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Statistical Theory of Multiple Meson Production with Angular Momentum Conservation.

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Summary. — The conventional statistical theory (with energy-momentum but without angular momentum conservation) is shown to give the same results (multiplicities and spectra) as one can obtain from a model which assumes the existence of an intermediate « hot spot » in the collision of two high-energy particles, if the « hot-spot » formation is statistically independent of its decay into many particles. The model is suggested by the formulae used in statistical theory, which in effect put the probability to produce n particles proportional to the probability to find n particles together in a volume Ω. Angular momentum conservation (together with energy and momentum conservation) can be satisfied by considering only states of n particles with prescribed values of total J, energy and momentum. By using the formalism of the density matrix and an explicit expression for the projection operators on states of given angular momentum one arrives at a modified form of the phase-space integral, which is simply related to the probability to produce n particles. The theory with angular momentum but without momentum conservation, as given by KOBA, is shown to be a special case easily derivable from the present formalism. The « classical » theory of angular momentum conservation by ERICSON is shown to be a limiting case of wide applicability, however. The conventional theory is strictly speaking valid only if all end particles are in s-states. The formulae have been derived for a spherical Gaussian shape for Ω. Contracted shapes can in principle be allowed for. A Monte-Carlo program is proposed to evaluate this phase-space integral. The method will allow to compute spectra of the longitudinal and the transverse momenta of the final particles. The effect on multiplicity of J-conservation is evaluated. Compared to the conventional theory multiplicity is increased by 10% in p-$\bar{\text{p}}$ annihilation and decreased by 10% in 6 GeV p-p collisions.

PART I

THEORY OF ANGULAR MOMENTUM CONSERVATION IN STATISTICAL MODELS OF MESON PRODUCTION

1. – The hypotheses of the statistical theory.

In the conventional statistical theories of meson production one takes as starting point the following formula for the probability W_n to produce n particles with masses m_1, m_2, ..., m_n in a high-energy collision with given total energy E, in the centre-of-mass system (c.m.s.)

$$\left\{\begin{array}{l} W_n = C f_{n,\tau,\ldots} \Omega^{n-1} \varrho_n(E;0)\,, \\ \varrho_n(E;0) = \int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_n \delta\Big(E - \sum_{i=1}^{n} \sqrt{m_i^2 + p_i^2}\Big)\, \delta\Big(\sum_{i=1}^{n} \boldsymbol{p}_i\Big)\,, \end{array}\right. \tag{1.1}$$

where $f_{n,\tau,\ldots}$ is a numerical coefficient depending on the spins and isospins of the particles, on their number and on the total isospin and Ω is a three-dimensional volume. C is a normalizing factor, such that $\sum_n W_n = 1$, where the sum is over all numbers and kinds of particles that can be produced in such a collision, without violating any of the strong-interaction conservation laws.

This formula can be derived from S-matrix theory, with suitable hypotheses as to the behaviour of the S-matrix elements as a function of total energy, isospin, etc. [1,2] (*).

We shall try in the present work to look at it also from a more intuitive point of view, and ask what physical assumptions are made in stating such a formula.

To arrive at a physical interpretation, let us assume that we have described the n particles in the end-states by a superposition of plane waves,

$$\Psi_f(\boldsymbol{p}_1, \ldots; \boldsymbol{x}_1, \ldots) = \exp\Big[i \sum_{i=1}^{n} (\boldsymbol{p}_i, \boldsymbol{x}_i)\Big]\,.$$

(1) R. HAGEDORN: *Nuovo Cimento*, **15**, 434 (1960).

(2) R. HAGEDORN: *Fortschr. d. Phys.*, **9**, 1 (1961).

(*) This formula, in its simplest form due to Fermi, has been derived and treated by many authors; see the above references for detailed references to previous literature.

We could then write formula (1.1) in the following form (omitting the factor $f_{n,r,\ldots}$)

$$W_n = C\Omega^{n-1}\int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_n \exp\left[-i\sum_{i=1}^{n}(\boldsymbol{p}_i, \boldsymbol{x}_i)\right]\cdot \cdot\delta\Big(E-\sum_{i=1}^{n}\sqrt{m_i^2+p_i^2}\Big)\,\delta\Big(\sum_{i=1}^{n}\boldsymbol{p}_i\Big)\exp i\Big[\sum_{i=1}^{n}(\boldsymbol{p}_i, \boldsymbol{x}_i)\Big] \tag{1.2}$$

$$= C\int_{\Omega}\mathrm{d}\boldsymbol{x}_1 \ldots \int_{\Omega}\mathrm{d}\boldsymbol{x}_n \int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_n \cdot \cdot |\Psi_f(\boldsymbol{p}_1, \ldots; \boldsymbol{x}_1, \ldots)|^2 \delta\Big(E-\sum_{i=1}^{n}\sqrt{m_i^2+p_i^2}\Big)\,\delta\Big(\sum_{i=1}^{n}\boldsymbol{p}_i\Big)\,\delta\Big(\sum_{i=1}^{n}\boldsymbol{x}_i\Big)\,. \tag{1.2'}$$

The last δ-function has been introduced essentially for dimensional reasons, and means that the average co-ordinate has been taken as 0; because the total momentum is zero a constraint of this type is necessary. If we interpret $|\Psi_f(\boldsymbol{p}_1, \ldots; \boldsymbol{x}_1, \ldots)|^2$ (which is strictly speaking equal to $|\exp[i(\boldsymbol{p}, \boldsymbol{x})]|^2 \equiv 1$), as the density of particles in the region of phase-space around $\boldsymbol{p}_1 \ldots$; $\boldsymbol{x}_1 \ldots$ the formula means that the probability to produce n particles is proportional to the probability to find n particles in a region of phase-space which corresponds to a volume Ω (for each particle) in configuration space times the volume of a complicated hypersurface (defined by the δ-functions) in momentum space. Because $|\Psi_f|^2 = 1$ the phase-space density is constant, and all states in the above defined region are equally probable. Another way of saying the same thing is: W_n is proportional to the sum of the probabilities to find the n particles—which are in a state described by $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$—in the volume Ω, where the sum is over all states with the given total energy E and total momentum zero, and all states in the sum have equal probability. Ω plays the role of an adjustable parameter.

How can one understand this result? Let us start, as in the derivation [1] of (1.1) from S-matrix theory by stating that

$$W_n = \sum_f |\langle f|S|i\rangle|^2\,, \tag{1.3}$$

where S is the S-matrix, $|i\rangle$ the initial state (two colliding particles) and $|f\rangle$ an n-particle state. The sum is over all n-particle final states.

Because we believe that time-invariance holds in strong interactions and because S is unitary we have

$$|\langle f|S|i\rangle|^2 = |\langle i_T|S|f_T\rangle|^2\,.$$

On the r.h.s. we have the value of the matrix element squared between an n-particle initial state and a two-particle final state, *i.e.* the probability of the

inverse reaction. We could hence also write

$$W_n = \sum_f |\langle i_T | S | f_T \rangle|^2 ,$$

and by comparing with (1.2′) we see that the sum of the probabilities to go from any n-particle state to a two-particle state is effectively put equal to the probability to find the n initial particles in a volume Ω; the *sum* of the squared matrix element, can be expressed—extracting the δ-functions for energy-momentum conservation—by the *average* of a reduced squared matrix element S'

$$W_n = \overline{|\langle i_T | S' | f_T \rangle|^2} \sum_f \delta\Big(E - \sum_i \sqrt{p_i^2 + m_i^2}\Big)\, \delta\Big(\sum_i \boldsymbol{p}_i\Big) . \tag{1.4}$$

The statistical hypothesis means then that *on the average* the reduced matrix element squared is equal to the probability to find the n initial particles (in the reverse reaction) in the volume Ω.

We see therefore that one has not made any detailed hypothesis on the S-matrix element between the given two-particle state and a particular n-particle state; the statistical theory as condensed in (1.1) makes only a statement about the average behaviour of the S-matrix elements.

One can now make a model which leads also to (1.2′) and shows therefore the same average behaviour of the transition probabilities as was assumed for the S-matrix: if one assumes that in the collision a « hot spot » is formed, which can afterwards decay in any of the allowed n-particle states, and that the probability of formation of the hot spot is independent of the probability of decay. One sees that this assumption is closely akin to the compound nucleus idea in nuclear physics.

Denoting the probability for hot-spot formation by $W_{2\to c}$ and for its decay by $W_{c\to n}$ the model means

$$W_n = W_{2\to c} W_{c\to n}$$

and for the reverse reaction

$$W_n = W_{n\to c} W_{c\to 2} .$$

If now in addition one makes the further physical assumption that $W_{n\to c} = C_n \cdot$ $\cdot$(probability to find the particle in Ω) [*i.e.*, probability in the reverse reaction to form a hot spot is proportional to the probability to find all particles together in a volume Ω] then we get

$$W_n = C_n W_{c\to 2}\, \Omega^{n-1} \varrho_n(E; 0) . \tag{1.5}$$

Here Ω has got evidently a very precise meaning: its linear extension has to be of the order of magnitude of the range of the strong interactions. This formula will only be identical with (1.1) if we assume furthermore that the proportionality constant C_n is in fact independent of n, *i.e.* that hot-spot formation depends only on all the particles being in Ω, whatever is their number (or their momenta for that matter).

One sees also that it is necessary to make the hot-spot model to arrive at (1.1): according to (1.1) W_n depends only on the probability for the n particles to be in Ω, which is physically only the first phase of the (reverse) process. The total probability W_n is only independent of the second phase (2 particles coming out of Ω) if the two phases are statistically independent.

We emphasize again that for the validity of (1.1) this model need not be true, in all its details. The true S-matrix should only on *the average* give the same results as this model. However, the contents of what one calls usually statistical theory are not exhausted by formula (1.1), which can only predict multiplicities, but no spectra or angular distribution. In order to get information about spectra, one usually takes the formula

$$W_n(p)\,\mathrm{d}p_n = C f_{n,T} \Omega^{n-1} \varrho_{n-1}\left(E - \sqrt{p_n^2 + m_n^2},\, p_n\right) 4\pi p_n^2\,\mathrm{d}p_n\,, \tag{1.6}$$

$$\varrho_{n-1}\left(E - \sqrt{p_n^2 + m_n^2},\, p_n\right) = \int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_{n-1} \int \mathrm{d}\boldsymbol{e}_n\, \delta\Big(E - \sum_{i=1}^{n} \sqrt{p_i^2 + m_i^2}\Big)\, \delta\Big(\sum_{i=1}^{n} \boldsymbol{p}_i\Big)\,, \tag{1.6'}$$

where $\boldsymbol{e}_n$ is the unit vector along $\boldsymbol{p}_n$. One gets (1.6) in effect from (1.6′) by omitting the integration over the magnitude of p_n, and takes $W_n(p_n)\,\mathrm{d}p_n$ to be the probability that p_n shall fall between p_n and $p_n + \mathrm{d}p_n$. One can now make the same kind of reasoning as before, by taking for the allowed n-particle states only those with given total energy-momentum *and* with the n-th particle momentum equal p_n:

$$\overline{|\langle f | S' | i \rangle|^2} \equiv S'(p_n)\,, \tag{1.7}$$

and in order to get (1.6) one must put this proportional to the average of the probabilities that n particles with total energy E in their c.m.s., the last particle having momentum p_n, will find themselves together in Ω, when the density in the allowed part of phase-space is uniform. One sees immediately that (1.6) demands that $S'(p_n)$ be in fact independent of p_n, (and equal to a constant of the form Ω^{n-1}), and here we take in effect the « hot-spot » model more seriously. One should point out that on the other hand the spectrum of particles within a given multiplicity n is independent of the assumption C_n=constant.

Formally, it is possible to go on in this vein, to leave, *e.g.* also the angles $\boldsymbol{e}_n$ out of the integration in (1.1) and derive a new formula where the S-matrix element squared averaged over a still smaller number of variables is put equal

to the corresponding average from the hot-spot model; this yields an isotropic distribution, as in the integrand of (1.6) no direction $\boldsymbol{e}_n$ is privileged:

$$(1.8) \qquad W_n(\boldsymbol{p}_n) = C f_{n,r_{\dots n}} \Omega^{n-1} \int \mathrm{d}\boldsymbol{p}_1 \dots \int \mathrm{d}\boldsymbol{p}_{n-1} \delta\Big(E - \sum_{i=1}^{n} \sqrt{p_i^2 + m_i^2}\Big)\, \delta\Big(\sum_{i=1}^{n} \boldsymbol{p}_i\Big) .$$

At this point we have to introduce explicitly angular momentum conservation. Formula (1.1) is not in contradiction with this conservation law, because the S-matrix element will take care of it by its dependence on the $\boldsymbol{p}_i$ (magnitude and direction); after averaging no explicit trace of this remains, except perhaps in the magnitude of Ω. The same cannot be said of (1.6) and *a fortiori* of (1.8), where the assumed independence of the reduced matrix element squared from p_n or $\boldsymbol{p}_n$ may be in contradiction with angular momentum conservation. This contradiction may be avoided by restricting the allowed n-particle states to those having given total energy E, momentum $\boldsymbol{P}_{\text{total}}$, angular momentum squared $j(j+1)$ and z component of angular momentum m (as we operate in the c.m.s. where $\boldsymbol{P}_{\text{total}} = 0$ these four quantum numbers can be specified simultaneously). E, $P = 0$, j and m are of course the same as in the initial state, and the states left out by restricting the averaging to those with fixed j and m would have contributed zero to the integral. Averaging then $|\langle i|S|f\rangle|^2$ over these more restricted final states (specified by E, $P = 0$, j, m and p_n) one shall get a different function $S''(p_n)$, because the region of integration is different. Taking this function independent of p_n is an assumption which involves no contradiction with angular momentum conservation. The hot-spot model which should yield the same average is obtained restricting here also the allowed initial states $|f_r\rangle$ for the reverse reaction to n-particle states having well-defined values, E, $P = 0$, j, m, and taking consequently the probability $W_{n\to c}$ for hot-spot formation proportional to the probability to find the n particles—with the quantum specified above—in the volume Ω. Taking the spectrum and the angular distribution from this hot-spot model will certainly give a better approximation to $W_n(p_n)$ or $W_n(\boldsymbol{p}_n)$ than formulae (1.6) or (1.6′); but there is of course no *a priori* reason that it will yield the whole story, *i.e.* that $S''(p_n)$ is really independent of p_n.

Specifying n-particle states with the quantum numbers given above, and finding an expression for this last probability is not a trivial problem; the next section is devoted to it.

2. – The density matrix.

In this section we shall answer the following question: « what is the probability W_n, that n particles are found inside a volume Ω, assuming that the n particles have well-defined total energy, total angular momentum and that their total momentum is zero? ».

Having stated that the n-particle system has sharp values of H, J^2, J_z and has $P=0$ is of course not sufficient to define the state of the system. We have seen on the other hand that the statistical theory of meson-production considers every state as equally probable, provided no conservation laws are violated. This then leads in a natural way to the description of the n-particle system by a density matrix. A density matrix can be written quite generally as

$$\varrho = \sum_{\substack{\alpha, E \\ j, j_3}} |\alpha, E, j, j_3, \boldsymbol{P}=0\rangle p_{\alpha, E, j, j_3} \langle \alpha, E, j, j_3, \boldsymbol{P}=0| . \tag{2.1}$$

We have already taken advantage in writing this that we shall only be interested in states with $\boldsymbol{P}=0$. The α stand for all the quantum numbers necessary to define the state completely in addition to E, j, j_3 and $\boldsymbol{P}=0$. Stating that only states with fixed E, j, j_3 (say E', j', j_3') are allowed means putting the weights $p_{\alpha, E, j, j_3}=0$ if E, j, j_3 have values different from E', j', j_3'. Stating that all allowed states have equal probability means putting

$$p_{\alpha, E', j', j_3'} = \text{constant}$$

independent of α (say C_n; it might in principle still depend on n). One could at this stage allow for final-state interaction by a suitable choice of the system of observables α, and by making $p_{E, \alpha', j', j_3'}$ dependent on α. We shall, however, stick here to the simple assumption

$$p_{\alpha, E, j, j_3} = C_n \, \delta(E-E') \, \delta(j-j') \, \delta(j_3-j_3') , \tag{2.2}$$

where C_n is a proportionality constant. This means that we assume for the density matrix in effect the simple form

$$\varrho = C_n \mathscr{P}_E \mathscr{P}_j \mathscr{P}_m \mathscr{P}_0 , \tag{2.3}$$

where $\mathscr{P}_E$, $\mathscr{P}_j$, $\mathscr{P}_m$ are projection operators on n-particle states having respectively total energy $= E$, total angular momentum $= \sqrt{j(j+1)}$, total angular momentum third component m, and $\mathscr{P}_0$ is the projection operator on total momentum $= 0$ states.

This density matrix allows us to state the probability to find a given observable A in a range of values $a_0 \ldots a_n$; as is well-known this is

$$\operatorname{Tr} \varrho \mathscr{P}_a ,$$

where $\mathscr{P}_a$ is the projection operator on the subspace of eigenstates of A having eigenvalues in the range $a_0 \ldots a_n$.

Our question now is to find the probability that the observable $\boldsymbol{x}_i$ (= position of i-th particle) will have an eigenvalue $\boldsymbol{\xi}_i$ inside Ω. The corresponding projection operator is simply

$$\int_{\Omega} \mathrm{d}\boldsymbol{\xi}_i \delta(\boldsymbol{x}_i - \boldsymbol{\xi}_i) ,$$

and the probability that the n particles are inside Ω is given by

$$W_{n,\Omega} = \int_{\Omega} \mathrm{d}\boldsymbol{\xi}_1 \dots \int_{\Omega} \mathrm{d}\boldsymbol{\xi}_n \operatorname{Tr} \varrho \, \delta(\boldsymbol{x}_1 - \boldsymbol{\xi}_1) \dots \delta(\boldsymbol{x}_n - \boldsymbol{\xi}_n) . \tag{2.4}$$

In order to work this out we have to introduce at this stage a complete system of states of n particles. For this we shall take products of plane waves; it will turn out that with this choice the mathematics are still manageable; other choices (*e.g.* spherical waves) are of course allowed in principle, but would make it hard in practice to retain rigorously momentum conservation in the formalism

$$|\boldsymbol{p}_1, \dots, \boldsymbol{p}_n\rangle = \exp\left[i \sum_{i=1}^{n} (\boldsymbol{p}_i, \boldsymbol{x}_i)\right] .$$

With this choice we have

$$\operatorname{Tr} \varrho \, \delta(\boldsymbol{x}_1 - \boldsymbol{\xi}_1) \dots \delta(\boldsymbol{x}_n - \boldsymbol{\xi}_n) = \int \mathrm{d}\boldsymbol{p}_1 \dots \int \mathrm{d}\boldsymbol{p}_n \langle \boldsymbol{p}_1 \dots \boldsymbol{p}_n | \varrho \prod_{i=1}^{n} \delta(\boldsymbol{x}_i - \boldsymbol{\xi}_i) | \boldsymbol{p}_1 \dots \boldsymbol{p}_n \rangle =$$

$$= \int \mathrm{d}\boldsymbol{p}_1 \dots \mathrm{d}\boldsymbol{p}_n \int \mathrm{d}\boldsymbol{x}_1 \dots \int \mathrm{d}\boldsymbol{x}_n \exp\left[-i \sum_{i=1}^{n} (\boldsymbol{p}_i, \boldsymbol{x}_i)\right] \varrho \prod_{i=1}^{n} \delta(\boldsymbol{x}_i - \boldsymbol{\xi}_i) \exp\left[i \sum_{i=1}^{n} (\boldsymbol{p}_i, \boldsymbol{x}_i)\right] ,$$

and consequently

$$W_{n,\Omega} = \int \mathrm{d}\boldsymbol{p}_1 \dots \int \mathrm{d}\boldsymbol{p}_n \int_{\Omega} \mathrm{d}\boldsymbol{\xi}_1 \dots \int_{\Omega} \mathrm{d}\boldsymbol{\xi}_n \exp\left[-i \sum_{i=1}^{n} (\boldsymbol{p}_i, \boldsymbol{\xi}_i)\right] \varrho \exp\left[i \sum_{i=1}^{n} (\boldsymbol{p}_i, \boldsymbol{\xi}_i)\right] . \tag{2.5}$$

(Strictly speaking, one should first compute the bra $\langle \boldsymbol{p}_1 \dots \boldsymbol{p}_n | \varrho$ and carry out the $\boldsymbol{x}_i$ integration afterwards; it will be clear that this does not alter the result.)

3. – The projection operators for momentum and energy.

The whole point of the present method lies in the use of a suitable form of the angular momentum projection operator. Because the operators $\boldsymbol{H}$, $\boldsymbol{P}$, J^2, J_3—of which our states are pure eigenstates—can be considered as infini-

tesimal generators of continuous groups, we are led to the following problem of group theory.

To find the projection operator on a given irreducible representation of a compact group (space-time translation group for H, $\boldsymbol{P}$; rotation group for J^2 and J_3). This problem can be solved quite generally for all finite and compact groups, and the answer is ([3,4])

$$\mathscr{P}_\Gamma = \frac{d^\Gamma}{h}\int g(A)\,\mathrm{d}A\,\chi^{\Gamma *}(A)A\,, \tag{3.1}$$

where
- A: an operator, element of the group,
- Γ: a representation of the group
- $\mathscr{P}_\Gamma$: the projection operator on Γ,
- d^Γ: the dimension of the representation,
- $\chi^\Gamma(A)$: the character of the element A in the representation Γ,
- $g(A)\,\mathrm{d}A$: the « volume element » in the integration of the space of the group elements,
- $h = \int g(A)\,\mathrm{d}A =$ the « volume » of the group (for a finite group: $h =$ number of elements of the group).

We recall the orthogonality property of the characters:

$$\int g(A)\chi^{\Gamma'*}(A)\chi^\Gamma(A)\,\mathrm{d}A = h\delta_{\Gamma,\Gamma'}$$

In order to make this clear we apply this formalism to a well-known case and we propose to derive the projection operator on one-particle states with momentum p' along the x-axis. We have then to build the projection operator $\mathscr{P}_{p'}$ on this representation of the group of all linear translations along the x-axis. The elements of the group are the operators of linear displacement:

$$A(a) = \exp\left[a\,\frac{\mathrm{d}}{\mathrm{d}x}\right],$$

because evidently $f(x+a) = \exp[a(\mathrm{d}/\mathrm{d}x)f(x)]$. This group is Abelian, all representations are one-dimensional functions of a and consequently the characters equal the representative functions

$$\chi^{(p)}(a) = \exp[ipa]$$

([3]) E. WIGNER: *Gruppentheorie* ... (Braunschweig, 1931).
([4]) V. HEINE: *Group Theory in Quantum Mechanics* (London, 1960).

whose orthogonality relations are

$$\int_{-\infty}^{+\infty} \mathrm{d}a\, \chi^{(p')*}(a)\, \chi^{(p)}(a) = \int_{-\infty}^{+\infty} \exp\left[i(p - p')a\right] \mathrm{d}a = 2\pi\delta(p - p') .$$

Consequently

$$\mathscr{P}_p = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}a \exp\left[- ipa\right] \exp\left[a \frac{\mathrm{d}}{\mathrm{d}x}\right] . \tag{3.2}$$

Let us apply this to a function with known Fourier transform

$$f(x) = \int_{-\infty}^{+\infty} \mathrm{d}k \exp\left[ikx\right] \varphi(k) , \tag{3.3}$$

$$\begin{aligned} \mathscr{P}_p f(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}a \int_{-\infty}^{+\infty} \mathrm{d}k \varphi(k) \exp\left[- ipa\right] \exp\left[a \frac{\mathrm{d}}{\mathrm{d}x}\right] \exp\left[ikx\right] = \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}a \int_{-\infty}^{+\infty} \mathrm{d}k \varphi(k) \exp\left[- i(p - k)a\right] \exp\left[ikx\right] = \\ &= \int_{-\infty}^{+\infty} \mathrm{d}k \varphi(k) \exp\left[ikx\right] \delta(p - k) = \end{aligned} \tag{3.4}$$

$$= \varphi(p) \exp\left[ipx\right] . \tag{3.5}$$

This gives the well-known result; comparing (3.4) with the definition of $f(x)$, we see also that one could put, when $\mathscr{P}_p$ is applied to an eigenfunction of momentum p_x (*i.e.* a representation of the translation group along the x-axis),

$$\mathscr{P}_p \exp\left[ikx\right] = \delta(p - k) \exp\left[ikx\right] . \tag{3.6}$$

We can now apply this immediately to our problem and write

$$\exp\left[- i \sum_{i=1}^{n} (\boldsymbol{p}_i, \boldsymbol{\xi}_i)\right] \mathscr{P}_E \mathscr{P}_0 = \exp\left[- i \sum_{i=1}^{n} (\boldsymbol{p}_i, \boldsymbol{\xi}_i)\right] \delta\Big(E - \sum_{i=1}^{n} \sqrt{p_i^2 + m_i^2}\Big)\, \delta\Big(\sum_{i=1}^{n} \boldsymbol{p}_i\Big) . \tag{3.7}$$

in order to get this result one should remember that for independent particles $H = \sum_{i=1}^{n} H_i$ (H_i = energy operator on i-th particle) and the time variation of

the plane waves is $\exp\left[-i\sum_i \varepsilon i t\right]$. A similar argument on time translations gives then the δ-function for the energy.

4. – The projection operators for angular momentum.

Were it not for the operators $\mathscr{P}_j \mathscr{P}_m$ this would lead at once to the result obtained by the traditional theory, without angular momentum conservation

$$W_{n,\Omega} \sim \Omega^n \int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_n \delta\Big(E - \sum_i \sqrt{p_i^2 + m_i^2}\Big)\, \delta\Big(\sum_i \boldsymbol{p}_i\Big)\,. \tag{4.1}$$

In order to have an explicit form of $\mathscr{P}_j$ we have to choose first suitable parameters to describe rotations. We shall use in the following two equivalent sets:

a) rotation axis, *i.e.* a unit vector $\boldsymbol{n}$; rotation angle ω, where $0 \leqslant \omega \leqslant \pi$

b) Euler angles: α, β, γ.

For each of those the factors that enter the integrand of the projection operator $\mathscr{P}_j$ are as follows (Table I) ([3,5,6]).

TABLE I.

	$\boldsymbol{n}, \omega$	α, β, γ
d^Γ	$2j+1$	$2j+1$
h	$2\pi^2$	$8\pi^2$
$g(A)$	$\sin^2 \omega/2$	$\sin \beta$
$\chi^\Gamma(A)$	$\sin (j+\frac{1}{2})\omega / \sin \omega/2$	$\sum_{m=-j}^{+j} D^{(j)}_{m,m}(\alpha, \beta, \gamma)$
A	$R(\boldsymbol{n}, \omega) = \exp[i\omega(\boldsymbol{n}, \boldsymbol{J})]$	$R(\alpha, \beta, \gamma) = \exp[-i\alpha J_z] \exp[-i\beta J_y] \exp[-i\gamma J_z]$
	$\boldsymbol{J} = \sum_{i=1}^{n} \boldsymbol{j}_i$	$(\boldsymbol{j}_i = -i\boldsymbol{r}_i \times \boldsymbol{\nabla}_i)$

(5) A. EDMONDS: *Angular Momentum in Quantum Mechanics* (Princeton, 1957).
(6) M. ROSE: *Elementary Theory of Angular Momentum* (New York, 1957).

We shall first construct the $\mathscr{P}_j$ operator with the $(\boldsymbol{n}, \omega)$ parameters, which gives the simpler form. Substituting in (3.1) we get (*)

$$(4.2)\qquad \exp\Big[-i\sum_{i=1}^{n}(\boldsymbol{p}_i, \boldsymbol{r}_i)\Big]\mathscr{P}_j \exp\Big[i\sum_{i=1}^{n}(\boldsymbol{p}_i, \boldsymbol{r}_i)\Big] =$$

$$= \frac{2j+1}{2\pi^2}\int \mathrm{d}\boldsymbol{n}\int_0^{\pi}\mathrm{d}\omega \sin^2\frac{\omega}{2}\frac{\sin(j+\frac{1}{2})\omega}{\sin\omega/2}\exp\Big[-i\sum_{i=1}^{n}(\boldsymbol{p}_i, \boldsymbol{r}_i)\Big]\cdot$$

$$\cdot\exp\Big[\omega(\boldsymbol{n}, \sum_i \boldsymbol{r}_i\times\boldsymbol{\nabla}_i)\Big]\exp\Big[i\sum_{i=1}^{n}(\boldsymbol{p}_i, \boldsymbol{r}_i)\Big].$$

Our next problem is then to express the effect of a finite rotation of the coordinate system on a plane wave, *i.e.* to compute

$$(4.3)\qquad \exp\Big[\omega(\boldsymbol{n}, \sum_{i=1}^{n}\boldsymbol{r}_i\times\boldsymbol{\nabla}_i)\Big]\exp\Big[i\sum_{i=1}^{n}(\boldsymbol{p}_i, \boldsymbol{r}_i)\Big] =$$

$$= \prod_{i=1}^{n}\exp\big[\omega(\boldsymbol{n}, \boldsymbol{r}_i\times_i)\big]\exp\big[i(\boldsymbol{p}\boldsymbol{\nabla}_i, \boldsymbol{r}_i)\big].$$

One can compute this by expanding the rotation operator in an exponential series

$$(4.4)\qquad \exp[\omega(\boldsymbol{n}, \boldsymbol{r}\times\boldsymbol{\nabla})] = \sum_{\nu=0}^{\infty}\frac{\omega^\nu}{\nu!}(\boldsymbol{n}, \boldsymbol{r}\times\boldsymbol{\nabla})^\nu.$$

To evaluate

$$(\boldsymbol{n}, \boldsymbol{r}\times\boldsymbol{\nabla})^\nu \exp[i(\boldsymbol{p}, \boldsymbol{r})]$$

is not trivial because of the non-vanishing commutators $[\boldsymbol{r}, \boldsymbol{\nabla}]$. The algebra involved is not complicated, however, and the result is

$$(4.5)\qquad \exp[\omega(\boldsymbol{n}, \boldsymbol{r}\times\boldsymbol{\nabla})]\exp[i(\boldsymbol{p}, \boldsymbol{r})] =$$

$$= \exp[i(\boldsymbol{p}, \boldsymbol{r})]\exp[i(\boldsymbol{r}, \boldsymbol{p}\times\boldsymbol{n})\sin\omega - i(\boldsymbol{r}, (\boldsymbol{p}\times\boldsymbol{n})\times\boldsymbol{n})(\cos\omega - 1)],$$

$$(4.6)\qquad = \exp[i(\boldsymbol{r}', \boldsymbol{p})] = \exp[i(\boldsymbol{r}, \boldsymbol{p}')],$$

(*) In this form we consider only orbital angular momenta; one could in principle introduce spinor plane waves for the nucleons, and make R also act on these spinors. This seems at the moment an unnecessary complication, as the bulk of the (high) angular momenta is certainly due to orbital motion of the fast product particles.

where

$$\boldsymbol{r}' = \boldsymbol{r} - (\boldsymbol{r}\times\boldsymbol{n})\,\sin\omega - [(\boldsymbol{r}\times\boldsymbol{n})\times\boldsymbol{n}]\,(\cos\omega - 1)\,, \tag{4.6a}$$

$$\boldsymbol{p}' = \boldsymbol{p} + (\boldsymbol{p}\times\boldsymbol{n})\sin\omega - [(\boldsymbol{p}\times\boldsymbol{n})\times\boldsymbol{n}](\cos\omega - 1)\,. \tag{4.6b}$$

This result is quite natural: $\boldsymbol{r}'$ is the vector $\boldsymbol{r}$ turned over an angle ω around n. One can alternatively consider the operation of rotation as affecting the field, with a stationary frame and this leads then to a rotation of the momentum vectors over an angle $|\omega|$ in the opposite direction. $\boldsymbol{p}'$ is nothing but the original vector $\boldsymbol{p}$ turned over an angle $-\omega$ around $\boldsymbol{n}$, (4.6*a*) and (4.6*b*) express indeed the result of performing a rotation $(n, \pm\omega)$ on a vector ([7]); the new vector is a sum of three mutually orthogonal vectors. One sees also that the *difference* between the turned vector and the original vector is orthogonal to the axis of rotation, and its length is

$$\begin{Bmatrix} |\boldsymbol{r}' - \boldsymbol{r}| \\ |\boldsymbol{p}' - \boldsymbol{p}| \end{Bmatrix} = \begin{Bmatrix} r \\ p \end{Bmatrix} |\sin\theta|[\sin^2\omega + (\cos\omega - 1)^2]^{\frac{1}{2}} = 2\begin{Bmatrix} r \\ p \end{Bmatrix} |\sin\theta| \sin\frac{\omega}{2}\,. \tag{4.7}$$

The matrix element of the projection operator for total angular momentum $j(j+1)$ between plane waves reads now

$$\exp\Big[-i\sum_i (\boldsymbol{p}_i, \boldsymbol{r}_i)\Big]\mathscr{P}_j \exp\Big[i\sum_i (\boldsymbol{p}_i, \boldsymbol{r}_i)\Big] = \tag{4.8}$$

$$= \frac{2j+1}{2\pi^2}\int \mathrm{d}\boldsymbol{n}\int_0^{\pi} \mathrm{d}\omega\,\sin\frac{\omega}{2}\,\sin(j+\tfrac{1}{2})\omega\,\exp\Big[i\sum_{i=1}^{n}(\boldsymbol{K}_i, \boldsymbol{r}_i)\Big]\,,$$

where

$$\boldsymbol{K}_i = \sin\omega(\boldsymbol{p}_i\times\boldsymbol{n}) - (\cos\omega - 1)[(\boldsymbol{p}_i\times\boldsymbol{n})\times\boldsymbol{n}]$$

is the difference between the turned vector $\boldsymbol{p}_i'$ and the original vector $\boldsymbol{p}_i$ (see Fig. 1).

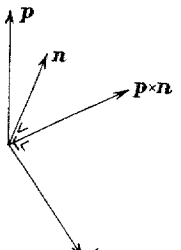

Fig. 1. – The vector $\boldsymbol{p}$, the rotation axis $\boldsymbol{n}$ and the triad of orthogonal vectors $\boldsymbol{n}$, $\boldsymbol{p}\times\boldsymbol{n}$, $(\boldsymbol{p}\times\boldsymbol{n})\times\boldsymbol{n}$ to which the turned vector is referred in formula (4.6*b*).

([7]) J. L. Synge: *Classical Dynamics*, in *Handb. d. Phys.*, vol. **3**/1 (Berlin, 1960).

5. – The projection operator for magnetic quantum numbers.

Here we are concerned with states having a well-defined value of J_z, the z-component of the total angular momentum. The operator J is the infinitesimal generator of rotations around the z-axis; one convinces oneself easily that in case of this Abelian group the projection operator is

$$\mathscr{P}_m = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \exp\left[-im\varphi\right] \exp\left[i\varphi J_z\right] . \tag{5.1}$$

The interesting thing, however, for high-energy collisions is the operator

$$\mathscr{P}, \mathscr{P}_m \qquad \text{with } m = 0.$$

If the z-axis is taken along the collision line the total angular momentum must be orthogonal to the z-axis, and hence only states with $J_z = 0$ can contribute to the final state. So we have to compute

$$\exp\Big[-i\sum_i (\boldsymbol{p}_i, \boldsymbol{r}_i)\Big]\, \mathscr{P}, \mathscr{P}_{m=0} \exp\Big[i\sum_i (\boldsymbol{p}_i, \boldsymbol{r}_i)\Big] . \tag{5.2}$$

One can do this by applying in succession the explicit forms (5.1) and (4.8) or else making use of the fact that $\mathscr{P}, \mathscr{P}_{m=0}$ is a projection operator on a particular line of the $2j+1$ dimensional irreducible representation of the rotation group. The general form of such an operator is [3,4]

$$\frac{d^\Gamma}{h}\int g(A)\, \mathrm{d}(A)\, \mathscr{D}^\Gamma_{m,m}(A) A , \tag{5.3}$$

where $\mathscr{D}^\Gamma_{m,m}$ is the diagonal element on the particular line m of the representative matrix. Formula (3.1) is a special case of this theorem, because

$$\chi^\Gamma = \sum_m \mathscr{D}^\Gamma_{m,m} .$$

Applied to the problem at hand this gives

$$\mathscr{P}, \mathscr{P}_{m\ 0} = \frac{2j+1}{2\pi^2}\int \mathrm{d}\boldsymbol{n}\int_0^\pi \mathrm{d}\omega \sin^2\frac{\omega}{2}\, \mathscr{D}^{(j)}_{0,0}(\boldsymbol{n}, \omega) \exp\Big[\omega\Big(\boldsymbol{n}, \sum_i \boldsymbol{r}_i \times \boldsymbol{\nabla}_i\Big)\Big] . \tag{5.4}$$

We see that the only difference with the operator P_j consists in the substitution of $\mathscr{D}^{(j)}_{0,0}(\boldsymbol{n}, \omega)$ for $\chi^{(j)}(\omega)$. The matrix element $\mathscr{D}^{(j)}_{0,0}$ has a particularly simple expression when the rotation is expressed with Euler angles

$$\mathscr{D}^{(j)}_{0,0}(\alpha, \beta, \gamma) = \mathscr{D}^{(j)}_{0,0}(0, \beta, 0) = P_j(\cos\beta)\,, \tag{5.5}$$

($P_j(\cos\beta)$ = Legendre polynomial of order j). As a function of $\boldsymbol{n}$, ω it is obtained by expressing β as a function of $\boldsymbol{n}$ and ω. We notice that

$$\cos\beta = \mathscr{D}^{(1)}_{0,0}(\boldsymbol{n}, \omega) = (\boldsymbol{e}_3,\ R(\boldsymbol{n}, \omega)\boldsymbol{e}_3)\,, \tag{5.6}$$

i.e. the unit vector along the z-axis, $\boldsymbol{e}_3$, turned over ω around $\boldsymbol{n}$, and projected on the z-axis.

By using (4.6*a*) one finds

$$\cos\beta = \cos^2\theta + \sin^2\theta\cos\omega\,,$$

where θ is the angle between n and the z-axis.
Consequently

$$\mathscr{D}^{(j)}_{0,0}(\boldsymbol{n}, \omega) = P_j(\cos^2\theta + \sin^2\theta\cos\omega) = \tag{5.7}$$

$$= (P_j(\cos\theta))^2 + 2\sum_{m=1}^{j}\frac{(j-m)!}{(j+m)!}(P(\cos\theta))^2\cos m\omega\,.$$

We see that the parameters $(\boldsymbol{n}, \omega)$ give simple results in the formula for $\mathscr{P}_j$ only; for expressing $\mathscr{P}_j\mathscr{P}_{m=0}$ it is easier to use the Euler angles.

6. – Statistical theory with J but without P conservation (Koba's theory).

In Section 2 it was pointed out that the formalism of the density matrix leaves one in principle free to choose any complete set of n-particle wave functions. The explicit form of $\mathrm{Tr}\,\varrho\mathscr{P}_a$ will of course depend on the chosen system.

In connection with the theory developed by KOBA ([8,9]) it is interesting to calculate the form of $\langle\mathscr{P}_j\mathscr{P}_m\rangle$ using spherical harmonics instead of plane

([8]) Z. KOBA: *Statistical theory of multiple particle production with angular momentum conservation*, preprint Warsaw, October 1960; to be published in *Acta Phys. Polon.*

([9]) Z. KOBA: *The angular momentum weight factor in the statistical theory of multiple production* - I, II, preprints February and April 1961; to be published in *Bulletin of the Polish Academy of Sciences.*

waves. A complete system of wave functions of a free scalar particle is then

$$|klm\rangle \equiv j_l(kr)\, Y_l^m(\theta, \varphi)\,,$$

($j_l(kr) =$ the l-th spherical Bessel function) and a diagonal matrix element of $\mathscr{P}_j \mathscr{P}_m$ is therefore

$$(6.1)\qquad \frac{2j+1}{8\pi^2} \int_0^\infty r_1^2\, \mathrm{d}r_1 \ldots \int_0^\infty r_n^2\, \mathrm{d}r_n\, |j_{l_1}(k_1 r_1) \ldots j_{l_n}(k_n r_n)|^2 \cdot$$
$$\cdot \int_0^\pi \sin\theta_1 \mathrm{d}\theta_1 \int_0^{2\pi} \mathrm{d}\varphi_1 \ldots \int_0^\pi \sin\theta_n \mathrm{d}\theta_n \int_0^{2\pi} \mathrm{d}\varphi_n \int_0^{2\pi} \mathrm{d}\alpha \int_0^\pi \mathrm{d}\beta \int_0^{2\pi} \mathrm{d}\gamma\, \sin\beta \cdot$$
$$\cdot \mathscr{D}^{(j)*}_{m,m}(\alpha, \beta, \gamma)\, Y^{m_1 *}_{l_1}(\theta_1 \varphi_1) \ldots Y^{m_n *}_{l_n}(\theta_n \varphi_n) R_{(\alpha\beta\gamma)}\, Y^{m_1}_{l_1}(\theta_1 \varphi_1) \ldots Y^{m_n}_{l_n}(\theta_n \varphi_n)\,.$$

Because the Y_l^m are orthonormal and belong to a well-defined irreducible representation of the rotation group

$$(6.2)\qquad \int_0^{2\pi} \mathrm{d}\varphi \int_0^\pi \sin\theta\, \mathrm{d}\theta\, Y_l^{m*} R_{(\alpha\beta\gamma)}\, Y_l^m = \mathscr{D}^{(l)}_{m,m}(\alpha\beta\gamma)$$

and the last two lines of formula (6.1) reduce to

$$(6.3)\qquad \int_0^{2\pi} \mathrm{d}\alpha \int_0^\pi \mathrm{d}\beta \int_0^{2\pi} \mathrm{d}\gamma\, \sin\beta\, \mathscr{D}^{(j)*}_{m,m}(\alpha\beta\gamma)\, \mathscr{D}^{(l_1)}_{m_1,m_1}(\alpha\beta\gamma) \ldots \mathscr{D}^{(l_n)}_{m_n,m_n}(\alpha\beta\gamma)\,.$$

If one chooses to *neglect momentum conservation* and consequently takes as density matrix

$$(6.4)\qquad \varrho' = C'_n\, \mathscr{P}_E \mathscr{P}_j \mathscr{P}_m$$

and compute in this case

$$(6.5)\qquad \mathrm{Tr}\, \varrho'\, \delta(\boldsymbol{x}_1 - \boldsymbol{\xi}_1) \ldots \delta(\boldsymbol{x}_n - \boldsymbol{\xi}_n)$$

the trace would involve summing over all m_i (as well as l, and integrals over k_i). Taking the expression (6.2) and summing over all m_i gives a certain coefficient $Z_m^{(j)}(l_1 \ldots l_n)$

$$(6.6)\qquad Z_m^{(j)}(l_1 \ldots l_n) = \int_0^{2\pi} \mathrm{d}\alpha \int_0^\pi \mathrm{d}\beta \int_0^{2\pi} \mathrm{d}\gamma\, \sin\beta\, \mathscr{D}^{(j)*}_{m,m} \sum_{m_1=-l_1}^{+l_1} \ldots \sum_{m_n=-l_n}^{+l_n} \mathscr{D}^{(l_1)}_{m_1,m_1} \ldots \mathscr{D}^{(l_n)}_{m_n,m_n}\,,$$

which can in principle be computed, as the $\mathscr{D}^{(j)}_{m\,m}(\alpha\beta\gamma)$ are well-known. This coefficient contains implicitly all the selection rules due to the addition of the the angular momenta $l_1, \ldots, l_n$ to a resultant j, with $m_1+m_2+\ldots+m_n = m$. If *e.g.* $l_1+\ldots+l_n<j$ or $l_n>j+l_1+\ldots+l_{n-1}$ so that the triangular conditions cannot be satisfied, then $Z^{(j)}_m(l_1 \ldots l_n)$ is automatically zero. This method has been used in two previous articles for computing « phase-space integrals » in isospin space, either by recursion using Clebsch-Gordon coefficients ([10]) or in closed form ([11]).

The phase space integral for a spherical Ω with radius R looks then as follows (up to a constant coefficient)

$$(6.7)\qquad \int_0^R \mathrm{d}r_1 \ldots \int_0^R \mathrm{d}r_n \int_0^\infty \mathrm{d}k_1 \ldots \int_0^\infty \mathrm{d}k_n \sum_{l_1=0}^\infty \ldots \sum_{l_n=0}^\infty Z^{(j)}_m(l_1 \ldots l_n) \prod_{i=1}^n k_i^2 r_i^2 |j_{l_i}(k_i r_i)|^2 \cdot \cdot \delta\Big(E - \sum_i \sqrt{k_i^2+m_i^2}\Big) .$$

Formula (6.7) is essentially Koba's formulation of angular momentum conservation in the statistical theory.

7. – Phase-space integral with angular momentum conservation.

The probability to find n particles in volume Ω, with specified E, $P=0$, j, and $m=0$ is then

$$(7.1)\qquad W_{n,\Omega}(E; j; m=0) = = C_n \int\ldots\int \mathrm{d}\boldsymbol{p}_1 \ldots \mathrm{d}\boldsymbol{p}_n \delta\Big(E - \sum_{i=1}^n \sqrt{p_i^2+m_i^2}\Big)\, \delta\Big(\sum_i \boldsymbol{p}_i\Big)\, G_j(\boldsymbol{p}_1 \ldots \boldsymbol{p}_n) ,$$

where

$$(7.2)\qquad G_j = \frac{2j+1}{2\pi^2} \int \mathrm{d}\boldsymbol{n} \int_0^\pi \mathrm{d}\omega \sin^2\frac{\omega}{2}\, \mathscr{D}^{(j)}_{0,0}(\boldsymbol{n}, \omega) \int_\Omega \mathrm{d}\boldsymbol{r}_1 \ldots \int_\Omega \mathrm{d}\boldsymbol{r}_n \exp\Big[i \sum_{i=0}^n (\boldsymbol{r}_i, \boldsymbol{K}_i)\Big] ,$$

or alternatively

$$(7.3)\qquad = \frac{2j+1}{8\pi^2} \int_0^{2\pi} \mathrm{d}\alpha \int_0^\pi \mathrm{d}\beta \int_0^{2\pi} \mathrm{d}\gamma \sin\beta P_j(\cos\beta) \int_\Omega \mathrm{d}\boldsymbol{r}_1 \ldots \int_\Omega \mathrm{d}\boldsymbol{r}_n \exp\Big[i \sum_{i=1}^n (\boldsymbol{r}_i, \boldsymbol{K}_i)\Big] ,$$

([10]) F. CERULUS: *Suppl. Nuovo Cimento*, **15**, 402 (1960).
([11]) F. CERULUS: *Nuovo Cimento*, **19**, 528 (1961).

and $\boldsymbol{K}$ is each time the difference between the turned vector $\boldsymbol{p}'$ and the original $\boldsymbol{p}$ (4.6).

The further development depends now on the assumption made about the interaction volume Ω. We shall here take a spherical shape, and replace the sharp cut-off by a Gaussian cut-off. We replace then

$$\int_{\Omega} \mathrm{d}\boldsymbol{r} \exp\left[i(\boldsymbol{K}, \boldsymbol{r})\right] \quad \text{by} \quad \int \mathrm{d}\boldsymbol{r} \exp\left[-\frac{r^2}{4a^2} + i(\boldsymbol{r}, \boldsymbol{K})\right]. \tag{7.4}$$

This form of cut-off is made for computational convenience. The mean square radius of the volume with this cut-off is

$$\langle r^2 \rangle = 6a^2 ,$$

i.e. we could take as radius

$$R = \sqrt{6}a = 2.45\, a . \tag{7.5}$$

The integral (7.4) is the Fourier transform of the form factor of Ω. For a spherical Gaussian we get

$$\int_{\Omega} \mathrm{d}\boldsymbol{r} \exp\left[i(\boldsymbol{K}, \boldsymbol{r}) \to 8\pi^{\frac{3}{2}} a^3 \exp\left[-a^2K^2\right] = \left(\frac{2\pi}{3}\right)^{\frac{3}{2}} R^3 \exp\left[-\frac{R^2K^2}{6}\right], \tag{7.6}$$

and

$$\int_{\Omega} \mathrm{d}\boldsymbol{r}_1 \ldots \int_{\Omega} \mathrm{d}\boldsymbol{r}_n \exp\left[i \sum_{i} (\boldsymbol{r}_i, \boldsymbol{K}_i)\right] = \left(\sqrt{\frac{2\pi}{3}}\, R\right)^{3n} \exp\left[-a^2\Big(\sum_{i=1}^{n} K_i^2\Big)\right] . \tag{7.7}$$

This expression depends only on the length of the vector $\boldsymbol{K}_i$. This length was already given for the parameters of rotation $\boldsymbol{n}$, ω (4.7).

For the Euler angles it is easily computed as follows:

$$\boldsymbol{K} = \boldsymbol{p}' - \boldsymbol{p} = \mathscr{D}^{(1)}(\alpha\beta\gamma)\boldsymbol{p} - \boldsymbol{p} ,$$

where $\mathscr{D}^{(1)}$ is the transformation matrix of a vector under a rotation $(\alpha\beta\gamma)$. Consequently

$$K^2 = (\mathscr{D}^{(1)}\boldsymbol{p})^2 + \boldsymbol{p}^2 - 2\,(\boldsymbol{p}, \mathscr{D}^{(1)}\boldsymbol{p}) = 2\big(p^2 - (\boldsymbol{p}, \mathscr{D}^{(1)}\boldsymbol{p})\big) . \tag{7.8}$$

Using the known formulae for the matrix $\mathscr{D}^{(1)}_{m,m'}(\alpha\beta\gamma)$ [5] one finds

$$(\boldsymbol{p}, \mathscr{D}^{(1)}\boldsymbol{p}) = p_z^2 \cos\beta + \boldsymbol{p}_\perp^2 \frac{1+\cos\beta}{2} \cos(\alpha\beta\gamma) , \tag{7.9}$$

where p_z and $\boldsymbol{p}_\perp$ are the components of $\boldsymbol{p}$ along the z-axis and perpendicular to the z-axis.

We are now led to the following expression for the probability to find n particles in a spherical Gaussian volume Ω

$$W_{n,\Omega}(E, j, 0) = C_n(2j+1)\left(\frac{1}{2}\sqrt{\frac{2\pi}{3}}\right)^n\left(\frac{4\pi}{3}R^3\right)^n\cdot \tag{7.10}$$
$$\cdot\int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_n \delta\Big(E - \sum_{i=1}^{n}\sqrt{p_i^2 + m_i^2}\Big)\,\delta\Big(\sum_i \boldsymbol{p}_i\Big)\mathscr{F}_j\,.$$

The change, with respect to the statistical theory without J-conservation is

a) a factor $2j+1$,

b) a factor $(\frac{1}{2}\sqrt{2\pi/3})$,

(this amounts in effect only to an unimportant change of the interaction volume, due to the introduction of a Gaussian cut-off);

c) a weight function $\mathscr{F}_j(p_1 \ldots p_n)$ in the integrand.

By analogy with the usual statistical theory we shall henceforth call « phase-space integral » the expression

$$\varrho^*_{m_1 \ldots m_n}(E, \boldsymbol{P}=0, j, m=0) \equiv \int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_n \delta\Big(E - \sum_i \sqrt{p_i^2 + m_i^2}\Big)\delta\Big(\sum_i \boldsymbol{p}_i\Big)\mathscr{F}_j, \tag{7.11}$$

also to be denoted for short by $\varrho^*_n(E, j, 0)$.

The weight function $\mathscr{F}_j$ is expressed as a triple integral over the parameters of the rotation

$$\mathscr{F}_j(\boldsymbol{p}_1 \ldots \boldsymbol{p}_n) = \frac{1}{8\pi^2}\exp\left[-a^2Q^2\right]\int_0^{2\pi}\mathrm{d}\alpha\int_0^{\pi}\mathrm{d}\beta\int_0^{2\pi}\mathrm{d}\gamma\,\sin\beta\,P_j(\cos\beta)\cdot \tag{7.12}$$
$$\cdot\exp\left[2a^2Z^2\cos\beta\right]\exp\left[a^2T^2(1+\cos\beta)\cos(\alpha+\gamma)\right],$$

where

$$Q^2 \equiv \sum_{i=1}^{n}\boldsymbol{p}_i^2 = Z^2 + T^2\,,$$

$$Z^2 \equiv \sum_{i=1}^{n} p_{zi}^2\,,$$

$$T^2 \equiv \sum_{i=1}^{n} p_{\perp i}^2\,.$$

We see from this that the weight function $\mathscr{F}_j$ depends on a single $\boldsymbol{p}_i$ only so to say through the *average* over the n perpendicular and the n longitudinal components of the momenta. The alternative form of $\mathscr{F}_j$ is

$$\mathscr{F}_j = \frac{1}{2\pi^2}\int \mathrm{d}\boldsymbol{n}\int_0^\pi \mathrm{d}\omega \sin^2\frac{\omega}{2}\,\mathscr{D}^{(j)}_{0,0}(\boldsymbol{n},\omega)\exp\left[-2a^2\Theta^2\right](1-\cos\omega)\,, \tag{7.13}$$

where

$$\Theta^2 = \sum_{i=1}^{n}(\boldsymbol{p}_i\times\boldsymbol{n})^2\,,$$

$$= \text{sum of the projection squared of } \boldsymbol{p}_i \text{ on a plane perpendicular to the axis of rotation } \boldsymbol{n}.$$

We shall not use this form for computation, but it is useful for qualitative discussion.

8. – The classical limit of the phase-space integral.

T. ERICSON [12] has been able to derive many results of statistical theory with angular momentum conservation by treating the $\boldsymbol{l}_i$-vectors of each single particle as classical vectors, *i.e.* by neglecting the fact that $\boldsymbol{r}$, $\boldsymbol{p}$ and $\boldsymbol{l}$ do not commute.

What happens in our case when we neglect these commutators? The rotation of a plane wave is then written as

$$\left\{\begin{aligned}\exp\left[\omega(\boldsymbol{n},\boldsymbol{r}\times\boldsymbol{\nabla})\right]\exp\left[i(\boldsymbol{p},\boldsymbol{r})\right] &= \exp\left[i\omega(\boldsymbol{n},\boldsymbol{r}\times\boldsymbol{p})+i(\boldsymbol{p},\boldsymbol{r})\right],\\ &= \exp\left[i(\boldsymbol{r},\boldsymbol{p}+\omega(\boldsymbol{p}\times\boldsymbol{n})\right],\end{aligned}\right. \tag{8.1}$$

so that now $\boldsymbol{K}=\omega(\boldsymbol{p}\times\boldsymbol{n})$.

$$\mathscr{F}_j^{(\text{Classical})} = \frac{1}{2\pi^2}\int \mathrm{d}\boldsymbol{n}\int_0^\pi \mathrm{d}\omega\,\mathscr{D}^{(j)}_{0,0}(\boldsymbol{n},\omega)\exp\left[-a^2\Theta^2\omega^2\right], \tag{8.2}$$

with $\mathscr{D}^{(j)}_{0,0}(\boldsymbol{n},\omega)$ as given in (5.7).

When is this a good approximation? In all such cases where

$$\exp\left[-a^2\Theta^2\omega^2\right] \approx \exp\left[-2a^2\Theta^2(1-\cos\omega)\right] \tag{8.3}$$

(12) T. ERICSON: *Nuovo Cimento*, **21**, 605 (1961).

as is easily seen by comparing (8.2) with (7.13); this means when $a^2\Theta^2$ is so big that moderate values of ω make the factor $\exp[-a^2\Theta^2\omega^2]$ already vanishingly small. The condition is then

$$a^2 \sum_{i=1}^{n} (\boldsymbol{p}_i \times \boldsymbol{n})^2 \gg 1 .$$

Because $\boldsymbol{n}$ is an integration variable (over the unit sphere) this is only sure if the p_i's are big enough, and with fairly spread out directions. The last point is true on the average, for n not too small, because configurations of $\boldsymbol{p}_i$ which cluster together occur very seldom and contribute therefore a negligible amount to the phase-space integral. The physical requirement for the classical picture to hold is then

a) $\langle p^2 \rangle \gg \dfrac{1}{na^2}$;

b) n rather large to assure the probable spreading of directions.

If we can neglect the particles' masses as compared to their energies the first condition can be stated as a condition on the total energy which has to be larger than

$$E_{\min} = n\sqrt{\langle p^2 \rangle} \gg \sqrt{n}\frac{1}{a} = \sqrt{6n}\,\frac{1}{R} . \tag{8.4}$$

We shall see in Section **9** the exact limits of validity of the classical approximation by comparing the exact end formula with the classical one. Eq. (8.4), however, shows already that in most practical cases condition *a*) will be fulfilled. For *e.g.* 6 particles and $R \approx 1.4 \cdot 10^{-13}$ cm one has $E_{\min} \approx 1$ GeV.

9. – Explicit form of the phase-space integrand.

We shall presently carry out the integrations in formula (7.12) for $\mathscr{F}_n$. The integrals over α and γ are simple and of the form

$$\int_0^{2\pi} \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\gamma \exp[A \cos(\alpha + \gamma)] = 4\pi^2 I_0(A) , \tag{9.1}$$

where $I_0(A) = J_0(iA)$, the Bessel function of pure imaginary argument. After

this we are left with one more integral

$$(9.2)\quad \mathscr{F}_j = \frac{1}{2}\int_0^\pi \mathrm{d}\beta \sin\beta P_j(\cos\beta)\exp[-2a^2Q^2]\exp[2a^2Z^2\cos\beta]I_0[a^2T^2(1+\cos\beta)]\,.$$

Remembering that $Q^2 = T^2 + Z^2$ the integrand can be written as a product of three factors

$$F_1(\cos\beta) = P_j(\cos\beta)\sin\beta\,,$$
$$F_2(\cos\beta) = \exp[-(2a^2Z^2 + a^2T^2)(1-\cos\beta)]\,,$$
$$F_3(\cos\beta) = \exp[-a^2T^2(1+\cos\beta)]I_0(a^2T^2(1+\cos\beta))\,.$$

We shall first transform the integral to get rid of the oscillating $P_j(\cos\theta)$. The factor $F_3(\cos\beta)$ can be expressed as a Laplace transform using the formula

$$(9.3)\qquad \exp[-bp]I_0(bp) = \frac{1}{\pi}\int_0^{2b}\exp[-pt](2bt-t^2)^{-\frac{1}{2}}\mathrm{d}t\,.$$

Putting for a moment

$$2a^2Z^2 + a^2T^2 = a^2(Q^2+Z^2) \equiv A\,,$$
$$a^2T^2 = a^2(Q^2-Z^2) \equiv B\,,$$

we have

$$(9.4)\quad \mathscr{F}_j = \frac{1}{2\pi}\int_0^{2B}\frac{\mathrm{d}t}{(2Bt-t^2)^{\frac{1}{2}}}\int_0^\pi \mathrm{d}\beta\sin\beta P_j(\cos\beta)\exp[-A(1-\cos\beta)-t(1+\cos\beta)]$$

$$(9.5)\qquad = \sqrt{\frac{1}{2\pi}}\exp[-A]\int_0^{2B}\frac{\mathrm{d}t}{(2Bt-t^2)^{\frac{1}{2}}}\exp[-t]\frac{1}{(A-t)^{\frac{1}{2}}}I_{j+\frac{1}{2}}(A-t)$$

$$(9.6)\qquad = 4\pi\sqrt{2\pi}\exp[-t_0](A-t_0)^{-\frac{1}{2}}\exp[-(A-t_0)]I_{j+\frac{1}{2}}(A-t_0)\exp[-B]I_0(B)\,,$$

where the last line is derived from the foregoing by using the mean value theorem, t_0 being a value between 0 and $2B$. For most practical purposes A is a large number, because the particles' momenta are larger than $\mu = 1/R$, except perhaps for very high multiplicities; it is of the order of $n(R^2\langle p\rangle^2/6)$.

Because $I_{j+\frac{1}{2}}(x)$ is a fast growing function of x (see further down for their general behaviour), the important contributions to the integral come from

small values of t; for $j+\frac{1}{2}<2A$ the width (*i.e.* where $\exp[-t](A-t)^{-\frac{1}{2}}\cdot I_{j+\frac{1}{2}}(A-t)=\frac{1}{2}A^{-\frac{1}{2}}I_{j+\frac{1}{2}}(A)$) it $t\approx\frac{1}{2}\log 2$, which is quite negligible compared to A. In most cases it will therefore be a good approximation to take simply $A-t_0\approx A$ and $\exp[-t_0]\approx 2^{-\frac{1}{2}}$. For extreme cases one will have to resort to a numerical evaluation.

Taking for the moment the approximate formula one gets

$$\mathscr{F}_j=\frac{1}{2\sqrt{\pi}}\exp[-a^2T^2]I_0(a^2T^2)\exp[-a^2(2Z^2+T^2)]\cdot \cdot I_{j+\frac{1}{2}}\big(a^2(2Z^2+T^2)\big)\frac{1}{a(2Z^2+T^2)^{\frac{1}{2}}}\,. \tag{9.7}$$

Before discussing how to calculate the phase-space integral with this function in the integrand we shall discuss the general behaviour of $\mathscr{F}_j$.

The functions $\exp[-x]I_\nu(x)$ are smooth, real functions of x and ν. For very small or very large values of x they can be easily approximated by the first term of a power-series expansion, or of an asymptotic series respectively. We have as a matter of definition

$$I_\nu(x)=\sum_{s=0}^{\infty}\frac{(\frac{1}{2}x)^{\nu+2s}}{s!\,(\nu+s)!}=i^{-\nu}J_\nu(ix)\,, \tag{9.8}$$

therefore for $x\ll 1$

$$\exp[-x]I_\nu(x)\approx\frac{(\frac{1}{2}x)^\nu}{\nu!}\left(1-x+\frac{(\frac{1}{2}x)^2}{\nu+1}+\ldots\right). \tag{9.9}$$

For $x\gg 1$ *and* $x\gg\nu$

$$\exp[-x]I_\nu(x)\approx\frac{1}{\sqrt{2\pi x}}\left(1-\frac{4\nu^2-1}{8x}+\ldots\right). \tag{9.10}$$

For moderate or large ν the following form is more accurate (Debye asymptotic form)

$$\exp[-x]I_\nu(x)\approx\sqrt{\frac{\operatorname{tgh}\gamma}{2\pi\nu}}\exp[-x+\nu(\operatorname{cotgh}\gamma-\gamma)]\cdot \cdot\left[1+\frac{1}{24\nu}\operatorname{tgh}\gamma(3-5\operatorname{tgh}^2\gamma)+\ldots\right], \tag{9.11}$$

with

$$\sinh\gamma\equiv\frac{\nu}{x}\qquad i.e.\qquad \operatorname{tgh}\gamma=\frac{\nu}{\sqrt{\nu^2+x^2}}\,.$$

For moderate ν and $\nu<\frac{1}{2}x$ formula (9.11) is approximately (up to terms $0(\nu^3/x^3)$)

$$\exp[-x]I_\nu(x)\approx\frac{1}{\sqrt{2\pi x}}\exp\left[-\frac{\nu^2}{2x}\right]\left(1-\frac{1}{4}\frac{\nu^2}{x^2}\right). \tag{9.12}$$

In Fig. 2 and 3 are shown the functions for $\nu=0,\ \frac{1}{2},\ \frac{3}{2}$ and further for $\nu=1,3,\ldots,11$. As may be seen from (9.8) only $\exp[-x]I_0(x)$ does not vanish

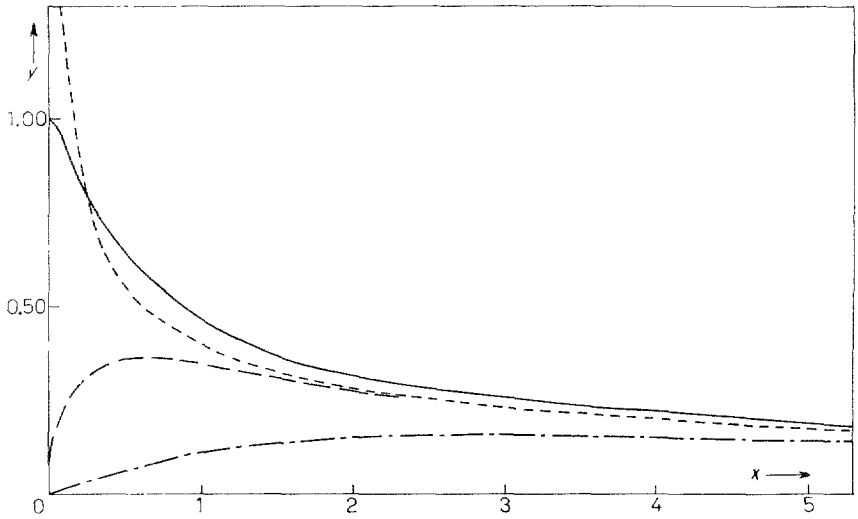

Fig. 2. – Graphs of the functions: $y = 1/\sqrt{2\pi x}$: - - - -, $y = \exp[-x]I_{\frac{1}{2}}(x)$: – – –, $y = \exp[-x]I_0(x)$: ———, $y = \exp[-x]I_{\frac{3}{2}}(x)$: –·–·–.

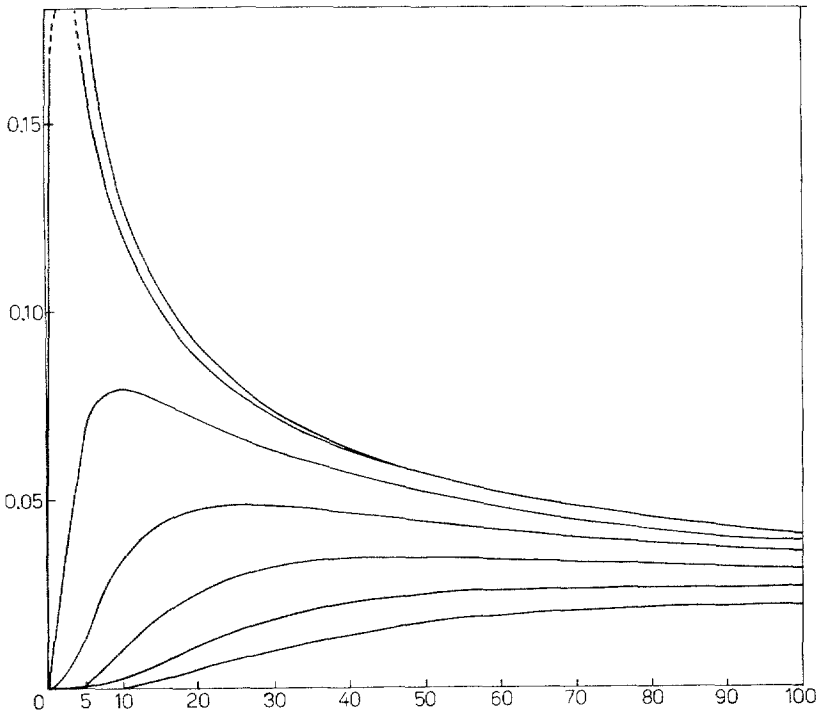

Fig. 3. – Graphs of the functions $\exp[-x]I_n(x)$ for $n = 0, 1, 3, 5, 7, 9, 11$ (counting from top to bottom).

for $x=0$ $(I_0(0)=1)$. All the other functions start from 0, and remain practically 0 up to $x\approx\nu$. From there they rise to a maximum around $x\approx\nu^2$. and beyond this point can be fairly well approximated by $\exp[-x]I_\nu(x)\approx$ $\approx 1/\sqrt{2\pi x}$.

To understand the formula one can think therefore of $\exp[-x]I_\nu(x)$ as a cut-off function: for *given* x it is zero (very roughly speaking) for $\nu > x^2$, for $\nu < x^2$ it is equal to $1/\sqrt{2\pi x}$; cf. *e.g.* Fig. 4.

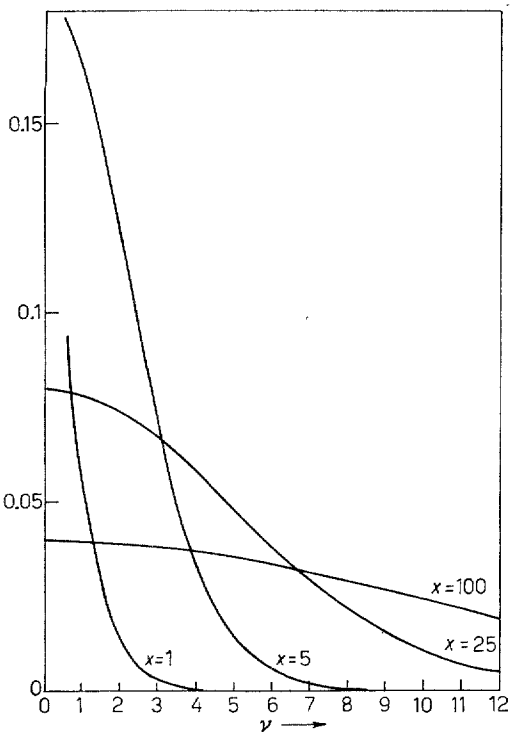

Fig. 4. – The functions $\exp[-x]I_\nu(x)$ plotted *vs.* the order ν, for different values of the variable x. The cut-off character is quite apparent.

The functions $I_{l+\frac{1}{2}}(x)$ have been tabulated ([13]). They can always be expressed as

$$I_{l+\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[A_n\left(\frac{1}{x}\right)\cosh(x)+A_{n-1}\left(\frac{1}{x}\right)\sinh x\right],$$

with $A_n(t)$ a n-th order polynomial in t, *e.g.* ([13])

$$I_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sinh x\,,$$

$$I_{\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[\cosh x-\frac{1}{x}\sinh x\right].$$

The approximation we have made in deriving formula (9.7) $a^2Q^2\gg\frac{1}{2}\log 2=0.346$ coincides more or less with the first of the two assumptions made by the classical approximation. From the discussion on p. 978 we know that it should be all right in practice.

How does formula (9.7) compare then to the classical approximation? Taking the asymptotic formulae (9.10) and (9.12) we find for

$$(9.13)\qquad \mathscr{F}_j\approx(2\pi)^{-\frac{3}{2}}2^{-\frac{1}{2}}(aT)^{-1}[a^2(2Z^2+T^2)]^{-1}\exp\left[-\frac{(j+\frac{1}{2})^2}{2a^2(2Z^2+T^2)}\right],$$

([13]) C. W. JONES: *A short table of the Bessel functions* $I_{n+\frac{1}{2}}(x)$, $(2/\pi)K_{n+\frac{1}{2}}(x)$, (Cambridge, 1952).

up to terms of order $(j/aQ)^2$ and comparing this with the formula (4.4) of ERICSON (remember $6a^2 = R^2$) [12]

$$\mathscr{F}_j^{(\text{Classical})} = (2\pi)^{-\frac{3}{2}} \frac{1}{(\frac{2}{9}R^2Q^2)^{\frac{3}{2}}} \exp\left[-\frac{j^2}{\frac{4}{9}R^2Q^2}\right], \tag{9.14}$$

one sees that both are equal if $Z^2 = 1/3Q^2 = 2/3T^2$ (neglecting the difference between j^2 and $j(j+1)+\frac{1}{4}$, which is to be expected from a classical approximation and the factor $1/\sqrt{2}$ which comes from taking $t_0 = \frac{1}{2}\log 2$). So $\mathscr{F}_j^{\text{Classical}}$ is strictly valid only for isotropic distribution of momenta; however, for angular momenta j small compared to RQ the difference is not important.

10. – Angular momentum zero.

This special case is of practical importance in annihilation of anti-nucleons at rest (together with the case $j = 1$). Going back to formulae (9.5) and (9.7) we find

$$\mathscr{F}_0 = 2^{-\frac{1}{2}}\pi^{-1} \exp[-a^2T^2] I_0(a^2T^2) \exp[-a^2(2Z^2+T^2)] \frac{\sinh[a^2(2Z^2+T^2)]}{a^2(2Z^2+T^2)}. \tag{10.1}$$

One knows that for very small momenta $|\boldsymbol{p}_i|$ (*i.e.* many particles sharing a small total energy (*), all end particles are necessarily in s-states, and the total angular momentum must be zero; this follows also at once from the exact formula (9.2), because there for

$$aT \to 0\,, \qquad aZ \to 0\,,$$

we have $\mathscr{F}_0 = 1$, (*i.e.* a constant independent of T and Z, and n) up to terms of order $a^2(Z^2+T^2)$. This leads to

$$W(E, 0, 0) = C_n \left(\frac{1}{2}\sqrt{\frac{2\pi}{3}}\right)^n \left(\frac{4\pi}{3}R^3\right)^n \int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_n \cdot \tag{10.2}$$

$$\cdot \delta\Big(E - \sum_i \sqrt{p_i^2 + m_i^2}\Big)\, \delta\Big(\sum_i \boldsymbol{p}_i\Big)\,,$$

i.e. essentially the expression one gets without introducing angular-momentum conservation; this is not surprising since all available states are necessarily

(*) A mathematically equivalent statement is: for very small a, *i.e.* very short radius of the interaction volume.

s-states, and hence no additional constraint is imposed on the phase-space by J-conservation.

Conversely, the expressions (10.2) or (1.1) are only strictly valid for s-states, although they may still be a good approximation when the outgoing particles can have *e.g.* p and d waves. The above hypothesis ($aQ \to 0$) is however very unrealistic in all cases of practical importance

For annihilation we have $\langle n \rangle \approx 4.7$ and $\langle \sqrt{p^2+\mu^2} \rangle \approx 0.4$ GeV. We find $a^2Q^2 \approx \frac{1}{6}R^2 \cdot n\big((E/n)^2 - \mu^2\big) \approx 5.51$ for the case $n = 6$, *e.g.*, which allows to use the formula (9.7). This may be further checked: numerical integration of (9.5) yields in the above mentioned case 12.93 as compared with 12.37 from (9.7). For $n < 6$ the approximation is still better.

As no direction is privileged, the classical approximation may therefore be used, which gives

$$\mathscr{F}_0 \approx \frac{1}{[(4\pi/9)R^2Q^2]^{\frac{3}{2}}} .$$

One sees how this will affect the multiplicity: the higher n, the lower Q, and the higher $\mathscr{F}_0$. Angular momentum conservation will therefore *increase* the multiplicity of pions in annihilation at rest. Also for $j = 1$ the above conclusion will be almost true.

Part II

THE VALUE OF THE PHASE-SPACE INTEGRAL WITH ANGULAR MOMENTUM CONSERVATION

11. – Problem.

In the previous part we derived the following expression for the phase-space integral

$$\varrho_n(E, j) = \int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_n \delta\Big(E - \sum_{i=1}^{n} \sqrt{p_i^2 + m_i^2}\Big) \delta\Big(\sum_{i=1}^{n} \boldsymbol{p}_i\Big) \mathscr{F}_j , \tag{11.1}$$

with

$$\mathscr{F}_j = 2^{-\frac{1}{2}} \exp\left[-\tfrac{1}{6}R^2T^2\right] I_0\left(\tfrac{1}{6}R^2T^2\right) \left[\tfrac{1}{6}R^2(2Z^2 + T^2)\right]^{-\frac{1}{2}} \cdot$$
$$\cdot \exp\left[-\tfrac{1}{6}R^2(2Z^2 + T^2)\right] I_{j+\frac{1}{2}}\left[\tfrac{1}{6}R^2(2Z^2 + T^2)\right] ,$$

$$Z^2 = \sum_{i=1}^{n} p_{z_i}^2 ,$$

$$T^2 = \sum_{i=1}^{n} \boldsymbol{p}_{\perp_i}^2 .$$

One has now to devise ways and means to compute this expression in practice. The simpler problem with $\mathscr{F}_j \equiv 1$ has been studied in the past ([14-17]), and it has appeared that there the big difficulty comes from the square-roots $\sqrt{p_i^2+m_i^2}$, which make a closed-form expression of the integral very difficult.

Two approximations based on the hypotheses

$$A) \quad \sqrt{p_i^2 + m_i^2} \approx p_i \,, \qquad \text{(ultrarelativistic)};$$

$$B) \quad \sqrt{p_i^2 + m_i^2} \approx m_i + \frac{p_i^2}{2m_i} \,, \qquad \text{(unrelativistic)};$$

have been proposed, which yield both fairly simple formulae, but whose practical importance is small, because neither $A)$ nor $B)$ is true in practice.

A Monte-Carlo method, which is free from the bias of $A)$ or $B)$ has been used to treat many problems, in statistical theory without angular momentum conservation ([15]). All the above mentioned methods make use of the fact that the integration over all directions of $\boldsymbol{p}_i$ can be done in closed form, because the integrand is isotropic.

Because $\mathscr{F}_j$ introduces the direction of the z-axis, the p_z and $p_\perp$ have now to be treated separately; one introduces cylindrical co-ordinates, p_z, $p_\perp$

$$(11.2) \qquad \varrho_n(E, j) = \prod_{i=1}^{n} \left[\int_0^\infty p_{\perp i} \, \mathrm{d}p_{\perp i} \int_{-\infty}^{+\infty} \mathrm{d}p_{zi} \int_0^{2\pi} \mathrm{d}\varphi_i \right] \mathscr{F}_j \cdot$$

$$\cdot \delta\Big(E - \sum_{i=1}^{n} \sqrt{p_{\perp i}^2 + p_{zi}^2 + m_i^2}\Big) \, \delta\Big(\sum_{i=1}^{n} \boldsymbol{p}_{\perp i}\Big) \, \delta\Big(\sum_{i=1}^{n} p_{zi}\Big) \,.$$

The angles φ_i appear only in $\delta(\sum_i \boldsymbol{p}_{\perp i})$. The angular integration is therefore, using the Fourier transform of the two-dimensional δ-function

$$(11.3) \qquad \frac{1}{(2\pi)^2} \int_{\text{plane}} \mathrm{d}\boldsymbol{\lambda} \prod_{i=1}^{n} \int_0^{2\pi} \exp\left[-i(\boldsymbol{\lambda}, \boldsymbol{p}_{\perp i})\right] \mathrm{d}\varphi_i \,:$$

now

$$\int_0^{2\pi} \mathrm{d}\varphi \, \exp\left[-i(\boldsymbol{\lambda}, \boldsymbol{p}_\perp)\right] = \int_0^{2\pi} \mathrm{d}\varphi \, \exp\left[-i\lambda p_\perp \cos\varphi\right] = 2\pi J_0(\lambda p_\perp) \,,$$

([14]) R. H. MILBURN: *Rev. Mod. Phys.*, **27**, 1 (1955).

([15]) S. Z. BELENKII, V. MAKZIMENKO, A. NIKISOV and I. ROSENTAL: *Fortsch. d. Phys.*, **6**, 524 (1958).

([16]) G. FIALHO: *Phys. Rev.*, **105**, 328 (1957).

([17]) F. CERULUS and R. HAGEDORN: *Suppl. Nuovo Cimento*, **9**, 646 (1958).

and the angular integration reduces to

$$(11.4) \qquad (2\pi)^{n-1}\int_0^\infty \lambda\,\mathrm{d}\lambda J_0(\lambda p_{\perp 1}) \ldots J_0(\lambda p_{\perp n})\,.$$

This expression is a well-known result in random-walk theory: there

$$P^{(2)}(r;\, a_1, \ldots, a_n) = \int_0^\infty \lambda J_0(r\lambda) \prod_{i=1}^n J_0(a_i\lambda)\,\mathrm{d}\lambda\,,$$

is the probability that the n vectors in a plane with lengths $a_1 \ldots a_n$ will add up to a resultant r, when all directions are equally probable. We shall assume that this function is known (cf. Appendix).

Similarly, we can interpret the p_z integration as a random-walk in one dimension: consider the integral

$$(11.5) \qquad J \equiv \int_{-\infty}^{+\infty} \mathrm{d}z_1 \ldots \int_{-\infty}^{+\infty} \mathrm{d}z_n F(z_1^2, \ldots, z_n^2)\,\delta(z_1 + z_2 \ldots + z_n - Z)$$

where we can split the integration regions in a negative and a positive part

$$(11.6) \qquad J = \sum_{(\sigma)} \int_0^\infty \mathrm{d}z_1 \ldots \int_0^\infty \mathrm{d}z_n F(z_1^2 \ldots z_n^2)\,\delta(\sigma_1 z_1 + \sigma_2 z_2 + \ldots + \sigma_n z_n - Z)\,,$$

where $\sum_{(\sigma)}$ means a sum over all combinations of $\sigma_i = \pm 1$ and $F(z_1^2 \ldots z_n^2)$ is an arbitrary even function of the $z_1, \ldots, z_n$. The function

$$(11.7) \qquad P^{(1)}(Z;\, z_1, \ldots, z_n) = (\tfrac{1}{2})^n \sum_{(\sigma)} \delta(\sigma_1 z_1 + \ldots + \sigma_n z_n - Z)\,,$$

is the probability that a particle moving along a line in successive steps of magnitude z_1, z_2, ... but random sign will reach after n steps the position Z. This is the random-walk function in one dimension. Here we shall also assume this function to be known (and give formulae for it in the Appendix).

We are then faced with the integral (putting $p = a$, $p_z = z$)

$$(11.8) \qquad 2^n(2\pi)^{n+1}\varrho_n(E, j) = \prod_{i=1}^n \left[\int_0^\infty a_i\,\mathrm{d}a_i \int_0^\infty \mathrm{d}z_i\right] \delta\Big(E - \sum_i \sqrt{a_i^2 + z_i^2 + m_i^2}\Big) \cdot$$
$$\cdot P^{(2)}(0;\, a_1, \ldots, a_n) P^{(1)}(0;\, z_1, \ldots, z_n)\,.$$

To this it seems feasible to apply a Monte-Carlo method similar in principle to the one used for $\varrho_n^*(E)$, *i.e.* the phase-space integral without angular momentum conservation.

12. – A Monte-Carlo method.

Let us first transform the integration $\int_0^\infty \mathrm{d}a \int_0^\infty \mathrm{d}z$ to polar co-ordinates: a and z are considered as rectangular co-ordinates in a plane, whereas

$$p = \sqrt{a^2 + z^2},$$

$$\theta = \operatorname{arctg} \frac{a}{z},$$

are corresponding polar co-ordinates. So we have now

$$(12.1) \qquad \varrho_n(E, j) = 2^{-1}\pi^{n-1} \int_0^\infty \dots \int_0^\infty \mathrm{d}p_1 \dots \mathrm{d}p_n \int_0^{\pi/2} \dots \int_0^{\pi/2} \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \delta\Big(E - \sum_i \sqrt{p_i^2 + m_i^2}\Big) \cdot$$
$$\cdot p_1^2 \dots p_n^2 \sin\theta_1 \dots \sin\theta_n P^{(2)}(0; p_1 \sin\theta_1, \dots, p_n \sin\theta_n) \cdot$$
$$\cdot P^{(1)}(0; p_1 \cos\theta_1 \dots p_n \cos\theta_n) \mathscr{F}_j(Z^2; T^2) .$$

One can go over to kinetic energies, instead of momenta by putting $t_i = \sqrt{p_i^2 + m_i^2} - m_i$ and write consequently

$$\mathrm{d}p_1 \dots \mathrm{d}p_n \, p_1^2 \dots p_n^2 \, P^{(2)}(0; p_1 \sin\theta_1, \dots) \, P^{(1)}(0; p_1 \cos\theta_1, \dots) \, \mathscr{F}_j(Z^2; T^2) ,$$

as function of the t_i and of the $\cos\theta_i$

$$\Phi(t_1, t_2, \dots, t_n; \cos\theta_1, \dots, \cos\theta_n) \, \mathrm{d}t_1 \dots \mathrm{d}t_n .$$

And, as proved in [17], $\varrho_n(E, j)$ can be written as

$$(12.2) \qquad \varrho_n(E, j) = 2^{-1}\pi^{n-1} \int_0^T \mathrm{d}T_1 \int_{T_1}^T \mathrm{d}T_2 \dots \int_{T_{n-2}}^T \mathrm{d}T_{n-1} \int_0^1 \mathrm{d}(\cos\theta_1) \dots \int_0^1 \mathrm{d}\cos\theta_n) \cdot$$
$$\cdot \Phi(T_1, T_2 - T_1, \dots, T - T_{n-1}; \cos\theta_1, \dots, \cos\theta_n) .$$

This integral is computed by Monte-Carlo by drawing N times two samples A) and B).

A) $n-1$ random numbers, according to a uniform probability distribution from the segment $(0, T)$. These are labelled in ascending order of magnitude

$$T_1,\ T_2,\ \ldots,\ T_{n-1}\,.$$

B) n random numbers, according to a uniform probability distribution from the segment $(0, 1)$. These are labelled (in the order in which they come)

$$\cos\theta_1,\ \ldots,\ \cos\theta_n\,.$$

For each sample one computes the expression

$$\Phi(T_1,\ \ldots,\ T-T_{n-1};\ \cos\theta_1,\ \ldots,\ \cos\theta_n)$$

adds up these numbers for all samples, takes the average value; this is proportional to $\varrho_n(E, j)$.

In the same way as in the method described in [17] one can compute here kinetic energy spectra; these can even now be split into transverse kinetic energy and longitudinal kinetic energy, giving in principle the answer of the statistical theory as to the distribution of transverse momenta in a high-energy collision.

13. – Estimates of the influence of angular momentum conservation.

As long as the method outlined in the previous chapter has not been put to work it will not be possible to have detailed predictions of the statistical theory. What we propose to do here is an approximation that should enable us to compare semi-quantitatively the results of the theory with and without angular momentum conservation.

Let us write the phase-space integral again, but taking out of the integration the mean value of the weight function $\mathscr{F}_j(p_1 \ldots p_n)$:

$$\varrho_n(E, j) = \langle \mathscr{F}_j \rangle \int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_n \delta\Big(E - \sum_{i=1}^{n} \sqrt{p_i^2 + m_i^2}\Big)\, \delta\Big(\sum_i \boldsymbol{p}_i\Big) = \langle \mathscr{F}_j \rangle \varrho_n^*(E)\,, \tag{13.1}$$

where $\varrho_n^*(E)$ is the phase-space integral as conventionally defined [17]. This mean value, which is precisely defined by formula (13.1), we are now going to replace by the function $\mathscr{F}_j(\langle p_1 \rangle, \langle p_2 \rangle \ldots)$ where we shall take

$$\langle p_i \rangle \equiv \sqrt{\langle \varepsilon_i \rangle^2 - m_i^2}\,,$$

whith $\langle \varepsilon_i \rangle$ the average value of the energy computed from the conventional phase-space integral; *e.g.*

$$(13.2) \qquad \varrho_n^*(E)\,\langle \varepsilon_1 \rangle = \int \mathrm{d}\boldsymbol{p}_1 \ldots \int \mathrm{d}\boldsymbol{p}_n \sqrt{p_i^2 + m_i^2}\,\delta\Big(E - \sum_i \sqrt{p_i^2 + m_i^2}\Big)\,\delta\Big(\sum_i \boldsymbol{p}_i\Big)\,.$$

This is, of course, an approximation the error of which could only be ascertained by comparison with an exact calculation. We shall use it, as mentioned, as an exploratory tool.

For two cases which have been computed previously (6 GeV p-p collisions [18] and N-$\bar{N}$ annihilation at rest [19]) we shall compute $\langle \mathscr{F}_j \rangle$ and see how the new results compare with the old ones. A source of uncertainty is that the old calculation gives $\langle \varepsilon_i \rangle$ all right, but averaged over all angles, whereas to compute $\langle \mathscr{F}_j \rangle$ we should know separately $\langle Z^2 \rangle = \langle \sum_{i=1}^{n} p_{zi}^2 \rangle$ and $\langle T^2 \rangle = \langle \sum_{i=1}^{n} p_{\perp i}^2 \rangle$. We shall *assume* here near-isotropy as the most probable configuration on which to base our estimate of $\langle \mathscr{F}_j \rangle$, at least for not too high values of j. This assumption is good as long as the classical approximation to $\mathscr{F}_j$, is valid, *i.e.* $a^2 Q^2 \gg j^2$. With $\mathscr{F}_j = 1$ (*i.e.* no angular momentum conservation) one gets of course that $\langle Z^2 \rangle = \frac{1}{3}\langle Q^2 \rangle$, because then no direction is privileged. We take therefore

$$(13.3) \qquad 2Z^2 + T^2 = Z^2 + Q^2 = \tfrac{4}{3} Q^2 = \tfrac{4}{3}\Big[\sum_{i=1}^{2} \langle p_N \rangle^2 + \sum_{i=1}^{n} \langle p_\pi \rangle^2\Big]\,,$$

for the p-p collisions, because we want to distinguish between nucleons and pions. In the annihilation only pions are taken into account in the final state.

The result of substituting then in the calculations for p-$\bar{\mathrm{p}}$ annihilation at rest the expression (13.1) instead of $\varrho^*(E)$ is that the average number of pions is increased:

$$\langle n_\pi \rangle \quad \Big(\text{with } J\text{-conservation and } \Omega = \frac{4\pi}{3}\lambda_\pi^3\Big) = 3.73\,,$$

as compared to

$$\langle n_\pi \rangle \quad \Big(\text{without } J\text{-conservation and } \Omega = \frac{4\pi}{3}\lambda_\pi^3\Big) = 3.40\,.$$

Alternatively, one can try to adjust Ω so as to have the observed multiplicity

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([18]) R. HAGEDORN: *Nuovo Cimento*, **15**, 246 (1960).
([19]) F. CERULUS: *Nuovo Cimento*, **14**, 827 (1959).

of ≈ 4.7

$$\Omega = 10\,\frac{4\pi}{3}\,\lambda\!\!\!^{-3}_{\pi} \text{ yields } \langle n_\pi \rangle \approx 4.7 \text{ without } J\text{-conservation},$$

$$\Omega = \ 5\,\frac{4\pi}{3}\,\lambda\!\!\!^{-3}_{\pi} \text{ yields } \langle n_\pi \rangle \approx 4.8 \text{ with } J\text{-conservation}.$$

All this is of course done *not* assuming any π-π isobar. For the p-p collisions at 6 GeV the results are displayed in Fig. 5, as a function of the total

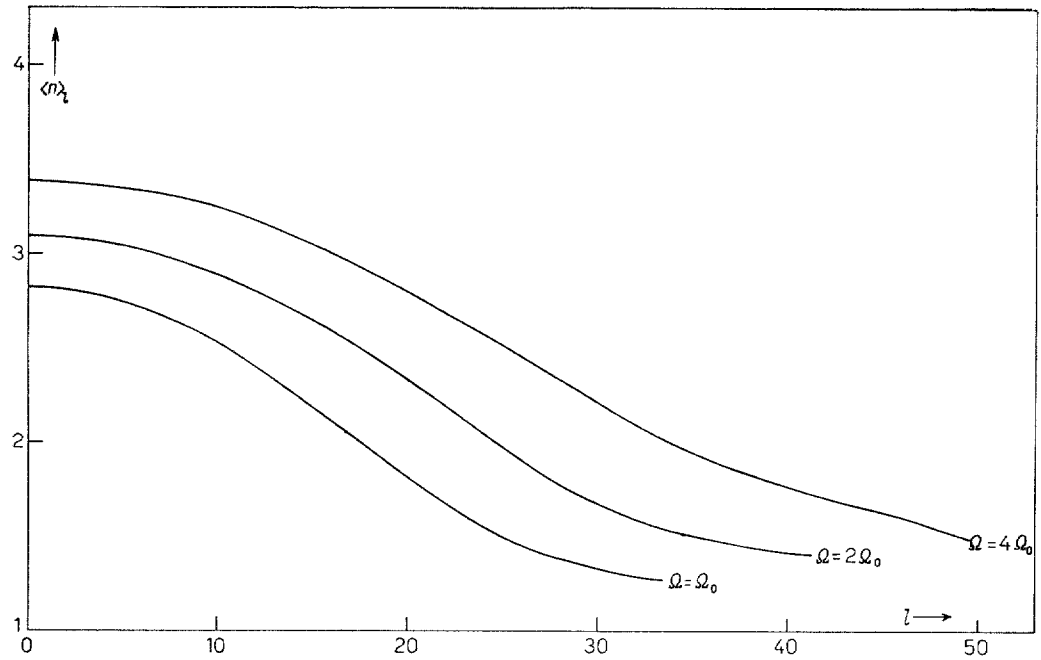

Fig. 5. – Average number of pions produced in a 6 GeV p-p collision, as a function of total angular momentum and of interaction volume $\Omega_0 = (4\pi/3)\lambda\!\!\!^{-3}_{\pi}$.

angular momentum l. To get the average multiplicity one has to sum over all l. If one takes the hot-spot model seriously enough all phase relationships between initial and final states are destroyed, and the only thing that counts is the statistical weight w_l of the l-th partial wave in the plane wave incident on the sphere Ω

$$w_l = \int_{\Omega} \mathrm{d}\xi_1 \int_{\Omega} \mathrm{d}\xi_2 \langle i \,|\, \mathscr{P}_l \mathscr{P}_{m=0}\, \delta(\boldsymbol{x}_1 - \boldsymbol{\xi}_1)\, \delta(\boldsymbol{x}_2 - \boldsymbol{\xi}_2) \,|\, i \rangle \,, \tag{13.4}$$

where $|i\rangle = \exp[i(\boldsymbol{k}, \boldsymbol{r}_1 - \boldsymbol{r}_2)]$ is the wave function of the two colliding par-

ticles in their c.m.s.; (13.4) is a special case ($n = 2$) of formula (2.5). w_l is akin to the « penetration factor » in nuclear physics. We obtain

$$w_l = 2^{-\frac{3}{2}}\pi^{\frac{1}{2}}(2l+1)(ak)^{-1}\exp[-4a^2k^2]I_{l+\frac{1}{2}}(4a^2k^2)\,, \tag{13.5}$$

where again we have taken the Gaussian cut-off (7.4). For the 6 GeV p-p collision this is shown in Fig. 6. The linear rise for small l corresponds to the factor $(2l+1)$ in the classical picture of the geometrical partial cross-section $\sigma_{r,l} = (2l+1)\pi\lambda\!\!\!^{-2}$, which is valid for $ak \ll 1$.

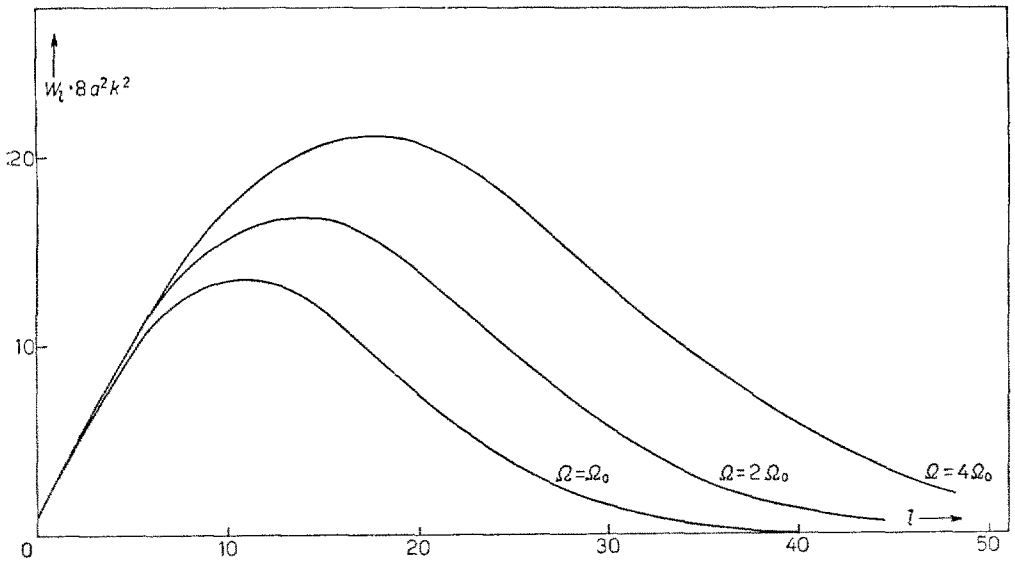

Fig. 6. – The penetration factors w_l of the partial waves for different interaction spheres; $\Omega_0 = (4\pi/3)\lambda\!\!\!^{-3}_{\pi}$; k corresponds to 6 GeV p-p collision in the c.m.s.

The average pion number then, resulting from folding $\langle n_\pi \rangle$ for given l with the w_l, is found to be

$$\langle n_\pi \rangle = 2.25 \qquad \text{with } J\text{-conservation}\,,$$

against

$$\langle n_\pi \rangle = 2.50 \qquad \text{without } J\text{-conservation}\,.$$

One sees that here the correction is in the opposite direction as in annihilation. This is easily understood because in a high-energy collision the high-angular momenta contribute most, and for those the exponential factor in $\mathscr{F}_l$ (9.13) makes itself felt.

The numbers just mentioned are obtained using $\Omega = (4\pi/3)\lambda\!\!\!^{-3}_{\pi} = \Omega_0$ and the existence of a π-$\mathcal{N}$ isobar; cf. [18]. In order to obtain $\langle n_\pi \rangle = 2.5$ with J-conservation one should take $\Omega = 2.0\,\Omega_0$.

14. – Conclusions.

Examination of the formulae used in conventional statistical theory has led us to interpret them in such a way that the probability to produce n particles in a high-energy collision can be simply related to the probability of finding n particles together in a volume Ω. The transition matrix elements squared will behave that way if the process proceeds via an intermediate « hot spot » and the formation and decay of this « hot spot » are statistically independent. The assumptions one really makes are, however, weaker than this, because the matrix element should only reproduce this behaviour on the average. As one asks however for more and more information (spectra, angular correlations and distributions) from the statistical theory, the averaging is done over less and less parameters and the assumptions are stronger and stronger.

Asking for angular distributions in the conventional theory is probably making too strong an assumption on the matrix elements, which—save for S waves—is in contradiction with angular momentum conservation.

By considering in the averaging process over the unobserved parameters of the final states only states of given total angular momentum this contradiction can be avoided. One has still of course to make hypotheses on $|\langle f|S|i\rangle|^2$, and these are most easily found by comparison with the hot-spot model.

The averaging over final states having specified values of J, E and $P = 0$ can be done using the projection operators on states with these quantum numbers. If one leaves out the P-conservation the Koba theory results.

The projection operators are given in integral form; the integrals converge and can be well approximated by fairly simple formulae in most cases of practical interest, where the available energy is several GeV and the radius of the interaction volume is $\approx \lambda\!\!\!^{-}_{\pi}$.

The theory has then given a modified form of the phase-space integral, which has to be computed to get detailed results.

As a further guide to methods of extracting information from the phase-space integral it will be useful to consider the so-called « classical » theory of T. Ericson; by a comparison with the present method it is seen that in many cases its results should be good approximations. Some qualitative features can, however, be extracted at once.

For high values of J, the phase-space integral will be depressed; also in this case there will be a marked correlation of the end particles, which will

tend to come out peaked backward-forward. For low values of J, as *e.g.* in $\mathcal{N}$-$\overline{\mathcal{N}}$ annihilation the phase-space integral is increased, no correlation is introduced by J-conservation.

Because the integrand of the phase-space integral depends only on the average momentum of all the particles (for a given multiplicity) the effect on the spectrum of a single particle will be fairly small. One could therefore expect that the spectra of particles (averaged over angles in the c.m.s.) from events with given multiplicity should not be much different from those obtained by the conventional theory.

The Monte Carlo method proposed should be able to obtain reliable spectra, both averaged over angles longitudinal and transverse. It is of course an interesting problem to find out how much backward-forward peaking J-conservation is able to give, or what magnitude of J one has to assume to be able to reproduce the experimentally observed peaking.

The assumptions of the hot-spot model are badly needed, finally, if one tries to estimate *e.g.* multiplicities from a high-energy collision where the initial state is a plane wave with definite phase relationship between the partial waves. Averaging over all angular momenta incoherently is only justified by the statistical independence of the two phases of the collision process. A consequence of this independence is that the angular distributions in this form of statistical theory are necessarily symmetric with respect to a plane perpendicular to the collision line; this feature is also clear from the phase-space integrand where only the squares of the longitudinal momenta appear.

Finally, the changes introduced by J-conservation in the multiplicity of $\mathcal{N}$-$\overline{\mathcal{N}}$ annihilation and of high-energy p-p collision (increasing the former, reducing the latter) make that one has now to take new values of the interaction volume Ω in order to get agreement with experiment; the long-standing discrepancy between the different values of Ω needed in each case is now reduced:

$$\Omega_{\text{annih}} = 2.5\, \Omega_{\text{collision}}$$

but has not disappeared completely.

* * *

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APPENDIX

Random-flight functions.

The probability that n ν-dimensional vectors of given lengths $a_1, a_2, \ldots, a_n$ have a resultant whose length lies between r and $r+\mathrm{d}r$ is [20]

$$(A.1) \qquad P^{(\nu)}(r)\,\mathrm{d}r = 2\left\{\Gamma\left(\frac{\nu}{2}\right)\right\}^{n-1}\int_0^\infty \left(\frac{tr}{2}\right)^{\nu/2} J_{(2/\nu)-1}(rt)\ldots\prod_{m=1}^{n}\frac{\{J_{(\nu/2)-1}(a_m t)\}}{(\frac{1}{2}a_m t)^{(\nu/2)-1}}\,\mathrm{d}t\,.$$

In a first approximation these $P^{(\nu)}(r)$ are Gaussians. It has been proved [21] for $a_1=a_2\ldots=a_n$ or more generally for all a's of the same order of magnitude [22] that integral (A_1) can be approximated, up to terms of order $1/n^2$ by the following expressions.

$$(A.2) \quad P^{(1)}(r)\,\mathrm{d}r = (2\pi)^{-1}A^{-1}\exp\left[-\frac{r^2}{2A^2}\right]\left[1-\frac{1}{24}\frac{B^4}{A^4}\left(3-6\frac{r^2}{A^2}+\frac{r^4}{A^4}\right)\right]\mathrm{d}r\,,$$

$$(A.3) \quad P^{(2)}(r)\,\mathrm{d}r = 2rA^{-2}\exp\left[-\frac{r^2}{A^2}\right]\left[1-\frac{1}{4}\frac{B^4}{A^4}\left(2-4\frac{r^2}{A^2}+\frac{r^4}{A^4}\right)\right]\mathrm{d}r\,,$$

$$(A.4) \quad P^{(3)}(r)\,\mathrm{d}r = 4\left(\frac{3}{2}\right)^{\frac{3}{2}}\pi^{-\frac{1}{2}}A^{-3}\exp\left[-\frac{3}{2}\frac{r^2}{A^2}\right]\left[1-\frac{1}{20}\frac{B^4}{A^4}\left(15-30\frac{r^2}{A^2}+9\frac{r^4}{A^4}\right)\right]\mathrm{d}r\,,$$

where, for $\nu=1, 2, 3$, one has defined

$$A^2 \equiv \sum_{i=1}^{n} a_i^2\,,$$

$$B^4 \equiv \sum_{i=1}^{n} a_i^4\,.$$

The function $P^{(1)}(r)$ given in (A.2) is *sensu stricto* not the probability to reach a point r on a line after n steps, which is a discontinuous function, but rather the density of the points that can be reached in n steps.

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(20) G. N. WATSON: *Bessel Functions* (Cambridge, 1945).
(21) F. G. TRICOMI: *Funzioni ipergeometriche confluenti* (Roma, 1954).
(22) F. CERULUS: to be published.

RIASSUNTO (*)

Si dimostra che la teoria statistica convenzionale (con conservazione dell'energia-impulso, ma senza conservazione del momento angolare) dà gli stessi risultati (molteplicità e spettri) che si possono ottenere da un modello che presuppone l'esistenza di un « hot-spot » intermedio nella collisione di due particelle di alta energia, se la formazione dell'« hot-spot » è statisticamente indipendente dal suo decadimento in numerose particelle. Questo modello è suggerito dalle formule usate nella teoria statistica, che in effetti pongono la probabilità di produrre n particelle proporzionale alla probabilità di trovare assieme n particelle in un volume Ω. La conservazione del momento angolare (assieme alla conservazione dell'energia e della quantità di moto) può essere soddisfatta prendendo in considerazione solo stati di n particelle con valori prescritti del J totale, dell'energia e della quantità di moto. Facendo uso del formalismo della matrice di densità e di una espressione esplicita degli operatori di proiezione su stati di dato momento angolare, si arriva ad una forma modificata dell'integrale dello spazio delle fasi, che è semplicemente collegata con la probabilità di produrre n particelle. Si mostra che la teoria con momento angolare, ma senza conservazione della quantità di moto, esposta da Koba, è un caso speciale facilmente derivabile dal presente formalismo. Si dimostra che la teoria « classica » della conservazione del momento angolare di T. Ericson è un caso limite che tuttavia ha una vasta applicabilità. Strettamente parlando la teoria convenzionale è valida solo se tutte le particelle finali sono nello stato s. Le formule sono state derivate per una forma sferica gaussiana di Ω. In linea di principio si possono ammettere forme contratte. Si propone un programma tipo Monte Carlo per valutare questo integrale dello spazio delle fasi. Il metodo permetterà di calcolare gli spettri degli impulsi longitudinali e trasversi delle particelle finali. Si calcolano gli effetti della conservazione di J sulla molteplicità. In confronto della teoria convenzionale la molteplicità è accresciuta del 10% nelle annichilazioni p-p e diminuito del 10% nelle collisioni p-p di 6 GeV.

(*) *Traduzione a cura della Redazione.*