ON PÓLYA FREQUENCY FUNCTIONS IV: THE FUNDAMENTAL SPLINE FUNCTIONS AND THEIR LIMITS(*)

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CONTENTS

Introduction

The present paper was written in 1945 and completed by 1947 (see the abstract [3]) but for no good reason has so far not been published. It appears now in a somewhat revised and improved form.

The paper is divided into two parts. The spline functions for equidistantly spaced knots were introduced in [9]. In Part I we discuss such functions, of

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degree $n-1$, for arbitrarily spaced knots. The possibility of such an extension was already implied by Curry in his review [2] of Schoenberg's paper [9].

The so-called fundamental spline functions are introduced and shown to be bell-shaped (Theorem 1). They are also shown to be the projections onto the x-axis of the volumes of appropriate n-dimensional simplices (Theorem 2). This geometric interpretation allows us to conclude, by means of Brunn's theorem, that the fundamental spline functions are logarithmically concave (Theorem 3). They are also shown to form a basis for all spline functions of degree $n-1$ and given knots (Theorem 4).

The present interest in spline functions is one of the reasons for the publication of this paper at the present time; in fact Theorem 4 is quoted and used in the paper [14] which is about to appear. An extension of the theory to the case of multiple knots (Section 3 and 4) has recently been added. This extension (Theorem 5), in particular Corollary 1 of Section 4, allows us to generalize to the case of multiple nodes the fundamental results of spline interpolation. This generalization of spline interpolation was stated in [14] and was applied to the construction of best quadrature formulae in the recent paper [15].

The fundamental spline functions $M_n(x)$ are frequency functions, which means that they are non-negative and their integral over the reals is unity. In Part II it is shown that the Pólya distribution functions are the only possible limits of a convergent sequence of distribution functions of the form

$$\int_{-\infty}^{x} M_n(t)dt, \quad \text{as } n \to \infty,$$

and conversely (Theorem 6). This was actually the way in which the Pólya frequency functions were originally discovered, although this approach is not mentioned in any of the previous papers of this series ([10], [11], [12]). Theorem 6 adds a fourth to the three previously known characterizations of Pólya frequency functions (See Section 5). Finally, it is shown that the fundamental spline functions themselves converge to the Pólya frequency function of the limit distribution (Theorem 7). The proof of Theorem 7 is shown to depend on a limit theorem (Theorem 8) concerning the class $\mathscr{P}_2$, of logarithmically concave frequency functions. A proof of Theorem 8, together with further extremum properties of the class $\mathscr{P}_2$, will be published elsewhere.

I. ON SPLINE FUNCTIONS

1. The fundamental spline functions $M_n(x)$. Let

$$\cdots < x_{-2} < x_{-1} < x_0 < x_1 < \cdots < x_\nu < \cdots \tag{1.1}$$

be a sequence of reals, such that $x_\nu \to \pm\infty$ as $\nu \to \pm\infty$, and let n be a natural number. By a spline function $S_n(x)$, of order n, or of degree $n-1$, having the knots (1.1), we mean a function of the class $C^{n-2}(-\infty,\infty)$, such that in each interval $(x_\nu, x_{\nu+1})$ it reduces to a polynomial of degree not exceeding $n-1$. $S_n(x)$ will also be referred to as an n-spline.

A remarkable example of an n-spline appears in connection with the integral representation of divided differences. We write

$$x_+^{n-1} = \begin{matrix} x^{n-1} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0, \end{matrix} \tag{1.2}$$

and define

$$M_n(x;y) = n(y-x)_+^{n-1}. \tag{1.3}$$

Finally, we consider the divided difference of order n of the function (1.3) with respect to the variable y and based on the values $y = x_0, x_1, \cdots, x_n$. Using Steffensen's notation, we denote this divided difference by $M_n(x; x_0, x_1, \cdots, x_n)$, which we will often abbreviate to $M_n(x)$ or even to $M(x)$. In terms of $\omega(x) = (x-x_0)\cdots(x-x_n)$, its explicit expression is

$$M_n(x; x_0, \cdots, x_n) = \sum_{\nu=0}^{n} \frac{n(x_\nu - x)_+^{n-1}}{\omega'(x_\nu)}. \tag{1.4}$$

Let us look at this function more closely. By (1.2) and (1.4) it is clear that $M_n(x)$ is an n-spline having the knots $x_0, \cdots, x_n$. Moreover, it evidently vanishes if $x > x_n$. However, it vanishes also if $x < x_0$, for then we may remove in (1.4) the subscript "+" and the sum then vanishes as a divided difference of order n of a polynomial of degree $n-1$.

By a fundamental theorem of Peano (see [4, Chap. III, Sec. 3.7]) applied to divided differences the following holds:

If $f(x) \in C^n$ *then*

$$f(x_0, x_1, \cdots, x_n) = \frac{1}{n!} \int_{x_0}^{x_n} M_n(x; x_0, \cdots, x_n) f^{(n)}(x)dx. \tag{1.5}$$

In particular, if $f(x) = x^n$ we obtain

$$\int_{-\infty}^{\infty} M_n(x; x_0, \cdots, x_n)dx = 1. \tag{1.6}$$

Further properties of $M(x) = M_n(x; x_0, \cdots, x_n)$ are as follows.(1)

Theorem 1. *The* v^{th} *derivative* $M^{(v)}(x)$ $(v = 0, \cdots, n-2)$ *has exactly* v *distinct simple zeros in the interval* (x_0, x_n). *In particular* $M(x) > 0$ *in* (x_0, x_n).

Proof. By (1.4)

$$M(x) = n(x_n - x)^{n-1} / \omega'(x_n) \text{ if } x_{n-1} < x < x_n ,$$

(1) From Lemma 6 (Section 6 below) we easily find (by expanding in powers of $s = ti$) for the mean and the standard deviation of the frequency function $M(x) = M_n(x; x_0, \ldots, x_n)$ the values

$$\mu_1 = \int_{-\infty}^{\infty} xM(x)dx = \frac{1}{n+1} \sum_{0}^{n} x_v,$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu_1)^2 M(x)dx = \frac{1}{(n+1)^2(n+2)} \sum_{i>j} (x_i - x_j)^2.$$

hence $M(x) > 0$ in this interval. We can therefore find three increasing values of x, the first and last being x_0 and x_n, such that the corresponding values of $M(x)$ have the signs $0, +, 0$. By the mean-value theorem we can find four increasing values of x, again x_0, x_n being the extreme ones, such that the corresponding values of $M'(x)$ have the signs 0, $+$, $-$, 0 (one change of sign). Continuing in like manner we finally obtain $n+1$ increasing values of x, such that the corresponding values of $M^{(n-2)}(x)$ have the signs 0, $+$, $-$, $+$, $-$, $\cdots$, 0 ($n-2$ changes of sign). On the other hand, differentiation of (1.4) gives

$$M^{(n-2)}(x) = (-1)^{n-2} \; n! \sum_0^n \frac{(x_\nu - x)_+}{\omega'(x_\nu)}$$

showing that the graph of this function is an ordinary (continuous) polygonal line having vertices at $x = x_0, \cdots, x_n$ and vanishing at the extreme points. As seen above, this graph must exhibit *at least* $n-2$ changes of sign. From this we easily conclude that the elements of the sequence $M^{(n-2)}(x_\nu)$, $(\nu = 1, \cdots, n-1)$ must be all different from zero and alternate in sign. Therefore $M^{(n-2)}(x)$ *has exactly* $n-2$ *simple zeros in* (x_0, x_n). The theorem now follows easily: We have seen above that $M^{(\nu)}$ has at least ν distinct zeros in (x_0, x_n). If it had more, or if some of them were not simple, then Rolle's theorem would imply that $M^{(n-2)}(x)$ has more than $n-2$ zeros, which we have shown not to be the case.

2. A geometric interpretation of $M_n(x)$. According to Hermite and Gennochi (see [8, p.16]) we can write

$$f(x_0, \cdots, x_n) = \int_{\tau_n} \cdots \int f^{(n)}(x_0 t_0 + \cdots + x_n t_n)\, dt_1 \cdots dt_n \tag{2.1}$$

where $t_0 = 1 - t_1 - \cdots - t_n$ and the integration is performed over the simplex

$$\tau_n \colon t_1 \geqq 0, \cdots, t_n \geqq 0, \quad 1 - \sum t_\nu \geqq 0.$$

This leads to the following interpretation of $M_n(x)$ which is hard to forget:

Theorem 2. *The fundamental spline function $M_n(x;x_0,\cdots,x_n)$ is the linear density function obtained by projecting orthogonally onto the x-axis the volume of an n-dimensional simplex σ_n, of volume unity, so located that its $n+1$ vertices project orthogonally into the points $x_0,x_1,\cdots,x_n$ of the x-axis, respectively.*

Proof: Let R^n be the space of points $X=(x,x',\cdots,x^{(n-1)})$ which is Cartesian and referred to orthogonal axes obtained by adjoining to the x-axis $n-1$ additional axes for the variables $x',\cdots,x^{(n-1)}$. Denoting by $\Phi(x)$ $(x_0 \leqq x \leqq x_n)$ the linear density described in the theorem we obtain, by Cavalieri's principle, the relation

$$\int_{\sigma_n}\cdots\int f^{(n)}(x)dxdx'\cdots dx^{(n-1)} = \int_{x_0}^{x_n} \Phi(x)f^{(n)}(x)dx. \tag{2.2}$$

Let $X_0,X_1,\cdots,X_n$ be the vertices of σ_n having coordinates given by

$$X_\nu = (x_\nu,x'_\nu,\cdots,x_\nu^{(n-1)}), \quad (\nu = 0,\cdots,n).$$

In the n-fold integral (2.2) we change to the new variables of integration $t_1,\cdots,t_n$ defined by the relations

$$\begin{aligned} x &= x_0(1-\textstyle\sum t_\nu) + x_1t_1 + \cdots + x_nt_n \\ &\vdots \\ x^{(n-1)} &= x_0^{(n-1)}(1-\textstyle\sum t_\nu) + x_1^{(n-1)}t_1 + \cdots + x_n^{(n-1)}t_n. \end{aligned} \tag{2.3}$$

The absolute value of the Jacobian is found to be

$$\left|\frac{\partial(x,\cdots,x^{(n-1)})}{\partial(t_1,\cdots,t_n)}\right| = \text{abs. val.} \begin{vmatrix} x_1-x_0, x_2-x_0,\cdots,x_n-x_0 \\ x'_1-x'_0,\cdots, \quad x'_n-x'_0 \\ \vdots \\ x_1^{(n-1)}-x_0^{(n-1)},\cdots,x_n^{(n-1)}-x_0^{(n-1)} \end{vmatrix}$$

$$= n!\,\mathrm{vol}\,\sigma_n = n!,$$

In view of the first relation (2.3), our integral (2.2) is transformed into the integral on the right side of (2.1) multiplied by $n!$. Thus (2.2) yields the relation

$$\int_{x_0}^{x_n} \Phi(x) f^{(n)}(x)\, dx = f(x_0, x_1, \cdots, x_n) \cdot n!$$

which is valid for all $f \in C^n$. Using the relation (1.5) and the continuity of $\Phi(x)$, we conclude that $\Phi(x) = M_n(x)$, which proves our theorem.

The above discussion implies that $M_n(x) = M_n(x; x_0, \cdots, x_n)$ is equal to the $(n-1)$-dimensional volume of the intersection of the simplex σ_n with the hyperplane orthogonal to the x-axis at the point x $(x_0 \leqq x \leqq x_n)$. Since σ_n is a convex body we may now use the famous theorem of H. Brunn (see [1, p. 71]) according to which

$(M_n(x))^{1/(n-1)}$ *is a concave function of* x *in* $x_0 \leqq x \leqq x_n$.

Writing

$$g(x) = (M_n(x))^{1/(n-1)} \tag{2.4}$$

we conclude that

$$g(x) > 0, \quad g''(x) \leqq 0 \quad \text{if } x_0 < x < x_n. \tag{2.5}$$

For the function

$$h(x) = \log g(x) = \frac{1}{n-1} \log M_n(x) \tag{2.6}$$

we now obtain

$$h'(x) = g'(x)/g(x), \quad h''(x) = (g''(x)g(x) - (g'(x))^2)/(g(x))^2 \leqq 0, \tag{2.7}$$

where the last inequality follows from (2.5). In view of (2.6) we have just established the following

Theorem 3. *The function*

$$\log M_n(x; x_0, \cdots, x_n) \tag{2.8}$$

is concave in the open interval (x_0, x_n).

It is, of course, desirable to give an analytic proof of Theorem 3 which does not use the geometry of convex domains and Brunn's theorem. This can be done by means of the variation diminishing properties of the class $\mathscr{P}_2$ of Section 8 and will be done in a future paper which will also contain a proof of Theorem 8 below.

3. Spline functions with multiple knots. It is easy and important to extend the above theory to the case when some of the knots (1.1), which so far were assumed distinct, become coalescent. The meaning of multiple knots, a priori devoid of sense, must be defined as follows.

Definition 1. *In* (1.1) *we assume that*

$$x_{-1} < x_0 = x_1 = \cdots = x_{r-1} < x_r, \tag{3.1}$$

while the spline function $S_n(x)$, *of Section* 1, *satisfies the condition*

$$S_n(x) \in C^{n-1-r}(x_{-1}, x_r). \tag{3.2}$$

We then say that $x = x_0$ *is an r-fold knot of* $S_n(x)$, *or a knot of multiplicity* r.

In particular, the case when $r = n$ *means that there are no continuity requirements whatever, at* $x = x_0$, *between the two polynomials defining* $S_n(x)$ *in the adjacent intervals* (x_{-1}, x_0) *and* (x_0, x_r). *The multiplicity* r *of a knot is therefore restricted by the inequality*

$$r \leqq n, \tag{3.3}$$

which we assume throughout this paper.

We now generalize our class of spline functions $S_n(x)$ by allowing in (1.1) any number of equality signs subject to the condition

$$x_\nu < x_{\nu+n} \text{ for all } \nu. \tag{3.4}$$

Most of our previous discussion remains valid: The formulae (1.4) and (1.6) still hold, but the right side of (1.4) must be replaced by the expression for divided differences for multiple arguments. As an illustration, we mention the extreme case when $x_0 = x_1 = \cdots = x_{n-1} < x_n = \xi$. From Newton's general interpolation formula we obtain

$$f(\xi) = f(x_0) + (\xi - x_0)f'(x_0) + \cdots + \frac{(\xi - x_0)^{n-1}}{(n-1)!} f^{(n-1)}(x_0) + (\xi - x_0)^n f(x_0, \cdots, x_0, \xi).$$

Equating its remainder with its familiar integral form we obtain

$$(\xi - x_0)^n f(x_0, \cdots, x_0, \xi) = \frac{1}{(n-1)!} \int_{x_0}^{\xi} (\xi - x)^{n-1} f^{(n)}(x)\, dx.$$

Comparing with (1.5) we obtain

$$M_n(x; x_0, \cdots, x_0, \xi) = \begin{cases} \dfrac{n(\xi - x)^{n-1}}{(\xi - x_0)^n} & \text{in } (x_0, \xi) \\ 0 & \text{outside } (x_0, \xi). \end{cases} \tag{3.5}$$

$M(x)$ is discontinuous at $x = x_0$ and Theorem 2, which remains valid, shows that the entire base of the simplex σ_n projects into x_0 and the opposite vertex into ξ. Theorem 1 needs modification; it can be shown that its conclusions concerning the zeros of $M^{(\nu)}(x)$ remain valid as long as these derivatives are continuous.

4. The fundamental n-splines form a basis. We return for the moment to the knots (1.1) and the n-splines which they define. The fundamental n-splines having consecutive knots $x_\nu, x_{\nu+1}, \cdots, x_{\nu+n}$ will now be denoted by

$$M_\nu(x) = M_n(x; x_\nu, x_{\nu+1}, \cdots, x_{\nu+n}). \tag{4.1}$$

Their interest is due to the following

Theorem 4. 1. *If* $0 < N < n$ *and if* $S(x)$ *is an* n*-spline having the knots* (1.1) *and such that*

$$S(x) = 0 \text{ everywhere outside the interval } (x_0, x_N), \tag{4.2}$$

then

$$S(x) = 0 \quad \text{for all } x. \tag{4.3}$$

2. *If* $N \geqq n$ *and* (4.2) *holds, then* $S(x)$ *can be uniquely represented in the form*

$$S(x) = \sum_{0}^{N-n} c_\nu M_\nu(x). \tag{4.4}$$

3. *An* n*-spline* $S(x)$ *vanishing if* $x < x_0$ *can be uniquely represented in the form*

$$S(x) = \sum_{0}^{\infty} c_\nu M_\nu(x). \tag{4.5}$$

4. *Every* n*-spline* $S(x)$ *can be uniquely represented in the form*

$$S(x) = \sum_{\nu=-\infty}^{\infty} c_\nu M_\nu(x), \tag{4.6}$$

where the c_ν *are constants, and conversely, any such series represents an* n*-spline.*

We shall not establish this theorem as it stands, but rather its generalization, Theorem 5 below, to the case of multiple knots. For this purpose we must now allow also equality relations between consecutive knots in (1.1), subject to the restriction (3.4).

We shall assume that the knots (1.1) are located at the distinct points

$$\cdots < y_{-1} < y_0 < y_1 < y_2 < \cdots \quad (y_n \to \pm\infty \text{ as } n \to \pm\infty), \tag{4.7}$$

where

$$y_i \text{ is a knot of multiplicity } \alpha_i, \ (\alpha_i \leqq n). \tag{4.8}$$

Our discussion requires a kind of "double book-keeping" where multiple knots (4.7) are also represented by the non-decreasing sequence $\{x_\nu\}$ with the correct multiplicities. To fix the ideas, we assume this representation such that

$$\cdots, x_0 = y_0, x_1 = x_2 = \cdots = x_{\alpha_1} = y_1, x_{\alpha_1+1} = y_2, \cdots.$$

Thus in terms of the original sequence (1.1) the knots are

$$(4.9)\qquad \cdots < x_{-\alpha_0+1} = \cdots = x_0 < x_1 = \cdots = x_{\alpha_1} < x_{\alpha_1+1} = \cdots = x_{\alpha_1+\alpha_2} < \cdots .$$

Again, as in (4.1), we associate with the knots (4.9) the sequence of fundamental n-splines

$$(4.10)\qquad M_j(x) = M_n(x; x_j, x_{j+1}, \cdots, x_{j+n}), \quad (-\infty < j < \infty).$$

The generalization of Theorem 4 is as follows.

Theorem 5. *Let $S(x)$ be an n-spline having the knots* (4.7), *of multiplicities as described by* (4.8).

1. *If*

$$(4.11)\qquad s = \sum_1^N \alpha_\nu \leqq n$$

and

$$(4.12)\qquad S(x) = 0 \textit{ everywhere outside the interval } (y_1, y_N),$$

then

$$(4.13)\qquad S(x) = 0 \textit{ for all } x.$$

2. *If*

$$(4.14)\qquad s = \sum_1^N \alpha_\nu \geqq n + 1$$

and (4.12) *holds, then $S(x)$ can be uniquely represented in the form*

(4.15) $$S(x) = \sum_{1}^{s-n} c_j M_j(x).$$

3. *If* $S(x) = 0$ *whenever* $x < y_1$, *then* $S(x)$ *can be uniquely represented in the form*

(4.16) $$S(x) = \sum_{1}^{\infty} c_j M_j(x).$$

4. *Every* $S(x)$ *can be uniquely represented in the form*

(4.17) $$S(x) = \sum_{-\infty}^{\infty} c_j M_j(x).$$

Proof: The four parts of the theorem are logically related in the order in which they are stated.

1. Let us construct an $S(x)$ such that (4.12) holds. By the continuity requirement (3.2), which defines the meaning of the multiplicity α_1 of the knot y_1, we conclude that

$$S(x) = a_1^1(x - y_1)^{n-1} + a_2^1(x - y_1)^{n-2} + \cdots + a_{\alpha_1}^1(x - y_1)^{n-\alpha_1} \text{ in } y_1 < x < y_2.$$

Proceeding in like fashion from each (y_i, y_{i+1}) to the next, new terms are added, until we arrive at the expression

$$S(x) = \sum_{i=1}^{\alpha_1} a_i^1(x - y_1)^{n-i} + \cdots + \sum_{i=1}^{\alpha_N} a_i^N(x - y_N)^{n-i} \quad \text{if } x > y_N.$$

By (4.12) $S(x)$ is supposed to vanish in this range, whence the identity

(4.18) $$\sum_{i=1}^{\alpha_1} a_i^1(x - y_1)^{n-i} + \cdots + \sum_{i=1}^{\alpha_N} a_i^N(x - y_N)^{n-i} = 0 \quad \text{for all } x.$$

Conversely, an identity (4.18) implies an $S(x)$ satisfying all conditions and defined by

(4.19) $$S(x) = \sum_{i=1}^{\alpha_1} a_i^1(x - y_1)_+^{n-i} + \cdots + \sum_{i=1}^{\alpha_N} a_i^N(x - y_N)_+^{n-i}, \quad (-\infty < x < \infty).$$

Let us assume for the moment a little more than (4.11), namely

$$s = \sum_{1}^{N} \alpha_\nu = n \tag{4.20}$$

and let us show that the identity (4.18) implies the vanishing of all coefficients a_i^ν. In other words, we have to show that (4.20) implies the linear independence of the n polynomials

$$\begin{array}{l} \frac{(x-y_1)^{n-1}}{(n-1)!}, \frac{(x-y_1)^{n-2}}{(n-2)!}, \cdots, \frac{(x-y_1)^{n-\alpha_1}}{(n-\alpha_1)!}, \\ \vdots \\ \frac{(x-y_N)^{n-1}}{(n-1)!}, \frac{(x-y_N)^{n-2}}{(n-2)!}, \cdots, \frac{(x-y_N)^{n-\alpha_N}}{(n-\alpha_N)!}. \end{array} \tag{4.21}$$

To decide this point we expand them in increasing powers of x and find that the $n \times n$ determinant of their coefficients is (up to a positive numerical factor) equal to the determinant $\mathscr{D}$ having N groups of α_i rows each, the i^{th} group being described by the formula

$$\mathscr{D} = \left| \begin{array}{llll} \frac{y_i^{n-1}}{(n-1)!}, & \frac{y_i^{n-2}}{(n-2)!}, & \cdots, & 1 \\ \frac{y_i^{n-2}}{(n-2)!}, & \frac{y_i^{n-3}}{(n-3)!}, & \cdots, & 0 \\ \vdots & & & \\ \frac{y_i^{n-\alpha_i}}{(n-\alpha_i)!}, & \frac{y_i^{n-\alpha_i-1}}{(n-\alpha_i-1)!}, & \cdots, & 0 \end{array} \right| \quad i = 1, 2, \cdots, N. \tag{4.22}$$

This is a generalization of the Vandermonde determinant, to which it reduces if all the α_i are equal to unity. That $\mathscr{D} \neq 0$ will now follow from the unicity of the solution

$$P(x) = A_0 \frac{x^{n-1}}{(n-1)!} + A_1 \frac{x^{n-2}}{(n-2)!} + \cdots + A_{n-1} \tag{4.23}$$

of Hermite's interpolation problem ([5, p. 432])

$$P(y_i) = f(y_i),\ P'(y_i) = f'(y_i), \cdots, P^{(\alpha_i - 1)}(y_i) = f^{(\alpha_i - 1)}(y_i) \tag{4.24}$$

$$(i = 1, 2, \cdots, N;\ \alpha_i + \cdots + \alpha_N = n).$$

Indeed, the relation $\mathscr{D} = 0$ would imply the existence of a polynomial (4.23), $P(x) \neq 0$, having an α_i-fold zero at $x = y_i$ $(i = 1, \cdots, N)$, hence a total of $\sum \alpha_i = n$ zeros whose number exceeds its degree. Our assertion is therefore established if (4.20) holds.

We now drop the assumption (4.20) and assume instead that (4.11) holds with the inequality sign. Given $S(x)$ satisfying (4.12), we then add beyond y_N additional fictitious simple knots $y'_{n+1}, y'_{n+2}, \cdots$ in sufficient number to produce the previous case of the equality (4.20). By the previous case we now know that $S(x)$ must vanish identically.

2. We need the following

Lemma 1. *If*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_N = n + 1 \tag{4.25}$$

then the fundamental spline function

$$M_1(x) = M_n(x; \overbrace{y_1, \cdots, y_1}^{\alpha_1}, \cdots, \overbrace{y_N, \cdots, y_N}^{\alpha_N}) \tag{4.26}$$

is precisely of the continuity class $C^{n-\alpha_i-1}$ in the neighborhood of the point $x = y_i$ $(i = 1, \cdots, N)$. "Precisely" means that it belongs to no higher class.

Proof: The construction of $M_1(x)$ is a special case of our problem when we have equality in (4.14), hence (4.25). By (4.19) it appears that $M_1(x)$ must be of the form

$$\begin{aligned} M_1(x) = {} & a_1^1(x - y_1)_+^{n-1} + \cdots + a_{\alpha_1}^1(x - y_1)_+^{n-\alpha_1} \\ & \vdots \\ & a_1^N(x - y_N)_+^{n-1} + \cdots + a_{\alpha_N}^N(x - y_N)_+^{n-\alpha_N}, \quad (-\infty < x < \infty), \end{aligned} \tag{4.27}$$

which expression should vanish identically on removing everywhere the subscript "+". The resulting identity (4.18) now expresses the linear *dependence* of the functions (4.21), and we wish to show that (up to a constant factor) only one such relation exists. This much is clear: In (4.27), or (4.18) we must have

$$(4.28) \qquad a^{\nu}_{\alpha_\nu} \neq 0 \text{ for } \nu = 1, \cdots, N,$$

for if one were to vanish, $a^1_{\alpha_1} = 0$ say, then we could diminish the multiplicity α_1 (of the knot y_1) by one unit and thereby fall back on the case (4.20) when we know that only trivial identities (4.18) are possible. Using (4.28) the same kind of reasoning shows that, up to factors, only one relation (4.18) exists.

Finally (4.27) and (4.28) show that

$$M_1^{(n-\alpha_\nu)}(y_\nu + 0) - M_1^{(n-\alpha_\nu)}(y_\nu - 0) = (n - \alpha_\nu)! a^{\nu}_{\alpha_\nu} \neq 0 \ (\nu = 1, \cdots, N)$$

which proves our lemma.

A proof of (4.15) under the assumptions (4.12) and (4.14), is now straightforward. Indeed, observe that by (4.12) and (4.8)

$$S(x) = (x - y_1)^{n-\alpha_1} \cdot \phi(x), \ (\phi \text{ is a polynomial}),$$

in a right-neighborhood of y_1. By Lemma 1 also

$$M_1(x) = c(x - y_1)^{n-\alpha_1} + \text{higher powers of } x - y_1, \ (c \neq 0),$$

in a right-neighborhood of y_1. We can therefore determine c_1 uniquely so that

$$S_1(x) = S(x) - c_1 M_1(x) = (x - y_1)^{n-\alpha_1+1} \phi_1(x), (y_1 < x < y_2).$$

However, again by Lemma 1,

$$M_2(x) = c'(x - y_1)^{n-\alpha_1+1} + \cdots \ (y_1 < x < y_2)$$

and a unique c_2 will produce

$$S_2(x) = S(x) - c_1 M_1(x) - c_2 M_2(x) = (x - y_1)^{n-\alpha_1+2} \phi_2(x), \quad (y_1 < x < y_2),$$

and so on.

Two cases are possible:

1. $\alpha_2 + \alpha_3 + \cdots + \alpha_N \leqq n$. The construction may continue until we obtain

(4.29) $S_t(x) = S(x) - c_1 M_1(x) - \cdots - c_t M_t(x) = (x - y_1)^{n-\alpha_1+t}\phi_t(x), \quad (y_1 < x < y_2),$

where $t = \alpha_1 + \alpha_2 + \cdots + \alpha_N - n \leqq \alpha_1$. However, $S_t(x)$ is a spline function which may be thought of, by (4.29), as having at $x = y_1$ a knot of multiplicity $\alpha_1 - t$. But then the sum of all multiplicities of knots of $S_t(x)$ is

$$(\alpha_1 - t) + \alpha_2 + \cdots + \alpha_N = n$$

and by Case 1 of Theorem 5 we conclude that $S_t(x) = 0$ for all x, while $t = s - n$.

2. $\alpha_2 + \alpha_3 + \cdots + \alpha_N > n$. In this case we may surely proceed until we reach

$$S_{\alpha_1-1}(x) = S(x) - c_1 M_1(x) - \cdots - c_{\alpha_1-1} M_{\alpha_1-1}(x) = (x - y_1)^{n-1}\phi(x),$$

in $y_1 < y < y_2$, where $\phi(x)$ is necessarily a constant. But then, with an appropriate unique c_{α_1}

$$S_{\alpha_1}(x) = S_{\alpha_1-1}(x) - c_{\alpha_1} M_{\alpha_1}(x) = 0 \quad \text{if } x < y_2 .$$

From this point we continue with

$$M_{\alpha_1+1}(x) = M_n(x; \overbrace{y_2, \cdots, y_2}^{\alpha_2}, y_3, \cdots)$$

using right-neighborhoods of y_2. We can surely continue this successive subtraction process until we reach the spline function

$$S_{s-n}(x) = S(x) - \sum_{1}^{s-n} c_j M_j(x)$$

and we wish to show that it vanishes for all values of x.

In order to describe and justify the termination of this process, let the integers r, α and β be defined by the relations

$$\alpha_{r+1} + \cdots + \alpha_N < n + 1 \leqq \alpha_r + \alpha_{r+1} + \cdots + \alpha_N, \quad (2 \leqq r < N),$$

$$\alpha + \alpha_{r+1} + \cdots + \alpha_N = n + 1, \quad (0 < \alpha \leqq n).$$

$$\alpha + \beta = \alpha_r.$$

The functions $S(x)$, $S_1(x), \cdots, S_{\alpha_1 - 1}(x)$ are near the point $x = y_1$ of the continuity classes $C^{n-\alpha_1-1}$, $C^{n-\alpha_1}, \cdots, C^{n-2}$, respectively. Likewise $S_{\alpha_1}(x), \cdots$, $S_{\alpha_1+\alpha_2-1}(x)$ vanish for $x < y_2$ and are near $x = y_2$ of the continuity classes $C^{n-\alpha_2-1}, C^{n-\alpha_2}, \cdots, C^{n-2}$ respectively. Finally

$$S_{\alpha_1+\cdots+\alpha_{r-1}}(x),\ S_{\alpha_1+\cdots+\alpha_{r-1}+1}(x), \cdots, S_{\alpha_1+\cdots+\alpha_{r-1}+\beta+1}(x)$$

vanish for $x < y_r$ and are near $x = y_r$ of the continuity classes

$$C^{n-\alpha_r-1}, C^{n-\alpha_r}, \cdots, C^{n-\alpha}, \text{ respectively.}$$

Observe now that by our definition of r, α and β we have

$$\alpha_1 + \cdots + \alpha_{r-1} + \beta + 1 = s - n.$$

Also that $S_{s-n}(x)$ is a spline function which may be thought of as having at $x = y_r$ a knot of multiplicity $\alpha - 1$. Thus the total sum of the multiplicities of its knots is $(\alpha - 1) + \alpha_{r+1} + \cdots + \alpha_N = n$. As $S_{s-n}(x)$ vanishes outside (y_r, y_N), we may apply the result of Case 1 and conclude that $S_{s-n}(x) = 0$ for all real x. This settles the Case 2 of Theorem 5.

3. The previous construction applies with the difference that it continues indefinitely resulting in (4.16). Clearly Case 2 is a special case of this when the c_j vanish for sufficiently large j.

4. To deal with this last case we need

Lemma 2. *The fundamental functions*

$$M_{-n+1}(x), M_{-n+2}(x), \cdots, M_0(x) \tag{4.30}$$

are linearly independent in the interval $(x_0, x_1) = (y_0, y_1)$ *and therefore form in this interval a basis for* π_{n-1}.[2]

[2] π_{n-1} denotes the class of polynomials of degree not exceeding $n - 1$.

Proof: Let us assume that

$$S(x) \equiv \sum_{-n+1}^{0} c_j M_j(x) = 0 \text{ in } (y_0, y_1) \tag{4.31}$$

and let us show that this implies that

$$c_j = 0 \ (j = -n+1, \cdots, 0). \tag{4.32}$$

Consider the restriction of $S(x)$ to the interval $(y_0, +\infty)$. By (4.31) this restriction is a spline function vanishing everywhere outside the interval (x_1, x_n) provided that we define it as $= 0$ if $x \leqq y_0$. It may therefore be thought of as an n-spline having at most n knots, namely $x_1, x_2, \cdots, x_n$; it must therefore vanish everywhere by Case 1 of Theorem 5. Thus (4.31) implies that $S(x) = 0$ in (y_0, ∞). Similarly it is shown that $S(x) = 0$ in the interval $(-\infty, y_1)$. Thus $S(x) = 0$ for all x and now (4.32) follows from the unicity in Case 2 of our theorem. This completes a proof of Lemma 2.

Let us finally consider an arbitrary n-spline $S(x)$. Let

$$S(x) = P(x) \text{ if } y_0 < x < y_1, \ (P \in \pi_{n-1}).$$

By Lemma 2 we can write

$$P(x) = \sum_{-n+1}^{0} c_j M_j(x)$$

which implies that

$$S^*(x) = S(x) - \sum_{-n+1}^{0} c_j M_j(x) \tag{4.33}$$

is an n-spline vanishing in the interval $(y_0, y_1) = (x_0, x_1)$. We may therefore write

$$S^*(x) = S_1(x) + S_0(x), \tag{4.34}$$

where S_1 and S_0 are n-splines vanishing in the intervals $(-\infty, x_1)$ and $(x_0, +\infty)$ respectively. By Case 3 of our theorem we may therefore write uniquely

$$S_1(x) = \sum_1^{\infty} c_j M_j(x), \quad S_0(x) = \sum_{-\infty}^{-n} c_j M_j(x).$$

Now (4.33) and (4.34) imply the desired representation (4.17). This completes a proof of Theorem 5.

In view of its relevance to spline interpolation we restate here the Case 2 of Theorem 5 as follows.

Corollary 1. *Let* $x_1, x_2, \cdots, x_s$, $(s > n)$, *be non-decreasing reals such that*

$$x_\nu < x_{\nu+n} \quad (1 \leqq \nu < \nu + n \leqq s).$$

This implies that at most n of the x_ν *may coalesce at a point. Then the fundamental spline functions*

$$M_j(x) = M_n(x; x_j, x_{j+1}, \cdots, x_{j+n}), \quad (j = 1, 2, \cdots, s-n), \tag{4.35}$$

are linearly independent. Every spline function $S(x)$, *of degree* $n-1$, *having the points* $x_1, \cdots, x_s$ *as knots, with multiplicities as they occur in this sequence, and such that*

$$S(x) = 0 \text{ everywhere outside } (x_1, x_s), \tag{4.36}$$

may be uniquely represented in the form

$$S(x) = \sum_1^{s-n} c_j M_j(x). \tag{4.37}$$

By means of Corollary 1 the proofs of the main results concerning spline interpolation as given in [14], extend easily to the case of interpolation with *multiple nodes*. By a *node* of an interpolation problem we mean a point where a functional value is given; a node is *multiple* if an appropriate number of consecutive derivatives of a function are also to be interpolated at that point.

Fig. 1 shows a sketch of the five fundamental quadratic spline functions for the case when $n = 3$, $s = 9$ and

$$x_1 = x_2 = 0,\ x_3 = 1,\ x_4 = 2,\ x_5 = x_6 = 3,\ x_7 = x_8 = x_9 = 4.$$

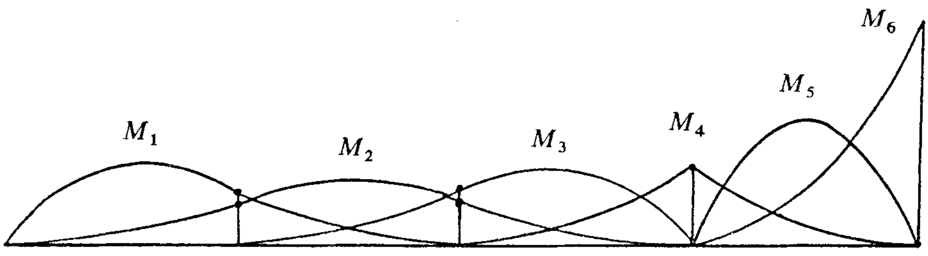

Figure 1

In terms of the expression (4.10) we may write

$$\begin{aligned} M_1(x) &= M_3(x;0,0,1,2) \\ M_2(x) &= M_3(x;0,1,2,3) \\ M_3(x) &= M_3(x;1,2,3,3) \\ M_4(x) &= M_3(x;2,3,3,4) \\ M_5(x) &= M_3(x;3,3,4,4) \\ M_6(x) &= M_3(x;3,4,4,4). \end{aligned}$$

All arcs between consecutive integral values of x are parabolic and are easily found explicitly. Fig. 1 illustrates nicely Lemma 1. By Theorem 5 we know that

$$S(x) = \sum_{1}^{6} c_j M_j(x)$$

represents the most general quadratic spline function in the interval $[0,4]$ satisfying the following conditions

1. It has the knots 1, 2, 3.

2. It belongs to the classes C^1, C^1, C^0 near the points $x = 1,2,3$, respectively.

3. $S(0) = 0$.

As illustrations of Cases 3 and 4 of Theorem 5 we mention the following identities. The knots being integral ($x_i = i$) and writing

$$M_j(x) = M_m(x;j,j+1,\cdots,j+m)$$

we find that

$$x_+^{m-1} = \sum_{j=0}^{\infty} (j+1)(j+2)\cdots(j+m-1)\,M_j(x),$$

$$x^{m-1} = \sum_{j=-\infty}^{\infty} (j+1)(j+2)\cdots(j+m-1)\,M_j(x).$$

II. THE LIMITS OF FUNDAMENTAL SPLINE FUNCTIONS

5. The problem and main results. Interesting results in Analysis are sometimes of the following nature: We are given a certain class $\{F\}$ of functions F and we are able to describe independently, in terms of structural properties, the *closure* of the class $\{F\}$ in terms of a suitable topology. A classical result of this nature is the following theorem:

Let

$$\left\{P_n(s) = \prod_{\nu=1}^{n} (1 + \alpha_{n,\nu} s)\right\} \tag{5.1}$$

be the class of real polynomials having only real zeros, so normalized that $P_n(0) = 1$. The closure of this class with respect to uniform convergence in every finite domain of the complex s-plane is identical with the Laguerre-Pólya class of entire functions described by

$$\Psi(s) = e^{-\gamma s^2 + \delta s} \prod_{1}^{\infty} (1 + \delta_\nu s) e^{-\delta_\nu s} \tag{5.2}$$

$$(\gamma \geqq 0,\ \delta,\ \delta_\nu \text{ real},\ \textstyle\sum \delta_\nu^2 < \infty),$$

(see [6, Chap. III, Section 3] where also original references are given).

By a distribution function we mean a monotone function $F(x)$ such that $F(-\infty) = 0$, $F(+\infty) = 1$. $F_n(x)$ and $F(x)$ being distribution functions, we say that $F_n(x)$ converges to $F(x)$ and write $F_n(x) \to F(x)$, provided that the relation holds for all x at which $F(x)$ is continuous. Our problem, closely related with the class of functions (5.2), is as follows. We consider the class of spline distribution functions (see (1.6))

$$F_n(x) = \int_{-\infty}^{x} M_n(x; x_{0,n}, x_{1,n}, \cdots, x_{n,n})\, dx, \tag{5.3}$$

the integrand being a fundamental n-spline (Sections 1 and 4). Concerning its knots we make the widest possible assumption, namely

$$x_{0,n} \leqq x_{1,n} \leqq \cdots \leqq x_{n,n},\ x_{0,n} < x_{n,n}. \tag{5.4}$$

We ask: *Assuming*

$$\lim_{n \to \infty} F_n(x) = F(x), \tag{5.5}$$

what is the nature of the limit distribution functions $F(x)$?

An answer to this question requires the class of Pólya distribution functions $F(x)$. These are distribution functions having a bilateral Laplace transform of the form

$$\int_{-\infty}^{\infty} e^{-sx} dF(x) = \frac{1}{\Psi(s)}, \tag{5.6}$$

where $\Psi(s)$ is described by (5.2). If $\Psi(s) = e^{\delta s}$, then $dF(x)$ has its entire unit mass located at $x = \delta$. If

$$\Psi(s) \neq e^{\delta s} \tag{5.7}$$

then

$$\int_{-\infty}^{\infty} e^{-sx} \Lambda(x) dx = \frac{1}{\Psi(s)}, \tag{5.8}$$

where $\Lambda(x)$ is a Pólya frequency function. The Pólya frequency functions may also be characterized by the structural property of being totally positive (see [10], [6]).

Theorem 6. *The limit $F(x)$ of a convergent sequence of spline distribution functions $F_n(x)$ is a Pólya distribution function. Conversely, every Pólya distribution function $F(x)$ is the limit of an appropriate sequence of spline distribution functions $F_n(x)$, of the form* (5.3), *as $n \to \infty$.*

The question arises as to when the limit relation (5.5) may be differentiated, so that we may infer that the fundamental spline functions M_n themselves converge. The answer will depend on the continuity of $\Lambda(x)$. Writing

$$\lambda(x) = \begin{matrix} e^{-x} & \text{if } x \geqq 0 \\ 0 & \text{if } x < 0 \end{matrix}$$

we know (see [10]) that the only discontinuous $\Lambda(x)$ is the one-sided exponential

$$\Lambda_0(x) = \frac{1}{|\delta_1|}\lambda\left(\frac{x-\delta+\delta_1}{\delta_1}\right) \quad (\delta_1 \neq 0) \tag{5.9}$$

having the transform

$$\int_{-\infty}^{\infty} e^{-sx}\Lambda_0(x)dx = 1/\{e^{\delta s}(1+\delta_1 s)e^{-\delta_1 s}\} \tag{5.10}$$

Theorem 7. *Let* (5.5) *and* (5.7) *hold, hence $dF(x) = \Lambda(x)dx$, where $\Lambda(x)$ is a Pólya frequency function. Then*

$$\lim_{n\to\infty} M_n(x; x_{0,n}, \cdots, x_{n,n}) = \Lambda(x), \tag{5.11}$$

uniformly for all x, provided that $\Lambda(x)$ is not of the form (5.9). *For $\Lambda(x) = \Lambda_0(x)$, again* (5.11) *holds uniformly for x outside an arbitrarily small neighborhood of the point $x = \delta - \delta_1$, where $\Lambda_0(x)$ is discontinuous.*

If $\Psi(s) = e^{\delta s}$ then

(5.12) $$\lim_{n\to\infty} M_n(x; x_{0,n}, \cdots, x_{n,n}) = 0$$

uniformly in x outside an arbitrarily small neighborhood of the point $x = \delta$.

Theorem 7 describes a new characterization of the Pólya frequency functions. It adds a fourth to the three previously known characterizations: 1. The special nature (5.8), (5.2), of their Laplace transforms. 2. Their totally positive character ([10]). 3. Their variation diminishing property ([11]).

6. A few auxiliary propositions.(3) Since Paul Levy's work of the 1920's, the characteristic functions are usually the most convenient tools in dealing with convergence problems for distribution functions. Here we also use some related but different integrals. Their use will be justified by two propositions. Below, $F_n(x)$ and $F(x)$ are arbitrary distribution functions.

Lemma 3. *If*

(6.1) $$F_n(x) \to F(x)$$

then the functions

(6.2) $$g_n(t) = \int_{-\infty}^{\infty} \left(1 + \frac{itx}{n}\right)^{-n} dF_n(x)$$

have the property

(6.3) $$\lim_{n\to\infty} g_n(t) = \int_{-\infty}^{\infty} e^{-itx} dF(x) \text{ locally uniformly.}$$

Lemma 4. *Let* $\{F_n(x)\}$ *be a sequence of distribution functions which are equi-continuous at infinity. By this we mean that for every positive* ε *there corresponds an* $A = A(\varepsilon)$ *such that*

(6.4) $$\int_{|x| \geq A} dF_n(x) < \varepsilon \text{ for all } n.$$

(3) In connection with Lemma 4 we acknowledge helpful conversations with Professor Joshua Chover.

Then the pointwise convergence of the sequence of functions (6.2), *for all real* t, *implies the convergence relation* (6.1) *and therefore the relation* (6.3) *holds by Lemma* 3.

Proof of Lemma 3: The assumption (6.1) implies that

$$(6.5)\qquad f_n(t) = \int_{-\infty}^{\infty} e^{-itx} dF_n(x) \to \int_{-\infty}^{\infty} e^{-itx} dF(x) \text{ locally uniformly.}$$

On the other hand

$$(6.6)\ |f_n(t) - g_n(t)| \leqq \int_{-A}^{A} \left| e^{-itx} - \left(1 + \frac{itx}{n}\right)^{-n} \right| dF_n(x) + 2(1 - F_n(A)) + 2F_n(-A).$$

Let ε and T be positive and let us choose A such that A and $-A$ are continuity points of $F(x)$ and such that

$$(6.7)\qquad 2(1 - F(A)) < \varepsilon, \quad 2F(-A) < \varepsilon.$$

Furthermore, let $N = N(\varepsilon, TA)$ be such that

$$(6.8)\qquad \left| e^{-z} - \left(1 + \frac{z}{n}\right)^{-n} \right| < \varepsilon \text{ for } |z| \leqq TA, \text{ if } n > N.$$

By (6.6), (6.7), (6.8) and $F_n(\pm A) \to F(\pm A)$ we conclude that

$$|f_n(t) - g_n(t)| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \text{ if } -T \leqq t \leqq T,$$

provided that n is sufficiently large. Now (6.5) implies (6.3) and Lemma 3 is established.

Proof of Lemma 4. The result will follow from the fact that a distribution function is uniquely defined by its characteristic function. Indeed, applying Helly's selection principle, we obtain an infinite sequence N of increasing natural numbers such that $F_n(x)$ converges pointwise for all real x, as $n \to \infty$, $n \in N$. Thus

$$(6.9)\qquad \lim F_n(x) = F(x), \text{ for all real } x, \text{ as } n \to \infty,\ n \in N.$$

$F(x)$ is not yet a distribution function as we may well have $F(\infty) - F(-\infty) < 1$. However, (6.9) and the equi-continuity assumption (6.4) easily imply that $F(-\infty) = 0$, $F(\infty) = 1$, by an argument which we may omit. Thus $F(x)$ is a distribution function and (6.1) holds if we restrict n to the sequence N. However, Lemma 3 is applicable and shows that

$$\lim g_n(t) = \int_{-\infty}^{\infty} e^{-itx} dF(x) \text{ as } n \to \infty,\ n \in N. \tag{6.10}$$

Since the entire sequence $\{g_n(t)\}$ converges pointwise, by assumption, it is clear that (6.10) holds for every t as $n \to \infty$ through all integers. The limit distribution $F(x)$ being *uniquely* defined by its characteristic function (6.10) we may have essentially only one such function, hence (6.1) holds. Finally, the relation (6.3) is implied by Lemma 3.

Lemma 5. *Let*

$$P_n(s) = \prod_{\nu=1}^{n} (1 + \alpha_{n,\nu} s)$$

be a sequence of real polynomials having only real zeros, $P_n(0) = 1$, *which converges uniformly on a segment*

$$-\rho \leqq t \leqq \rho \ (\rho > 0;\ s = it)$$

of the imaginary axis. Then the sequence $\{P_n(s)\}$ *converges locally uniformly throughout the s-plane to an entire function* $\Psi(s)$ *of the class described by* (5.2).

This result was apparently novel in 1947 but is no longer so now (see [6, Theorem 3.4 on page 47]). For related recent results see [7].

In this section we simplify our notation for the knots (5.4) by dropping the second subscript and denoting them by $x_0, x_1, \cdots, x_n$.

Lemma 6. *The relation*

$$(6.11)\quad 1\Big/\prod_{\nu=0}^{n}\left(1+\frac{x_\nu}{n+1}ti\right) = \int_{-\infty}^{\infty}\left(1+\frac{xti}{n+1}\right)^{-n-1} M_n(x;x_0,\cdots,x_n)\,dx$$

holds for all t which are not purely imaginary.

Proof: Let $f(x) = 1/(1-ax)$, where a is a constant. By induction we verify that

$$f(x_0,x_1,\cdots,x_n) = a^n\Big/\prod_{0}^{n}(1-ax_\nu)$$

while

$$f^{(n)}(x) = n!a^n(1-ax)^{-n-1}\,.$$

The relation (1.5) now implies

$$1\Big/\prod_{0}^{n}(1-ax_\nu) = \int_{-\infty}^{\infty}(1-ax)^{-n-1}M_n(x;x_0,\cdots,x_n)dx.$$

Setting $a = -it/(n+1)$ we obtain (6.11).

7. A proof of Theorem 6. Let $F_n(x)$ denote the spline distribution function (5.3), our assumption being that

$$F_n(x) \to F(x),$$

where $F(x)$ is a distribution function. By Lemma 3 we conclude that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\left(1+\frac{itx}{n+1}\right)^{-n-1} M_n(x)dx = \int_{-\infty}^{\infty} e^{-itx}\,dF(x),$$

locally uniformly for all real t and by Lemma 6 we may rewrite this as

$$(7.1)\qquad \lim_{n\to\infty} 1\Big/\prod_{\nu=0}^{n}\left(1+\frac{x_{\nu,n}}{n+1}ti\right) = \int_{-\infty}^{\infty} e^{-itx}dF(x),$$

locally uniformly for all real t.

Let us select a positive ρ such that

$$\int_{-\infty}^{\infty} e^{-itx} dF(x) \neq 0 \qquad \text{if } -\rho \leqq t \leqq \rho .$$

In this range we may form the reciprocals of both sides of (7.1) and conclude that

$$\lim_{n\to\infty} \prod_{\nu=0}^{n} \left(1 + \frac{x_{\nu,n}}{n+1} ti\right) = 1 \bigg/ \int_{-\infty}^{\infty} e^{-itx} dF(x), \tag{7.2}$$

uniformly in the interval $-\rho \leqq t \leqq \rho$. Now Lemma 5 becomes applicable and implies the relation

$$\lim_{n\to\infty} \prod_{\nu=0}^{n} \left(1 + \frac{x_{\nu,n}}{n+1} s\right) = \Psi(s), \tag{7.3}$$

locally uniformly in the complex s-plane, where $\Psi(s)$ is an entire function of the form (5.2). The zeros of $\Psi(s)$ being all real, we now realize that the limit (7.1) is different from zero for all real t and therefore that (7.2) holds for all real t. By (7.2) and (7.3) we obtain

$$\int_{-\infty}^{\infty} e^{-itx} dF(x) = \frac{1}{\Psi(it)} \qquad (-\infty < t < \infty)$$

and a comparison with (5.6) shows that $F(x)$ is a Pólya distribution function. This concludes a proof of the direct part of Theorem 6.

Conversely, let $F(x)$ be a Pólya distribution function having a transform defined by (5.6) and (5.2). The easy converse part of the Laguerre-Pólya theorem asserts the existence of reals $x_{0,n}, x_{1,n}, \cdots, x_{n,n}$ satisfying (5.4), defined for a sequence N of values of n, and such that (7.3) holds locally uniformly in the s-plane. With these reals $x_{\nu,n}$ as knots, we construct the spline distribution function $F_n(x)$, defined by (5.3), and wish to show that

$$F_n(x) \to F(x), \qquad (n \in N). \tag{7.4}$$

Let us choose a positive number a such that

$$\Psi(s) \neq 0 \qquad \text{if } -a < \operatorname{Re} s < a.$$

From (5.6), (7.3) and Lemma 6 we conclude that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left(1 + \frac{x}{n+1} s\right)^{-n-1} M_n(x)dx = \int_{-\infty}^{\infty} e^{-sx} dF(x), \tag{7.5}$$

where $M_n(x) = M_n(x; x_{0,n}, \cdots, x_{n,n})$, holds locally uniformly in the same strip $-a < \operatorname{Re} s < a$. In particular, (7.5) holds uniformly in a certain circular neighborhood of $s = 0$. The expansions in powers of s of these integrals are readily obtained in terms of the moments of the distributions and we find

$$\int_{-\infty}^{\infty} e^{-sx} dF(x) = \sum_{0}^{\infty} (-1)^\nu \frac{\mu_\nu}{\nu!} s^\nu$$

$$\int_{-\infty}^{\infty} \left(1 + \frac{xs}{n+1}\right)^{-n-1} M_n(x)dx = \sum_{0}^{\infty} (-1)^\nu \frac{(n+1)(n+2)\cdots(n+\nu)}{(n+1)^\nu} \frac{\mu_\nu^{(n)}}{\nu!} s^\nu,$$

where μ_ν and $\mu_\nu^{(n)}$ are the moments of $F(x)$ and $F_n(x)$, respectively. These expansions and the uniform convergence relation (7.5) show that

$$\lim_{n\to\infty} \mu_\nu^{(n)} = \mu_\nu \quad \text{for all } \nu, \qquad n \in N.$$

In particular for $\nu = 2$ we conclude that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x^2 M_n(x)dx = \int_{-\infty}^{\infty} x^2 dF(x), \qquad (n \in N). \tag{7.6}$$

A proof of (7.4) now follows readily. Indeed, on the one hand (7.5) shows that

$$\lim_{\substack{n \to \infty \\ n \in N}} \int_{-\infty}^{\infty} \left(1 + \frac{itx}{n+1}\right)^{-n-1} dF_n(x) = \int_{-\infty}^{\infty} e^{-itx} dF(x) \text{ locally uniformly.} \tag{7.7}$$

On the other hand (7.6) implies that the left side of the inequality

$$A^2 \int_{|x| \geqq A} dF_n(x) \leqq \int_{-\infty}^{\infty} x^2 M_n(x) dx$$

remains below a constant K for all n and all positive A. Thus

$$\int_{|x| \geqq A} dF_n(x) < KA^{-2} \text{ for all } A > 0,$$

which evidently implies the equi-continuity condition (6.4) required by Lemma 4. In view of this and (7.7), Lemma 4 implies our conclusion (7.4), completing our proof of Theorem 6.

8. On frequency functions which are logarithmically concave and proof of Theorem 7. It was pointed out that Pólya frequency functions are *totally positive*. A wider class of functions is formed by the so-called *multiply positive functions* of a certain order (see [13]). Here we are concerned with the widest such class, namely the class of functions which are *twice-positive*. The precise definition is as follows.

Definition 2. *We say that a frequency function $f(x)$, i.e. a non-negative finite-valued function whose integral over the reals is unity, belongs to the class $\mathscr{P}_2$ of twice-positive functions, provided that*

$$\begin{vmatrix} f(x_1 - t_1) & f(x_1 - t_2) \\ f(x_2 - t_1) & f(x_2 - t_2) \end{vmatrix} \geqq 0 \quad \text{if } x_1 < x_2, t_1 < t_2. \tag{8.1}$$

It was shown in [10, p. 337] that an equivalent definition(4) is as follows: A frequency function $f(x)$ is an element of the class $\mathscr{P}_2$ if and only if it is of the form

$$f(x) = e^{-\phi(x)} \qquad (-\infty < x < \infty), \tag{8.2}$$

the function $\phi(x)$ having the following properties

1. $-\infty < \phi(x) \leqq +\infty$
2. There is an interval (α,β) $(-\infty \leqq \alpha < \beta \leqq +\infty)$ such that

$$\phi(x) = \begin{cases} \text{a continuous convex function in } \alpha < x < \beta \\ +\infty \text{ outside } [\alpha,\beta] \end{cases} \tag{8.3}$$

while

$$\phi(\alpha+0) \leqq \phi(\alpha) \leqq +\infty, \ \phi(\beta-0) \leqq \phi(\beta) \leqq +\infty, \tag{8.4}$$

in case that α, respectively β, are finite.

Examples of elements of the class $\mathscr{P}_2$ are all Pólya frequency functions, e.g. e^{-x^2}, $e^{-|x|}$and the one-sided exponential (5.9). From the properties (8.2), (8.3) and (8.4), it is clear that if $f(x) \in \mathscr{P}_2$, then $f(x)$ is everywhere continuous with the possible exception of at most two points, α and (or) β, where $f(x)$ may have discontinuities of the first kind.

We denote by $\overline{\mathscr{P}}_2$ the class $\mathscr{P}_2$ enlarged by including also the Dirac functions $f(x) = \delta_\xi(x)$. These are the frequency functions corresponding to the distribution functions

$$F(x) = (x-\xi)_+^0. \tag{8.5}$$

Theorem 8. *Let $\{f_n(x)\}$ be a sequence of frequency functions,*

(4) This definition describes functions $f(x)$ which are appropriately called "logarithmically concave". However, this (descriptive) definition of the class $\mathscr{P}_2$ is not as searching, in some ways, as the first (structural) definition (8.1). For instance, the important fact that *the class $\mathscr{P}_2$ is closed with respect to the operation of convolution* is not at all apparent from the descriptive definition, while it is easily derived from the property (8.1) (see [10, the proof of Lemma 5, 341–342]).

$$f_n(x) \in \mathscr{P}_2 \qquad (n = 1, 2, \cdots). \tag{8.6}$$

Let $F_n(x)$ denote the corresponding distribution functions and let us assume that

$$F_n(x) \to F(x), \tag{8.7}$$

where $F(x)$ is a distribution function.

Then $dF(x) = f(x)dx$, where

$$f(x) \in \overline{\mathscr{P}}_2 . \tag{8.8}$$

Moreover the relation (8.7) *may be differentiated giving*

$$\lim_{n \to \infty} f_n(x) = f(x), \tag{8.9}$$

which holds uniformly for all x, provided that we remove arbitrarily small neighborhoods of the one or two possible discontinuity points of $f(x)$.

A proof of this theorem will be given in a future paper devoted to a study of the class $\mathscr{P}_2$. Here we only observe that Theorem 8 implies Theorem 7 immediately. Indeed, the alternative definition (8.2), (8.3), (8.4) and Theorem 3 shows that

$$M_n(x; x_0, x_1, \cdots, x_n) \in \mathscr{P}_2 .$$

9. Examples and Applications. By Theorems 6 and 7 we know that the relation

$$\lim_{n \to \infty} \prod_{\nu=0}^{n} \left(1 + \frac{x_{\nu,n}}{n+1} s \right) = \Psi(s) \tag{7.3}$$

implies either the relation

$$\lim_{n \to \infty} M_n(x; x_{0,n}, \cdots, x_{n,n}) = \Lambda(x) \tag{5.11}$$

or

$$\lim_{n \to \infty} M_n(x; x_{0,n}, \cdots, x_{n,n}) = 0, \quad (|x - \xi| \geqq \eta), \tag{5.12}$$

depending on whether the limit frequency $\Lambda(x)$ is a Pólya frequency function or the Dirac function $\delta_\xi(x)$. Here we wish to illustrate this result by a few concrete examples.

1. Let $\{x_\nu\}$ be an increasing sequence, $x_n \to \xi$, and let us choose the knots

$$x_{\nu,n} = x_\nu \qquad (\nu = 0, 1, \cdots, n).$$

As it is easy to show that (7.3) holds, with $\Psi(s) = e^{\xi s}$, we conclude by Theorem 7 that

$$\lim_{n\to\infty} M_n(x; x_0, x_1, \cdots, x_n) = \delta_\xi(x).$$

2. Let

$$x_{0,n} = x_{1,n} = \cdots = x_{n-1,n} = 0, \; x_{n,n} = n.$$

Now (7.3) holds, with $\Psi(s) = 1 + s$, the corresponding Pólya frequency function being

$$\Lambda(x) = \lambda(x) = \begin{cases} e^{-x} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

By Theorem 7 we conclude that

$$\lim_{n\to\infty} M_n(x; \overbrace{0, \cdots, 0}^{n}, n) = \lambda(x)$$

uniformly in x if we remove a neighborhood of $x = 0$, a result which is also easily verified directly from the explicit expression of the fundamental spline function (3.5).

3. The easiest way to obtain in (7.3) the limit

$$\Psi(s) = e^{-s^2/4},$$

which corresponds to

$$\Lambda(x) = \frac{1}{\sqrt{\pi}} e^{-x^2},$$

is to set $n+1=2k$ and to write

$$\prod_{\nu=0}^{n}\left(1+\frac{x_{\nu,n}}{n+1}s\right)=\left(1-\frac{s}{2\sqrt{k}}\right)^k\left(1+\frac{s}{2\sqrt{k}}\right)^k=\left(1-\frac{s^2}{4k}\right)^k.$$

Indeed, Theorem 7 shows that

$$\lim_{k\to\infty} M_{2k-1}(x;\overbrace{-k^{1/2},\cdots,-k^{1/2}}^{k},\overbrace{k^{1/2},\cdots,k^{1/2}}^{k})=\frac{1}{\sqrt{\pi}}e^{-x^2}, \tag{9.1}$$

uniformly for all real x.

4. The normal frequency function appears as the limit of spline functions also under different circumstances. Let $n=2k$ and let us consider the frequency function with equidistant knots

$$M_{2k}(x) = M_{2k}(x;-k,-k+1,\cdots,k-1,k).$$

Setting $f(x)=e^{itx}$ in (1.5) we easily obtain its well-known characteristics function which turns out to be

$$\int_{-\infty}^{\infty} M_{2k}(x)e^{itx}dx = \left(\frac{\sin t/2}{t/2}\right)^{2k} = \left(1-\frac{t^2}{24}+\cdots\right)^{2k}.$$

Appropriate changes of variables now produce the relation

$$\sqrt{\frac{k}{3}}\int_{-\infty}^{\infty} M_{2k}\left(x\sqrt{\frac{k}{3}}\right)e^{itx}dx=\left(1-\frac{t^2}{4.2k}+\cdots\right)^{2k}. \tag{9.2}$$

Notice that the integrand in (9.2) is a spline function of degree $2k-1$ having $2k+1$ knots at the points x obtained from

$$x\sqrt{\frac{k}{3}}=\nu,\quad (\nu=-k,\cdots,k).$$

Since its total area is unity, we must have

$$\sqrt{\frac{k}{3}}M_{2k}\left(x\sqrt{\frac{k}{3}};-k,\cdots,k\right)=M_{2k}\left(x;-k\sqrt{\frac{3}{k}},\cdots,k\sqrt{\frac{3}{k}}\right).$$

On the other hand, the right hand side of (9.2) converges to $\exp(-t^2/4)$. From the fundamental limit theorem for characteristic functions and Theorem 7 (which assures us that we can work with the frequency functions) we conclude that

$$\lim_{k\to\infty} M_{2k}(x;-k(3/k)^{1/2},(-k+1)(3/k)^{1/2},\cdots,k(3/k)^{1/2})=\frac{1}{\sqrt{\pi}}e^{-x^2}. \tag{9.3}$$

We now apply our theory (in reverse order), to conclude that

$$\lim_{k\to\infty}\prod_{\nu=-k}^{k}\left(1+\frac{\nu\sqrt{3}\,s}{\sqrt{k}\,(2k+1)}\right)=e^{-s^2/4}. \tag{9.4}$$

However, the relation (9.4) is easily shown to hold locally uniformly, and then we conclude again that (9.3) holds, this time without the appeal to characteristic functions.

The relation (9.3) has been noticed before in different notations, e.g. by F. Tricomi [16].

5. Let us now consider the Pólya frequency function

$$\Lambda(x)=\frac{1}{\pi\cosh x}$$

having the transform $1/\Psi(x)$, where $\Psi(s)=\cos\frac{\pi s}{2}$. From

$$\cos\frac{\pi s}{2}=\prod_{1}^{\infty}\left(1-\frac{s^2}{(2\nu-1)^2}\right)$$

we conclude that

$$\lim_{k\to\infty}\prod_{\nu=-k+1}^{k}\left(1+\frac{s}{2\nu-1}\right)=\cos\frac{\pi s}{2}.$$

Theorem 7 shows that if we set $n=2k-1$ and select the $2k$ knots

$$\frac{2k}{2\nu - 1} \quad (\nu = -k+1, \cdots, k)$$

we obtain, uniformly for all x, the relation

(9.5)

$$\lim_{k\to\infty} M_{2k-1}\left(x; -\frac{2k}{1}, -\frac{2k}{3}, \cdots, -\frac{2k}{2k-1}, \frac{2k}{2k-1}, \frac{2k}{2k-3}, \cdots, \frac{2k}{1}\right) = \frac{1}{\pi \cosh x}.$$

It is instructive to compare the formulae (9.3) and (9.5). The knots in (9.3) are equidistant. In (9.5), however, their density is higher near the origin in order to produce the function (9.5) which decays so much more slowly than the normal function.

6. As a last example, we consider the Pólya frequency function

$$\Lambda(x) = e^{-x-e^{-x}}$$

having the transform $1/\Psi(s) = s\Gamma(s)$. By Gauss' formula

$$1/(s\Gamma(s)) = \lim_{k\to\infty} \frac{(s+1)(s+2)\cdots(s+k)}{k!} k^{-s}$$

$$= \lim_{k\to\infty} \left(1+\frac{s}{1}\right)\left(1+\frac{s}{2}\right)\cdots\left(1+\frac{s}{k}\right)\left(1-\frac{s\log k}{k}\right)^k = \Psi(s).$$

On comparing with (7.3) we see that we have $2k$ knots, hence $n = 2k-1$, and that

$$\lim_{k\to\infty} M_{2k-1}\Big(x; \overbrace{-2\log k, \cdots, -2\log k}^{k}, \frac{2k}{k}, \frac{2k}{k-1}, \cdots, \frac{2k}{1}\Big) = e^{-x-e^{-x}}.$$

Notice the k-fold negative knot which $\to -\infty$ very slowly in order to produce a limit which tends to zero very fast as $x \to -\infty$. We also notice that in the ever increasing interval

$$-2\log k \leqq x \leqq 2$$

the spline function is a polynomial of degree $2k-2$, of the form

$$(x + 2\log k)^{k-1} \cdot \pi_{k-1}(x),$$

converging uniformly to $\exp(-x - e^{-x})$. Here $\pi_{k-1}(x)$ denotes an appropriate polynomial of degree $k-1$.

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