EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR SOME NONLINEAR ELLIPTIC EQUATIONS

By

LUCIO BOCCARDO, THIERRY GALLOUËT AND LUIGI ORSINA

1 Introduction and statement of results

In this paper we consider nonlinear boundary value problems whose simplest model is the following:

$$\begin{cases} -\Delta u + u\,|\nabla u|^2 = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set in $\boldsymbol{R}^N$, $N \geq 2$, and μ is a Radon measure on Ω.

We are interested in existence and nonexistence of solutions for (1.1). More precisely, we prove the existence of a solution u in $H_0^1(\Omega)$ for problem (1.1) if and only if the measure μ does not charge the sets of capacity zero in Ω. The main tool of our proof will be a characterization result, proved by the authors in [6], which states that every measure which is zero on sets of zero capacity is an element of $L^1(\Omega) + H^{-1}(\Omega)$ (see also Theorem 2.3, below). We also prove that if we consider a sequence $\{u_n\}$ of solutions of (1.1) with $L^\infty(\Omega)$ data μ_n converging to a nonzero measure which is singular with respect to the capacity (for example, a Dirac mass), then u_n converges to zero as n tends to infinity. For semilinear problems, an analogous result can be found in [10].

Our result is closely related to the work of H. Brezis and L. Nirenberg (see [11]), where (as a particular case of more general results) it is proved that if μ is a bounded $L^\infty(\Omega)$ function and u is a smooth solution of (1.1) in $\Omega\backslash K$, with K a closed set of zero capacity, then u is smooth in the whole of Ω; that is to say, u cannot be singular on sets of zero capacity.

The link between the two results is due to the fact that, as a consequence of a theorem of [17] (see Proposition 3.1 below), every bounded Radon measure μ on Ω can be decomposed in a unique way as the sum of a measure μ_0 which is zero on the sets of zero capacity and a measure λ which is concentrated on a set E of

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 73 (1997)

zero capacity. Hence, problem (1.1) implies that

$$-\Delta u + u|\nabla u|^2 = \mu_0 \qquad \text{in } \Omega \backslash E.$$

The result of [11] states that if μ_0 is an $L^\infty(\Omega)$ function, then u is a solution of the problem with datum μ_0 in the whole Ω; that is to say, the term λ does not play any role. Our result states that if we look for solutions in the whole of Ω, then we have to take into account only the term μ_0, without the $L^\infty(\Omega)$ restriction on μ_0.

Problem (1.1), and some variants of it, have been widely studied in the literature. For example, if the nonlinear lower order term is independent of ∇u, and behaves like $|u|^{q-1}u$, with $q > 1$, there are existence and nonexistence results depending on the measure μ; see, e.g., [10], [12], [13], [1], [18], [20]. If the nonlinear term is as in (1.1), there are existence results if μ belongs to $H^{-1}(\Omega)$ (see [2], [7] and [16]) or to $L^1(\Omega)$ (see [4], [5], and [21]).

Since we are going to consider more general nonlinear boundary value problems, we now state the assumptions that will hold throughout the paper.

Let Ω be a bounded, open subset of $\boldsymbol{R}^N$, $N \geq 2$. Let p be a real number such that $1 < p \leq N$, and let p' be its Hölder conjugate exponent (i.e., $1/p + 1/p' = 1$).

Let $a : \Omega \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ be a Carathéodory function (i.e., $a(x, \cdot)$ is continuous on $\boldsymbol{R}^N$ for almost every x in Ω, and $a(\cdot, \xi)$ is measurable on Ω for every ξ in $\boldsymbol{R}^N$) such that

$$a(x,\xi)\cdot\xi \geq \alpha\,|\xi|^p\,, \tag{1.2}$$

for almost every $x \in \Omega$ and every $\xi \in \boldsymbol{R}^N$, where α is a positive constant;

$$|a(x,\xi)| \leq \ell(x) + \beta\,|\xi|^{p-1}\,, \tag{1.3}$$

for almost every $x \in \Omega$ and every $\xi \in \boldsymbol{R}^N$, where β is a positive constant and ℓ belongs to $L^{p'}(\Omega)$; and

$$[a(x,\xi) - a(x,\eta)]\cdot(\xi - \eta) > 0\,, \tag{1.4}$$

for almost every $x \in \Omega$ and every ξ and η in $\boldsymbol{R}^N$ with $\xi \neq \eta$.

Let

$$A(u) = -\mathrm{div}\,(a(x, \nabla u))\,.$$

By (1.2), (1.3) and (1.4), A is a monotone and coercive differential operator acting between $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$; hence, it is surjective on $W^{-1,p'}(\Omega)$ (see [19]).

Let $g : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}$ be a Carathéodory function (i.e., $g(\cdot, s, \xi)$ is measurable in Ω for any (s, ξ) in $\boldsymbol{R} \times \boldsymbol{R}^N$, and $g(x, \cdot, \cdot)$ is continuous in $\boldsymbol{R} \times \boldsymbol{R}^N$ for almost every $x \in \Omega$) such that

$$|g(x,s,\xi)| \le b(|s|)\,[|\xi|^p + d(x)]\,, \tag{1.5}$$

with b a real valued, positive, increasing, continuous function, and d a nonnegative function in $L^1(\Omega)$;

$$g(x,s,\xi)\,\mathrm{sgn}(s) \ge \rho\,|\xi|^p\,, \tag{1.6}$$

for almost every x in Ω, every ξ in $\boldsymbol{R}^N$, and every s in $\boldsymbol{R}$ such that $|s| \ge \sigma$, where σ and ρ are two positive real numbers.

For every compact subset K of Ω, the p-capacity of K with respect to Ω is defined as

$$\mathrm{cap}_p(K,\Omega) = \inf\left\{\int_\Omega |\nabla u|^p\,dx : u \in C_0^\infty(\Omega),\ u \ge \chi_K\right\},$$

where χ_K is the characteristic function of K; we use the convention that $\inf\emptyset = +\infty$. The p-capacity of any open subset U of Ω is then defined by

$$\mathrm{cap}_p(U,\Omega) = \sup\left\{\mathrm{cap}_p(K,\Omega),\ K \text{ compact},\ K \subseteq U\right\},$$

and the p-capacity of any subset $B \subseteq \Omega$ by

$$\mathrm{cap}_p(B,\Omega) = \inf\left\{\mathrm{cap}_p(U,\Omega),\ U \text{ open},\ B \subseteq U\right\}.$$

We denote by $\mathcal{M}_{\mathrm{b}}(\Omega)$ the space of all signed measures on Ω, i.e., the space of all σ-additive set functions μ with values in $\boldsymbol{R}$ defined on the Borel σ-algebra. Note that if μ belongs to $\mathcal{M}_{\mathrm{b}}(\Omega)$, then $|\mu|$ (the total variation of μ) is a bounded positive measure on Ω. We denote by $\mathcal{M}_0^p(\Omega)$ the space of all measures μ in $\mathcal{M}_{\mathrm{b}}(\Omega)$ such that $\mu(E) = 0$ for every set such that $\mathrm{cap}_p(E,\Omega) = 0$. Examples of measures in $\mathcal{M}_0^p(\Omega)$ are $L^1(\Omega)$ function and measures in $W^{-1,p'}(\Omega)$.

Define, for s and k in $\boldsymbol{R}$, with $k \ge 0$, $T_k(s) = \max(-k, \min(k,s))$ and $G_k(s) = s - T_k(s)$.

Our result is the following.

Theorem 1.1. *Let μ be a measure in $\mathcal{M}_{\mathrm{b}}(\Omega)$. Then there exists a solution u of*

$$\begin{cases} A(u) + g(x,u,\nabla u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

in the sense that u belongs to $W_0^{1,p}(\Omega)$, $g(x,u,\nabla u)$ belongs to $L^1(\Omega)$, and

$$\int_\Omega a(x,\nabla u)\cdot\nabla v\,dx + \int_\Omega g(x,u,\nabla u)\,v\,dx = \int_\Omega v\,d\mu\,, \tag{1.8}$$

for every v in $C_0^\infty(\Omega)$, if and only if μ belongs to $\mathcal{M}_0^p(\Omega)$.

Remark 1.2. Note that the solution of (1.7) given by the previous theorem belongs to $W_0^{1,p}(\Omega)$ even if μ belongs to $L^1(\Omega)$: this is in sharp contrast with the results in the case $g \equiv 0$, since the solutions of

$$\begin{cases} A(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

are known to belong only to $W_0^{1,q}(\Omega)$ for every $q < N(p-1)/(N-1)$ (see [3]). The better regularity of the solution of (1.7) is due to assumption (1.6). Indeed, even if the nonlinear lower order term has a growth of order p with respect to the gradient, but does not satisfy (1.6), we may not have solutions in $W_0^{1,p}(\Omega)$. To show this, let us consider the following example. Let $p = 2$, $\Omega = \{x \in \boldsymbol{R}^N : |x| < 1\}$, and $a(x,\xi) = \xi$, so that $A(u) = -\Delta u$, the Laplacian. Let $m = \frac{1}{2}N - 1$ and let $u(\rho) = \rho^{-m} - 1$, where $\rho = |x|$. Then u is solution of

$$-\Delta u + \frac{|\nabla u|^2}{1+u} = f(\rho),$$

where

$$f(\rho) = \frac{(N-2)^2}{2\rho^{m+2}}.$$

It is easily seen that u does not belong to $H_0^1(\Omega)$, while f belongs to $L^s(\Omega)$, for every s in $\left[1, \dfrac{2N}{N+2}\right)$.

Remark 1.3. The result of Theorem 1.1 explains the restriction $p \le N$. Indeed, if $p > N$, then there are no nonempty sets of zero p-capacity; in other words, every measure in $\mathcal{M}_b(\Omega)$ is in $\mathcal{M}_0^p(\Omega)$. Moreover, due to the Sobolev embeddings, every measure in $\mathcal{M}_b(\Omega)$ is in $W^{-1,p'}(\Omega)$.

2 Proof of Theorem 1.1

In the following, we denote by c any constant which depends on the various quantities of the problem but not on n. The value of c may vary from line to line.

We begin with an existence result for problem (1.7) in the case in which the datum μ is regular.

Theorem 2.1. *Let f be an $L^\infty(\Omega)$ function, and let F be an element of $(L^s(\Omega))^N$, with $s > N/(p-1)$. Then there exists a solution u in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of*

$$\begin{cases} A(u) + g(x,u,\nabla u) = f - \operatorname{div}(F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

in the sense that

$$\int_\Omega a(x,\nabla u)\cdot\nabla v + \int_\Omega g(x,u,\nabla u)\,v = \int_\Omega f\,v + \int_\Omega F\cdot\nabla v\,,$$

for every v in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Remark 2.2. We remark that the result of the preceding theorem does not exist in the literature. Indeed, existence results in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ have been given, under the same assumptions on f and F as in Theorem 2.1, for the problem

$$A(u) + g(x,u,\nabla u) + \alpha_0\,|u|^{p-2}\,u = f - \operatorname{div}(F)\,,$$

with $\alpha_0 > 0$ (see, for example, [9] and the references contained therein), or under a sign assumption on g: namely,

$$g(x,s,\xi)\,s \ge 0\,,$$

for almost every x in Ω, for every s in $\boldsymbol{R}$, for every ξ in $\boldsymbol{R}^N$ (see [2]). In our case, (1.6) gives a sign condition on g only for large values of s.

Proof of Theorem 2.1. For the sake of simplicity, we give a proof of this result in the case $f \equiv 0$; the case of f different from zero can be dealt with by means of minor technical modifications.

Let n be in $\boldsymbol{N}$ and let

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}\,|g(x,s,\xi)|}\,.$$

Then $g_n(x,s,\xi)$ is bounded, satisfies (1.5) and, thanks to (1.6), is such that

$$g_n(x,s,\xi)\operatorname{sgn}(s) \ge 0\,, \tag{2.2}$$

for almost every x in Ω, for every ξ in $\boldsymbol{R}^N$, and for every s in $\boldsymbol{R}$ with $|s| \ge \sigma$.

Since g_n is bounded, by classical results (see for example [19]) there exists a solution u_n in $W_0^{1,p}(\Omega)$ of

$$\begin{cases} A(u_n) + g_n(x,u_n,\nabla u_n) = -\operatorname{div}(F) & \text{in } \Omega,\\ u_n = 0 & \text{on } \partial\Omega,\end{cases}$$

in the sense that

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla v\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)\,v\,dx = \int_\Omega F\cdot\nabla v\,dx\,, \tag{2.3}$$

for every v in $W_0^{1,p}(\Omega)$.

As proved in [8], [9], if the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$, then it is possible to extract a subsequence which converges strongly in $W_0^{1,p}(\Omega)$ to a solution of (2.1). Thus, the proof of the theorem will be achieved if we prove an $L^\infty(\Omega)$ *a priori* estimate on the sequence $\{u_n\}$.

To do this, we choose $v = G_k(u_n)$ as test function in (2.3), with $k \geq \sigma$. We obtain

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla G_k(u_n)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n) G_k(u_n)\, dx = \int_\Omega F \cdot \nabla G_k(u_n)\, dx.$$

Since $G_k(s)$ has the same sign as s, and since $G_k(s)$ is different from zero only where $|s| \geq k \geq \sigma$, (2.2) then implies

$$\int_\Omega g_n(x, u_n, \nabla u_n)\, G_k(u_n)\, dx \geq 0\,.$$

On the other hand, setting

$$A_{k,n} = \{|u_n| \geq k\}\,,$$

we have, by the Young inequality,

$$\int_\Omega F \cdot \nabla G_k(u_n)\, dx \leq c \int_{A_{k,n}} |F|^{p'}\, dx + \frac{\alpha}{2} \int_\Omega |\nabla G_k(u_n)|^p\, dx\,.$$

Thus, using (1.2) we have

$$\frac{\alpha}{2} \int_\Omega |\nabla G_k(u_n)|^p\, dx \leq c \int_{A_{k,n}} |F|^{p'}\, dx\,.$$

From now on we will follow the method introduced by G. Stampacchia in order to prove $L^\infty(\Omega)$ *a priori* estimates for solutions of elliptic equations (see [22]). Since $|F|$ belongs to $L^s(\Omega)$, and $s > p'$, we have, by the Hölder inequality,

$$\int_{A_{k,n}} |F|^{p'}\, dx \leq \|F\|_{(L^s(\Omega))^N} \operatorname{meas}(A_{k,n})^{1-p'/s}\,.$$

On the other hand, by the Sobolev embedding, we have

$$\int_\Omega |\nabla G_k(u_n)|^p\, dx \geq c \left(\int_\Omega |G_k(u_n)|^{p^*}\, dx \right)^{p/p^*}\,,$$

where $p^* = Np/(N-p)$ if $p < N$, and is any real number greater than $Ns(N-1)/(Ns - N - s)$ if $p = N$. Thus, we have

$$\left(\int_\Omega |G_k(u_n)|^{p^*}\, dx \right)^{p/p^*} \leq c \operatorname{meas}(A_{k,n})^{1-p'/s}\,.$$

Choosing $h > k$, and using the fact that $|G_k(u_n)| \geq h - k$ on $A_{h,n}$, we have

$$(h-k)^p \operatorname{meas}(A_{h,n})^{p/p^*} \leq c \operatorname{meas}(A_{k,n})^{1-p'/s}, \qquad \forall h > k \geq \sigma,$$

that is,

$$\operatorname{meas}(A_{h,n}) \leq \frac{c}{(h-k)^{p^*}} \operatorname{meas}(A_{k,n})^{\frac{p^*}{p}\left(1-\frac{p'}{s}\right)}, \qquad \forall h > k \geq \sigma.$$

Since, by our choice of p^* and s, we have

$$\frac{p^*}{p}\left(1 - \frac{p'}{s}\right) > 1,$$

a well-known result by G. Stampacchia (see [22], Lemme 4.1) implies that there exists a constant M (independent on n), such that

$$\operatorname{meas}(A_{k,n}) = 0, \qquad \forall k \geq \sigma + M,$$

that is to say,

$$\|u_n\|_{L^\infty(\Omega)} \leq \sigma + M.$$

This fact concludes the proof of the theorem. □

The next result is a decomposition theorem for measures in $\mathcal{M}_b(\Omega)$, proved in [6], Theorem 2.1.

Theorem 2.3. *Let* $1 < p < +\infty$ *and let* $\mu \in \mathcal{M}_b(\Omega)$. *Then* $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ *if and only if* $\mu \in \mathcal{M}_0^p(\Omega)$.

Finally, we state a technical lemma, whose proof is straightforward.

Lemma 2.4. *Let a and b be two nonnegative real numbers, and let*

$$\varphi(s) = s\,e^{\theta s^2},$$

with $\theta = b^2/4a^2$. *Then*

$$a\,\varphi'(s) - b\,|\varphi(s)| \geq a/2, \qquad s \in \mathbf{R}. \tag{2.4}$$

Using the previous results, we can prove Theorem 1.1.

Proof of Theorem 1.1. It is clear that if there exists a solution u of (1.7), with u in $W_0^{1,p}(\Omega)$ and $g(x, u, \nabla u)$ in $L^1(\Omega)$, then, since $a(x, \nabla u)$ belongs to $(L^{p'}(\Omega))^N$ thanks to assumption (1.3), μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$ and so is in $\mathcal{M}_0^p(\Omega)$ by Theorem 2.3.

On the other hand, suppose that μ belongs to $\mathcal{M}_0^p(\Omega)$. By Theorem 2.3, μ can be decomposed as $f - \operatorname{div}(F)$, with $f \in L^1(\Omega)$ and F in $(L^{p'}(\Omega))^N$.

Let $\{f_n\}$ be a sequence of $L^\infty(\Omega)$ functions that converges to f strongly in $L^1(\Omega)$, and let $\{F_n\}$ be a sequence of $(L^\infty(\Omega))^N$ functions that converges to F strongly in $(L^{p'}(\Omega))^N$.

By Theorem 2.1, there exists a solution u_n of

$$\left\{ \begin{array}{ll} A(u_n) + g(x, u_n, \nabla u_n) = f_n - \operatorname{div}(F_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{2.5}$$

in the sense that u_n belongs to $W_0^{1,p}(\Omega)$, $g(x, u_n, \nabla u_n)$ belongs to $L^1(\Omega)$, and

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla v \, dx + \int_\Omega g(x, u_n, \nabla u_n)\, v \, dx = \int_\Omega f_n \, v \, dx + \int_\Omega F_n \cdot \nabla v \, dx\,, \tag{2.6}$$

for every $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Let us choose

$$v = \varphi(T_\sigma(u_n))\,,$$

as test function in (2.6), where σ is given by 1.6, and $\varphi(s)$ is as in Lemma 2.4, with $a = \alpha/2$ and $b = b(\sigma)$ ($b(s)$ is given by (1.5)).

Using (1.2) and the Young inequality, and writing $\varphi'_\sigma = \varphi'(T_\sigma(u_n))$ and $\varphi_\sigma = \varphi(T_\sigma(u_n))$ for simplicity, we obtain

$$\begin{aligned} \alpha \int_\Omega |\nabla T_\sigma(u_n)|^p \varphi'_\sigma \, dx &+ \int_\Omega \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \\ &\le \varphi(\sigma) \int_\Omega |f_n| \, dx + \varphi'(\sigma) \int_\Omega |F_n|^{p'} \, dx + \frac{\alpha}{2} \int_\Omega |\nabla T_k(u_n)|^p \varphi'_\sigma \, dx\,, \end{aligned}$$

so that, since $\{f_n\}$ is bounded in $L^1(\Omega)$, and $\{|F_n|\}$ is bounded in $L^{p'}(\Omega)$,

$$\frac{\alpha}{2} \int_\Omega |\nabla T_k(u_n)|^p \varphi'_\sigma \, dx + \int_\Omega \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \le c\,(\varphi(\sigma) + \varphi'(\sigma))\,. \tag{2.7}$$

We now have

$$\begin{aligned} \int_\Omega \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx &= \int_{\{|u_n| < \sigma\}} \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \\ &\quad + \int_{\{|u_n| \ge \sigma\}} \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx\,. \end{aligned}$$

Using (1.5), we have

$$\left| \int_{\{|u_n| < \sigma\}} \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \right| \le b(\sigma) \left(\int_\Omega |\nabla T_\sigma(u_n)|^p \varphi_\sigma \, dx + \varphi(\sigma) \, \|d\|_{L^1(\Omega)} \right)\,,$$

while, using (1.6), we get

$$\int_{\{|u_n| \ge \sigma\}} \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \ge \rho \, \varphi(\sigma) \int_{\{|u_n| \ge \sigma\}} |\nabla u_n|^p \, dx\,.$$

We thus obtain

$$\int_\Omega |\nabla T_\sigma(u_n)|^p \left[\frac{\alpha}{2}\varphi'_\sigma - b(\sigma)|\varphi_\sigma|\right] dx + \rho\,\varphi(\sigma)\int_{\{|u_n|\geq\sigma\}} |\nabla u_n|^p\,dx \leq c\,,$$

so that, by (2.4),

$$\int_\Omega |\nabla T_\sigma(u_n)|^p\,dx + \int_{\{|u_n|\geq\sigma\}} |\nabla u_n|^p\,dx \leq c\,.$$

This proves that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$; hence, there exist a function $u \in W_0^{1,p}(\Omega)$ and a subsequence, still denoted by $\{u_n\}$, such that u_n converges to u weakly in $W_0^{1,p}(\Omega)$ and almost everywhere in Ω.

The next step of the proof is the strong convergence of u_n to u in $W_0^{1,p}(\Omega)$.

We begin proving that we have

$$\lim_{k\to+\infty}\sup_{n\in\mathbf{N}}\int_{\{|u_n|\geq k\}} |\nabla u_n|^p\,dx = 0\,. \tag{2.8}$$

We choose $v = \psi_{k-1}(u_n)$ as test function in (2.6), where

$$\psi_{k-1}(s) = T_1(G_{k-1}(s))\,,$$

and $k \geq \sigma + 1$. We get, again using (1.2) and the Young inequality,

$$\begin{aligned}\alpha\int_\Omega |\nabla\psi_{k-1}(u_n)|^p\,dx &+ \int_\Omega \psi_{k-1}(u_n)\,g(x,u_n,\nabla u_n)\,dx \\ &\leq \int_{\{|u_n|\geq k-1\}} |f_n|\,dx + \int_{\{k-1\leq|u_n|\leq k\}} |F_n|^{p'}\,dx + \frac{\alpha}{2}\int_\Omega |\nabla\psi_{k-1}(u_n)|^p\,dx\,.\end{aligned}$$

Using (1.6) and the fact that $\psi_{k-1}(s)$ has the same sign as s if $|s| > \sigma$ and is zero if $|s| \leq \sigma$, we have

$$g(x,u_n,\nabla u_n)\,\psi_{k-1}(u_n) \geq |g(x,u_n,\nabla u_n)|\,\chi_{\{|u_n|\geq k\}}\,,$$

so that we obtain, dropping positive terms,

$$\int_{\{|u_n|\geq k\}} |g(x,u_n,\nabla u_n)|\,dx \leq \int_{\{|u_n|\geq k-1\}} |f_n|\,dx + \int_{\{k-1\leq|u_n|\leq k\}} |F_n|^{p'}\,dx\,.$$

Since $\{u_n\}$ is bounded in $L^1(\Omega)$, we have

$$\lim_{k\to+\infty}\sup_{n\in\mathbf{N}} \operatorname{meas}(\{|u_n| \geq k-1\}) = 0\,.$$

Thus, since f_n is strongly compact in $L^1(\Omega)$, and $|F_n|$ is strongly compact in $L^{p'}(\Omega)$, we have

$$\lim_{k\to+\infty}\sup_{n\in\mathbf{N}}\int_{\{|u_n|\geq k-1\}} |f_n|\,dx + \int_{\{k-1\leq|u_n|\leq k\}} |F_n|^{p'}\,dx = 0\,,$$

and so

$$\lim_{k\to+\infty}\sup_{n\in\mathbf{N}}\int_{\{|u_n|\geq k\}}|g(x,u_n,\nabla u_n)|\,dx=0\,. \tag{2.9}$$

Using the fact that $k\geq\sigma$ and (1.6), we see that (2.9) implies (2.8).

The next step is the proof that for every $k\geq\sigma$, the sequence $T_k(u_n)$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega)$. In order to do this, we shall closely follow the outline of [5]. Let $k\geq\sigma$ be fixed, and choose as test function in (2.6)

$$v=\varphi(T_k(u_n)-T_k(u))\,,$$

where $\varphi(s)$ is as in Lemma 2.4, with $a=1$ and $b=b(k)/\alpha$. Such a function is admissible since it belongs to $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. We thus have

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))\varphi'(T_k(u_n)-T_k(u))dx \tag{A}$$

$$+\int_\Omega g(x,u_n\,\nabla u_n)\,\varphi(T_k(u_n)-T_k(u))\,dx \tag{B}$$

$$=\int_\Omega f_n\,\varphi(T_k(u_n)-T_k(u))\,dx \tag{C}$$

$$+\int_\Omega F_n\cdot\nabla(T_k(u_n)-T_k(u))\,\varphi'(T_k(u_n)-T_k(u))\,dx\,. \tag{D}$$

Here and in the following, we write ε_n for any quantity which converges to zero as n tends to infinity.

In order to deal with (A)–(D), we write, for simplicity,

$$\varphi_n'=\varphi'(T_k(u_n)-T_k(u))\,,\qquad \varphi_n=\varphi(T_k(u_n)-T_k(u))\,.$$

We shall often use the fact that, in the weak$*$ topology of $L^\infty(\Omega)$, and almost everywhere in Ω, we have

$$\lim_{n\to+\infty}\varphi_n=\varphi(0)=0\,,\qquad \lim_{n\to+\infty}\varphi_n'=\varphi'(0)=1\,. \tag{2.10}$$

Since f_n is strongly compact in $L^1(\Omega)$, (2.10) implies

$$(\mathrm{C})=\varepsilon_n\,. \tag{2.11}$$

Moreover, since $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p}(\Omega)$, and F_n is strongly compact in $(L^{p'}(\Omega))^N$, (2.10) implies

$$(\mathrm{D})=\varepsilon_n\,. \tag{2.12}$$

We can then decompose (A) as

$$\text{(E)} \qquad \int_\Omega a(x, T_k(u_n)) \cdot \nabla(T_k(u_n) - T_k(u))\, \varphi_n'\, dx$$

$$\text{(F)} \qquad + \int_\Omega a(x, G_k(u_n)) \cdot \nabla(T_k(u_n) - T_k(u))\, \varphi_n'\, dx\,.$$

Since $\nabla T_k(u_n)$ is zero where $\nabla G_k(u_n)$ is different from zero, and conversely, we have

$$\text{(F)} = - \int_\Omega a(x, G_k(u_n)) \cdot \nabla T_k(u)\, \varphi_n'\, dx\,.$$

Since $\nabla T_k(u) \equiv 0$ on the set $\{|u| \geq k\}$, we have that

$$\nabla T_k(u)\, \chi_{\{|u_n| \geq k\}} \to 0\,, \quad \text{almost everywhere in } \Omega.$$

In view of the fact that $\nabla T_k(u)$ belongs to $(L^p(\Omega))^N$, the Lebesgue theorem implies

$$\nabla T_k(u)\, \chi_{\{|u_n| \geq k\}} \to 0\,, \quad \text{strongly in } (L^p(\Omega))^N,$$

and so, since $a(x, \nabla G_k(u_n))$ is bounded in $(L^{p'}(\Omega))^N$ by (1.3), we have

$$\text{(2.13)} \qquad \text{(F)} = \varepsilon_n\,.$$

As for (E), we can decompose it as

$$\text{(G)} \qquad \int_\Omega [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla(T_k(u_n) - T_k(u))\, \varphi_n'\, dx$$

$$\text{(H)} \qquad + \int_\Omega a(x, \nabla T_k(u)) \cdot \nabla(T_k(u_n) - T_k(u))\, \varphi_n'\, dx\,.$$

Since $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p}(\Omega)$, and $a(x, \nabla T_k(u))$ belongs to $(L^{p'}(\Omega))^N$ by (1.3), (2.10) implies

$$\text{(2.14)} \qquad \text{(H)} = \varepsilon_n\,.$$

Thus, putting together (2.13) and (2.14), we have

$$\text{(2.15)} \quad \text{(A)} = \int_\Omega [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla(T_k(u_n) - T_k(u))\, \varphi_n'\, dx + \varepsilon_n\,.$$

We now deal with (B), decomposing it as

$$\text{(I)} \qquad \int_{\{|u_n| \geq k\}} g(x, u_n, \nabla u_n)\, \varphi_n\, dx$$

$$\text{(J)} \qquad + \int_{\{|u_n| < k\}} g(x, u_n, \nabla u_n)\, \varphi_n\, dx\,.$$

Using the fact that $k \geq \sigma$ and assumption (1.6), and observing that on the set $\{u_n \geq k\}$ we have

$$\varphi_n = \varphi(k - T_k(u)) \geq 0$$

while on the set $\{u_n \leq -k\}$ we have

$$\varphi_n = \varphi(-k - T_k(u)) \leq 0\,,$$

we obtain

$$(\mathrm{I}) \geq 0\,,$$

so that we can drop it. Using (1.5), we have

$$|(\mathrm{J})| \leq b(k) \int_\Omega d(x)\,|\varphi_n|\,dx + b(k) \int_\Omega |\nabla T_k(u_n)|^p\,|\varphi_n|\,dx\,,$$

where b is the function given in (1.5). Since d belongs to $L^1(\Omega)$, we have using (2.10)

$$\int_\Omega d(x)\,|\varphi_n|\,dx = \varepsilon_n\,,$$

so that by (1.2)

$$|(\mathrm{J})| \leq \frac{b(k)}{\alpha} \int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)\,|\varphi_n|\,dx + \varepsilon_n\,.$$

Now add and subtract to the above inequality the term

$$\int_\Omega a(x, \nabla T_k(u)) \cdot \nabla(T_k(u_n) - \nabla T_k(u))\,|\varphi_n|\,dx\,,$$

which converges to zero as n tends to infinity since $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p}(\Omega)$ and $a(x, \nabla T_k(u))$ belongs to $(L^{p'}(\Omega))^N$), and the term

$$\int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla T_k(u)\,|\varphi_n|\,dx\,,$$

which is an ε_n since $a(x, \nabla T_k(u_n))$ is bounded in $(L^{p'}(\Omega))^N$ and φ_n converges to zero by (2.10). We thus get

$$|(\mathrm{J})| \leq \frac{b(k)}{\alpha} \int_\Omega [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla(T_k(u_n) - T_k(u))\,|\varphi_n|\,dx\; + \varepsilon_n\,.$$

Putting this inequality together with (2.15), we thus obtain that

$$\int_\Omega [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla(T_k(u_n) - T_k(u))[\varphi_n' - \frac{b(k)}{\alpha}|\varphi_n|]\,dx$$

is an ε_n. Hence, by (2.4),

$$\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))] \cdot \nabla(T_k(u_n) - T_k(u))\,dx = \varepsilon_n\,.$$

This fact and the assumptions on a imply, by a result in [8] (see also [14]), that $T_k(u_n)$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega)$.

Now let E be a measurable subset of Ω. Then

$$\int_E |\nabla u_n|^p\,dx = \int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p\,dx + \int_{E\cap\{|u_n|>k\}} |\nabla u_n|^p\,dx\,.$$

Let $\varepsilon > 0$ be fixed. Since

$$\int_{E\cap\{|u_n|>k\}} |\nabla u_n|^p\,dx \le \int_{\{|u_n|>k\}} |\nabla u_n|^p\,dx\,,$$

(2.8) implies that there exists $k \ge \sigma$ such that

$$\int_{E\cap\{|u_n|>k\}} |\nabla u_n|^p\,dx \le \frac{\varepsilon}{2}\,, \qquad n \in \boldsymbol{N}\,.$$

Once k is fixed, since we have

$$\int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p\,dx \le \int_E |\nabla T_k(u_n)|^p\,dx\,,$$

the strong compactness of $T_k(u_n)$ in $W_0^{1,p}(\Omega)$ implies that there exists $\delta > 0$ such that if meas $(E) < \delta$, then

$$\int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p\,dx \le \frac{\varepsilon}{2}\,, \qquad n \in \boldsymbol{N}\,.$$

Thus, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if meas $(E) < \delta$, then

$$\int_E |\nabla u_n|^p\,dx \le \varepsilon\,, \qquad n \in \boldsymbol{N}\,,$$

that is to say, the sequence $\{|\nabla u_n|^p\}$ is equi-integrable. Since, up to a subsequence still denoted by u_n, ∇u_n is almost everywhere convergent to ∇u (as a consequence of the strong convergence of truncates), u_n converges strongly to u in $W_0^{1,p}(\Omega)$.

In order to pass to the limit in the approximate equation, we now show that

$$g(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \qquad \text{strongly in } L^1(\Omega).$$

Since $g(x,u_n,\nabla u_n)$ converges almost everywhere to $g(x,u,\nabla u)$, it remains to prove the equi-integrability of the sequence $\{|g(x,u_n,\nabla u_n)|\}$. Once again, if E is a measurable subset of Ω, we have

$$\begin{aligned}\int_E |g(x,u_n,\nabla u_n)|dx &= \int_{E\cap\{|u_n|\le k\}} |g(x,u_n,\nabla u_n)|dx \\ &\quad + \int_{E\cap\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|dx\,.\end{aligned}$$

Let $\varepsilon > 0$ be fixed. Since

$$\int_{E\cap\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx \le \int_{\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx\,,$$

we can use (2.9) in order to choose $k \ge \sigma$ such that

$$\int_{E\cap\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2}\,, \qquad n \in \boldsymbol{N}\,.$$

Moreover, using (1.3), we have

$$\int_{E\cap\{|u_n|\le k\}} |g(x,u_n,\nabla u_n)|\,dx \le b(k)\int_E [d(x) + |\nabla T_k(u_n)|^p]\,dx\,,$$

and, since d belongs to $L^1(\Omega)$ and $T_k(u_n)$ is strongly compact in $W_0^{1,p}(\Omega)$, there exists $\delta > 0$ such that if meas$(E) < \delta$, then

$$\int_{E\cap\{|u_n|\le k\}} |g(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2}\,, \qquad n \in \boldsymbol{N}\,.$$

Thus, as before, we have proved that $\{|g(x,u_n,\nabla u_n)|\}$ is equi-integrable, and this allows us to pass to the limit in (2.6), in order to obtain (1.8). □

Remark 2.5. Since there exists a solution of (1.7) in the sense (1.8) if and only if μ belongs to $\mathcal{M}_0^p(\Omega)$, that is to say if and only if μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$, the class of admissible test functions in (1.8) can be extended by means of an easy density argument in order to consider test functions v in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 2.6. The result of Theorem 1.1 refines the result of Theorem 2.3 in the sense that it states that if μ is a measure in $\mathcal{M}_0^p(\Omega)$, then there exists a function u in $W_0^{1,p}(\Omega)$ such that

$$\mu = A(u) + g(x,u,\nabla u)\,,$$

with $g(x,u,\nabla u)$ in $L^1(\Omega)$.

If, for example, $p = 2$, $a(x,\xi) = \xi$, and $g(x,s,\xi) = \text{sgn}(s)\,|\xi|^2$, then for every nonnegative measure μ in $\mathcal{M}_0^2(\Omega)$ there exists a function u in $H_0^1(\Omega)$ such that

$$\mu = -\text{div}\,(\nabla u) + |\nabla u|^2\,.$$

It is also easy to see, taking for instance $T_k(u^-)$ as test function in (1.8), that such a function u is nonnegative.

Remark 2.7. The same result of Theorem 1.1 holds true if the function a depends also on u, and satisfies the following assumptions:

$$a(x,s,\xi)\cdot\xi \ge \alpha\,|\xi|^p\,,$$

$$|a(x,s,\xi)| \le \ell(x) + \beta\,[|s|^{p-1} + |\xi|^{p-1}]\,,$$

$$[a(x,s,\xi) - a(x,s,\eta)]\cdot(\xi-\eta) > 0\,,$$

for almost every $x \in \Omega$, for every s in $\boldsymbol{R}$, for every ξ, η in $\boldsymbol{R}^N$ ($\xi \neq \eta$), where α and β are positive constants, and ℓ belongs to $L^{p'}(\Omega)$.

3 A nonexistence result

We recall that if μ is a Radon measure, and E is a Borel subset of Ω, the *restriction* of μ to E is the measure $\lambda = \mu \llcorner E$ defined by

$$\lambda(B) = \mu(E \cap B)\,,$$

for every Borel subset B of Ω. We say that a measure λ is *concentrated* on a Borel set E if $\lambda = \lambda \llcorner E$.

We recall the following result, proved in [17], Lemma 2.1.

Proposition 3.1. *Let μ be a measure in $\mathcal{M}_b(\Omega)$, and let $1 < p \le N$. Then μ can be decomposed in a unique way as $\mu_0 + \lambda$, where*

(1) *μ_0 belongs to $\mathcal{M}_0^p(\Omega)$;*

(2) *$\lambda = \mu \llcorner E$, and $\mathrm{cap}_p(E,\Omega) = 0$.*

Given a measure μ in $\mathcal{M}_b(\Omega)$, we can decompose it, by the previous proposition, as $\mu_0 + \lambda$. The result of Theorem 1.1 then states that problem (1.7) has a solution with datum μ if and only if $\lambda = 0$. Suppose now that $\mu_0 = 0$, so that $\mu = \lambda$ is singular with respect to the p-capacity. What happens if we try (as in the proof of Theorem 1.1) to approximate the measure λ with a sequence $\{f_n\}$ of $L^\infty(\Omega)$ functions and solve the corresponding problems with data f_n? If we take the sequence $\{f_n\}$ bounded in $L^1(\Omega)$, then the same technique used in the proof of Theorem 1.1 yields that the corresponding sequence of solutions $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$; hence, it converges weakly in $W_0^{1,p}(\Omega)$ to some function u. Clearly, u cannot be a solution of the problem with datum λ, since such a solution does not exist. What can we say about u? The answer is given by the following theorem, under a sign condition on the lower order nonlinearity g.

Theorem 3.2. *Let λ be a positive measure in $\mathcal{M}_b(\Omega)$, concentrated on a set E such that $\mathrm{cap}_p(E,\Omega) = 0$, and let $\{f_n\}$ be a sequence of nonnegative $L^\infty(\Omega)$ functions such that*

$$\lim_{n\to+\infty} \int_\Omega f_n\,\varphi\,dx = \int_\Omega \varphi\,d\lambda \qquad \forall \varphi \in C^0(\bar{\Omega})\,.$$

Suppose that, in addition to assumptions (1.5) *and* (1.6), *g also satisfies*

$$g(x,s,\xi)\,s \geq 0\,, \tag{3.1}$$

for almost every x in Ω, for every s in $\mathbf{R}$, and for every ξ in $\mathbf{R}^N$. Let u_n be a solution of the equation

$$\begin{cases} A(u_n) + g(x,u_n,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

in the sense (2.6) *(with $F \equiv 0$). Then there exists $k > 0$ (depending on g and α), such that*

$$T_k(u_n) \to 0 \qquad \text{strongly in } W_0^{1,p}(\Omega).$$

Moreover, u_n converges weakly to zero in $W_0^{1,p}(\Omega)$, and

$$\lim_{n\to+\infty} \int_\Omega g(x,u_n,\nabla u_n)\,\varphi\,dx = \int_\Omega \varphi\,d\lambda \qquad \forall \varphi \in C_0^1(\Omega)\,.$$

In the following we denote by ε_δ and $\varepsilon_{n,\delta}$ respectively any real numbers such that

$$\lim_{\delta\to 0^+} \varepsilon_\delta = 0\,, \qquad \lim_{\delta\to 0^+}\lim_{n\to+\infty} \varepsilon_{n,\delta} = 0\,.$$

Before giving the proof of the theorem, we need to construct as in [15] a suitable collection of cut-off functions.

Lemma 3.3. *Let λ be a nonnegative measure in $\mathcal{M}_b(\Omega)$ which is concentrated on a set E of zero p-capacity. Then for every $\delta > 0$ there exists a $C_0^\infty(\Omega)$ function ψ_δ such that*

$$\int_\Omega |\nabla\psi_\delta|^p\,dx = \varepsilon_\delta\,, \qquad 0 \leq \psi_\delta \leq 1\,, \qquad \int_\Omega (1-\psi_\delta)\,d\lambda = \varepsilon_\delta\,. \tag{3.3}$$

Proof. Since λ belongs to $\mathcal{M}_b(\Omega)$, for every $\delta > 0$ there exists a compact set $K_\delta \subseteq E$ such that $\lambda(E\backslash K_\delta) \leq \delta$. Since K_δ is compact, and its p-capacity is zero, there exists a $C_0^\infty(\Omega)$ function ψ_δ which is equal to 1 in a neighbourhood of K_δ and that satisfies both the first and second conditions of (3.3). As for the third, we have

$$0 \leq \int_\Omega (1-\psi_\delta)\,d\lambda = \int_{E\backslash K_\delta} (1-\psi_\delta)\,d\lambda \leq \lambda(E\backslash K_\delta) \leq \delta\,.$$

This completes the proof. □

Remark 3.4. If E is compact, we can choose $K_\delta = E$ for every $\delta > 0$. As a consequence of (3.3), we have that ψ_δ converges to zero strongly in $W_0^{1,p}(\Omega)$, almost everywhere in Ω, and in the weak$*$ topology of $L^\infty(\Omega)$.

Proof of Theorem 3.2. Since f_n is nonnegative, the sign assumption (3.1) on g implies that u_n is also nonnegative. As in the proof of Theorem 1.1, it is easy to see, using the assumptions on a and g, that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exists a subsequence, again denoted by u_n, a function u in $W_0^{1,p}(\Omega)$, and an element G in $(L^{p'}(\Omega))^N$ such that

$$u_n \to u \qquad \text{weakly in } W_0^{1,p}(\Omega) \text{ and almost everywhere in } \Omega,$$

$$a(x, \nabla u_n) \to G \qquad \text{weakly in } (L^{p'}(\Omega))^N;$$

the latter convergence is due to the fact that, by (1.3), $a(x, \nabla u_n)$ is bounded in $(L^{p'}(\Omega))^N$.

Let $k > 0$ be such that

$$b(k)\, k \leq \alpha/2\,, \tag{3.4}$$

where b is the function that appears in (1.5). Such a k exists since b is continuous.

We now choose as test function in (2.6) the function

$$v = (k - T_k(u_n))\, \psi_\delta\,,$$

which is admissible since it belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We obtain

$$\text{(A)} \qquad - \int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, \psi_\delta \, dx$$

$$\text{(B)} \qquad + \int_\Omega [a(x, \nabla u_n) \cdot \nabla \psi_\delta]\, (k - T_k(u_n))\, dx$$

$$\text{(C)} \qquad + \int_\Omega g(x, u_n, \nabla u_n)\, (k - T_k(u_n))\, \psi_\delta \, dx$$

$$\text{(D)} \qquad = \int_\Omega f_n\, (k - T_k(u_n))\, \psi_\delta \, dx\,.$$

Since $k - T_k(u_n)$ converges to $k - T_k(u)$ both in the weak* topology of $L^\infty(\Omega)$ and almost everywhere in Ω, we have that $\nabla\psi_\delta\,(k - T_k(u_n))$ converges to $\nabla\psi_\delta(k - T_k(u))$ strongly in $(L^p(\Omega))^N$, and so

$$\text{(B)} = \int_\Omega G \cdot \nabla \psi_\delta\, (k - T_k(u))\, dx + \varepsilon_n = \varepsilon_{n,\delta}\,, \tag{3.5}$$

where the last passage is due to the first equality of (3.3). We then have, by (1.5) and since $k - T_k(u_n) = 0$ for $u_n > k$,

$$|\text{(C)}| \leq \int_{\{0 \leq u_n \leq k\}} b(u_n)\, (k - T_k(u_n))\, \psi_\delta\, [d(x) + |\nabla u_n|^p]\, dx\,.$$

Thus,

$$\int_{\{0\le u_n\le k\}} b(u_n)\,(k-T_k(u_n))\,\psi_\delta\, d(x)\,dx \le \frac{\alpha}{2}\int_\Omega d(x)\,\psi_\delta\,dx = \varepsilon_\delta\,,$$

by the choice of k and since ψ_δ converges to zero in the weak$*$ topology of $L^\infty(\Omega)$ as a consequence of (3.3). Moreover, again by the choice of k,

$$\int_{\{0\le u_n\le k\}} b(u_n)\,(k-T_k(u_n))\,\psi_\delta\,|\nabla u_n|^p\,dx \le \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx\,.$$

We thus have

$$|(\mathrm{B})| \le \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx + \varepsilon_\delta\,.$$

Using (1.2), we also have

$$(\mathrm{A}) \ge -\alpha\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx\,.$$

Thus, since (D) is nonnegative,

$$\alpha\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx \le \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx + \varepsilon_{n,\delta}\,,$$

that is to say,

$$\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx = \varepsilon_{n,\delta}\,. \tag{3.6}$$

The next step consists in choosing as test function in (2.6) the function

$$v = T_k(u_n)(1-\psi_\delta)\,,$$

which is again admissible. We get

$$\begin{aligned}
&(\mathrm{E}) && \int_\Omega a(x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,(1-\psi_\delta)\,dx\\
&(\mathrm{F}) && -\int_\Omega [a(x,\nabla u_n)\cdot\nabla\psi_\delta]\,T_k(u_n)\,dx\\
&(\mathrm{G}) && +\int_\Omega g(x,u_n,\nabla u_n)\,T_k(u_n)\,(1-\psi_\delta)\,dx\\
&(\mathrm{H}) && =\int_\Omega f_n\,T_k(u_n)\,(1-\psi_\delta)\,dx\,.
\end{aligned}$$

The term (G) is nonnegative by (3.1), so we drop it. Reasoning as before, we have

$$(\mathrm{F}) = \varepsilon_{n,\delta}\,,$$

while

$$(\mathrm{H}) \leq k \int_\Omega f_n\,(1-\psi_\delta)\,dx = k \int_\Omega (1-\psi_\delta)\,d\lambda + \varepsilon_n = \varepsilon_{n,\delta}\,,$$

by the third part of (3.3). Thus, using (1.2) in (E), we get

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p\,(1-\psi_\delta)\,dx = \varepsilon_{n,\delta}\,. \tag{3.7}$$

Putting together (3.6) and (3.7), we easily obtain

$$\int_\Omega |\nabla T_k(u_n)|^p\,dx = \varepsilon_n\,,$$

that is to say, $T_k(u_n)$ converges strongly to zero in $W_0^{1,p}(\Omega)$. Since the limit is independent of the choice of subsequence, the sequence $T_k(u_n)$ converges to zero strongly in $W_0^{1,p}(\Omega)$. Thus, $u = 0$, and so u_n converges weakly to zero in $W_0^{1,p}(\Omega)$.

In order to prove the second part of the theorem, observe that from the strong convergence to zero of $T_k(u_n)$ follows the almost everywhere convergence to zero of ∇u_n, and this implies that $G \equiv 0$. Now we choose a test function φ in $C_0^1(\Omega)$ in (2.6); we obtain

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla\varphi\,dx + \int_\Omega g(x,u_n,\nabla u_n)\,\varphi\,dx = \int_\Omega f_n\,\varphi\,dx\,. \tag{3.8}$$

Since $G = 0$, we have

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla\varphi\,dx = \varepsilon_n\,,$$

while

$$\int_\Omega f_n\,\varphi\,dx = \int_\Omega \varphi\,d\lambda + \varepsilon_n\,.$$

Thus, from (3.8) we obtain by subtracting

$$\lim_{n\to+\infty} \int_\Omega g(x,u_n,\nabla u_n)\,\varphi\,dx = \int_\Omega \varphi\,d\lambda\,,$$

for every φ in $C_0^1(\Omega)$, and this concludes the proof of the theorem. □

Remark 3.5. The result of the previous theorem also holds, with minor technical modifications in the proof, if λ is a signed measure. Results of this kind, concerning problems where the lower order term is independent of the gradient of u and behaves like $|u|^{q-1}u$, can also be found in [20].

REFERENCES

[1] P. Baras and M. Pierre, *Singularités éliminables pour des équations semi-linéaires*, Ann. Inst. Fourier (Grenoble) **34** (1984), 185–206.

[2] A. Bensoussan, L. Boccardo and F. Murat, *On a nonlinear P.D.E. having natural growth terms and unbounded solutions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **5** (1988), 347–364.

[3] L. Boccardo and T. Gallouët, *Nonlinear elliptic equations with right hand side measures*, Comm. Partial Differential Equations **17** (1992), 641–655.

[4] L. Boccardo and T. Gallouët, *Strongly nonlinear elliptic equations having natural growth terms and L^1 data*, Nonlinear Anal. **19** (1992), 573–579.

[5] L. Boccardo, T. Gallouët and F. Murat, *A unified presentation of two existence results for problems with natural growth*, in *Progress in Partial Differential Equations: The Metz Surveys, 2 (1992)*, Pitman Res. Notes Math. Ser., 296, Longman Sci. Tech., Harlow, 1993, pp. 127–137.

[6] L. Boccardo, T. Gallouët and L. Orsina, *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 539–551.

[7] L. Boccardo, F. Murat and J.P. Puel, *Existence de solutions non bornées pour certaines équations quasi-linéaires*, Portugal. Math. **41** (1982), 507–534.

[8] L. Boccardo, F. Murat and J. P. Puel, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Ann. Mat. Pura Appl. (4) **152** (1988), 183–196.

[9] L. Boccardo, F. Murat and J. P. Puel, *L^∞-estimates for some nonlinear partial differential equations and application to an existence result*, SIAM J. Math. Anal. **23** (1992), 326–333.

[10] H. Brezis, *Nonlinear elliptic equations involving measures*, in *Contributions to Nonlinear Partial Differential Equations (Madrid, 1981)*, Res. Notes in Math., 89, Pitman, Boston, Mass.–London, 1983, pp. 82–89.

[11] H. Brezis and L. Nirenberg, *Removable singularities for nonlinear elliptic equations*, Topol. Methods Nonlinear Anal., to appear.

[12] L. Brezis and W. Strauss, *Semi-linear second-order elliptic equations in L^1*, J. Math. Soc. Japan **25** (1973), 565–590.

[13] H. Brezis and L. Veron, *Removable singularities for some nonlinear elliptic equations*, Arch. Rational Mech. Anal. **75** (1980/81), 1–6.

[14] F. E. Browder, *Existence theorems for nonlinear partial differential equations*, in *Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968)*, Amer. Math. Soc., Providence, R.I., 1970, pp. 1–60.

[15] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, *Renormalized solutions for elliptic equations with general measure data*, preprint.

[16] T. Del Vecchio, *Strongly nonlinear problems with Hamiltonian having natural growth*, Houston J. Math. **16** (1990), 7–24.

[17] M. Fukushima, K. Sato and S. Taniguchi, *On the closable part of pre-Dirichlet forms and the fine supports of underlying measures*, Osaka J.Math. **28** (1991), 517–535.

[18] T. Gallouët and J. M. Morel, *Resolution of a semilinear equation in L^1*, Proc. Roy. Soc. Edinburgh **96** (1984), 275–288.

[19] J. Leray and J.-L. Lions, *Quelques résultats de Višik sur les problèmes elliptiques semi-linéaires par les méthodes de Minty et Browder*, Bull. Soc. Math. France **93** (1965), 97–107.

[20] L. Orsina and A. Prignet, *Nonexistence of solutions for some nonlinear elliptic equations involving measures*, preprint.

[21] A. Porretta, *Some remarks on the regularity of solutions for a class of elliptic equations with measure data*, Potential Anal., to appear.

[22] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), 189–258.

Lucio Boccardo
Dipartimento di Matematica
Università di Roma I
P.le A. Moro 2
00185, Roma, Italy

Thierry Gallouët
ENS-Lyon
69364 Lyon Cedex 7, France

Luigi Orsina
Dipartimento di Matematica
Università di Roma I
P.le A. Moro 2
00185, Roma, Italy

(Received May 28, 1997)