

$$\begin{aligned} \dot{I}(t) &= 2\mathrm{Re} \int \bar{u}\dot{u}\gamma(x) dx \\ &\overset{\text{from equation}}{=} 2\mathrm{Im} \int \bar{u}\Delta u \gamma \\ &= 2\mathrm{Im} \int \bar{u}\nabla u . \nabla\gamma. \end{aligned}$$

Hence

$$|\dot{I}| < CR^{-1} \int |u|\,|\nabla u| < CR^{-1}(\|\phi\|_2^2 + \|H(\phi)\|_{H^1}^2)$$

and

$$\int_{|x-a|<2R} |u|^2(t_2) > I(t_2) > I(t_1) - (t_2 - t_1) \sup_{t_1<t<t_2} |\dot{I}(t)|$$

$$> \int_{|x-a|<R} |u(t_1)|^2 - \frac{C}{R}(t_2 - t_1),$$

as claimed.

More generally, the same argument yields that for any set $S \subset \mathbb{R}^3$

$$\int_{\text{dist}\,(x,S)\le 2R} |u(t_2)|^2 dx \ge \int_{\text{dist}\,(x,S)\le R} |u(t_1)|^2 dx - C\frac{t_2 - t_1}{R}. \tag{3.2}$$

4 Decay estimates

The key lemma is the following

Lemma 1. *Fix* $0 < \varepsilon$, A. *Then for*

$$T > T(\varepsilon, A, \|\phi\|_2, \|\phi\|_{H^1}) \tag{4.1}$$

there is an interval $[T_1, T_2] \subset [0, T]$ *such that*

$$T_2 - T_1 > A \tag{4.2}$$

and

$$\|u\|_{L^p[T_1,T_2]} < \varepsilon. \tag{4.3}$$

The main novelty here compared with [L-S] is that the estimate is uniform without a decay assumption on ϕ.

Proof. Assume the statement false. We then perform an iterative construction leading to a contradictory statement $\|u(\tau)\|_2 > \|\phi\|_2$ for some $\tau \in [0, T]$.

Recall the inequality

$$\|\psi\|_p \lesssim \|\psi\|_2^{1-3(1/2-1/p)} \|\psi\|_{H^1}^{3(1/2-1/p)}$$

($p < 6$) and which may be localized to unit cubes.

If $\{Q_\alpha\}$ is a partition of $\mathbb{R}^3$ in unit cubes, we have

$$\begin{aligned}\int_A^{2A} |u(x,t)|^p dxdt &\lesssim A \sup_{0<t<A} \sum_\alpha \|u(t)\|_{L^2(Q_\alpha)}^{3-p/2} \|u(t)\|_{H^1(Q_\alpha)}^{3p/2-3} \\ &\lesssim A \sup_{\substack{0<t<A \\ \alpha}} \|u(t)\|_{L^2(Q_\alpha)}^{3-p/2}\end{aligned} \tag{4.4}$$

$\left(\frac{10}{3} \le p < 6\right)$.

Assuming $2A < T$, $\int_A^{2A} |u|^p dxdt > \varepsilon^p$ by assumption and thus

$$\|u(\tau_1)\|_{L^2[|x-a_1|<1]} > \left(\frac{\varepsilon^p}{A}\right)^{1/(3-p/2)} \equiv \gamma \tag{4.5}$$

for some $\tau_1 \in [A, 2A]$ and $a_1 \in \mathbb{R}^3$.

Denote by M a constant, which we choose sufficiently large with respect to the data ϕ, ε, A. It follows from (2.3) that

$$\int_{4\tau_1}^{\tau} \frac{1}{t-\tau_1}\left[\int_{|x-a_1|<2M(t-\tau_1)} |u(t)|^p dx\right] dt < 2M \iint \frac{|u|^p}{|x-a_1|} < 2MC.$$

Hence, letting $\tau = 4\tau_1 e^{M^2}$, we have $\tau_2' \in [4\tau_1, 4\tau_1 e^{M^2}]$ such that

$$\int_{\tau_2'}^{\tau_2'+A} \int_{|x-a_1|<2M(t-\tau_1)} |u|^p \lesssim \frac{AMC}{\log e^{M^2}} \lesssim \frac{A}{M}. \tag{4.6}$$

Assume $4\tau_1 e^{M^2} < T$; then $\int_{\tau_2'}^{\tau_2'+A} \int |u|^p > \varepsilon^p$, and (4.6) implies

$$\int_{\tau_2'}^{\tau_2'+A} \int_{|x-a_1|>2M|t-\tau_1|} |u(t)|^p > \varepsilon^p - \frac{CA}{M}. \tag{4.7}$$

Repeating (4.4) then gives some $\tau_2 \in [\tau_2', \tau_2' + A]$ and $a_2 \in \mathbb{R}^3$ such that

$$|a_1 - a_2| > 2M|\tau_2 - \tau_1|, \tag{4.8}$$

$$\|u(\tau_2)\|_{L^2[|x-a_2|<1]} > \gamma - C\frac{1}{M}. \tag{4.9}$$

Also, from (3.1), (4.5),

$$\begin{aligned} \|u(\tau_2)\|_{L^2[|x-a_1|<M(\tau_2-\tau_1)]} &> \|u(\tau_1)\|_{L^2[|x-a_1|<\frac{1}{2}M(\tau_2-\tau_1)]} - 1/M^{1/2} \\ &> \gamma - 1/M^{1/2}. \end{aligned} \tag{4.10}$$

Thus, by (4.8), (4.9), (4.10),

$$\|u(\tau_2)\|_{L^2[\text{dist}\,(x,\{a_1,a_2\})<M(\tau_2-\tau_1)]} > \sqrt{2}\gamma - C/M^{1/2}. \tag{4.11}$$

Next, repeat the construction. By (2.3)

$$\begin{aligned} &\int_{4\tau_2}^{\tau} \frac{1}{t-\tau_2}\left[\int_{\text{dist}\,(x,\{a_1,a_2\})<2M(t-\tau_2)} |u(x,t)|^p\, dx\right] dt \\ &\quad < 2M \iint |u|^p \left(\frac{1}{|x-a_1|} + \frac{1}{|x-a_2|}\right) < 4CM, \end{aligned}$$

and hence there is $\tau_3' \in [4\tau_2, 4\tau_2 e^{M^2}]$ for which

$$\int_{\tau_3'}^{\tau_3'+A} \int_{\text{dist}\,(x_1\{a_1,a_2\})<2M(t-\tau_2)} |u|^p \lesssim A/M. \tag{4.12}$$

For $4\tau_2 e^{M^2} < T$, the hypothesis implies

$$\int_{\tau_3'}^{\tau_3'+A} \int_{\text{dist}\,(x,\{a_1,a_2\})>2M(t-\tau_2)} |u|^p \, dxdt > \varepsilon^p - \frac{C}{M},$$

and we get $\tau_3 \in [\tau_3', \tau_3' + A]$, $a_3 \in \mathbb{R}^3$ satisfying

$$|a_3 - a_1| > 2M(\tau_3 - \tau_2) \qquad |a_3 - a_2| > 2M(\tau_3 - \tau_2), \tag{4.13}$$

$$\|u(\tau_3)\|_{L^2[|x-a_3|<1]} > \gamma - \frac{C}{M}. \tag{4.14}$$

Also, by (3.2), (4.11),

(4.15)
$$\begin{aligned}\|u(\tau_3)\|_{L^2[\text{dist}\,(x,\{a_1,a_2\})<M(\tau_3-\tau_2)]} &> \|u(\tau_2)\|_{L^2[\text{dist}\,(x,\{a_1,a_2\})<\frac{1}{2}M(\tau_3-\tau_2)]} - 1/M^{1/2}\\ &> \|u(\tau_2)\|_{L^2[\text{dist}\,(x,\{a_1,a_2\})<M(\tau_2-\tau_1)]} - 1/M^{1/2}\\ &> \sqrt{2}\gamma - C/M^{1/2}\end{aligned}$$

and (4.13), (4.14), (4.15) imply

$$\|u(\tau_3)\|_{L^2[\text{dist}\,(x,\{a_1,a_2,a_3\})<M(\tau_3-\tau_2)]} > \sqrt{3}\gamma - C/M^{1/2}. \tag{4.16}$$

The continuation of the process is now clear.

After ℓ steps, one gets τ_ℓ satisfying

$$\tau_\ell < 5^\ell e^{\ell M^2} \tag{4.17}$$

and, in particular,

$$\|\phi\|_2 = \|u(\tau_\ell)\|_2 > \sqrt{\ell}\gamma - C\ell/M^{1/2}, \tag{4.18}$$

where

$$\gamma \sim \left(\frac{\varepsilon^p}{A}\right)^{\frac{1}{3-p/2}}.$$

In order to get a contradiction, take

$$\ell \gtrsim \left(\frac{A}{\varepsilon^p}\right)^{\frac{2}{3-p/2}}, \tag{4.19}$$

$$M \gtrsim \ell^2, \tag{4.20}$$

and, by (4.17), (4.19), (4.20), the condition

$$2\tau_\ell < T$$

becomes

$$T > \exp\left(\frac{A}{\varepsilon}\right)^{C(p)}. \tag{4.21}$$

Thus (4.21) gives the dependence on ε, A in (4.1).

Recall that, since $p < 6$ is H^1-subcritical, the IVP

$$\begin{cases} iu_t - \Delta u + u|u|^{p-2} = 0 \\ u(0) = \phi \in H^1 \end{cases}$$

is locally wellposed and

$$u \in L^\infty_{H^1}(\mathbb{R}) \cap L^{10}_{10}[I]$$

with bounds depending only on $\|\phi\|_{H^1}$ and the size $|I|$ of the time interval I. It follows in particular from Lemma 1 and interpolation that (4.3) in Lemma 1 may be replaced by

$$\|u\|_{L^q_{L^q_x}[T_1,T_2]} < \varepsilon \quad \text{for all fixed } 2 < q < 10 \tag{4.22}$$

and also

$$\|u\|_{L^q_{L^r_x}[T_1,T_2]} < \varepsilon \quad \text{for all fixed } 2 < r < 6, \quad q < \infty. \tag{4.23}$$

At this stage, we shall mainly repeat the analysis from [G-V].

Fix $r > 6, r \approx 6$ and define

$$k(t) = \|u(t)\|_r, \tag{4.24}$$

$$k_0(t) = \|e^{it\Delta}\phi\|_r, \quad \phi = u(0). \tag{4.25}$$

Lemma 2 (*cf. Lemma 5.8 in [G-V]*).

$$k \le k_0 + \mu * [\min(k^{1+\beta}, k^{1+\beta'})], \tag{4.26}$$

where

$$\mu(t) = C\min(|t|^{-\delta}, |t|^{-\delta'}), \qquad t > 0$$
$$= 0, \qquad t \le 0$$

and $0 < \delta' < 1 < \delta$, $-1 < \beta' < 0 < \beta$.

Choose then $q < \infty$ large enough and (r, q) sub-admissible. Hence from the preceding

$$\|k_0\|_{L^q(\mathbb{R})} < C, \tag{4.27}$$

$$\sup_{|I|=1} \|k\|_{L^q[I]} < C, \tag{4.28}$$

and, moreover, as a consequence of (4.26),

$$\sup_{\substack{|I|=1 \\ I\subset[T',T'']}} \|k\|_{L^q[I]} \leq \sup_{\substack{|I|=1 \\ I\subset[T',T'']}} \|k_0\|_{L^q[I]} + C \sup_{\substack{|I|=1 \\ I\subset[T'-A,T'']}} \|k\|^{1+\beta}_{L^q[I]} + CA^{-\gamma} \tag{4.29}$$

and

$$\|k\|_{L^q[T',T'']} \leq \|k_0\|_{L^q[T',T'']} + C\|k\|_{L^q[T'-A,T'']}\Bigg(\sup_{I\subset[T'-A,T'']} \|k\|^{\beta}_{L^q[I]}\Bigg)$$

$$+ CA^{-\gamma}\|k\|_{L^q[0,T'']} \tag{4.30}$$

$$\leq C + C\left\{A^{-\gamma} + \sup_{I\subset[T'-A,T'']} \|k\|^{\beta}_{L^q[I]}\right\}\|k\|_{L^q[0,T'']} \tag{4.31}$$

for some $\gamma > 0$ (the constant C depends only on the parameters and $\|\phi\|_{H^1}$).

In the next construction, we estimate inductively $\|k\|_{L^q[0,T^{(j)}]}$ along a finite sequence of times $0 = T^{(0)} < T^{(1)} < \cdots < T^{(j')}$, where again j' will be uniformly bounded.

The method will be based on Lemma 1., (4.27), (4.28), (4.29), (4.31).

Fix $\varepsilon > 0$ sufficiently small and A sufficiently large (depending on $\|\phi\|_{H^1}$). Choose $T^{(1)}$ maximal such that

$$\|k_0\|_{L^q[I]} < \varepsilon \quad \text{for all } I \subset [0, T^{(1)}], \quad |I| = 1 \tag{4.32}$$

(thus $T^{(1)} \geq 1$ and $\|k_0\|_{L^q[T^{(1)}-1, T^{(1)}]} = \varepsilon$).

By Lemma 1, there is an interval $[T_1, T_2]$ such that

$$T_2 - T_1 > A, \tag{4.33}$$

$$\|k\|_{L^q[I]} \leq \|k\|_{L^q[T_1,T_2]} < \varepsilon \quad \text{for } I \subset [T_1, T_2], \quad |I| = 1 \tag{4.34}$$

and T_2 is bounded by (4.1), i.e.,

$$T_2 < C(A, \varepsilon, \|\phi\|_{H^1}). \tag{4.35}$$

(One deduces (4.34) by interpolation between (4.22), (4.23).)

If $T_2 + A \geq T^{(1)}$, (4.28) and (4.35) imply of course that

$$\|k\|_{L^q[0,T^{(1)}]} < C(A, \varepsilon, \|\phi\|_{H^1}).$$

Assume $T^{(1)} > T_2 + A$. Applying (4.29) with $T' = T_2, T'' < T^{(1)}$ yields

$$\begin{aligned}\sup_{\substack{|I|=1 \\ I\subset[T_2,T'']}} \|k\|_{L^q[I]} &\leq \varepsilon + C \sup_{I\subset[T_1,T'']} \|k\|^{1+\beta}_{L^q[I]} + CA^{-\gamma} \\ &\leq \varepsilon + C\varepsilon^{1+\beta} + CA^{-\gamma} + C \sup_{I\subset[T_2,T'']} \|k\|^{1+\beta}_{L^q[I]}\end{aligned}$$

and consequently

$$\sup_{I\subset[T_2,T^{(1)}]} \|k\|_{L^q[I]} < C(\varepsilon + A^{-\gamma}). \tag{4.36}$$

Substituting (4.36) in (4.31), we next get

$$\begin{aligned}\|k\|_{L^q[0,T^{(1)}]} &\leq \|k\|_{L^q[0,T_2+A]} + \|k\|_{L^q[T_2+A,T^{(1)}]}, \\ \|k\|_{L^q[T_2+A,T^{(1)}]} &\leq C + C[(\varepsilon + A^{-\gamma})^\beta + A^{-\gamma}]\, \|k\|_{L^q[0,T^{(1)}]},\end{aligned}$$

so that for ε sufficiently small and A sufficiently large, one finds

$$\|k\|_{L^q[0,T^{(1)}]} \leq 2\|k\|_{L^q[0,T_2+A]} + C < C(A, \varepsilon, \|\phi\|_{H^1}). \tag{4.37}$$

The inductive step is now clear. Assume $T^{(j)}$ obtained such that

$$\|k\|_{L^q[0,T^{(j)}]} < C(\varepsilon, A, \|\phi\|_{H^1}). \tag{4.38}$$

By construction,

$$\|k_0\|_{L^q[T^{(j)},T^{(j)}-1]} = \varepsilon. \tag{4.39}$$

Choose $T^{(j+1)} \geq T^{(j)} + 1$ maximal such that

$$\|k_0\|_{L^q[I]} < \varepsilon \quad \text{for all } I \subset [T^{(j)}, T^{(j+1)}], \quad |I| = 1. \tag{4.40}$$

By Lemma 1 applied on $[T^{(j)}, \infty[$, there are $T^{(j)} < T_1 < T_2$ such that

$$T_2 - T_1 > A, \tag{4.41}$$

$$T_2 < T^{(j)} + C(A, \varepsilon, \|\phi\|_{H^1}), \tag{4.42}$$

$$\|k\|_{L^q[I]} < \varepsilon \quad \text{for all } I \subset [T_1, T_2]. \tag{4.43}$$

If $T^{(j+1)} < T_2 + A$, there is the obvious estimate

$$\|k\|_{L^q[0,T^{(j+1)}]} \leq \|k\|_{L^q[0,T^{(j)}]} + \|k\|_{L^q[T^{(j)},\, T_2+A]} < C(\varepsilon, A, \|\phi\|_{H^1})$$

from (4.38), (4.28), (4.42).

Assume $T^{(j+1)} > T_2 + A$. Apply first (4.29) with $T' = T_2$, $T'' \leq T^{(j+1)}$ to get again by (4.40), (4.43)

$$\sup_{\substack{|I|=1 \\ I\subset[T_2,T'']}} \|k\|_{L^q[I]} \leq \varepsilon + C\varepsilon^{1+\beta} + CA^{-\gamma} + C \sup_{I\subset[T_2,T'']} \|k\|^{1+\beta}_{L^q[I]}$$

and hence

$$\sup_{I\subset[T_2,T^{(j+1)}]} \|k\|_{L^q[I]} < C(\varepsilon + A^{-\gamma}). \tag{4.44}$$

Hence, by (4.38), (4.31), (4.44)

$$\begin{aligned}\|k\|_{L^q[0,\,T^{(j+1)}]} &\leq C(A, \varepsilon, \|\varphi\|_{H^1}) + \|k\|_{L^q[T_2+A,T^{(j+1)}]} \\ &\leq C(A,\, \varepsilon, \|\phi\|_{H^1}) + C\big((\varepsilon + A^{-\gamma})^\beta + A^{-\gamma}\big)\, \|k\|_{L^q[0,T^{(j+1)}]},\end{aligned}$$

which implies an estimate (4.38) with $T^{(j)}$ replaced by $T^{(j+1)}$.

Since, by construction, for each j one has (4.39)

$$\|k_0\|_{L^q[T^{(j)}-1,T^{(j)}]} = \varepsilon,$$

(4.27) implies a uniform bound

$$j' < C\varepsilon^{-q} \tag{4.45}$$

on the maximal number of possible "stopping times".

Consequently, we have proved a uniform bound

$$\|u\|_{L^q_{L^r_x}(\mathbb{R})} < C(\|\phi\|_{H^1}) \tag{4.46}$$

for some $q < \infty$.

5 Completion of the argument

Once (4.46) has been obtained for some r, $q < \infty$, Lemma (5.12) of [G-V] permits us to get

$$\|u\|_{L^q_{L^r_x}(\mathbb{R})} < C(\|\phi\|_{H^1}) \tag{5.1}$$

for all admissible pairs (r, q).

This statement thus answers the uniformity question in the context of Prop. 5.2 of [G-V] affirmatively. Recalling (4.21), observe however that the bound obtained in (5.1) depends exponentially on $\|\phi\|_{H^1}$.

From the integral equation, one has for $t > t_0$

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) + i\int_{t_0}^{t} e^{i(t-\tau)\Delta}(u|u|^{p-2})(\tau)d\tau;$$

hence

$$\|D_x u(t)\|_{L_x^{10/3}} \le \|e^{i(t-t_0)\Delta}[D_x u(t_0)]\|_{L_x^{10/3}} + \int_{t_0}^{t} \frac{1}{|t-\tau|^{3/5}} \| \, |D_x u| \, |u|^{p-2} \|_{L_x^{10/7}} \, dx,$$

and from Strichartz' and Young's inequalities

$$\begin{aligned} \|D_x u\|_{L^{10/3}_{[t_0,t_1]}} &\le C\|u(t_0)\|_{H^1} + C\| \, |D_x u| \, |u|^{p-2}\|_{L^{10/7}[t_0,t_1]} \\ &\le C\|\phi\|_{H^1} + C\|D_x u\|_{L^{10/3}[t_0,t_1]} \|u\|^{p-2}_{L^{5(p-2)/2}[t_0,t_1]}. \end{aligned} \tag{5.2}$$

Since $\big(5(p-2)/2, \, 5(p-2)/2\big)$ is admissible,

$$\|u\|_{L_{x,t}^{5(p-2)/2}} < C(\|\phi\|_{H^1}). \tag{5.3}$$

Assume that the interval $[t_0, t_1]$ is such that

$$\|u\|_{L^{5(p-2)/2}[t_0,t_1]} < c_1 \tag{5.4}$$

for a sufficiently small constant c_1 (depending only on the parameters).

Substituting (5.4) in (5.2) implies that

$$\|D_x u\|_{L^{10/3}[t_0,t_1]} \le C\|\varphi\|_{H^1}. \tag{5.5}$$

Now, from (5.3), $[0, \infty[$ may be broken up into at most $K < C(\|\phi\|_{H^1})$ consecutive intervals $I_1, \ldots, I_K$ such that for each $k = 1, \ldots, K$

$$\|u\|_{L^{5(p-2)/2}[I_k]} < c_1 \tag{5.6}$$

and hence, from the preceding,

$$\|D_x u\|_{L^{10/3}[I_k]} \le C\|\varphi\|_{H^1}. \tag{5.7}$$

Adding up these contributions then gives a uniform estimate

$$\|D_x u\|_{L_{x,t}^{10/3}} < C(\|\phi\|_{H^1}). \tag{5.8}$$

6 Scattering

Recall the construction of the wave maps. From the integral equation

$$u(t) = e^{it\Delta}(\Omega_+\varphi) - i\int_{t}^{\infty} e^{i(t-\tau)\Delta}(u|u|^{p-2})(\tau)d\tau, \tag{6.1}$$

where

$$\Omega_+\varphi = \varphi + i\int_0^\infty e^{-i\tau\Delta}(u|u|^{p-2})(\tau)d\tau. \tag{6.2}$$

Then

$$\|\Omega_+\varphi\|_{H^1} \leq H(\varphi) \quad \text{and} \quad \|u(t) - e^{it\Delta}(\Omega_+\varphi)\|_{H^1} \overset{t\to\infty}{\longrightarrow} 0. \tag{6.3}$$

The key point is the fact that

$$\Big\| \int_t^\infty e^{-i\tau\Delta}(u|u|^{p-2})(\tau)d\tau\Big\|_{H^1_x} \overset{t\to\infty}{\longrightarrow} 0. \tag{6.4}$$

To get this last estimate, take $\psi \in L^2$, $\|\psi\|_2 = 1$ and consider

$$\left\langle \psi, D_x\left[\int_t^\infty e^{-i\tau\Delta}(u|u|^{p-2})(\tau)d\tau\right]\right\rangle$$

bounded by

$$\int_t^\infty \int |e^{-i\tau\Delta}\psi|\, |D_x u|\, |u|^{p-2}dx d\tau$$

and hence, from Hölder's inequality, Strichartz' inequality and (5.8),

$$< C\|e^{-i\tau\Delta}\psi\|_{L^{10/3}_{x,\tau}}\, \|D_x u\|_{L^{10/3}_{x,\tau}}\, \|u\|^{p-2}_{L^{5/2(p-2)}[t,\infty]},$$
$$C(\|\phi\|_{H^1})\, \|u\|^{p-2}_{L^{5/2(p-2)}[t,\infty]} \to 0 \quad \text{for } t\to\infty.$$

The preceding is well-known and was recalled here for later reference in the context of H^s-data.

7 Smooth solutions

We next consider data $\phi \in H^s$, $s > 1$.

For simplicity, let $p = 4$ and consider the IVP

$$\begin{cases} iu_t + \Delta u - u|u|^2 = 0, \\ u(0) = \phi \in H^s. \end{cases} \tag{7.1}$$

This problem is globally wellposed; we are interested in the H^s-behaviour of $u(t)$ for $t \to \infty$.

Remark. If one considers other nonlinearities $u|u|^{p-2}$, $p < 6$, some restrictions on s need to be made depending on the smoothness of the nonlinearity.

Proposition 3. *Considering the IVP (7.1) with $s \geq 1$, one has*

$$\|D_x^s u\|_{L^{10/3}} \leq C(\|\phi\|_{H^s}) \tag{7.2}$$

and

$$\|u(t)\|_{H^s} \leq C(\|\phi\|_{H^s}) \quad \textit{for all time.} \tag{7.3}$$

Proof. The main difference with the case $s = 1$ is the fact that (7.3) is not implied by a conserved quantity.

Let again

$$\mathbb{R} = I_1 \cup \cdots \cup I_K$$

be a partition of $\mathbb{R}$ in a bounded number of intervals I_k satisfying (5.6), i.e.,

$$\|u\|_{L^5_{x,t}[I_k]} < c_1 \quad \text{for } k = 1, \ldots, K. \tag{7.4}$$

We proceed by induction. Assume (7.2), (7.3) valid for t restricted to $I_1 \cup \cdots \cup I_k$. Let $I_{k+1} = [t_0, t_1]$. Repeating estimate (5.2) gives

$$\|D_x^s u\|_{L^{10/3}[t_0,t_1]} \leq C\|u(t_0)\|_{H^s} + C\|D_x^s u\|_{L^{10/3}[t_0,t_1]}\|u\|^2_{L^5[t_0,t_1]} \tag{7.5}$$

$$\leq C(\|\phi\|_{H^s}) + Cc_1^2\|D_x^s u\|_{L^{10/3}[t_0,t_1]} \tag{7.6}$$

by (7.4) and the assumption. Hence

$$\|D_x^s u\|_{L^{10/3}[I_{k+1}]} \leq C(\|\phi\|_{H^s}), \tag{7.7}$$

and (7.2) thus holds on $I_1 \cup \cdots \cup I_{k+1}$.

For $t \in I_1 \cup \cdots \cup I_{k+1}$, the integral equation again gives

$$\|u(t)\|_{H^s} \leq \|\varphi\|_{H^s} + \left\| D_x^s \left[\int_0^t e^{-i\tau\Delta}(u|u|^2)(\tau)d\tau \right] \right\|_2, \tag{7.8}$$

and the second term of (7.8) is bounded by

$$\begin{aligned} &\int_0^t \int |e^{i\tau\Delta}\psi|\, |D_x^s u|\, |u|^2 dx d\tau \qquad (\|\psi\|_2 \leq 1) \\ &\leq \|e^{i\tau\Delta}\psi\|_{L^{10/3}}\|D_x^s u\|_{L^{10/3}[I_1\cup\cdots\cup I_{k+1}]}\|u\|^2_{L^5} \\ &\leq C(\|\phi\|_{H^s}) \end{aligned} \tag{7.9}$$

using (7.7) and (5.3).

Thus (7.3) also holds for $t \in I_1 \cup \cdots \cup I_{k+1}$, which proves Proposition 3.

Returning to the wave maps, we get

Proposition 4. *For* $u(0) = \varphi \in H^s$, $s \geq 1$

$$\|\Omega_+ \varphi\|_{H^s} < C(\|\phi\|_{H^s}) \tag{7.10}$$

and

$$\|u(t) - e^{it\Delta}(\Omega_+ \varphi)\|_{H^s} \overset{t\to\infty}{\longrightarrow} 0. \tag{7.11}$$

Proof. Recalling Section 6, we need to bound

$$\begin{aligned} &\left\| \int_t^\infty e^{-i\tau\Delta}(u|u|^2)(\tau)d\tau \right\|_{H^s} \\ &\leq \int_t^\infty \int |e^{i\tau\Delta}\psi|\, |D_x^s u|\, |u|^2 dx d\tau \qquad (\|\psi\|_2 \leq 1) \\ &\leq C\|e^{i\tau\Delta}\psi\|_{L^{10/3}_{x,t}} \|D_x^s u\|_{L^{10/3}_{x,t}} \|u\|^2_{L^5_{x,t}[t,\infty]} \\ &< C(\|\phi\|_{H^s}) \|u\|^2_{L^5_{x,t}[t,\infty]} \end{aligned} \tag{7.12}$$

by Strichartz' inequality and (7.2). From (5.3), (7.12) is bounded by $C(\|\phi\|_{H^s})C(\|\phi\|_{H^1})$ and tends to 0 for $t \to \infty$.

8 Remark

The main purpose of the preceding was to obtain estimates independent of a decay assumption on the data $u(0) = \phi$ and which are uniform in $\|\phi\|_{H^s}$, $s \geq 1$. The main ingredient is Lemma 1, which is a slightly refined version of the Morawetz–Strauss apriori inequality. The space dimension is $d = 3$ here. If we assume moreover decay on the data, more precisely

$$|x|\varphi \in L^2(\mathbb{R}^d), \tag{8.1}$$

then it is well-known that the pseudo-conformal conservation law (applicable in any dimension) yields a more powerful tool for deriving apriori bounds. Thus considering the equation

$$iu_t + \Delta u - u|u|^{p-2} = 0, \tag{8.2}$$

one has for smooth solutions the apriori inequality (cf. [C])

(8.3)

$$\|(x + 2it\nabla)u(t)\|_2^2 + \frac{8t^2}{p}\|u(t)\|_p^p = \|x\varphi\|_2^2 + 4\frac{4 - d(p-2)}{p}\int_0^t s \int |u(s,x)|^p dx ds.$$

Hence for

$$p \geq 2 + \frac{4}{d} \tag{8.4}$$

(8.3) implies the apriori estimate

$$\|u(t)\|_p^p \leq \frac{\|x\varphi\|_2^2}{t^2}. \tag{8.5}$$

Assume in addition to (8.1) that

$$\varphi \in H^s(\mathbb{R}^d) \quad \text{with } p - 2 \leq \frac{4}{(d-2s)_+} \tag{8.6}$$

(the case of equality $p - 2 = 4/(d - 2s)$ corresponds to the H^s-critical case).

Then there is always a local solution on a nontrivial time-interval $[0, T^*[$, $T^* > 0$ and

$$T^* > c(\|\varphi\|_{H^s}) \tag{8.7}$$

in the subcritical case $p - 2 < 4/(d - 2s)$. For $T < T^*$, this local solution satisfies

$$\|D^s u\|_{L^{2(d+2)/d}_{x,t}[0,T]} < \infty \tag{8.8}$$

and

$$\|u\|_{L^q_{x,t}[0,T]} < \infty \quad \text{for } 2 \leq q \leq 2(d+2)/(d-2s). \tag{8.9}$$

In particular, for $T < T^*$,

$$\|u\|_{L^p_{x,t}[0,T]} < \infty; \tag{8.10}$$

and (8.5) implies therefore that

$$\|u\|_{L^p_{x,t}[0,T^*[} < \infty \tag{8.11}$$

with a uniform bound in the subcritical case by $C(\|\phi\|_{H^s}, \|x\varphi\|_{L^2})$.

For $d = 1, 2$, in the subcritical and critical case, one may then show that the local solution extends to a global one, i.e., $T^* = \infty$,

$$\|D^s_x u\|_{L^{2(d+2)/d}_{x,t}(\mathbb{R})} < \infty, \tag{8.12}$$

$$\sup_{t\in\mathbb{R}} \|D^s_x u(t)\| < \infty, \tag{8.13}$$

and scattering in H^s-space

$$\|u(t) - e^{it\Delta}\Omega_+(\varphi)\|_{H^s} \overset{t\to\infty}{\longrightarrow} 0. \tag{8.14}$$

The same statement holds for $d = 3$ when $p < 6$.

To derive these facts from the integral equation, one mainly needs an apriori bound on

$$\|u\|_{L^{(d+2)(p-2)/2}_{x,t}} < C. \tag{8.15}$$

We sketch the argument, distinguishing $d = 1, 2, 3$.

(i) $d = 1$.

From the integral equation, we get

$$\|u(t)\|_\infty \le \|e^{it\Delta}\varphi\|_\infty + \int_0^t \frac{1}{|t-\tau|^{1/2}} \|u(\tau)\|_{p-1}^{p-1} d\tau, \tag{8.16}$$

$$\|e^{it\Delta}\varphi\|_\infty \le C|t|^{-1/2}\|\varphi\|_1 \le C|t|^{-1/2}\|(1+|x|)\varphi\|_2. \tag{8.17}$$

The second term in (8.16) is bounded by

$$\int_0^{\frac{t}{2}\wedge 1} + \int_{\frac{t}{2}\wedge 1}^{t\wedge 1} + \int_1^t \le Ct^{-1/2} + \int_{\frac{t}{2}\wedge 1}^t \frac{1}{|t-\tau|^{1/2}} \frac{1}{\tau^{2(p-3)/(p-2)}} d\tau. \tag{8.18}$$

Hence

$$\|u(t)\|_\infty < C|t|^{-1/2} + C(1\wedge |t|)^{-(3p-10)/2(p-2)}$$

and, in particular, bounded for t away from zero.

By interpolating (8.10), (8.18), it follows that (8.15) holds for t away from 0. By (8.9), this establishes (8.15).

(ii) $d = 2$.

We proceed similarly, but replace the L^∞_x-norm by the L^q_x-norm for large q.

(iii) $d = 3$, $\frac{10}{3} < p < 6$.

Choose $r < 6, r \approx 6$. From the integral equation, it follows that

$$\|u(t)\|_r \le \|e^{it\Delta}\varphi\|_r + \int_0^t \frac{1}{|t-\tau|^{3(1/2-1/r)}} \|u(\tau)\|_{r'(p-1)}^{p-1} d\tau \tag{8.19}$$

where $3(1/2 - 1/r) < 1$; and since $r' > \frac{6}{5}$

$$\|e^{it\Delta}\varphi\|_r \le C|t|^{-3(1/2-1/r)}\|\varphi\|_{r'} < C|t|^{-3(1/2-1/r)}\|(1+|x|)\varphi\|_2. \tag{8.20}$$

Since $p < 6$, one may choose $r < 6$ with $r'(p-1) < p$. The second term in (8.19) then admits a bound

$$\int_0^{\frac{t}{2}\wedge 1} + \int_{\frac{t}{2}\wedge 1}^{t\wedge 1} + \int_1^t < C|t|^{-3(1/2-1/r)} + \int_{\frac{t}{2}\wedge 1}^t \frac{1}{|t-\tau|^{3(1/2-1/r)}} \frac{1}{\tau^{2(p-3+2/r)/(p-2)}} d\tau;$$

hence

$$\|u(t)\|_r < C|t|^{-3(1/2-1/r)} + C(1 \wedge t)^{-(p-3+2/r)/(p-2)} \tag{8.21}$$

and $u \in L^r_{x,t}$ for t away from 0.

If we assume

$$\tfrac{5}{2}(p-2) < 6, \tag{8.22}$$

then (8.15) may again be deduced for t away from 0 by interpolating (8.21), (8.5). Thus, by (8.9), this establishes (8.15) in the case $p < \frac{22}{5}$.

To proceed when $p \geq \frac{22}{5}$ is a bit more complicated. Fix $t_0 > 0$ such that on $[0, t_0]$ wellposedness holds and, in particular,

$$D^s_x u \in L^{10/3}_{x,t}[0, t_0] \tag{8.23}$$

and

$$\|u(t)\|_{H^s} < C \quad \text{for } t \leq t_0. \tag{8.24}$$

Let

$$\gamma = 3\left(\frac{1}{r} - \frac{2}{5(p-2)}\right) > 0. \tag{8.25}$$

From the integral equation

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) + i\int_{t_0}^t e^{i(t-\tau)\Delta}(u|u|^{p-2})(\tau)d\tau$$

we get from Hölder's inequality

$$\|D^\gamma_x u(t)\|_r \leq \|D^\gamma_x[e^{i(t-t_0)\Delta}u(t_0)]\|_r \tag{8.26}$$

$$+\int_{t_0}^t \frac{1}{|t-\tau|^{3(1/2-1/r)}}\|D^\gamma_x u(\tau)\|_r \, \|u(\tau)\|^{p-2}_{(p-2)r/(r-2)}d\tau. \tag{8.27}$$

Since $2 < (p-2)r/(r-2) < p$, interpolation yields

$$(8.27) < C\int_{t_0}^t \frac{1}{|t-\tau|^{3(1/2-1/r)}}\,\frac{1}{\tau^{2(p-4/r')/(p-2)}}\|D^\gamma_x u(\tau)\|_r d\tau. \tag{8.28}$$

Since $3(1/2 - 1/r) < 1$, one may restrict t to a nontrivial interval $[t_0, t_1]$ such that

$$\|(8.28)\|_{L^{5(p-2)/2}[t_0,t_1]} \leq o(1)\|D^\gamma_x u\|_{L^{5(p-2)/2}_{[t_0,t_1]}L^r_x}. \tag{8.29}$$

Then, from (8.26), (8.27), the definition of γ (8.25) and (8.24), it follows that

$$\begin{aligned}\|D_x^\gamma u\|_{L^{5(p-2)/2}_{[t_0,t_1]}L^r_x} &\le C\|D_x^\gamma[e^{i(t-t_0)\Delta}u(t_0)]\|_{L_t^{5(p-2)/2}L^r_x}\\ &\le C\|D_x^s[e^{i(t-t_0)\Delta}u(t_0)]\|_{L^{10/3}_{x,t}}\\ &\le C\|u(t_0)\|_{H^s} < C.\end{aligned} \tag{8.30}$$

From (8.25), (8.30)

$$\|u\|_{L^{5(p-2)/2}_{x,t}[t_0,t_1]} < C, \tag{8.31}$$

$$\|u\|_{L^{5(p-2)/2}_{x,t}[0,t_1]} < C. \tag{8.32}$$

Applying the integral equation, one may then extend (8.23), (8.24) to $[0,t_1]$. By iterating the procedure any finite interval may be covered.

To establish (8.15) for $t\in\mathbb{R}$, replace t_0 in the preceding by a sufficiently large time such that the kernel

$$\frac{1}{|t-\tau|^{3(1/2-1/r)}|\tau|^{2(p-4/r')/(p-2)}}|_{\tau,t>t_0} \tag{8.33}$$

in (8.28) has a small norm acting on $L^{5(p-2)/2}[t_{0,\infty}[$. One then gets

$$\|D_x^\gamma u\|_{L^{5(p-2)/2}_{[t_0,\infty[}L^r_x} < C \tag{8.34}$$

and (8.15) on $[t_0,\infty[$, completing the argument.

In the subcritical case, bounds are uniform.

9 Estimates below energy norm

We next consider the IVP

$$\begin{cases} iu_t-\Delta u+u|u|^2=0\\ u(0)=\phi\in H^s, \quad s<1\end{cases} \tag{9.1}$$

where ϕ is general (not necessarily small) data. We will first carry out the analysis from [B_2] related to the 2D NLS

$$iu_t-\Delta u+u|u|^2=0 \tag{9.2}$$

to the 3D setting of (9.1). Results of this type may be expected in any H^1-subcritical situation ($p<6$). As in [B_2], the cubic nonlinearity permits us to rely on a fairly straightforward quartic Strichartz-type inequality, however.

Improved Strichartz inequality

Lemma 5 *Assume*

$$\text{supp}\,\widehat{\psi}_1 \subset B(0,2M_1)\backslash B(0,M_1), \quad \text{supp}\,\widehat{\psi}_2 \subset B(0,2M_2)\backslash B(0,M_2) \tag{9.3}$$

and

$$M_1 \leq M_2.$$

Then

$$\|(e^{it\Delta}\psi_1)(e^{it\Delta}\psi_2)\|_2 \leq C\frac{M_1}{M_2^{1/2}}\|\psi_1\|_2\,\|\psi_2\|_2. \tag{9.4}$$

Proof. Since, from Strichartz' inequality,

$$\begin{aligned}\|(e^{it\Delta}\psi_1)(e^{it\Delta}\psi_2)\|_2 \leq \|e^{it\Delta}\psi_1\|_4\,\|e^{it\Delta}\psi_2\|_4 &\leq C\|\psi_1\|_{H^{1/4}}.\|\psi_2\|_{H^{1/4}}\\ &< C(M_1M_2)^{1/4}\,\|\psi_1\|_2\,\|\psi_2\|_2,\end{aligned}$$

we may assume $M_1 \ll M_2$.

Write then

$$(e^{it\Delta}\psi_1)(e^{it\Delta}\psi_2) = \int \widehat{\psi}(\xi_1)\widehat{\psi}(\xi_2)e^{i[(\xi_1+\xi_2)x+(|\xi_1|^2+|\xi_2|^2)t]}d\xi_1 d\xi_2.$$

From Parseval's identity and Cauchy-Schwarz, it follows that

$$\begin{aligned}\|(e^{it\Delta}\psi_1)(e^{it\Delta}\psi_2)\|_2^2 &= \int d\xi d\lambda \left| \int \widehat{\psi}_1(\xi_1)\widehat{\psi}_2(\xi-\xi_1)\delta_0(|\xi_1|^2+|\xi-\xi_1|^2-\lambda)d\xi_1\right|^2\\ &\leq \|\psi_1\|_2^2\|\psi_2\|_2^2\int \delta_0(|\xi_1|^2+|\xi-\xi_1|^2-\lambda)d\xi_1\\ &\lesssim \frac{M_1^2}{M_2}\|\psi_1\|_2^2\|\psi_2\|_2^2,\end{aligned}$$

which proves (9.4).

Recall the definition of the spaces $X_{s,b} = X_{s,b}[I]$ (I = time interval):

$$\|u\|_{X_{s,b}[I]} = \left(\int |\hat{u}(\xi,\lambda)|^2(1+|\xi|^{2s})(1+|\lambda-\xi^2|^{2b})d\xi d\lambda\right)^{1/2} \tag{9.5}$$

provided

$$u(x,t) = \int \hat{u}(\xi,\lambda)e^{i(x.\xi+\lambda t)}d\xi d\lambda \quad \text{for } t \in I \tag{9.6}$$

(in fact, in (9.5) we take the infinum over representations (9.6)).

From (9.4) one easily deduces

Corollary 6 *For*

$$0 \le \rho < \tfrac{1}{2}, \tag{9.7}$$

$$\|D_x^\rho(u_1 u_2)\|_2 \le C\|u_1\|_{X_{\frac{1}{2}+\rho,\frac{1}{2}+}} \|u_2\|_{X_{0,\frac{1}{2}+}}. \tag{9.8}$$

Assume $\frac{1}{2} \le \rho < \sigma + \frac{1}{2}, \sigma \le 1$. Then

$$\|D_x^\rho(u_1 u_2)\|_2 \le C\|u_1\|_{X_{\sigma,\frac{1}{2}+}} \|u_2\|_{X_{\rho+\frac{1}{2}-\sigma,\frac{1}{2}+}}. \tag{9.9}$$

10 Decomposition of the data

Let $\phi \in H^s$ and decompose

$$\phi = \phi_0 + \psi_0 \quad \text{where } \phi_0 = \int_{|\xi|<N_0} \widehat{\phi}_0(\xi) e^{ix.\xi} d\xi. \tag{10.1}$$

Hence

(10.2)

$$\|\phi_0\|_{H^1} \lesssim N_0^{1-s} \quad \text{and} \quad (s > \tfrac{3}{4}) \quad H(\phi_0) = \frac{1}{2}\int |\nabla \phi_0|^2 + \frac{1}{4}\int |\phi_0|^4 \lesssim N_0^{2(1-s)}.$$

Consider the IVP

$$\begin{cases} i\dot{u}_0 - \Delta u_0 + u_0|u_0|^2 = 0 \\ u_0(0) = \phi_0 \end{cases} \tag{10.3}$$

for $t \in I = [0, \delta]$.

Estimate

$$\begin{aligned} \|D_x^{3/10}(e^{it\Delta}\phi_0)\|_{L_t^5 L_x^{10/3}[I]} &< |I|^{1/5} \sup_t \|D_x^{3/10}(e^{it\Delta}\phi_0)\|_{L_x^{10/3}} \\ &\lesssim |I|^{1/5}\|\phi_0\|_{H^{9/10}} \\ &\lesssim |I|^{1/5} N_0^{\frac{9}{10}(1-s)}. \end{aligned} \tag{10.4}$$

Take

$$|I| = \delta = N_0^{-(1-s)\frac{9}{2}-}, \tag{10.5}$$

so that by (10.4)

$$\|D_x^{3/10}(e^{it\Delta}\phi_0)\|_{L_t^5 L_x^{10/3}[I]} = o(1). \tag{10.6}$$

Write the integral equation for u_0,

$$u_0(t) = e^{it\Delta}\phi_0 + i\int_0^t e^{i(t-\tau)\Delta}(u_0|u_0|^2)(\tau)d\tau. \tag{10.7}$$

Hence, from the decay and Young's inequality

$$\|D_x^{3/10}u_0\|_{L_t^5L_x^{10/3}[I]} \tag{10.8}$$

$$\leq \|D_x^{3/10}(e^{it\Delta}\phi_0)\|_{L_t^5L_x^{10/3}[I]} + \left\|\int_0^t \frac{1}{|t-\tau|^{3/5}}\|D_x^{3/10}(u_0|u_0|^2(\tau))\|_{L_x^{10/7}}\right\|_{L_t^5[I]}$$

$$< o(1) + \left\|\int \frac{1}{|t-\tau|^{3/5}}\|D_x^{3/10}u_0(\tau)\|_{L_x^{10/3}}\ \|u_0(\tau)\|_{L_x^5}^2 d\tau\right\|_{L_t^5[I]}$$

$$< o(1) + \|D_x^{3/10}u_0\|_{L_t^5L_x^{10/3}}\|u_0\|_{L_{x,t}^5[I]}^2.$$

Also

$$\|u_0\|_{L_{x,t}^5[I]} \lesssim \|D_x^{3/10}u_0\|_{L_t^5L_x^{10/3}[I]}. \tag{10.9}$$

Substituting (10.9) in (10.8), we obtain

$$\|D_x^{3/10}u_0\|_{L_t^5L_x^{10/3}[I]} = o(1) \tag{10.10}$$

and

$$\|u_0\|_{L_{x,t}^5[I]} = o(1). \tag{10.11}$$

From $\|\phi_0\|_2 < C$, $\|\phi_0\|_{H^1} \lesssim N_0^{1-s}$, (10.7) and (10.11), one may further deduce that

$$|||u_0|||_{X_{0,\frac{1}{2}+}[I]} \leq 2\|\phi_0\|_2 < C \tag{10.12}$$

and

$$|||u_0|||_{X_{1,\frac{1}{2}+}[I]} \leq 2\|\phi_0\|_{H^1} \lesssim N_0^{1-s} \tag{10.13}$$

(we assume $|I| \leq 1$).

11 Estimates related to the difference equation

Writing $u = u_0 + v$, consider next the difference equation for v

$$\begin{cases} i\dot{v} - \Delta v + 2|u_0|^2v + 2u_0^2\overline{v} + 2\overline{u}_0v^2 + 2u_0|v|^2 + |v|^2v = 0 \\ v(0) = \psi_0 \end{cases} \tag{11.1}$$

and write

$$v = e^{it\Delta}\psi_0 + w. \tag{11.2}$$

Recall that

$$\|\psi_0\|_{H^s} \le C, \quad \|\psi_0\|_{L^2} \lesssim N_0^{-s}. \tag{11.3}$$

By (10.11), (11.3) and the integral equation form of (11.1), we get

$$\|v\|_{X_{0,\frac{1}{2}+}[I]} \lesssim N_0^{-s} + \|v\|_{X_{0,\frac{1}{2}+}}[o(1) + o(1)\|v\|_{X_{\frac{1}{2},\frac{1}{2}+}} + \|v\|^2_{X_{\frac{1}{2}+,\frac{1}{2}+}}], \tag{11.4}$$

$$\begin{aligned} \|v\|_{X_{s,\frac{1}{2}+}[I]} \le & C + \|v\|_{X_{s,\frac{1}{2}+}}[o(1) + o(1)\|v\|_{X_{\frac{1}{2},\frac{1}{2}+}} + \|v\|^2_{X_{\frac{1}{2}+,\frac{1}{2}+}}] \\ & + \|u_0\|_{X_{s,\frac{1}{2}+}}[o(1)\|v\|_{X_{\frac{1}{2},\frac{1}{2}+}} + \|v\|^2_{X_{\frac{1}{2}+,\frac{1}{2}+}}], \end{aligned} \tag{11.5}$$

while from interpolation (we assume $s > \frac{1}{2}$) and (10.12), (10.13)

$$\|u_0\|_{X_{s,\frac{1}{2}+}} \le \|u_0\|^{1-s}_{X_{0,\frac{1}{2}+}} \|u_0\|^s_{X_{1,\frac{1}{2}+}} \lesssim N_0^{(1-s)s},$$

$$\|v\|_{X_{\frac{1}{2},\frac{1}{2}+}} \le \|v\|^{1-\frac{1}{2s}}_{X_{0,\frac{1}{2}+}} \|v\|^{\frac{1}{2s}}_{X_{s,\frac{1}{2}+}}. \tag{11.6}$$

Thus (11.4)–(11.6) imply that (for $s > 1/\sqrt{2}$)

$$\|v\|_{X_{0,\frac{1}{2}+}[I]} \lesssim N_0^{-s}, \tag{11.7}$$

$$\|v\|_{X_{s,\frac{1}{2}+}[I]} \lesssim C. \tag{11.8}$$

Next, estimate

$$\begin{aligned} & \|D_x w\|_{L_t^\infty L_x^2} \\ & \le \sup_{\|\psi\|_2 \le 1} \int_I |\langle e^{i\tau\Delta}\psi,\, D_x[2|u_0|^2 v + 2u_0^2\bar{v} + 2\bar{u}_0 v^2 + +2u_0|v|^2 + v|v|^2]\rangle| d\tau \\ & \le \sup_{\|W\|_{X_{0,\frac{1}{2}+}} \le 1} \int |\langle W, D_x[2|u_0|^2 v + 2u_0^2\bar{v} + 2\bar{u}_0 v^2 + 2u_0|v|^2 + v|v|^2]\rangle| dt. \end{aligned} \tag{11.9}$$

Fix $\rho = 1/2-$. Then by (9.8), (9.9), (11.7) and (11.8) we have

$$\int |\langle W, D_x(|u_0|^2 v)\rangle| \le \iint |W|\,|u_0|\,|v|\,|D_x u_0| + \iint |D_x^\rho(W u_0)|\,|D_x^{1-\rho}(v.u_0)|,$$

$$\begin{aligned} \int |W|\,|u_0|\,|v|\,|D_x u_0| & \le \|W\|_{10/3}\,\|u_0\|_{10}\,\|v\|_{10/3}\|D_x u_0\|_{10/3} \\ & \lesssim \|u_0\|^2_{X_{1,\frac{1}{2}+}}\,\|v\|_{X_{0,\frac{1}{2}+}} \\ & \overset{(10.13),(11.7)}{\lesssim} N_0^{2-3s} \end{aligned} \tag{11.10}$$

and

$$\tag{11.11}\begin{aligned}\int |D_x^\rho(Wu_0)|\,|D_x^{1-\rho}(vu_0)| &\le \|D_x^\rho(Wu_0)\|_2\,\|D_x^{1-\rho}(vu_0)\|_2\\ &\lesssim \|u_0\|_{X_{\frac12+\rho,\frac12+}}\cdot\|v\|_{X_{\frac12-\rho,\frac12+}}\,\|u_0\|_{X_{1,\frac12+}}\\ &\lesssim N_0^{2-3s+}.\end{aligned}$$

Hence

$$\tag{11.12}\int |\langle W, D_x(|u_0|^2 v)\rangle| < N_0^{2-3s+}$$

and, similarly,

$$\tag{11.13}\int |\langle W, D_x(u_0^2\overline{v})\rangle| < N_0^{2-3s+}.$$

For the v-quadratic contribution we have, applying (9.8) and (9.9),

$$\tag{11.14}\begin{aligned}\int |\langle W, D(\overline{u}_0 v^2)\rangle| &< \int |W|\,Du_0|\,|v|^2 + \|D^\rho(W\overline{u}_0)\|_2 . \|D^{1-\rho}(v^2)\|_2\\ &< C\|u_0\|_{X_{1,\frac12+}}\,\|v\|^2_{X_{\frac12,\frac12+}} + \|u_0\|_{X_{\frac12+\rho,\frac12+}}\,\|v\|_{X_{s,\frac12+}}\,\|v\|_{X_{\frac32-\rho-s,\frac12+}}\\ &< N_0^{2-3s+}\end{aligned}$$

by (10.13), (11.7) and (11.8).

Similarly,

$$\tag{11.15}\int |\langle W, D(u_0|v|^2)\rangle| < N_0^{2-3s+}.$$

For the v-cubic contribution

$$\tag{11.16}\begin{aligned}\int |\langle W, D(v|v|^2)\rangle| &< \int |\langle W,(Dv)v\overline{v}\rangle| + |\langle W, vD(|v|^2)\rangle|\\ &< \|D^{s-\frac12}(Wv)\|_2\,\|D^{\frac32-s}(v^2)\|_2 + \|D^{s-\frac12}(W\overline{v})\|_2\,\|D^{\frac32-s}(|v|^2)\|_2\\ &< \|v\|_{X_{s,\frac12+}}\,\|v\|_{X_{s,\frac12+}}\,\|v\|_{X_{2(1-s),\frac12+}}\\ &< N_0^{2-3s},\end{aligned}$$

applying (9.9) and (9.10). Recall that we have assumed that s satisfies

$$\tag{11.17} s > \frac{1}{\sqrt{2}} > \frac{2}{3}.$$

Thus, from (11.9), (11.12), (11.13), (11.14), (11.15) and (11.16) we have

$$\sup_{t\in I} \|w(t)\|_{H^1} < N_0^{2-3s+}. \tag{11.18}$$

At time $t_1 = \delta$, we then write

$$\begin{aligned} u(t_1) &= u_0(t_1) + (e^{it_1\Delta}\psi_0) + w(t_1) \\ &= \phi_1 + \psi_1 \end{aligned}$$

with

$$\begin{cases} \phi_1 = u_0(t_1) + w(t_1), \\ \psi_1 = e^{it_1\Delta}\psi_0. \end{cases} \tag{11.19}$$

Thus the pair (ϕ_0, ψ_0) is replaced by (ϕ_1, ψ_1).

ψ_1 is similar to ψ_0.

The Hamiltonian increment when replacing ϕ_0 by ϕ_1 is bounded by

$$\begin{aligned} |H(\phi_1) - H(\phi_0)| =& |H(\phi_1) - H(u_0(t_1))| \quad \text{(from Hamiltonian conservation)} \\ \leq& (\|u_0(t_1)\|_{H^1} + \|w(t_1)\|_{H^1})\|w(t_1)\|_{H^1} \\ &+ (\|u_0(t_1)\|_6 + \|w(t_1)\|_6)^3 \|w(t_1)\|_2 \\ <& N_0^{1-s} N_0^{2-3s+} + N_0^{3(1-s)-s} \end{aligned}$$

(by (10.13), (11.2), (11.7) and (11.18))

$$< N_0^{3-4s+}. \tag{11.20}$$

By (10.5), iteration of the procedure leads to the condition (cf. [B2])

$$T.N_0^{9/2(1-s)} N_0^{3-4s+} < N_0^{2(1-s)},$$
$$TN_0^{\frac{11-13s}{2}} < 1.$$

Hence, we require

$$s > \frac{11}{13} \tag{11.21}$$

and take

$$N_0 = N_0(T) = T^{\frac{2}{13s-11}+}. \tag{11.22}$$

Our conclusion is thus

Proposition 7 *The IVP*

$$\begin{cases} iu_t - \Delta u + u|u|^2 = 0 \\ u(0) = \phi \in H^s \end{cases} \tag{11.23}$$

is globally wellposed for $s > \frac{11}{13}$ *and the solution* u *has the form*

$$u(t) = e^{it\Delta}\phi + v(t), \qquad \|v(t)\|_{H^1} < (1+|t|)^{\frac{2(1-s)}{13s-11}+}. \tag{11.24}$$

12 Scattering below energy norm

Recall first the Morawetz inequality

(12.1)

$$\int_0^T |u(0,t)|^2 dt + \int_0^T \int \frac{|\nabla u|^2 - u_r^2}{r} dx dt + \int_0^T \int \frac{|u|^4}{r} dx dt < C \sup_{0\le t\le T} \|u(t)\|^2_{H^{1/2}}.$$

In this section, we restrict ourselves to the radially symmetric case.

Lemma 8 *If* ϕ *is radially symmetric*

$$\sup r|\phi(r)| \lesssim \|\phi\|_{H^{1/2+}}. \tag{12.2}$$

Proof. Set $v = r\phi$. Then

$$\|v\|_{L^2_{(dr)}} = \left(\int |\phi|^2 r^2 dr\right)^{1/2} \sim \|\phi\|_2$$

and

$$\|v'\|_{L^2(dr)} \le \|r\phi'\|_{L^2(dr)} + \|\phi\|_{L^2(dr)} \sim \|\nabla\phi\|_2 + \left\|\frac{\phi}{|x|}\right\|_2. \tag{12.3}$$

To estimate $\|\phi/|x|\|_2$, write

$$\left\|\frac{\phi}{|x|}\right\|_2^2 \sim \sum_{K \text{ dyadic}} K^2\left(\int_{|x|\sim K^{-1}} |\phi|^2 dx\right).$$

By Fourier decomposition, denoting

$$\phi_L = \int_{|\xi|\sim L} \widehat{\phi}(\xi)\, e^{ix\xi} d\xi$$

we may write

$$\int_{|x|\sim K^{-1}} |\phi|^2 \lesssim \int_{|x|\sim K^{-1}} \Big|\sum_{L<K} \phi_L\Big|^2 + \sum_{L>K} \int_{|x|\sim K^{-1}} |\phi_L|^2$$
$$< \sum_{L<K} \left(\log \frac{K}{L}\right)^3 K^{-3} \|\phi_L\|^2_\infty + \sum_{L>K} L^{-2} \|\nabla \phi_L\|_2^2.$$

Thus

$$K^2 \int_{|x|\sim K^{-1}} |\phi|^2 dx \lesssim \sum_{L<K} \left(\log \frac{K}{L}\right)^3 \frac{L}{K} \|\nabla \phi_L\|_2^2 + \sum_{L>K} \left(\frac{K}{L}\right)^2 \|\nabla \phi_L\|_2^2,$$

and summing over dyadic K values ($K = 2^k, k \in \mathbb{Z}$) yields the bound $\sum_{L \text{ dyadic}} \|\nabla \phi_L\|_2^2$.

Hence

$$\left\| \frac{\phi}{|x|} \right\|_2 \lesssim \|\phi\|_{H^1}. \tag{12.4}$$

From (5.3),

$$\|v'\|_{L^2(dr)} \lesssim \|\phi\|_{H^1}. \tag{12.5}$$

Estimate

$$\big| |v|^2(r_1) - |v|^2(r_2) \big| \lesssim \int_{r_1}^{r_2} |v|\,|v'| \lesssim \|v\|_2 \, \|v'\|_2 < \|\phi\|_2 \, \|\phi\|_{H^1}.$$

Hence

$$|v(r)|^2 \lesssim \|\phi\|_2^2 + \|\phi\|_2 \, \|\phi\|_{H^1},$$

and the lemma follows from interpolation.

Theorem 9 *Consider the IVP in 3D*

$$\begin{cases} iu_t - \Delta u + u|u|^2 = 0 \\ u(0) = \phi \in H^s, \quad s > \frac{5}{7} \end{cases} \tag{12.6}$$

where ϕ is a radial function. Then there is global wellposedness and for all time

$$\|u(t) - e^{it\Delta}\phi\|_{H^1} < C(\|\phi\|_{H^s}). \tag{12.7}$$

Proof. Denote

$$B_1(t) = \sup_{0 \le t' \le t} \|u(t') - e^{it'\Delta}\phi\|_{H^1}. \tag{12.8}$$

Hence, from (12.1) and (12.2),

$$\int_0^T \int |u|^5 dx dt < \sup_{t<T} \|u(t)\|_{H^{1/2+}}^3 < C\big(B_1(T)^{3/2+} + 1\big). \tag{12.9}$$

We now recall the proof of Proposition 7.

Fix a cutoff N_0 and define

$$\phi_0 = \int_{|\xi|<N_0} \widehat{\phi}(\xi) e^{ix\xi} d\xi \quad \text{and} \quad \psi_0 = \int_{|\xi|>N_0} \widehat{\phi}(\xi)\, e^{ix.\xi} d\xi.$$

Now consider a sequence of times

$$0 = t_0 < t_1 < t_2 < \cdots < t_J = T, \tag{12.10}$$

where for each time t_j

$$u(t_j) = \phi_j + \psi_j \tag{12.11}$$

with

$$H(\phi_j) < H(\phi_{j-1}) + N_0^{3-4s+}, \tag{12.12}$$

$$\|\phi_j\|_2 < C, \quad H(\phi_j) \lesssim N_0^{2(1-s)}, \tag{12.13}$$

$$\psi_j = e^{it_j\Delta}\psi_0. \tag{12.14}$$

In the proof of Proposition 7, we chose $I_j = [t_j, t_{j+1}]$ with

$$|I_j| = N_0^{-9/2(1-s)-} \tag{12.15}$$

(and, in particular, bounded by 1).

Considering the solution u_0 of the IVP on I_j

$$\begin{cases} i\dot{u}_0 - \Delta u_0 + u_0|u_0|^2 = 0 \\ u_0(t_j) = \phi_j \end{cases} \tag{12.16}$$

the choice (12.15) implies, by condition (12.13), that

$$\|u_0\|_{L^5_{x,t}[I_j]} = N_0^{0-}. \tag{12.17}$$

Assume I_j such that (12.17) holds. Assume also $|I_j| \leq 1$. We get from the integral formula

$$u_0(t) = e^{it\Delta}\phi_j + i\int_{t_j}^{t} e^{i(t-\tau)\Delta}(u|u|^2)(\tau)d\tau$$

that

$$\|D_x u_0\|_{L^{10/3}_{x,t}[I_j]} \lesssim \|\phi_j\|_{H^1} + \|D_x u_0\|_{L^{10/3}_{x,t}[I_j]}\|u_0\|^2_{L^5_{x,t}[I]}; \tag{12.18}$$

hence

$$\|D_x u_0\|_{L^{10/3}_{x,t}[I_j]} \lesssim C\|\phi_j\|_{H^1} \lesssim N_0^{1-s}. \tag{12.19}$$

Also

$$\|u_0\|_{L^{10}_{x,t}[I_j]} \lesssim \|D_x^{3/5} u_0\|_{L^{10}_t L^{10/3}_x[I_j]} \lesssim N_0^{1-s}. \tag{12.20}$$

Furthermore (since we assume $|I_j| \le 1$),

$$\|u_0\|_{X_{0,\frac{1}{2}+}[I_j]} < C, \tag{12.21}$$

$$\|u_0\|_{X_{1,\frac{1}{2}+}[I_j]} \lesssim N_0^{1-s}. \tag{12.22}$$

Writing $u = u_0 + v$, where v satisfies the difference equation on I_j with initial value

$$v(t_j) = \psi_j, \tag{12.23}$$

the bounds (11.7), (11.8) hold again. Thus

$$\|v\|_{X_{0,\frac{1}{2}+}[I_j]} \lesssim N_0^{-s}, \tag{12.24}$$

$$\|v\|_{X_{s,\frac{1}{2}+}[I_j]} < C. \tag{12.25}$$

Also, writing

$$v = e^{i(t-t_j)\Delta}\psi_j + w, \tag{12.26}$$

one has (11.18)

$$\|w\|_{X_{1,\frac{1}{2}+}[I_j]} < N_0^{2-3s+}. \tag{12.27}$$

Write

$$u(t_{j+1}) = \phi_{j+1} + \psi_{j+1} \tag{12.28}$$

with

$$\phi_{j+1} = u_0(t_{j+1}) + w(t_{j+1}), \tag{12.29}$$

$$\psi_{j+1} = e^{it_{j+1}\Delta}\psi_0; \tag{12.30}$$

the same conclusion leading to (11.20) implies that

$$|H(\phi_{j+1}) - H(\phi_j)| \lesssim N_0^{3-4s+}. \tag{12.31}$$

If $|I_j| > 1$, one starts by rescaling to a time interval of unit length, considering

$$\tilde{u}(x,t) = u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \qquad (\lambda = |I_j|^{-1/2}). \tag{12.32}$$

Letting $I_j = [0, t_1]$ we then have the decomposition

$$\tilde{u}(0) = \tilde{\phi}_0 + \tilde{\psi}_0 \tag{12.33}$$

with

$$\|\tilde{\phi}_0\|_2 = \lambda^{-1/2}\|\phi_0\|_2 < \lambda^{-1/2}, \tag{12.34}$$

$$\|\tilde{\phi}_0\|_{H^1} = \lambda^{1/2}\|\phi_0\|_{H^1} \lesssim \lambda^{1/2}N_0^{1-s}, \tag{12.35}$$

$$\|\tilde{\psi}_0\|_2 = \lambda^{-1/2}\|\psi_0\|_2 \lesssim \lambda^{-1/2}N_0^{-s}, \tag{12.36}$$

$$\|\tilde{\psi}_0\|_{H^s} = \lambda^{s-\frac{1}{2}}\|\psi_0\|_{H^s} \lesssim \lambda^{s-\frac{1}{2}}. \tag{12.37}$$

This gives (condition (12.17) remains preserved)

$$\tilde{u}(1) = \tilde{u}_0(1) + \tilde{w}(1) + e^{i\Delta}\tilde{\psi}_0 = \tilde{\phi}_1 + e^{i\Delta}\tilde{\psi}_0 \tag{12.38}$$

where

$$\|\tilde{w}(1)\|_{H^1} < N_0^{2-3s+}\lambda^{1/2}, \tag{12.39}$$

$$H(\tilde{\phi}_1) < H(\tilde{\phi}_0) + \lambda N_0^{3-4s+}. \tag{12.40}$$

Hence, scaling back, we get again

$$u(t_1) = \phi_1 + e^{it_1\Delta}\psi_0 \tag{12.41}$$

with

$$H(\phi_1) < H(\phi_0) + N_0^{3-4s+}. \tag{12.42}$$

Since J is the number of steps (= number of intervals), one needs to fulfil the condition

$$CJN_0^{3-4s+} < N_0^{2(1-s)}, \tag{12.43}$$

where the constant C depends on $\|\phi\|_{H^s}$.

One may then conclude that

$$\sup_{j\le J}\|u(t_j) - e^{it_j\Delta}\psi_0\|_{H^1} = \sup_{j\le J}\|\phi_j\|_{H^1} \lesssim N_0^{1-s} \tag{12.44}$$

and hence

$$B_1(T) \lesssim N_0^{1-s} + \|\phi_0\|_{H^1} \sim N_0^{1-s}. \tag{12.45}$$

It remains to derive an estimate on J from (12.9) and (12.17).

Since on I_j

$$u = u_0 + v,$$

we have

$$\begin{aligned}\|u\|_{L^5_{x,t}[I_j]} &\geq \|u_0\|_{L^5_{x,t}[I_j]} - \|v\|_{L^5_{x,t}[I_j]} \\ &\overset{(12.17),(12.24),(12.25)}{>} N_0^{0-} - CN_0^{\frac{1}{2}-s} = N_0^{0-}.\end{aligned} \tag{12.46}$$

Consequently, by (12.9) and (12.45),

$$\begin{gathered} JN_0^{0-} < \sum_{j=0}^{J-1} \|u\|^5_{L^5_{x,t}[I_j]} < CB_1(T)^{3/2} + C < CN_0^{\frac{3}{2}(1-s)}, \\ J < CN_0^{\frac{3}{2}(1-s)+}, \end{gathered} \tag{12.47}$$

provided (12.43) holds.

Substituting (12.47) in (12.43) gives the condition

$$CN_0^{\frac{3}{2}(1-s)+3-4s+} < N_0^{2(1-s)} \tag{12.48}$$

satisfied for

$$s > \frac{5}{7} \tag{12.49}$$

and a choice of N_0 only depending on the data $\|u(0)\|_{H^s}$, not on T.

This proves Theorem 9.

From (12.9) and (12.7), we also get

$$\int_0^\infty \int |u|^5 dx dt < \infty. \tag{12.50}$$

Since from the integral equation, for $I = [t_0, t_1]$,

$$\begin{aligned}\|D_x^s u\|_{L^{10/3}_{x,t}[I]} &\leq \|D_x^s (e^{it\Delta} u(t_0))\|_{L^{10/3}} + \|D_x^s u\|_{L^{10/3}_{x,t}[I]} \|u\|^2_{L^5_{x,t}[I]} \\ &< C + \|D_x^s u\|_{L^{10/3}_{x,t}[I]} \|u\|^2_{L^5_{x,t}[I]}\end{aligned} \tag{12.51}$$

by (12.7), partitioning time as $\mathbb{R}_+ = \bigcup I_\alpha$ into finitely many intervals I_α satisfying

$$\|u\|_{L^5_{x,t}[I_\alpha]} = o(1) \tag{12.52}$$

(which is possible by (12.50)) permits us to conclude that

$$\|D_x^s u\|_{L^{10/3}_{x,t}} < C(\|\phi\|_{H^s}). \tag{12.53}$$

By writing

$$\Omega_+ \varphi = \varphi + i \int_0^\infty e^{-i\tau\Delta} (u|u|^2)(\tau) d\tau$$

it follows that

$$\|u(t) - e^{it\Delta}(\Omega_+\varphi)\|_{H^s} \le \left\| \int_t^\infty e^{-i\tau\Delta}(u|u|^2)(\tau)d\tau \right\|_{H^s} \lesssim \|D_x^s u\|_{L_{x,t}^{10/3}[t,\infty]} \|u\|^2_{L^5_{x,t}[t,\infty]} \overset{t\to\infty}{\to} 0. \tag{12.54}$$

Consequently, we have

Theorem 10 *In the context of Theorem 9, there is also scattering in H^s-space.*

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