SOLUTIONS OF THE FIRST AND SECOND PAINLEVÉ EQUATIONS ARE MEROMORPHIC

By

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1 Introduction

The Painlevé differential equations have attracted mathematical research since the celebrated Stockholm lectures by P. Painlevé in 1895; cf. [10]. Originally, the interest came from the role these six equations play as prototypes of second order differential equations which lack so-called movable singularities. Later on, the Painlevé equations, together with certain related partial differential equations, found important applications in physics; see [13], pp. 427–429 and the original references cited therein. A widely accepted property of the first, second and fourth Painlevé differential equations

(1.1) $$w'' = z + 6w^2,$$

(1.2) $$w'' = \alpha + zw + 2w^3, \qquad \alpha \in \mathbb{C},$$

(1.3) $$ww'' = \tfrac{1}{2}(w')^2 + \tfrac{3}{2}w^4 + 4zw^3 + 2(z^2 - \alpha)w^2 + \beta, \qquad \alpha,\ \beta \in \mathbb{C},$$

is that all of their solutions are meromorphic in the sense that all local solutions are meromorphic and have meromorphic continuations into the whole complex plane. In fact, a number of attempts at proof may be found in the mathematical literature, including some standard references. However, a careful look at these proofs reveals heuristic argumentation and incomplete reasoning.

To put this part of the theory of complex differential equations on a solid basis, we offer below what we believe to be a rigorous proof for the meromorphic nature of the solutions of (1.1) and (1.2). As the reader will observe, a lot of technical

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reasoning is needed to complete the ideas found in the existing literature. Our main contribution, §4, is a new method of estimating the coefficients of the power series of a locally-defined inverse function of a solution to a differential equation (in our case, a Painlevé equation), resulting in upper bounds for the moduli of the coefficients that satisfy a novel periodicity condition with respect to the index n.

This article has been organized as follows. In §2, we recall, in rough chronological order, the existing proofs for the meromorphic nature of solutions of (1.1), pointing out their essential gaps. In §§3–4, we provide our version of proof, based on the reasoning found in the book [7] by E. Hille. Actually, this reasoning originates with P. Painlevé [11] (cf. [12]) and is essentially repeated in the books by E. Ince [8], W. Golubew [4], and E. Hille [7]. In the final §5, we describe the difficulties which must be overcome in order to apply our reasoning to (1.2). Thus, our paper is devoted to proving the following two theorems.

Theorem 1. *All local solutions to Painlevé's first equation can be analytically continued to single-valued meromorphic solutions in the complex plane.*

Theorem 2. *All local solutions to Painlevé's second equation can be analytically continued to single-valued meromorphic solutions in the complex plane.*

For the convenience of the reader, we offer in the proof of Theorem 1 below a complete reasoning, although this requires repeating a few passages from the existing literature. The proof contains certain complicated calculations; these have been carried out using Mathematica, Version 2.2.1.

It seems likely to us that the reasoning presented here for Painlevé's first and second equation can be adapted to prove that the solutions of Painlevé's fourth equation and, after a suitable change of variables involving the logarithm, those of Painlevé's third equation are also single-valued and meromorphic in the plane.

2 Historical background

As mentioned above, the recognition of Theorem 1 as an established mathematical fact originates from the proof offered by P. Painlevé in [11], pp. 227–238. We briefly describe the essentials of this proof for the following reasons: (1) to make it easier for the reader to follow the complete proof in §§3–4; (2) to point out the gaps in the original proof in [11]; and (3) to make it easier to explain how these gaps propagate in the subsequent literature, with various attempts to fill them.

So, let $w(z)$ be a local solution to (1.1) around $z_0 \in \mathbb{C}$, and let $B(z_0, R)$ be the largest disc around z_0 such that $w(z)$ has a meromorphic continuation over the whole $B(z_0, R)$. If $R < \infty$, there must be at least one non-algebraic singularity, say a, of $w(z)$ on the boundary $S(z_0, R)$ of $B(z_0, R)$. This leads to a contradiction as follows. If $|w(z)|$ remains bounded on the line segment $[z_0, a]$ as $z \to a$, then $|w'(z)|$ also stays bounded on $[z_0, a]$. The Cauchy majorant principle now implies that $w(z)$ has to be analytic around a. Therefore, one may assume that $|w(z)|$ is unbounded on $[z_0, a]$ as $z \to a$. Painlevé now introduces the transformation

$$\begin{cases} w = v^{-2}, \\ w' = -2v^{-3} - \frac{1}{2}zv - \frac{1}{2}v^2 + uv^3, \end{cases}$$

and the auxiliary functions

$$u = w^{\frac{3}{2}}(w' + \tfrac{1}{2}w^{-1}) + 2w^3 + \tfrac{1}{2}zw \tag{2.1}$$

and

$$V = (w')^2 - 4w^3 - 2zw + \frac{w'}{w} + z.$$

Assuming that $|V(z)|$ remains bounded on $[z_0, a]$, Painlevé derives a contradiction from the observation that one of the two branches of $u(z)$ also remains bounded on $[z_0, a]$. However, we do not see any immediate reason why the possible case of an unbounded branch of $u(z)$ could be left out of consideration. As we shall see in §3, a short separate reasoning is, in fact, needed. The next case, in which $|V(z)|$ is unbounded on $[z_0, a]$ while $|w(z)| \geq \varepsilon > 0$ on $[z_0, a]$, was treated by Painlevé in a heuristic way only. However, a rigorous completion may easily be found, as shown by E. Hille [7], p. 702. The final case, in which $|V(z)|$ is unbounded on $[z_0, a]$ while $\liminf_{[z_0,a]\ni z\to a} |w(z)| = 0$, needs a path modification of $[z_0, a]$ to $\tilde{\Gamma}$, starting at z_0 and ending at a, such that $\liminf_{\tilde{\Gamma}\ni z\to a} |w(z)| \geq \varepsilon > 0$, that there are no zeros or poles of $w(z)$ on $\tilde{\Gamma}$ and that $\tilde{\Gamma}$ is of finite length. To achieve this, Painlevé considers the inverse function $z(w)$ and its differential equation

$$z''(w) = -\big(6w^2 + z(w)\big)\big(z'(w)\big)^3.$$

As appears below, it is necessary to work out a lower bound for the radius of convergence of the power series expansion of $z(w)$ around w_0 close to the origin of the w-plane. Such a consideration was left out in the original memoir [11] of Painlevé.

The gaps in the proof offered by Painlevé, pointed out above, are well-known. In fact, several subsequent presentations of Theorem 1 contain various attempts to fill them. We briefly recall some of these presentations.

The classical proof in the book [8], pp. 347–351, by E. Ince essentially repeats the proof offered by Painlevé, apart from a slightly different path modification.

However, the estimate for the radius of convergence of the power series expansion of $z(w)$ is omitted here also.

The book [4] by W. Golubew offers another attempt to overcome the difficulties related to this path modification. Unfortunately, the attempt is unsuccessful. In fact, it is enough to remark that the application of the usual mean value theorem in the complex plane on p. 165 remains unjustified. On the other hand, Golubew seems be the first to have tried to treat the difficulty of the two possible branches of u in (2.1). Unfortunately, there is an error of sign on [4], p. 162; see the extreme right term of the second equation in the formula (7). We offer a less ambiguous treatment which also corrects the error.

The book [7] by E. Hille repeats, essentially, the presentation offered by E. Ince. The proof given by N. Erugin in the survey article [3], pp. 267–275, leaves aside the branch problem of u. Concerning the path modification, Erugin refers to a Poincaré–Lyapunov theorem; see [3], p. 277. We have been unable to see how this reasoning could be used to settle the radius-of-convergence problem.

The paper of Joshi and Kruskal [9] contains the interesting idea of trying to establish the possibility of the analytic continuation of a local solution by defining a suitable operator on a function space and showing that it has a fixed point in a small neighbourhood of a singularity (which singularity would then be shown to be a pole). The fixed point is an analytic function which, one hopes, will be related to the analytic continuation of the initially given local solution. Joshi and Kruskal identify as an essential problem the proof that such a connection between the two functions actually exists. To prove it, they define each of the two functions by means of two parameters (which, in the case of the given solution, are the values of the function and its derivative at a given point). They wish to show that there is a one-to-one correspondence between certain pairs of complex numbers (parameters) so obtained. However, since the functions must necessarily be considered as defined in a large set, not only in a small neighbourhood of a single point, and since the only justification for invertibility that they offer [9, p. 206] is the suggestion that the Jacobian determinant of an implicitly defined function is non-zero, it seems that this proof also fails to be logically complete.

A number of further presentations may be found in the literature, for example, [1], [2], [5], [13]. We omit details, since these references either repeat the defective arguments described above or offer reasoning that is clearly heuristic.

3 Proof of Theorem 1: Pairs of differential equations and Cauchy's estimates

Write $B(z,r) = \{ \zeta \in \mathbb{C} : |\zeta - z| < r \}$ and $S(z,r) = \{ \zeta \in \mathbb{C} : |\zeta - z| = r \}$.

Consider an arbitrary local solution $w(z)$ to Painlevé's first equation (1.1) with $w(z_0) = w_0$, defined in a small disk centred at the point $z_0 \in \mathbb{C}$. We may assume that $w_0 \neq \infty$, for if this is initially not the case, we may move the point z_0 slightly to achieve this. Let R be the radius of the largest disk centred at z_0 to which $w(z)$ can be continued as a single-valued meromorphic function. Of course, this extended function then satisfies (1.1) in $B(z_0, R)$. We seek to prove that $R = \infty$, which means that $w(z)$ extends to a single-valued meromorphic function in the whole plane. To do this, we assume that R is finite and derive a contradiction.

So suppose that R is finite and write $D = B(z_0, R)$. If, for each $z_1 \in S(z_0, R)$, there exists $\delta > 0$ such that $w(z)$ can be continued as a single-valued meromorphic function to the domain $D \cup B(z_1, \delta)$, then we may cover $S(z_0, R)$ by finitely many such disks (by reducing some of the radii, if necessary) so that any disk overlaps exactly two others. The union of D with these disks covers a disk $D_1 = B(z_0, R_1)$ for some $R_1 > R$. If two disks B and B' of the form $B(z_1, \delta)$ overlap, then the extensions of w to $D \cup B$ and $D \cup B'$ agree in $D \cap B \cap B' \neq \emptyset$ so that they agree in $B \cap B'$. We conclude that w may be analytically (more precisely, meromorphically) continued without restriction in D_1. Since D_1 is simply connected, it follows from the monodromy theorem that we have defined w as a single-valued meromorphic function in D_1. This contradicts the definition of R.

Therefore, there exists a point $a \in S(z_0, R)$ for which there is no $\delta > 0$ such that $w(z)$ can be continued as a single-valued meromorphic function from D to $D \cup B(a, \delta)$.

We now introduce a number of auxiliary functions related to $w(z)$, following Painlevé. We start with the pair $u(z)$, $v(z)$ defined by

$$\begin{cases} w = v^{-2}, \\ w' = -2v^{-3} - \frac{1}{2}zv - \frac{1}{2}v^2 + uv^3. \end{cases} \tag{3.1}$$

These functions satisfy a system of first-order differential equations

$$\begin{cases} u' = \frac{1}{8}z^2v + \frac{3}{8}zv^2 + \frac{1}{4}v^3 - zuv^3 - \frac{5}{4}uv^4 + \frac{3}{2}u^2v^5, \\ v' = 1 + \frac{1}{4}zv^4 + \frac{1}{4}v^5 - \frac{1}{2}uv^6. \end{cases} \tag{3.2}$$

Since the functions on the right hand side of (3.2) are holomorphic functions (in fact, polynomials) of z, u, v for all finite values of the variables z, u, v, the equations (3.2) have unique solutions which at a given point $z = z_0$ assume pre-assigned values u_0 and v_0. In particular, this is the case if $v_0 = 0$.

The second equation (3.1) gives

$$u = w^{3/2}(w' + \tfrac{1}{2}w^{-1}) + 2w^3 + \tfrac{1}{2}zw = vw^2\left(w' + \frac{1}{2w}\right) + 2w^3 + \frac{zw}{2}, \tag{3.3}$$

so that $u(z)$ is not a single-valued function around points where $w = 0$ or $w = \infty$. If $z = z_1$ is a double pole of $w(z)$ and if we choose the square root so that

$$[w(z)]^{3/2} = (z - z_1)^{-3} + \cdots,$$

then $u(z)$ is holomorphic at $z = z_1$. As is well known (see [7], p. 700), the function

$$V \equiv (w')^2 - 4w^3 - 2zw + \frac{w'}{w} + z \tag{3.4}$$

is also holomorphic at a pole of $w(z)$. We also use the function

$$W \equiv \frac{V'}{V} = \frac{4w^3 - (w')^2 + w^2 + zw}{w[w(w')^2 - 4w^4 + w' - 2zw^2 + zw]} \tag{3.5}$$

below.

We now continue with the proof of the absence of singularities other than poles. Let z_0, R, and a be as before. We join z_0 and a by an arc Γ of finite length contained in $B(z_0, R)$, such that Γ avoids the zeros and poles of $w(z)$. This implies that $w(z)$, $u(z)$, $v(z)$ and $V(z)$ are holomorphic along Γ except at $z = a$. A number of situations may arise as we approach a along Γ; it is our object to show that the unfavourable ones are excluded and that $z = a$ is either a regular point or a pole of $w(z)$. We initially join z_0 and a by a line segment. The assumption that Γ avoids the zeros and poles of $w(z)$ can be satisfied by modifying Γ by arbitrarily small amounts close to the zeros and poles of $w(z)$ on the line segment. Since these changes can be made as small as we please, we shall ignore them in what follows and assume that Γ is a line segment. Later on, this segment will have to be modified for other reasons.

Following the reasoning in [7], we divide the proof below into five subcases, the first three of which are included in this section, §4 consisting of the remaining two.

Case I. We first show that if at least one of $|w(z)|$ or $|w'(z)|$ is bounded along Γ, then $w(z)$ is analytic at $z = a$, so that the point a cannot be a singularity. In fact, the relations

$$w(z) = w(z_0) + \int_{z_0}^{z} w'(t)\,dt,$$

$$w'(z) = w'(z_0) + 6\int_{z_0}^{z} [w(t)]^2\,dt + \tfrac{1}{2}(z^2 - z_0^2),$$

where the integrals are taken along Γ, show that boundedness on Γ of one of the functions $w(z)$ and $w'(z)$ implies that of the other. Now, if $|w(z)| < A$ and $|w'(z)| < A$ on Γ, for some $A < \infty$, then at every point $z_1 \in \Gamma$ there is a disk $\{ z : |z - z_1| < \rho \}$ in which $w(z)$ is holomorphic; and since $|z_1| < R + |z_0|$, the number ρ has a positive lower bound r. Now if $|z_1 - a| < r$, it follows that $w(z)$ is holomorphic at $z = a$ as asserted. This follows by the well-known Cauchy estimates, applied to the system $w' = g$, $g' = 6w^2 + z$ with unknown functions $w(z)$ and $g(z)$; see, e.g., [6], p. 17.

For later reference, we point out that this argument shows that $w(z)$ is holomorphic at $z = a$ provided that there exists a sequence $z_n \in \Gamma$ with $z_n \to a$, such that $|w(z_n)| < A$ and $|w'(z_n)| < A$ for all n and for some $A < \infty$.

Case II. Next we prove that if $|w(z)|$ is unbounded on Γ while $|u(z)|$ stays bounded, then $z = a$ is a pole of $w(z)$. For this assumption means that for some fixed $A < \infty$ and for any pre-assigned $\varepsilon > 0$, there are points z_1 on Γ in any neighbourhood of $z = a$ where $|u(z_1)| < A$ and $|v(z_1)| < \varepsilon$. At such a point z_1, the system (3.2) has a solution $u(z)$, $v(z)$ which takes on pre-assigned values $u(z_1)$, $v(z_1)$ at $z = z_1$ such that the functions $u(z)$, $v(z)$ are holomorphic in a disk $|z - z_1| < r$, where r has a positive lower bound, which we can take to be a fixed constant independent of z_1 (under the conditions $z_1 \in \Gamma$, $|u(z_1)| < A$, $|v(z_1)| < \varepsilon$). This follows from [6], Hilfssatz 2.2. If $|z_1 - a| < r$, then $u(z)$ and $v(z)$ are holomorphic at $z = a$. Hence $w = 1/v^2$ is well-defined in a neighbourhood of a. Since $\liminf |v(z)| = 0$ as $z \to a$ along Γ and $\lim_{z \to a} v(z)$ exists, it follows that $v(a) = 0$. Thus $w(z)$ has a pole at $z = a$ (and hence a double pole, by the standard local power series analysis; see [7], p. 698).

Remark. The assumptions in Case II will not be used as a logical subcase as such (for example, the possibility that w and u are both unbounded on Γ will not be considered later). The only reason for including Case II here is that various other cases can be reduced to it. Cases I, III, IV and V provide a list that logically exhausts all the possibilities.

Case III. We now assume that there is a sequence of points $\{z_n\}$ on Γ such that $z_n \to a$, $|w(z_n)| \to \infty$, and $|V(z_n)| < A$ for some finite A. Note that $|v(z_n)| \to 0$.

From (3.4) and the second equation in (3.2) we obtain

$$
\begin{aligned}
V - z &= (w')^2 - 4w^3 - 2zw + w'/w = \frac{4[(v')^2-1] - 2zv^4 - 2v'v^5}{v^6} \\
&= -\frac{v^4}{4} - \frac{1}{4v^6}(8zv^4 + 16 - (2uv^6 - 4 - zv^4)^2) \\
&= -\frac{v^4}{4} - \frac{1}{4v^6}(-z^2v^8 + 2uv^6(2zv^4 + 8 - 2uv^6)) \\
&= -\frac{v^4}{4} + \frac{z^2v^2}{4} - \tfrac{1}{2}u(2zv^4 + 8 - 2uv^6) \\
&= -\frac{1}{4w^2} + \frac{z^2}{4w} + \frac{u^2}{w^3} - 4u - \frac{zu}{w^2}.
\end{aligned}
\tag{3.6}
$$

Since $v(z_n) \to 0$, we see from the second line of (3.6) that $2uv^6 - 4 \to \pm 4$, so that uv^6 tends to 0 or 4 on z_n as $z_n \to a$. In the case of $uv^6 \to 0$, we consider the last expression in (3.6) to obtain

$$
u^2 - uw^3(zw^{-2} + 4) + \varphi w^3 = 0, \tag{3.7}
$$

where

$$
\varphi = \frac{z^2}{4w} - (V - z) - \frac{1}{4w^2}
$$

is bounded on the sequence z_n. Solving (3.7) for u, we obtain

$$
u = \frac{w^3}{2}\left(zw^{-2} + 4 \pm (zw^{-2} + 4)\sqrt{1 - 4\varphi w^{-3}/(zw^{-2} + 4)^2}\right).
$$

Since φ is bounded on z_n, we see that

$$
\begin{aligned}
&\sqrt{1 - 4\varphi w^{-3}/(zw^{-2} + 4)^2} \\
&\quad = 1 - 2\varphi w^{-3}/(zw^{-2} + 4)^2 + O\Big(\big(\varphi w^{-3}/(zw^{-2} + 4)^2\big)^2\Big) \\
&\quad = 1 - \frac{\varphi}{8w^3} + O(w^{-5}).
\end{aligned}
$$

Therefore,

$$
u = \frac{w^3}{2}(zw^{-2} + 4)\left(1 \pm \left(1 - \frac{\varphi}{8w^3} + O(w^{-5})\right)\right).
$$

Clearly, $uv^6 \to 0$ corresponds to the minus sign here; and we obtain

$$
u = \tfrac{1}{16}(zw^{-2} + 4)(\varphi + O(w^{-2})) = \varphi/4 + O(w^{-2}),
$$

which remains bounded on z_n. Hence the argument used in Case II shows that $z = a$ is a pole of $w(z)$.

We now proceed to consider the case $uv^6 \to 4$. By the second last expression of (3.6), $u(2zv^4 + 8 - 2uv^6)$ also remains bounded; and so

$$
u = \frac{4}{v^6} + \frac{z}{v^2} + O(1).
$$

We now define a new auxiliary function g by

$$g = u - \frac{4}{v^6} - \frac{z}{v^2},$$

which remains bounded on z_n. Also now

$$u = g + \frac{4}{v^6} + \frac{z}{v^2}$$

and

$$g' = u' + \frac{24v'}{v^7} - \frac{1}{v^2} + \frac{2zv'}{v^3}.$$

Using the second equation in (3.2), we now express v' in terms of g and v (simply writing u in terms of g and v) and obtain

$$v' = -1 - \frac{zv^4}{4} + \frac{v^5}{4} - \frac{gv^6}{2}. \tag{3.8}$$

We note that $v'(z_n) \to -1$. Using this expression for v', we obtain a formula for g' in terms of g and v, by using the first equation in (3.2) for u' and then writing u in terms of v and g. The result is

$$g' = \tfrac{1}{8}z^2v - \tfrac{3}{8}zv^2 + \tfrac{1}{4}v^3 + zgv^3 - \tfrac{5}{4}gv^4 + \tfrac{3}{2}g^2v^5. \tag{3.9}$$

Now (3.8) and (3.9) form a pair of differential equations for $v(z)$ and $g(z)$. Since g and v remain bounded on $\{z_n\}$, the same reasoning as in Case II shows that v extends to an analytic function in a neighbourhood of a with $v(a) = 0$, so that $w(z) = 1/v(z)^2$ has a pole at $z = a$. This completes the proof in Case III.

4 Proof of Theorem 1: Estimates for the power series of an inverse function

Case IV. Since Case III completes the proof in the situation where $|w(z)|$ is unbounded on Γ while $|V(z)|$ is bounded on Γ, the only remaining possibility is that both $|w(z)|$ and $|V(z)|$ are unbounded on Γ. We split this case into two sub-cases. We first assume that $|w(z)| \geq \varepsilon > 0$ on Γ; under this assumption, we show the existence of a sequence $\{z_n\}$ on Γ such that $z_n \to a$, $|w(z_n)| \to \infty$ and $|V(z_n)| \to 0$. Hence this sub-case reduces to Case III.

Now if $|V(z)|$ is unbounded on Γ, the function $W(z)$ of (3.5) must also be unbounded, since

$$V(z) = V(z_0) \exp\left[\int_{z_0}^{z} W(t)\, dt\right].$$

Hence there is a sequence $\{z_n\}$ on Γ with $z_n \to a$ and $|W(z_n)| \to \infty$. Let us now examine the two sequences $\{w(z_n)\}$ and $\{w'(z_n)\}$. They cannot both be bounded because then the argument of Case I would apply, and then $w(z)$ would be holomorphic at $z = a$, which contradicts the assumption that $w(z)$ is unbounded on Γ. If now $\{w(z_n)\}$ is bounded, $\{w'(z_n)\}$ must be unbounded and we may assume that $\lim_{n\to\infty} |w'(z_n)| = \infty$. Then (3.5) gives

$$\limsup_{n\to\infty} |W(z_n)| \le \left[\liminf_{n\to\infty} |w(z_n)|\right]^{-2},$$

and this is finite by the assumption that $|w(z)| \ge \varepsilon > 0$ on Γ. This contradicts the assumption that $|W(z_n)| \to \infty$.

It follows that $\{|w(z_n)|\}$ is an unbounded sequence. Again, we may assume, after passing to a suitable subsequence and renumbering its elements, that we have a sequence $\{z_n\}$ such that

$$|w(z_n)| \to \infty, \qquad |W(z_n)| \to \infty.$$

We can now eliminate w' between (3.4) and (3.5) and obtain the expression

$$VW = g_1 + g_3 V \pm (g_2 + g_4 V)^{1/2},$$

where

$$g_1 = 1 + z(1-w)w^{-2} - \tfrac{1}{2}w^{-4}, \qquad g_3 = -w^{-2},$$
$$g_2 = 4w^{-3} + 2zw^{-5} - zw^{-6} + \frac{1}{4}w^{-8}, \qquad g_4 = w^{-6}.$$

It follows that V is a root of the quadratic equation

$$(W-g_3)^2V^2 - [2g_1(W-g_3) + g_4]V + g_1^2 - g_2 = 0.$$

As $z_n \to a$, we have

$$g_1 \to 1, \quad g_2 \to 0, \quad g_3 \to 0, \quad g_4 \to 0, \quad W \to \infty;$$

and it follows that both roots of the quadratic equation tend to 0, that is,

$$\lim_{n\to\infty} V(z_n) = 0,$$

which reduces this situation to Case III.

Case V. (1) In the second sub-case, $V(z)$ and $w(z)$ are unbounded on Γ but

$$\liminf_{z\to a,\, z\in\Gamma} |w(z)| = 0.$$

The only properties of Γ that matter in Case IV are that Γ is of finite length and avoids the zeros and poles of $w(z)$. We call such an arc Γ from z_0 to a admissible.

We are free to modify Γ to another arc $\tilde{\Gamma}$ as long as these properties are preserved. Thus we can reduce this second sub-case to the first, if it is possible to find an admissible arc $\tilde{\Gamma}$ on which $\liminf_{z\to a,\, z\in\tilde{\Gamma}} |w(z)| > 0$. This is indeed possible, but the proof is far from elementary.

Let $\varepsilon > 0$ be a fixed small positive number. There is then by assumption a sequence of points $\{z_n\}$ on Γ such that $|w(z)| < \varepsilon$ for $z \in \Gamma_k$, where Γ_k is the arc of Γ joining z_{2k-1} to z_{2k}, for $k \geq 1$, while $|w(z)| \geq \varepsilon$ on the remaining subarcs of Γ. Consider now the sequence $\{|w'(z_n)|\}$. None of its infinite subsequences can be bounded, for, if there were a bounded subsequence, then the method of Case I would apply (since also $|w(z_n)| \leq \varepsilon$ for all n) and $w(z)$ would be holomorphic at $z = a$, which contradicts the assumption that $|w(z)|$ is unbounded on Γ. Hence, given any small $\kappa > 0$, we may assume that

$$|w'(z_n)| > 1/\kappa \qquad \text{for all } n > N.$$

We set $w(z_n) = w_n$, $w'(z_n) = 1/\zeta_n$ and consider the differential equation for $z = f(w)$,

$$-f''(w) = (6w^2 + f(w))(f'(w))^3 \tag{4.1}$$

at $w = w_{2k-1}$ with

$$f(w_{2k-1}) = z_{2k-1}, \qquad f'(w_{2k-1}) = \zeta_{2k-1}.$$

In fact, the inverse function $f(w)$ of $w(z)$ exists in a small neighbourhood of w_n, since $w'(z_n) \neq 0$.

(2) We now interrupt the actual proof of Theorem 1 for a lengthy reasoning about this inverse function. To this end, consider a point b with $w(b) = w_1$ and $w'(b) \neq 0$. The inverse function $f(w)$ exists in a small neighbourhood of w_1, taking w_1 to b. There is a power series expansion

$$f(w) = \sum_{n=0}^{\infty} a_n (w - w_1)^n \tag{4.2}$$

valid in some disk $B(w_1, r) = \{\, w : |w - w_1| < r \,\}$ centred at w_1. From now on we write $\zeta = w - w_1$. Let R_1 be the radius of convergence of the series (4.2). It then follows that the function $f(w)$ given by the series (4.2) satisfies (4.1) in the whole disk $B(w_1, R_1)$. Of course, if $R_1 = \infty$, then $B(w_1, R_1)$ is the whole complex plane, but this naturally cannot occur when f is the inverse function of w and w satisfies (1.1).

Observe that if f' has a zero of order $m \geq 1$ at some point $w \in B(w_1, R_1)$, then f'' has a zero of order $m - 1$ at w. Hence the left hand side of (4.1) has a zero of order $m-1$ at w, while the right hand side of (4.1) has a zero of order at least $3m$ at w. Since $3m > m-1$, this is impossible, and we conclude that f' has no zeros in $B(w_1, R_1)$.

We shall prove the following auxiliary result.

Lemma 1. *Suppose that the function $w(z)$ satisfies (1.1) in a neighbourhood of the point b and that $w(b) = w_1$ and $w'(b) \neq 0$. Let $f(w)$ be the inverse function of w, given by (4.2) in a small neighbourhood of w_1 and taking w_1 onto b. There exists an absolute constant $C > 1$ such that the following holds. Suppose that $M > 1$, $\varepsilon \in (0,1)$, and that $|b| = |a_0| \leq M$, $|w_1| \leq \varepsilon$, $|a_1| = |f'(w_1)| \leq M\varepsilon/8$, and $M^3\varepsilon < 1$. Then the coefficients a_k in (4.2) satisfy (4.5) below for all $k \geq 1$, and the radius of convergence R_1 of the series (4.2) satisfies $R_1 \geq 1/(C(M\varepsilon)^{2/3})$.*

Let $\eta \in (0,1)$ and $M > 1$. Then there exists $\varepsilon_0 \in (0,1)$ such that if $0 < \varepsilon \leq \varepsilon_0$ and all the above assumptions are satisfied, then for all $z, w \in B(w_1, 2\varepsilon)$ (and hence, in particular, for all $z, w \in B(0, |w_1|)$), we have $|(f'(z)/f'(w)) - 1| < \eta$.

To prove Lemma 1, we first examine the power series expansions in $\zeta = w - w_1$ of the two sides of (4.1). The left hand side is

$$-f''(w) = -\sum_{n=0}^{\infty} n(n-1)a_n\zeta^{n-2} = -\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\zeta^n.$$

For the right hand side, note first that

$$w^2 = (w_1+\zeta)^2 = w_1^2 + 2w_1\zeta + \zeta^2.$$

We have

$$f'(w) = \sum_{n=0}^{\infty} na_n\zeta^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}\zeta^n,$$

so that

$$\left(f'(w)\right)^3 = \sum_{n=0}^{\infty}\zeta^n \sum_{k+l+p=n}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1},$$

where k, l, p take non-negative integer values (as do all indices to be considered).

Hence also

$$f(w)\left(f'(w)\right)^3 = \sum_{n=0}^{\infty}\zeta^n \sum_{k+l+p+q=n}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}a_q.$$

Next, we have

$$
\begin{aligned}
6w^2(f'(w))^3 &= 6(w_1^2 + 2w_1\zeta + \zeta^2)(f'(w))^3 \\
&= 6w_1^2 \sum_{n=0}^{\infty} \zeta^n \sum_{k+l+p=n} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1} \\
&\quad + 12w_1 \sum_{n=0}^{\infty} \zeta^n \sum_{k+l+p=n-1} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1} \\
&\quad + 6 \sum_{n=0}^{\infty} \zeta^n \sum_{k+l+p=n-2} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&-\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\zeta^n \\
&\quad = \sum_{n=0}^{\infty} \zeta^n \sum_{k+l+p+q=n} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}a_q \\
&\qquad + 6w_1^2 \sum_{n=0}^{\infty} \zeta^n \sum_{k+l+p=n} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1} \\
&\qquad + 12w_1 \sum_{n=0}^{\infty} \zeta^n \sum_{k+l+p=n-1} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1} \\
&\qquad + 6 \sum_{n=0}^{\infty} \zeta^n \sum_{k+l+p=n-2} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}.
\end{aligned} \tag{4.3}
$$

Of course, if $n = 0$, then the sum involving $k+l+p = n-1$ is (vacuous and hence) zero, and if $n = 0$ or $n = 1$, then the sum involving $k + l + p = n - 2$ is zero.

Comparing the coefficients of ζ^n on both sides of (4.3), we conclude that for each $n \geq 0$, we have

$$
\begin{aligned}
-(n+1)(n+2)a_{n+2} &= \sum_{k+l+p+q=n} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}a_q \\
&\quad + 6w_1^2 \sum_{k+l+p=n} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1} \\
&\quad + 12w_1 \sum_{k+l+p=n-1} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1} \\
&\quad + 6 \sum_{k+l+p=n-2} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}.
\end{aligned} \tag{4.4}
$$

Note that $a_1 = f'(w_1) \neq 0$. Let M and ε be as in the statement of Lemma 1.

Recall that $|a_0| \le M$, and suppose that C is a number with $C > 1$ such that

$$|a_k/a_1| \le 8D_k C^{k-1}(M\varepsilon)^{(2k/3)-1}/(k+1)^3 \tag{4.5}$$

for $1 \le k \le n+1$, for some $n \ge 0$, where we define for each $k \ge 1$,

$$\begin{aligned} D_k &= (M\varepsilon)^2 && \text{if } k \text{ is divisible by } 3, \\ D_k &= (M\varepsilon)^{1/3} && \text{if } k \text{ is of the form } k = 3j+1, \\ D_k &= M(M\varepsilon)^{5/3} && \text{if } k \text{ is of the form } k = 3j+2. \end{aligned}$$

We shall prove that, if C is a sufficiently large absolute constant, (4.5) holds also for $k = n+2$. Since (4.5) holds trivially for $k = 1$, it follows that (4.5) holds for all $k \ge 1$.

We mention in passing that our induction proof could easily be modified to show that

$$|a_k| \le D_k C^{k-1}(M\varepsilon)^{2k/3}/(k+1)^3$$

for all $k \ge 1$. In the present situation this conclusion follows from (4.5), since we are assuming that $|a_1| \le M\varepsilon/8$. However, when applying this general method to some other differential equation, it might be more convenient to obtain an upper bound directly for $|a_k|$ rather than for a quotient such as $|a_k/a_1|$, so we mention that our method can be used for that also. Our reason for estimating $|a_k/a_1|$ is that this will be needed to prove (4.8) below, which is the purpose for considering the inverse function f of w in the first place.

We proceed to prove (4.5) for $k = n+2$. Considering in the first sum those terms where $q = 0$ and $q \ge 1$ separately (because of the different upper bounds for $|a_0|$ and for $|a_q/a_1|$, $q \ge 1$, given by (4.5)) and using (4.4), we obtain

$$\begin{aligned} (n+1)(n+2)|a_{n+2}| \le &\sum_{k+l+p+q=n} (k+1)(l+1)(p+1)|a_{k+1}a_{l+1}a_{p+1}a_q| \\ &+ 6|w_1|^2 \sum_{k+l+p=n} (k+1)(l+1)(p+1)|a_{k+1}a_{l+1}a_{p+1}| \\ &+ 12|w_1| \sum_{k+l+p=n-1} (k+1)(l+1)(p+1)|a_{k+1}a_{l+1}a_{p+1}| \\ &+ 6 \sum_{k+l+p=n-2} (k+1)(l+1)(p+1)|a_{k+1}a_{l+1}a_{p+1}| \end{aligned}$$

$$
\begin{aligned}
&\le |a_1|^4 \sum_{\substack{k+l+p+q=n \\ q\ge 1}} (k+1)(l+1)(p+1)|a_{k+1}a_{l+1}a_{p+1}a_q|/|a_1|^4 \\
&\quad + (M+6|w_1|^2)|a_1|^3 \sum_{k+l+p=n} (k+1)(l+1)(p+1)|a_{k+1}a_{l+1}a_{p+1}|/|a_1|^3 \\
&\quad + 12|w_1||a_1|^3 \sum_{k+l+p=n-1} (k+1)(l+1)(p+1)|a_{k+1}a_{l+1}a_{p+1}|/|a_1|^3 \\
&\quad + 6|a_1|^3 \sum_{k+l+p=n-2} (k+1)(l+1)(p+1)|a_{k+1}a_{l+1}a_{p+1}|/|a_1|^3 \\
&\le 8^4|a_1|^4 C^{n-1}(M\varepsilon)^{2(n-3)/3} \sum_{\substack{k+l+p+q=n \\ q\ge 1}} \frac{D_{k+1}D_{l+1}D_{p+1}D_q}{((k+2)(l+2)(p+2))^2(q+1)^3} \\
&\quad + 8^3|a_1|^3(M+6|w_1|^2)C^n(M\varepsilon)^{(2n-3)/3} \sum_{k+l+p=n} \frac{D_{k+1}D_{l+1}D_{p+1}}{((k+2)(l+2)(p+2))^2} \\
&\quad + 12|w_1|8^3|a_1|^3C^{n-1}(M\varepsilon)^{(2n-5)/3} \sum_{k+l+p=n-1} \frac{D_{k+1}D_{l+1}D_{p+1}}{((k+2)(l+2)(p+2))^2} \\
&\quad + 6C^{n-2}8^3|a_1|^3(M\varepsilon)^{(2n-7)/3} \sum_{k+l+p=n-2} \frac{D_{k+1}D_{l+1}D_{p+1}}{((k+2)(l+2)(p+2))^2}. \qquad (4.6)
\end{aligned}
$$

We define, for $n \ge 0$, the expressions

$$A(n) = \sum_{\substack{k+l+p+q=n \\ q\ge 1}} \big((k+2)(l+2)(p+2)\big)^{-2}(q+1)^{-3}$$

and

$$C(n) = \sum_{k+l+p=n} \big((k+2)(l+2)(p+2)\big)^{-2}.$$

Thus $A(0) = 0$, corresponding to an empty sum. We define $A(n) = C(n) = 0$ if n is a negative integer.

Suppose now that n is divisible by 3 and write $X(k)$ for the residue class of k modulo 3, so that $X(k) \in \{0, 1, 2\}$.

In the first sum on the extreme right of (4.6), $k + l + p + q = n$, so that $(k+1)+(l+1)+(p+1)+q = n+3$, and hence $X(k+1)+X(l+1)+X(p+1)+X(q) = 0$ modulo 3. Since $M\varepsilon < 1 < M$ and $M^3\varepsilon < 1$, the largest possible value of $D_{k+1}D_{l+1}D_{p+1}D_q$ is $(M\varepsilon)^3$, obtained when one of $X(k+1)$, $X(l+1)$, $X(p+1)$, $X(q)$ is equal to 0 and all others are equal to 1. (Note that each of $k+1$, $l+1$, $p+1$, q is ≥ 1.) More precisely, apart from permutations, there are 5 possibilities for $(X(k+1), X(l+1), X(p+1), X(q))$; they are $(0,0,0,0)$, $(0,0,1,2)$, $(0,1,1,1)$, $(0,2,2,2)$ and $(1,1,2,2)$. In these cases, the value of $D_{k+1}D_{l+1}D_{p+1}D_q$ is $(M\varepsilon)^8$,

$M(M\varepsilon)^6$, $(M\varepsilon)^3$, $M^3(M\varepsilon)^7$, $M^2(M\varepsilon)^4$, respectively. Since $M\varepsilon < 1$ and $M^3\varepsilon < 1$, one can verify that the largest value among these is $(M\varepsilon)^3$.

In the second sum, we have $X(k+1)+X(l+1)+X(p+1)=0$ modulo 3; the largest value of $D_{k+1}D_{l+1}D_{p+1}$ is $M\varepsilon$, obtained when all of $X(k+1)$, $X(l+1)$, $X(p+1)$ are equal to 1. More precisely, apart from permutations, there are 4 possibilities for $(X(k+1), X(l+1), X(p+1))$, namely $(0,0,0)$, $(1,1,1)$, $(2,2,2)$ and $(0,1,2)$. In these cases, the value of $D_{k+1}D_{l+1}D_{p+1}$ is $(M\varepsilon)^6$, $M\varepsilon$, $M^3(M\varepsilon)^5 = M^8\varepsilon^5$ and $M(M\varepsilon)^4 = M^5\varepsilon^4$, respectively.

In the third sum, we have $X(k+1)+X(l+1)+X(p+1)=2$ modulo 3; the largest value of $D_{k+1}D_{l+1}D_{p+1}$ is $(M\varepsilon)^{8/3}$, obtained when one of $X(k+1)$, $X(l+1)$, $X(p+1)$ is equal to 0 while the others are equal to 1. More precisely, apart from permutations, there are 3 possibilities for $(X(k+1), X(l+1), X(p+1))$; they are $(0,0,2)$, $(0,1,1)$ and $(1,2,2)$. In these cases, the value of $D_{k+1}D_{l+1}D_{p+1}$ is $M(M\varepsilon)^{17/3}$, $(M\varepsilon)^{8/3}$ and $M^2(M\varepsilon)^{11/3}$, respectively.

In the fourth sum, we have $X(k+1)+X(l+1)+X(p+1)=1$ modulo 3; the largest value of $D_{k+1}D_{l+1}D_{p+1}$ is $M(M\varepsilon)^{7/3}$, obtained when one of $X(k+1)$, $X(l+1)$, $X(p+1)$ is equal to 2, while the others are equal to 1. More precisely, apart from permutations, there are 3 possibilities for $(X(k+1), X(l+1), X(p+1))$; they are $(0,0,1)$, $(0,2,2)$, $(1,1,2)$. In these cases, the value of $D_{k+1}D_{l+1}D_{p+1}$ is $M(M\varepsilon)^{13/3}$, $M^2(M\varepsilon)^{16/3}$ and $M(M\varepsilon)^{7/3}$, respectively.

Hence, when 3 divides n, we have

$$
\begin{aligned}
&(n+1)(n+2)|a_{n+2}| \\
&\quad \le 8^4|a_1|^4 C^{n-1}(M\varepsilon)^{(2(n-3)/3)+3} \sum_{\substack{k+l+p+q=n \\ q\ge 1}} ((k+2)(l+2)(p+2))^{-2}(q+1)^{-3} \\
&\qquad + 8^3|a_1|^3(M+6|w_1|^2)C^n(M\varepsilon)^{((2n-3)/3)+1} \sum_{k+l+p=n} ((k+2)(l+2)(p+2))^{-2} \\
&\qquad + 12|w_1|8^3|a_1|^3C^{n-1}(M\varepsilon)^{((2n-5)/3)+(8/3)} \sum_{k+l+p=n-1} ((k+2)(l+2)(p+2))^{-2} \\
&\qquad + 6C^{n-2}8^3|a_1|^3(M\varepsilon)^{((2n-7)/3)+(7/3)}M \sum_{k+l+p=n-2} ((k+2)(l+2)(p+2))^{-2} \\
&\quad \le 8^4|a_1|^4C^{n-1}(M\varepsilon)^{2(n-3)/3}(M\varepsilon)^3A(n) \\
&\qquad + 8^3|a_1|^3(M+6|w_1|^2)C^n(M\varepsilon)^{(2n-3)/3}M\varepsilon C(n) \\
&\qquad + 12|w_1|8^3|a_1|^3C^{n-1}(M\varepsilon)^{(2n-5)/3}(M\varepsilon)^{8/3}C(n-1) \\
&\qquad + 6C^{n-2}8^3|a_1|^3(M\varepsilon)^{(2n-7)/3}M(M\varepsilon)^{7/3}C(n-2).
\end{aligned}
$$

Suppose next that n is of the form $n = 3j+1$ for some non-negative integer j, and consider the right hand side of (4.6). In the first sum, $k+l+p+q=n$, so that

$(k+1)+(l+1)+(p+1)+q = n+3$, and hence $X(k+1)+X(l+1)+X(p+1)+X(q) = 1$ modulo 3. The largest possible value of $D_{k+1}D_{l+1}D_{p+1}D_q$ is $(M\varepsilon)^{4/3}$, obtained when $X(k+1)$, $X(l+1)$, $X(p+1)$, $X(q)$ are all equal to 1. More precisely, apart from permutations, there are 5 possibilities for $(X(k+1), X(l+1), X(p+1), X(q))$; they are $(0,0,0,1)$, $(0,0,2,2)$, $(0,1,1,2)$, $(1,1,1,1)$ and $(1,2,2,2)$. In these cases, the value of $D_{k+1}D_{l+1}D_{p+1}D_q$ is $(M\varepsilon)^{19/3}$, $M^2(M\varepsilon)^{22/3}$, $M(M\varepsilon)^{19/3}$, $(M\varepsilon)^{4/3}$ and $M^3(M\varepsilon)^{16/3}$, respectively.

In the second sum, we have $X(k+1)+X(l+1)+X(p+1) = 1$ modulo 3; the largest value of $D_{k+1}D_{l+1}D_{p+1}$ is $M(M\varepsilon)^{7/3}$, obtained when one of $X(k+1)$, $X(l+1)$, $X(p+1)$ is equal to 2 while the others are equal to 1.

In the third sum, we have $X(k+1)+X(l+1)+X(p+1) = 0$ modulo 3; and the largest value of $D_{k+1}D_{l+1}D_{p+1}$ is $M\varepsilon$, obtained when $X(k+1)$, $X(l+1)$, $X(p+1)$ are all equal to 1.

In the fourth sum, we have $X(k+1)+X(l+1)+X(p+1) = 2$ modulo 3; and the largest value of $D_{k+1}D_{l+1}D_{p+1}$ is $(M\varepsilon)^{8/3}$, obtained when one of $X(k+1)$, $X(l+1)$, $X(p+1)$ is equal to 0 while the others are equal to 1.

Hence, when n is of the form $n = 3j+1$, (4.6) yields

$$\begin{aligned}(n+1)(n+2)|a_{n+2}| \le\ & 8^4|a_1|^4C^{n-1}(M\varepsilon)^{2(n-3)/3}(M\varepsilon)^{4/3}A(n)\\ &+ 8^3|a_1|^3(M+6|w_1|^2)C^n(M\varepsilon)^{(2n-3)/3}M(M\varepsilon)^{7/3}C(n)\\ &+ 12|w_1|8^3|a_1|^3C^{n-1}(M\varepsilon)^{(2n-5)/3}M\varepsilon C(n-1)\\ &+ 6C^{n-2}8^3|a_1|^3(M\varepsilon)^{(2n-7)/3}(M\varepsilon)^{8/3}C(n-2).\end{aligned}$$

Suppose then that n is of the form $n = 3j+2$ for some non-negative integer j, and consider the right hand side of (4.6). In the first sum, $k+l+p+q = n$, so that $(k+1)+(l+1)+(p+1)+q = n+3$; hence $X(k+1)+X(l+1)+X(p+1)+X(q) = 2$ modulo 3. The largest possible value of $D_{k+1}D_{l+1}D_{p+1}D_q$ is $M(M\varepsilon)^{8/3}$, obtained when one of $X(k+1)$, $X(l+1)$, $X(p+1)$, $X(q)$ is equal to 2, while all the others are equal to 1. More precisely, apart from permutations, there are 5 possibilities for $(X(k+1), X(l+1), X(p+1), X(q))$, namely $(0,0,0,2)$, $(0,0,1,1)$, $(0,1,2,2)$, $(1,1,1,2)$ and $(2,2,2,2)$. In these cases, the value of $D_{k+1}D_{l+1}D_{p+1}D_q$ is $M(M\varepsilon)^{23/3}$, $(M\varepsilon)^{14/3}$, $M^2(M\varepsilon)^{17/3}$, $M(M\varepsilon)^{8/3}$ and $M^4(M\varepsilon)^{20/3}$, respectively.

In the second sum, we have $X(k+1)+X(l+1)+X(p+1) = 2$ modulo 3; the largest value of $D_{k+1}D_{l+1}D_{p+1}$ is $(M\varepsilon)^{8/3}$, obtained when one of $X(k+1)$, $X(l+1)$, $X(p+1)$ is equal to 0 while the others are equal to 1.

In the third sum, we have $X(k+1)+X(l+1)+X(p+1) = 1$ modulo 3; the largest value of $D_{k+1}D_{l+1}D_{p+1}$ is $M(M\varepsilon)^{7/3}$, obtained when one of $X(k+1)$, $X(l+1)$, $X(p+1)$ is equal to 2 while the others are equal to 1.

In the fourth sum, we have $X(k+1)+X(l+1)+X(p+1)=0$ modulo 3; the largest value of $D_{k+1}D_{l+1}D_{p+1}$ is $M\varepsilon$, obtained when $X(k+1)$, $X(l+1)$, $X(p+1)$ are all equal to 1.

Hence, when n is of the form $n=3j+2$, (4.6) yields

$$\begin{aligned}(n+1)(n+2)|a_{n+2}| \le\ & 8^4|a_1|^4C^{n-1}(M\varepsilon)^{2(n-3)/3}M(M\varepsilon)^{8/3}A(n)\\ &+8^3|a_1|^3(M+6|w_1|^2)C^n(M\varepsilon)^{(2n-3)/3}(M\varepsilon)^{8/3}C(n)\\ &+12|w_1|8^3|a_1|^3C^{n-1}(M\varepsilon)^{(2n-5)/3}M(M\varepsilon)^{7/3}C(n-1)\\ &+6C^{n-2}8^3|a_1|^3(M\varepsilon)^{(2n-7)/3}M\varepsilon C(n-2).\end{aligned}$$

We next estimate $A(n)$ and $C(n)$. We have $A(0)=0$ and $C(0)=1/64$.

Suppose that $n\ge 1$. Consider first $A(n)$. For given k, l, p, $q\ge 0$ with $k+l+p+q=n$, write $s=\max\{k,l,p,q\}$. Then $s\ge n/4$. Thus we have (since $2(q+1)^3\ge(q+2)^2$ for all $q\ge 1$)

$$\begin{aligned}A(n)&=\sum_{\substack{k+l+p+q=n\\ q\ge 1}}((k+2)(l+2)(p+2))^{-2}(q+1)^{-3}\\ &\le 2\sum_{n/4\le s\le n}(s+2)^{-2}\sum_{k=0}^{\infty}(k+2)^{-2}\sum_{l=0}^{\infty}(l+2)^{-2}\sum_{p=0}^{\infty}(p+2)^{-2}\\ &\le 2(\pi^2/6-1)^3n/(n/4+2)^2<9n/(n+8)^2.\end{aligned}$$

Consider now $C(n)$. For given k, l, $p\ge 0$ with $k+l+p=n$, write $s=\max\{k,l,p\}$. Then $s\ge n/3$. Thus

$$\begin{aligned}C(n)&=\sum_{k+l+p=n}((k+2)(l+2)(p+2))^{-2}\\ &\le\sum_{n/3\le s\le n}(s+2)^{-2}\sum_{k=0}^{\infty}(k+2)^{-2}\sum_{l=0}^{\infty}(l+2)^{-2}\\ &\le(\pi^2/6-1)^2n/(n/3+2)^2<4n/(n+6)^2.\end{aligned}$$

Let us now return to the case when 3 divides n, so $n\ge 3$. Using the above estimates for $A(n)$ and $C(n)$, we obtain

$$\begin{aligned}(n+1)(n+2)|a_{n+2}|&\le 8^4|a_1|^4C^{n-1}(M\varepsilon)^{2(n-3)/3}(M\varepsilon)^3 9n/(n+8)^2\\ &+8^3|a_1|^3(M+6|w_1|^2)C^n(M\varepsilon)^{(2n-3)/3}M\varepsilon 4n/(n+6)^2\\ &+12|w_1|8^3|a_1|^3C^{n-1}(M\varepsilon)^{(2n-5)/3}(M\varepsilon)^{8/3}4(n-1)/(n+5)^2\\ &+6C^{n-2}8^3|a_1|^3(M\varepsilon)^{(2n-7)/3}M(M\varepsilon)^{7/3}4(n-2)/(n+4)^2.\end{aligned}$$

Here $D_{n+2} = M(M\varepsilon)^{5/3}$ and hence, since $|a_1| \le (M\varepsilon)/8$,

$$
\begin{aligned}
&|a_{n+2}/a_1|/(8C^{n+1}D_{n+2}(M\varepsilon)^{(2n+1)/3}/(n+3)^3) \\
&\quad \le 8^3|a_1|^3C^{-2}(n+3)^3(M\varepsilon)^{-1}9n/(M(n+8)^2(n+1)(n+2)) \\
&\qquad + \left(\frac{8|a_1|}{M\varepsilon}\right)^2 \{(M+6|w_1|^2)C^{-1}(n+3)^3 4n/(M(n+6)^2(n+1)(n+2)) \\
&\qquad\qquad + 12|w_1|\varepsilon C^{-2}(n+3)^3 4(n-1)/((n+5)^2(n+1)(n+2)) \\
&\qquad\qquad + 6C^{-3}(n+3)^3 4(n-2)/((n+4)^2(n+1)(n+2))\} < 1,
\end{aligned}
$$

provided that C is at least as large as a certain positive absolute constant.

Suppose then that n is of the form $n = 3j+1 \ge 4$. In the same way as above, we deduce that

$$
\begin{aligned}
(n+1)(n+2)|a_{n+2}| &\le 8^4|a_1|^4C^{n-1}(M\varepsilon)^{2(n-3)/3}(M\varepsilon)^{4/3}9n/(n+8)^2 \\
&+ 8^3|a_1|^3(M+6|w_1|^2)C^n(M\varepsilon)^{(2n-3)/3}M(M\varepsilon)^{7/3}4n/(n+6)^2 \\
&+ 12|w_1|8^3|a_1|^3C^{n-1}(M\varepsilon)^{(2n-5)/3}M\varepsilon 4(n-1)/(n+5)^2 \\
&+ 6C^{n-2}8^3|a_1|^3(M\varepsilon)^{(2n-7)/3}(M\varepsilon)^{8/3}4(n-2)/(n+4)^2.
\end{aligned}
$$

Now $D_{n+2} = (M\varepsilon)^2$; and so

$$
\begin{aligned}
&|a_{n+2}/a_1|/(8C^{n+1}D_{n+2}(M\varepsilon)^{(2n+1)/3}/(n+3)^3) \\
&\quad \le 8^3|a_1|^3(M\varepsilon)^{-3}C^{-2}(n+3)^3 9n/((n+8)^2(n+1)(n+2)) \\
&\qquad + \left(\frac{8|a_1|}{M\varepsilon}\right)^2 \left\{\frac{(M+6|w_1|^2)C^{-1}(n+3)^3M(M\varepsilon)4n}{(n+6)^2(n+1)(n+2)}\right. \\
&\qquad + 12|w_1|C^{-2}(M\varepsilon)^{-1}(n+3)^3 4(n-1)/((n+5)^2(n+1)(n+2)) \\
&\qquad \left. + 6C^{-3}(n+3)^3 4(n-2)/((n+4)^2(n+1)(n+2))\right\} < 1,
\end{aligned}
$$

provided that C is at least as large as a certain absolute constant, since $|w_1| \le \varepsilon$ and $M^3\varepsilon < 1$.

The above leaves out the case $n = 1$. In this case, we have

$$
\begin{aligned}
(n+1)(n+2)|a_{n+2}| &\le 8^4|a_1|^4C^{n-1}(M\varepsilon)^{2(n-3)/3}(M\varepsilon)^{4/3}9n/(n+8)^2 \\
&+ 8^3|a_1|^3(M+6|w_1|^2)C^n(M\varepsilon)^{(2n-3)/3}M(M\varepsilon)^{7/3}4n/(n+6)^2 \\
&+ 12|w_1|C^{n-1}8^3|a_1|^3(M\varepsilon)^{(2n-5)/3}M\varepsilon(1/64)
\end{aligned}
$$

so that (with $n = 1$)

$$\begin{aligned}
&|a_{n+2}/a_1|/(8C^{n+1}D_{n+2}(M\varepsilon)^{(2n+1)/3}/(n+3)^3) \\
&\quad \le 8^3|a_1|^3(M\varepsilon)^{-3}64C^{-2}(1/54) \\
&\quad + \left(\frac{8|a_1|}{M\varepsilon}\right)^2 \{(M+6|w_1|^2)C^{-1}\cdot 64M(M\varepsilon)(2/3)(1/49) \\
&\qquad\qquad + 12|w_1|C^{-1}(M\varepsilon)^{-1}/6\} < 1
\end{aligned}$$

if C is at least as large as a certain absolute constant.

Suppose then that n is of the form $n = 3j + 2 \ge 5$. In the same way as above, we deduce that

$$\begin{aligned}
(n+1)(n+2)|a_{n+2}| &\le 8^4|a_1|^4C^{n-1}(M\varepsilon)^{2(n-3)/3}M(M\varepsilon)^{8/3}9n/(n+8)^2 \\
&+ \left(\frac{8|a_1|}{M\varepsilon}\right)^3 \{(M+6|w_1|^2)C^n(M\varepsilon)^{2(n+3)/3}(M\varepsilon)^{8/3}4n/(n+6)^2 \\
&\qquad + 12|w_1|C^{n-1}(M\varepsilon)^{2(n+2)/3}M(M\varepsilon)^{7/3}4(n-1)/(n+5)^2 \\
&\qquad + 6C^{n-2}(M\varepsilon)^{2(n+1)/3}M\varepsilon 4(n-2)/(n+4)^2\}.
\end{aligned}$$

Now $D_{n+2} = (M\varepsilon)^{1/3}$, and so

$$\begin{aligned}
&|a_{n+2}/a_1|/(8C^{n+1}D_{n+2}(M\varepsilon)^{(2n+1)/3}/(n+3)^3) \\
&\quad \le 8^3|a_1|^3C^{-2}M9n(n+3)^3/((n+8)^2(n+1)(n+2)) \\
&\quad + \left(\frac{8|a_1|}{M\varepsilon}\right)^2 \left\{\frac{(M+6|w_1|^2)C^{-1}(M\varepsilon)^3 4n(n+3)^3}{(n+6)^2(n+1)(n+2)}\right. \\
&\quad + 12|w_1|C^{-2}M(M\varepsilon)^2 4(n-1)(n+3)^3/((n+5)^2(n+1)(n+2)) \\
&\quad \left. + 6C^{-3}(M\varepsilon)^{1/3}4(n-2)(n+3)^3/((n+4)^2(n+1)(n+2))\right\} < 1
\end{aligned}$$

under similar conditions as above.

This leaves out the case $n = 2$. When $n = 2$, we have

$$\begin{aligned}
(n+1)(n+2)|a_{n+2}| &\le 8^4|a_1|^4C^{n-1}(M\varepsilon)^{2(n-3)/3}M(M\varepsilon)^{8/3}9n/(n+8)^2 \\
&+ \left(\frac{8|a_1|}{M\varepsilon}\right)^3 \{(M+6|w_1|^2)C^n(M\varepsilon)^{2(n+3)/3}(M\varepsilon)^{8/3}4n/(n+6)^2 \\
&\qquad + 12|w_1|C^{n-1}(M\varepsilon)^{2(n+2)/3}M(M\varepsilon)^{7/3}4(n-1)/(n+5)^2 \\
&\qquad + 6C^{n-2}(M\varepsilon)^{2(n+1)/3}M\varepsilon(1/64)\},
\end{aligned}$$

so that (with $n = 2$)

$$\begin{aligned}
&|a_{n+2}/a_1|/(8C^{n+1}D_{n+2}(M\varepsilon)^{(2n+1)/3}/(n+3)^3) \\
&\quad \le 8^3|a_1|^3C^{-2}M9n(n+3)^3/((n+8)^2(n+1)(n+2)) \\
&\qquad + \left(\frac{8|a_1|}{M\varepsilon}\right)^2 \left\{\frac{(M+6|w_1|^2)C^{-1}(M\varepsilon)^3 4n(n+3)^3}{(n+6)^2(n+1)(n+2)}\right. \\
&\qquad + 12|w_1|C^{-2}M(M\varepsilon)^2 4(n-1)(n+3)^3/((n+5)^2(n+1)(n+2)) \\
&\qquad \left. + 6C^{-3}(M\varepsilon)^{1/3}(1/64)(n+3)^3/((n+1)(n+2))\right\} < 1
\end{aligned}$$

under similar conditions as above. This completes the proof of (4.5) for $k = n+2$, and it follows that (4.5) holds for all $k \ge 1$.

It follows from (4.5) that the radius of convergence R_1 of the series (4.2) satisfies

$$1/R_1 = \limsup_{n\to\infty} |a_n|^{1/n} \le C(M\varepsilon)^{2/3},$$

so that

$$R_1 \ge C^{-1}/(M\varepsilon)^{2/3}. \tag{4.7}$$

We mention in passing that since C can be taken to be an absolute constant, we can get as large a value for R_1 as we like by considering situations where M is fixed while ε is taken to be small enough.

Now suppose that $\eta \in (0,1)$ and $M > 1$ are given, and let the assumptions of the first paragraph of Lemma 1 be satisfied. Let $C > 1$ be an absolute constant for which the conclusions of the first paragraph of Lemma 1 are satisfied. There exists $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$, we have $C(M\varepsilon)^{2/3} < 1$, $3\varepsilon C(M\varepsilon)^{2/3} < 1$, and

$$(M\varepsilon)^{2/3}C(2\varepsilon)/(1 - 2\varepsilon C(M\varepsilon)^{2/3}) < \eta/3.$$

Fix ε_0 by this condition and consider any $\varepsilon \in (0, \varepsilon_0]$ in what follows. Now $a_1 = f'(w_1)$, and if $w \in B(w_1, R_1)$, we have

$$f'(w) = f'(w_1) + \sum_{n=2}^{\infty} n a_n \zeta^{n-1}.$$

Thus

$$\frac{f'(w)}{f'(w_1)} - 1 = \sum_{n=2}^{\infty} n(a_n/a_1)\zeta^{n-1}.$$

Hence by (4.5), if $|\zeta| = |w - w_1| \le 2\varepsilon$, then

$$\left|\frac{f'(w)}{f'(w_1)} - 1\right| \le \sum_{n=2}^{\infty} n|a_n/a_1|\,|\zeta|^{n-1}$$
$$\le \frac{(M\varepsilon)^{1/3}}{M\varepsilon}\sum_{n=2}^{\infty} C^{n-1}|\zeta|^{n-1}(M\varepsilon)^{2n/3} = \frac{(M\varepsilon)^{2/3}C|\zeta|}{1 - C|\zeta|(M\varepsilon)^{2/3}} < \frac{\eta}{3}.$$

Then for all $z, w \in B(w_1, 2\varepsilon)$ and hence, in particular, for all $z, w \in B(0, |w_1|)$, we have

$$\left|\frac{f'(z)}{f'(w)} - 1\right| = \left|\frac{f'(z)}{f'(w_1)}\frac{f'(w_1)}{f'(w)} - 1\right|$$
$$= \left|\left(\frac{f'(z)}{f'(w_1)} - 1\right)\left(\frac{f'(w_1)}{f'(w)} - 1\right) + \left(\frac{f'(z)}{f'(w_1)} - 1\right) + \left(\frac{f'(w_1)}{f'(w)} - 1\right)\right|$$
$$\le \frac{\eta}{3}\left(1 + \left|\frac{f'(w_1)}{f'(w)} - 1\right|\right) + \left|\frac{f'(w_1)}{f'(w)} - 1\right|$$
$$= \frac{\eta}{3} + \left(1 + \frac{\eta}{3}\right)\left|\frac{1 - \frac{f'(w)}{f'(w_1)}}{1 + \left(\frac{f'(w)}{f'(w_1)} - 1\right)}\right|$$
$$\le \frac{\eta}{3} + \left(1 + \frac{\eta}{3}\right)\frac{\eta/3}{1 - (\eta/3)} = \frac{2\eta/3}{1 - (\eta/3)} < \eta. \tag{4.8}$$

This completes the proof of Lemma 1.

(3) We now return to the proof of Theorem 1. Note that if b is chosen to be a point in the disk $B(z_0, R)$, then $|b - z_0| < R$, so that we may use the fixed value $M = \max\{2, |z_0| + R\}$, independent of the particular choice of b, to satisfy $|b| \le M$. We fix a number $\eta \in (0, 1)$ satisfying $(\pi/2)(1 + \eta)/(1 - \eta) < 1.6$. Then we choose $\varepsilon \in (0, 1)$ so that all the conditions in Lemma 1 are satisfied, including those required to apply the last paragraph of Lemma 1 with this η (note that, by construction, we then have $3\varepsilon C(M\varepsilon)^{2/3} < 1$).

We now continue the discussion of Case V using this choice of ε. Assume that the points z_n, which we have defined in Case V, have been chosen using this ε. We apply Lemma 1 to $b = z_{2k-1}$. Since $w'(z_n) \to \infty$, it is clear that, for all sufficiently large values of k, the assumptions of Lemma 1 will be satisfied.

The estimate (4.8) allows one to follow the argument as given in [7], pp. 703–704, for the path modification of Γ to $\tilde{\Gamma}$. Thus, for each sufficiently large value of k, we replace the subarc Γ_k of Γ by the arc $f_k(Q_k)$, where f_k is the branch of the inverse function of $w(z)$ mapping a neighbourhood of $w(z_{2k-1})$ onto a neighbourhood of z_{2k-1} and Q_k is the shorter arc of the circle $S(0, \varepsilon)$ joining $w(z_{2k-1})$ to $w(z_{2k})$, the latter point being the first point on $w(\Gamma)$ after $w(z_{2k-1})$

with modulus ε. Note that, as a branch of an inverse function, f_k is necessarily one-to-one in its domain of definition. Applying Lemma 1 to $f = f_k$, we see that the radius of convergence of the power series of f_k with base point $w(z_{2k-1})$ is at least $1/(C(M\varepsilon)^{2/3}) > 3\varepsilon$, so that $w(\Gamma_k) \subset B(0,\varepsilon) \subset B(w(z_{2k-1}), 3\varepsilon)$. It follows that the paths $f_k(w(\Gamma_k))$ and $f_k(Q_k)$ have the same end points, namely, z_{2k-1} and z_{2k}. Hence this modification leads to a continuous path. We may continue w analytically as f^{-1} in the simply connected domain $D_k = f_k(B(w(z_{2k-1}), 3\varepsilon))$, which contains $f_k(Q_k)$. Since $f_k(w(z)) = z$ close to z_{2k-1}, and since $w(\Gamma_k) \subset B(w(z_{2k-1}), 3\varepsilon)$, we further have $\Gamma_k = f_k(w(\Gamma_k)) \subset D_k$. Hence the analytic continuation of the function w along the original arc Γ_k leads to the same result (namely, to the point $w(z_{2k})$) as the continuation of w along the new arc $f_k(Q_k)$. Performing this operation for all large k, we obtain a new path $\tilde{\Gamma}$.

Next, it follows from the estimate for $|f'| = |f_k'|$ given by Lemma 1 that the length of the new part of the modified path $\tilde{\Gamma}$ between the points $w_1 = w(z_{2k-1})$ and $w_2 = w(z_{2k})$, where $|w_1| = |w_2| = \varepsilon$, is at most

$$(\pi/2)|w_2 - w_1||f'(w_1)|(1+\eta).$$

On the other hand, the length of the corresponding original part Γ_k of the path Γ was at least

$$|f(w_2) - f(w_1)| \geq |f'(w_1)||w_2 - w_1|(1-\eta).$$

Hence the new length divided by the old length is at most $(\pi/2)(1+\eta)/(1-\eta) < 1.6$.

We now construct a Riemann surface S by following the modified path $\tilde{\Gamma}$ and by surrounding it with a sufficiently small neighbourhood in which the function w remains single-valued. This seems necessary since, a priori, branches of the modified path might intersect outside $B(z_0, R)$, so that following the path might give rise to different definitions of w at the points of intersection. On S, we will be on different leaves (layers) above such points of self-intersection, so that the function $w(z)$ will be well-defined on S.

Applying the argument in Case IV to S, we now conclude, considering a point of the modified path $\tilde{\Gamma}$ close enough to a, that w extends meromorphically to a neighbourhood of a. This contradicts our assumption and so completes the proof of Theorem 1.

5 Proof of Theorem 2

The proof of Theorem 2 follows closely that of Theorem 1. Therefore we indicate only the major differences, leaving a number of details to the reader.

To derive a counterpart to the equations (3.2), we first consider the equation

$$w''(z) = 2w(z)^3 + zw + \alpha, \tag{1.2}$$

where α is a complex parameter, close to a pole z_1 of w. Hence, we are looking for a pair of analytic functions u, v defined in a neighbourhood of z_1 and satisfying equations analogous to (3.2). A familiar analysis shows that the pole of w is simple and that the residue, which we denote by μ, is ± 1. In particular, $\mu^2 = 1$, which will be a useful relation. Furthermore, with $\zeta = z - z_1$, we have

$$w(z) = \frac{\mu}{\zeta} - \frac{\mu z_1}{6}\zeta - \frac{\alpha + \mu}{4}\zeta^2 + D\zeta^3 + \frac{\mu + 3\alpha}{72} z_1 \zeta^4 + O(\zeta^5), \tag{5.1}$$

where D is an arbitrary complex parameter.

We now set $v = 1/w$. If we want to express

$$w' = \frac{-\mu}{\zeta^2} - \frac{\mu z_1}{6} - \frac{\alpha + \mu}{2}\zeta + O(\zeta^2) \tag{5.2}$$

in terms of v, starting with a term involving $1/v^2$, it seems inevitable that the number μ, which can vary from one pole to another, must be introduced into the equations, unlike the case of Painlevé's first equation (1.1), where no such parameters are involved. Otherwise, we proceed similarly to §§3–4. By a simple computation,

$$v = \mu\zeta + \frac{\mu z_1}{6}\zeta^3 + \frac{\alpha + \mu}{4}\zeta^4 + \left(\frac{\mu z_1^2}{36} - D\right)\zeta^5 + \frac{3\alpha + 5\mu}{72} z_1 \zeta^6 + O(\zeta^7). \tag{5.3}$$

We invert the series (5.3) by the usual formulas for formal power series and obtain

$$\zeta = \mu v - \frac{\mu z_1}{6} v^3 - \frac{1 + \alpha\mu}{4} v^4 + O(v^5) = \mu v \left(1 - \frac{z_1}{6} v^2 - \frac{\alpha + \mu}{4} v^3 + O(v^4)\right).$$

Thus

$$\frac{1}{\zeta} = \frac{\mu}{v}\left(1 + \frac{z_1}{6} v^2 + \frac{\alpha + \mu}{4} v^3 + O(v^4)\right) \tag{5.4}$$

and

$$\frac{1}{\zeta^2} = \frac{1}{v^2}\left(1 + \frac{z_1}{3} v^2 + \frac{\alpha + \mu}{2} v^3 + O(v^4)\right). \tag{5.5}$$

Inserting these formulas (5.4) and (5.5) into the formula (5.2) for w', we obtain

$$w' = \frac{-\mu}{v^2}\left(1 + \frac{z_1}{3} v^2 + \frac{\alpha + \mu}{2} v^3 + O(v^4)\right) - \frac{\mu z_1}{6} - \frac{\alpha + \mu}{2}\mu v + O(v^2)$$

or, after writing $z_1 = z - \zeta$ and expressing ζ in terms of v again in the constant term,

$$w' = \frac{-\mu}{v^2} - \frac{\mu z}{2} + \frac{v}{2} - (1 + \alpha\mu)v + O(v^2).$$

The above has been presented to motivate the fact that we now define the function u by

$$w' = \frac{-\mu}{v^2} - \frac{\mu z}{2} - (\tfrac{1}{2} + \alpha\mu)v + uv^2. \tag{5.6}$$

Thus

$$v' = -v^2 w' = \mu + \frac{\mu z}{2} v^2 + (\tfrac{1}{2} + \alpha\mu)v^3 - uv^4. \tag{5.7}$$

Now by (1.2),

$$w'' = 2w^3 + zw + \alpha = \frac{2}{v^3} + \frac{z}{v} + \alpha.$$

By (5.6),

$$w'' = (w')' = \frac{2\mu v'}{v^3} - \frac{\mu}{2} - (\tfrac{1}{2} + \alpha\mu)v' + u'v^2 + 2uvv'.$$

Combining these equations and substituting the right hand side of (5.7) for v' in the result, we see, after some calculations, that

$$u' = \frac{\mu z}{2}(\tfrac{1}{2} + \alpha\mu) - \mu zuv + (\tfrac{1}{2} + \alpha\mu)^2 v - 3(\tfrac{1}{2} + \alpha\mu)uv^2 + 2u^2v^3. \tag{5.8}$$

Now, (5.7) and (5.8) form the system corresponding to (3.2) in §3. Solving for u in (5.6) and replacing v by $1/w$, we get

$$u = \frac{w}{2} + w'w^2 + \mu\left(w^4 + \frac{zw^2}{2} + \alpha w\right).$$

Multiplying both sides of (1.2) by $2w'$ and rearranging terms, we see that

$$\left((w')^2 - w^4 - 2\alpha w - zw^2\right)' = -w^2.$$

By (5.1),

$$w^2 = \frac{1}{\zeta^2} - \frac{z_1}{3} + O(\zeta),$$

so that the integral of $-w^2$ is equal to $1/\zeta + (z_1/3)\zeta + O(\zeta^2)$, apart from a possible additive constant. On the other hand, $-w'/w = 1/\zeta + \phi$, where ϕ is holomorphic, close to z_1. We see that the function

$$V(z) = (w')^2 - w^4 - 2\alpha w - zw^2 + \frac{w'}{w} \tag{5.9}$$

is holomorphic in a neighbourhood of z_1. This definition now corresponds to (3.4). Note that V is analytic in any domain where w is defined (as a single-valued function) and where w does not take the values 0 and ∞. Observe also that μ is not needed in the definition of V.

Now (compare (3.5)), using also (1.2), we obtain

$$\begin{aligned} W \equiv \frac{V'}{V} &= \frac{w^2(2w'w'' - 4w'w^3 - 2\alpha w' - w^2 - 2zww') + w''w - (w')^2}{w^2((w')^2 - w^4 - 2\alpha w - zw^2 + (w'/w))} \\ &= \frac{w^4 - (w')^2 + zw^2 + \alpha w}{w(w(w')^2 - w^5 + w' - 2\alpha w^2 - zw^3)}. \end{aligned} \tag{5.10}$$

We are now ready to proceed as in §§3–4. So let $w(z)$ be an arbitrary local solution to (1.2) with $w(z_0) = w_0 \neq \infty$, and let R be the radius of the largest disk centred at z_0 to which $w(z)$ can be continued as a single-valued meromorphic function. Assuming R is finite, let $a \in S(z_0, R)$ be a point for which there is no $\delta > 0$ such that $w(z)$ can be continued as a single-valued meromorphic function from $B(z_0, R)$ to $B(z_0, R) \cup B(a, \delta)$. We sketch below how this leads to a contradiction, which shows that we actually have $R = \infty$.

To argue in a way analogous to that used in §3, replace (3.1) by $w = v^{-1}$ and (5.6), (3.2) by (5.7) and (5.8), (3.4) by (5.9) and (3.5) by (5.10). We do not have a pole of w available to determine μ, but we may choose μ to be 1 or -1 in (5.6), as long as we then use the same value of μ in all subsequent calculations. We may now start exactly as in §3, repeating Case I and Case II with obvious modifications only. Thus we omit further details concerning Cases I and II.

In Case III, we have a sequence $\{z_n\}$ on Γ such that $z_n \to a$, $|w(z_n)| \to \infty$, $|V(z_n)| < A$ for some finite A, and $|v(z_n)| \to 0$. By (5.7) and (5.9), making use of $w = v^{-1}$, we obtain

$$\begin{aligned} V &= \frac{(v')^2 - 1 - 2\alpha v^3 - zv^2 - v'v^3}{v^4} \\ &= \frac{z^2}{4} - 2\mu u + \alpha z v - \mu z u v^2 - 2\alpha\mu u v^3 + (\alpha^2 - \tfrac{1}{4})v^2 + u^2 v^4 \\ &= \frac{z^2}{4} + \alpha z v + (\alpha^2 - \tfrac{1}{4})v^2 + \frac{uv^4}{v^4}(uv^4 - 2\mu - \mu z v^2 - 2\alpha\mu v^3) \\ &= \frac{z^2}{4} + \alpha z v + (\alpha^2 - \tfrac{1}{4})v^2 + u(uv^4 - 2\mu - \mu z v^2 - 2\alpha\mu v^3) \\ &= \frac{z^2}{4} + \alpha z v + (\alpha^2 - \tfrac{1}{4})v^2 + u(-2\mu - \mu z v^2 - 2\alpha\mu v^3) + u^2 v^4. \end{aligned} \tag{5.11}$$

Solving the quadratic equation for u obtained from the last expression of (5.11), we get

$$2uv^4 = 2\mu + \mu z v^2 + 2\alpha\mu v^3 \pm \sqrt{\Phi},$$

where

$$\Phi = (2\mu + \mu z v^2 + 2\alpha\mu v^3)^2 + [4V - z^2 - 4\alpha z v - 4(\alpha^2 - \tfrac{1}{4})v^2]v^4.$$

Therefore, uv^4 remains bounded on $\{z_n\}$ as $n \to \infty$. Since $|v(z_n)| \to 0$, the third last expression of (5.11) implies that either $uv^4 \to 0$ or $uv^4 \to 2\mu \neq 0$ on $\{z_n\}$ as $n \to \infty$.

If $uv^4 \to 0$, then $u^2v^4 = o(|u|)$ and (5.11) gives $u(-2\mu + o(1)) = O(1)$. Hence u is bounded on z_n. Thus this situation reduces to Case II.

Assuming now that $uv^4 \to 2\mu$, we conclude from the next to last expression of (5.11) that

$$u(uv^4 - 2\mu - \mu z v^2 - 2\alpha\mu v^3)$$

remains bounded on $\{z_n\}$, and so

$$h = v^{-4}(uv^4 - 2\mu - \mu z v^2 - 2\alpha\mu v^3)$$

remains bounded as well. Writing

$$h = u - \frac{2\mu}{v^4} - \frac{\mu z}{v^2} - \frac{2\alpha\mu}{v}, \tag{5.12}$$

we find

$$h' = u' - \frac{\mu}{v^2} + \left(\frac{2\alpha\mu}{v^2} + \frac{2\mu z}{v^3} + \frac{8\mu}{v^5}\right)v'. \tag{5.13}$$

Expressing u in terms of h and v from (5.13), we rewrite (5.7) in the form

$$v' = -\mu - \frac{1}{2}\mu z v^2 + \left(\frac{1}{2} - \alpha\mu\right)v^3 - hv^4. \tag{5.14}$$

Substituting (5.14) into (5.13), and (5.8) for u', expressing u in terms of h and v from (5.12), and noting that $\mu^2 = 1$, we obtain finally

$$h' = \left(\frac{\alpha}{2} - \frac{\mu}{4}\right)z + \left(\tfrac{1}{4} - \alpha\mu + \alpha^2 + h\mu z\right)v + \left(3\alpha h\mu - \tfrac{3}{2}h\right)v^2 + 2h^2v^3. \tag{5.15}$$

Now, (5.14) and (5.15) is a pair of differential equations for $v(z)$ and $h(z)$. Since h and v remain bounded on $\{z_n\}$, the same reasoning as in Case II in §3 shows that $w(z) = 1/v(z)$ has a simple pole at $z = a$.

Case IV now proceeds in a way similar to that in §4. We assume that there is a sequence $\{z_n\}$ on Γ such that $z_n \to a$, $|w(z_n)| \to \infty$, and $|W(z_n)| \to \infty$. From (5.9) and (5.10), we obtain

$$\begin{aligned} w^2VW &= w^4 - (w')^2 + zw^2 + \alpha w = -(V - w'/w) - \alpha w \\ &= (w'/w) - V - \alpha w, \end{aligned}$$

so that $w' = w^3VW + wV + \alpha w^2$ and $w'/w = w^2VW + V + \alpha w$. Substituting these into (5.9), we obtain

$$(w^6W^2 + 2w^4W + w^2)V^2 + (2\alpha w^5W + 2\alpha w^3 + w^2W)V + (\alpha^2 - 1)w^4 - zw^2 - \alpha w = 0,$$

which may be written as a formal quadratic equation

$$h_1V^2 + h_2V + h_3 = 0.$$

Note that $h_1 = w^2(w^2W+1)^2$. A calculation now shows that h_3/h_1 and h_2/h_1 tend to zero on the sequence z_n as $n \to \infty$, so that also $V(z_n) \to 0$; hence this case reduces to Case III.

The counterpart of Case V remains to be verified. To this end, consider (1.2) at a point b with $w(b) = w_1$ and $w'(b) \neq 0$. Then in a small neighbourhood of w_1, $w(z)$ has an inverse function taking w_1 to b. We denote this inverse function not by $z(w)$ but by $f(w)$. There is a power series expansion

$$f(w) = \sum_{n=0}^{\infty} a_n(w-w_1)^n \tag{5.16}$$

valid in some disk $B(w_1, r) = \{ w : |w - w_1| < r \}$ centred at w_1. We write from now on $\zeta = w - w_1$.

We can write (1.2) in the form

$$-f''(w) = (2w^3 + wf(w) + \alpha)(f'(w))^3 \tag{5.17}$$

in $B(w_1, r)$. Let now R_2 be the radius of convergence of the series (5.16). Note that the series (5.16) is completely determined by the function $w(z)$ in a small neighbourhood of the point b. It then follows that the function $f(w)$ given by the series (5.16) satisfies (5.17) in the whole disk $B(w_1, R_2)$. Since f arises from (1.2), we have $R_2 < \infty$.

If f' has a zero of order $m \geq 1$ at some point $w \in B(w_1, R_2)$, then f'' has a zero of order $m-1$ at w. Hence the left hand side of (5.17) has a zero of order $m-1$ at w, while the right hand side of (5.17) has a zero of order at least $3m$ at w. Since $3m > m - 1$, this is impossible; so we conclude that f' has no zeros in $B(w_1, R_2)$.

The following statement is the counterpart of Lemma 1.

Lemma 2. *Suppose that the function $w(z)$ satisfies (1.2) in a neighbourhood of the point b and that $w(b) = w_1$ and $w'(b) \neq 0$. Let $f(w)$ be the inverse function of w, given by (5.16) in a small neighbourhood of w_1 and taking w_1 to b. There exists an absolute constant $C > 1$ such that the following holds. Suppose that $M > 1$, $\delta \in (0,1)$, and that $|b| = |a_0| \leq M$, $|\alpha| \leq M$, $|2w_1^3 + w_1a_0 + \alpha| \leq M$, $|6w_1^2 + a_0| \leq M$, $|w_1| \leq \delta$, $|a_1| = |f'(w_1)| \leq M\delta/8$, and $M^3\delta < 1$. Then the coefficients a_k in (5.16) satisfy (5.19) below for all $k \geq 1$, and the radius of convergence R_2 of the series (5.16) satisfies $R_2 \geq 1/(C(M\delta)^{1/2})$.*

Let $\eta \in (0,1)$ and $M > 1$ be preassigned numbers. Then there exists a number $\delta_0 \in (0,1)$ such that if $0 < \delta \leq \delta_0$ and if all of the above assumptions are satisfied, *then for all* $z, w \in B(w_1, 2\delta)$ *(and hence, in particular, for all* $z, w \in B(0, |w_1|)$*), we have* $|(f'(z)/f'(w)) - 1| < \eta$.

To prove Lemma 2, we examine the power series expansions in $\zeta = w - w_1$ of the two sides of (5.17). The left hand side is

$$-f''(w) = -\sum_{n=0}^{\infty} n(n-1)a_n\zeta^{n-2} = -\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\zeta^n.$$

For the right hand side, note first that

$$w^3 = (w_1 + \zeta)^3 = w_1^3 + 3w_1^2\zeta + 3w_1\zeta^2 + \zeta^3.$$

We have

$$f'(w) = \sum_{n=0}^{\infty} na_n\zeta^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}\zeta^n,$$

so that

$$\big(f'(w)\big)^3 = \sum_{n=0}^{\infty}\zeta^n \sum_{k+l+p=n} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1},$$

where k, l, p take non-negative integer values (as do all indices to be considered).

Hence also

$$f(w)\big(f'(w)\big)^3 = \sum_{n=0}^{\infty}\zeta^n \sum_{k+l+p+q=n} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}a_q$$

and

$$\begin{aligned} wf(w)\big(f'(w)\big)^3 &= (w_1+\zeta)f(w)\big(f'(w)\big)^3 \\ &= w_1\sum_{n=0}^{\infty}\zeta^n \sum_{k+l+p+q=n} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}a_q \\ &\quad + \sum_{n=1}^{\infty}\zeta^n \sum_{k+l+p+q=n-1} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}a_q. \end{aligned}$$

Next, we have

$$\begin{aligned}(2w^3+\alpha)\big(f'(w)\big)^3 &= (2(w_1^3+3w_1^2\zeta+3w_1\zeta^2+\zeta^3)+\alpha)\big(f'(w)\big)^3\\ &= (2w_1^3+\alpha)\sum_{n=0}^{\infty}\zeta^n \sum_{k+l+p=n}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}\\ &\quad + 6w_1^2\sum_{n=1}^{\infty}\zeta^n \sum_{k+l+p=n-1}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}\\ &\quad + 6w_1\sum_{n=2}^{\infty}\zeta^n \sum_{k+l+p=n-2}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}\\ &\quad + 2\sum_{n=3}^{\infty}\zeta^n \sum_{k+l+p=n-3}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}.\end{aligned}$$

We conclude that

$$\begin{aligned}&-\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\zeta^n\\ &\quad= w_1\sum_{n=0}^{\infty}\zeta^n \sum_{k+l+p+q=n}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}a_q\\ &\qquad + \sum_{n=1}^{\infty}\zeta^n \sum_{k+l+p+q=n-1}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}a_q\\ &\qquad + (2w_1^3+\alpha)\sum_{n=0}^{\infty}\zeta^n \sum_{k+l+p=n}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}\\ &\qquad + 6w_1^2\sum_{n=1}^{\infty}\zeta^n \sum_{k+l+p=n-1}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}\\ &\qquad + 6w_1\sum_{n=2}^{\infty}\zeta^n \sum_{k+l+p=n-2}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}\\ &\qquad + 2\sum_{n=3}^{\infty}\zeta^n \sum_{k+l+p=n-3}(k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}.\end{aligned}\tag{5.18}$$

Comparing the coefficients of ζ^n on both sides of (5.18), and taking separately the terms with $q = 0$ in the first two terms on the right hand side of (5.18), we conclude that for each $n \geq 0$, we have

$$
\begin{aligned}
-(n+1)(n+2)a_{n+2} &= w_1 \sum_{\substack{k+l+p+q=n \\ q\geq 1}} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}a_q \\
&+ \sum_{\substack{k+l+p+q=n-1 \\ q\geq 1}} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}a_q \\
&+ (2w_1^3 + w_1 a_0 + \alpha) \sum_{k+l+p=n} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1} \\
&+ (6w_1^2 + a_0) \sum_{k+l+p=n-1} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1} \\
&+ 6w_1 \sum_{k+l+p=n-2} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1} \\
&+ 2 \sum_{k+l+p=n-3} (k+1)(l+1)(p+1)a_{k+1}a_{l+1}a_{p+1}.
\end{aligned}
$$

Here and later, an empty sum is considered to be equal to zero.

Note that $a_1 = f'(w_1) \neq 0$. Let M and δ be as in the statement of Lemma 2. Suppose that the inequalities involving M, δ, α, a_0, and w_1, given in the assumption of Lemma 2, are satisfied, and suppose that C is a number with $C > 1$ such that

$$
|a_k/a_1| \leq 8E_k C^{k-1}(M\delta)^{(k-2)/2}/(k+1)^3 \tag{5.19}
$$

for $1 \leq k \leq n+1$, for some $n \geq 0$, where we define for each $k \geq 1$,

$$
\begin{aligned}
E_k &= (M\delta)^2 && \text{if } k \text{ is divisible by 4,} \\
E_k &= (M\delta)^{1/2} && \text{if } k \text{ is of the form } k = 4j+1, \\
E_k &= M(M\delta)^2 && \text{if } k \text{ is of the form } k = 4j+2, \\
E_k &= M(M\delta)^{3/2} && \text{if } k \text{ is of the form } k = 4j+3.
\end{aligned}
$$

We claim that then, if C is a sufficiently large absolute constant, (5.19) holds also for $k = n+2$. Since (5.19) holds trivially for $k = 1$, it follows that (5.19) holds for all $k \geq 1$.

That (5.19) is valid for all $k \geq 1$ can be proved by an induction similar to that used in the proof of Lemma 1. Hence we omit the details. We only mention that the counterparts of the conditions $C(M\varepsilon)^{2/3} < 1$, $3\varepsilon C(M\varepsilon)^{2/3} < 1$, and

$$
(M\varepsilon)^{2/3}C(2\varepsilon)/(1 - 2\varepsilon C(M\varepsilon)^{2/3}) < \eta/3
$$

needed for the proof of the second paragraph of Lemma 1 are $C(M\delta)^{1/2} < 1$, $3\delta C(M\delta)^{1/2} < 1$, and

$$
(M\delta)^{1/2}C(2\delta)/(1 - 2\delta C(M\delta)^{1/2}) < \eta/3.
$$

We shall now take the view that Lemma 2 has been proved.

Note that in Lemma 2, the upper bound for $|a_k|$ has period 4, while in Lemma 1, that upper bound had period 3. This is connected to the difference in the exponents 2/3 of $M\varepsilon$ and 1/2 of $M\delta$ in the lower bound for the radius of convergence of the power series for f.

We may now finish the proof of Theorem 2 in the same way as we dealt with Case V of Theorem 1 in §4, part **(3)**. Briefly, we note that we only consider points $b \in B(z_0, R)$ so that if $M = 6+|z_0|+R+|\alpha| > 1$, then the assumptions of Lemma 2 will be satisfied provided that $|w_1| \leq \delta$ and $0 < \delta < 1$. We fix a number $\eta \in (0,1)$ satisfying $(\pi/2)(1+\eta)/(1-\eta) < 1.6$. After that, we choose $\delta \in (0,1)$ so that all the conditions in Lemma 2 are satisfied, including those required to apply the last paragraph of Lemma 2 with this η (note that, by construction, we then have $3\delta C(M\delta)^{1/2} < 1$).

We are assuming that $V(z)$ and $w(z)$ are unbounded on Γ but

$$\liminf_{z \to a,\, z \in \Gamma} |w(z)| = 0.$$

We have fixed $\delta \in (0,1)$ above. Thus there exists a sequence of points $\{z_n\}$ on Γ such that $|w(z)| < \delta$ for $z \in \Gamma_k$, where Γ_k is the arc of Γ from z_{2k-1} to z_{2k}, for $k \geq 1$, while $|w(z)| \geq \delta$ on the remaining subarcs of Γ. As in the proof of Theorem 1, we see that $|w'(z_n)| \to \infty$.

We apply Lemma 2 to $b = z_{2k-1}$. Since $w'(z_n) \to \infty$, it is clear that for all sufficiently large values of k, the assumptions of Lemma 2 will be satisfied.

For each sufficiently large value of k, we replace the subarc Γ_k of Γ by the arc $f_k(Q_k)$, where f_k is the branch of the inverse function of $w(z)$ mapping a neighbourhood of $w(z_{2k-1})$ onto a neighbourhood of z_{2k-1} and Q_k is the shorter arc of the circle $S(0,\delta)$ joining $w(z_{2k-1})$ to $w(z_{2k})$, the latter point being the first point on $w(\Gamma)$ after $w(z_{2k-1})$ with modulus δ. Applying Lemma 2 to $f = f_k$, we see that the radius of convergence of the power series of f_k with base point $w(z_{2k-1})$ is at least $1/(C(M\delta)^{1/2}) > 3\delta$, so that $w(\Gamma_k) \subset B(0,\delta) \subset B(w(z_{2k-1}), 3\delta)$. It follows that the paths $f_k(w(\Gamma_k))$ and $f_k(Q_k)$ have the same end points, namely, z_{2k-1} and z_{2k}. Hence this modification leads to a continuous path. Performing this operation for all large k, we obtain a new path $\tilde{\Gamma}$.

Next, it follows in the same way as in the proof of Theorem 1 that for each modified part of Γ, the new length divided by the old length is at most $(\pi/2)(1+\eta)/(1-\eta) < 1.6$.

We now construct a Riemann surface S which amounts to a small neighbourhood of the modified path. The proof is completed in the same way as for Theorem 1. This completes our sketch of the proof of Theorem 2.

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