LOWERING TOPOLOGICAL ENTROPY

By

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Abstract. The main result we prove in this paper is that for any finite dimensional dynamical system (with topological entropy h), and for any factor with strictly lower entropy h', there exists an intermediate factor of entropy h'' for every $h'' \in [h', h]$. Two examples, one of them minimal, show that this is not the case for infinite dimensional systems.

1. Introduction

The problem considered here was suggested in the paper [7]. It can be stated as follows:

Problem (1.1) Given a dynamical system (X, T) with positive topological entropy, what can be said about the range of values of $h_{\text{top}}(Y)$ as Y ranges over the factors of (X, T); in particular, when does such a dynamical system have factors with arbitrarily small entropy?

In what follows, by a *dynamical system* (X, T) we mean a compact metric space X, and a continuous bijection $T: X \to X$. A *factor* of (X, T) is a system (Y, S) with a continuous surjection $\pi: X \to Y$ such that $S \circ \pi = \pi \circ T$. This surjection is called the *factor transformation*.

A natural dynamical system in this context is the system $[0, 1]^{\mathbf{Z}}$ (the infinite dimensional cube with the usual Tychonoff topology) together with the shift σ. The Hilbert cube is universal for compact metric spaces since every such space can be imbedded in it. Dynamically it is universal in the following sense:

Proposition (1.2) *Any dynamical system (X, T) has a non-trivial factor imbedded in the (bi-)infinite dimensional cube.*

Proof Let $f: X \to [0, 1]$ be any continuous function. Define $\hat{f}: X \to [0, 1]^{\mathbf{Z}}$ by

$$\hat{f}: x \mapsto (\ldots, f \circ T^{-1}(x), f(x), f \circ T(x), \ldots). \tag{1.2.1}$$

Set $Y = \hat{f}(X) \subset [0, 1]^{\mathbf{Z}}$ and $S = \sigma|_Y$. It is clear that (Y, S) is a factor, with $\hat{f}$ the factor map, since $\hat{f}$ is continuous and $\hat{f} \circ T = \sigma \circ \hat{f}$. □

In the next section we show that this system has no finite entropy factors.

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The system $[0,1]^{\mathbf{Z}}$ has infinite dimension. In Sections 3 and 4 we prove that if X has *finite* dimension then for any factor Y' of X with $h_{\text{top}}(Y') < h_{\text{top}}(X)$ we can find, for any $h \in (h_{\text{top}}(Y'), h_{\text{top}}(X))$, a factor Y of X compatible with Y' with $h_{\text{top}}(Y) = h$. By compatible we mean that if $\phi_{Y,X}: X \to Y$ and $\phi_{Y',X}: X \to Y'$ are the factor transformations from X to Y and Y' respectively then we can find a factor transformation $\phi_{Y',Y}: Y \to Y'$ such that

$$\phi_{Y',Y} \circ \phi_{Y,X} = \phi_{Y',X}.$$

The proof of this result will be divided into two parts. In Section 3 we use dimension theory to prove some facts about any finite dimensional dynamical system. In Section 4 we show that a consequence of these facts is that we can construct the intermediate factors described above.

This theorem has a few interesting corollaries. First note that if we take Y' to be the trivial factor consisting of a single point then the above theorem gives a complete answer to Problem (1.1) for finite dimensional systems:

For any finite dimensional system (X,T) *the range of values of* $h_{\text{top}}(Y)$ *as* Y *ranges over the factors of* (X,T) *is* $[0, h_{\text{top}}(X)]$.

Thus in particular there are no finite dimensional prime systems (i.e. dynamical systems with no non-trivial factors) with positive entropy.

Also note that in particular the theorem gives a connection between finite dimension and finite topological entropy: while it is very easy to construct a finite dimensional dynamical system with infinite entropy, the theorem ensures us that any finite dimensional dynamical system has a factor with finite entropy.

In Section 5 we will give a construction of an infinite dimensional *minimal* system with no finite entropy factors. An interesting open problem that remains in this context is whether there exists an (infinite dimensional) finite entropy system with (say) a positive lower bound on the topological entropy of its factors. We believe (at least if we require that the system be minimal) that no such system exists. Note that for the finite dimensional case we did not need any condition on the entropy.

After the preparation of the initial version of this manuscript, S. Glasner has shown that the existence of a minimal system with no finite entropy factors gives a positive answer to a question posed by H. Furstenberg ([3]) whether there exist a pair of minimal dynamical systems which have a common almost one-to-one extension but no common factors. With his permission we present this result in the last section.

This paper essentially contains the author's M.Sc. thesis, conducted under the guidance of Professor Benjamin Weiss from the Hebrew University. The author is grateful to B. Weiss for introducing him to this problem, for many enlightening discussions about this and related topics, and for helping him bring this paper into what is hoped to be a readable form.

2. Factors of the system $[0,1]^{\mathbf{Z}}$

For the one sided shift, a proof, due to J. Ashley, that the system $[0,1]^{\mathbf{N}}$ has no finite entropy factors, is given in [7]. However, new ideas must be used to prove the same for the two sided shift, and in particular one must use a classical topological result: the Brouwer fixed point theorem. Let $H \stackrel{\text{def}}{=} [0,1]^{\mathbf{Z}}$.

Theorem (2.1) *Let (Y,T) be a non-trivial factor of $([0,1]^{\mathbf{Z}},\sigma)$. Then there exists an N such that $([0,1]^{\mathbf{Z}},\sigma)$ is a factor of (Y,T^N).*

The following lemma is a simple consequence of the Brouwer fixed point theorem:

Lemma (2.2) *Let $F=(f_1,\dots,f_n)\colon[0,1]^n\to[0,1]^n$ be a continuous map such that for every $1\le i\le n$,*

$$f_i(x_0,\dots,x_{i-1},1,x_{i+1},\dots,x_n)>\frac{9}{10}\quad\text{and}\quad f_i(x_0,\dots,x_{i-1},0,x_{i+1},\dots,x_n)<\frac{1}{10}.$$

Then

$$\left[\tfrac{1}{4},\tfrac{3}{4}\right]^n\subset F([0,1]^n). \tag{2.2.1}$$

Proof Suppose $x_0\in[1/4,3/4]^n$ but not in the image of F. For any $x\in[0,1]^n$ define $G(x)$ as the point where the ray from $F(x)$ to x_0 intersects the boundary of $[0,1]^n$. This point is well defined since x_0 is not in the image of F, and clearly G so defined is continuous. But the conditions on F guarantees that any point on the boundary of the cube $[0,1]^n$ is *not* mapped to itself, and since the image of G is contained in the boundary of $[0,1]^n$ we see that G is a continuous function $G\colon[0,1]^n\to[0,1]^n$ with no fixed points, in violation of the Brouwer fixed point theorem. □

Proof of the Theorem Without loss of generality one can assume that the factor is of the form (1.2.1). That is, the factor is of the form $(Y,\sigma|_Y)$ where $Y\subset[0,1]^{\mathbf{Z}}$ and the factor mapping $\hat{\phi}$ is generated from the continuous function $\phi\colon[0,1]^{\mathbf{Z}}\to[0,1]$ by

$$\hat{\phi}\colon x\mapsto(\dots,\phi\circ T^{-1}(x),\phi(x),\phi\circ T(x),\dots).$$

We can normalize ϕ so that 0 and 1 will both be in its image. (Since $[0,1]^{\mathbf{Z}}$ is connected this means that ϕ is onto the interval $[0,1]$.)

From the (uniform) continuity of ϕ it follows that there is an N_0 such that for any two points in the infinite dimensional cube, x and y,

$$x_k = y_k \quad \text{for every } -N_0 \le k \le N_0 \quad \Longrightarrow \quad |\phi(x) - \phi(y)| < \frac{1}{10}. \tag{2.1.1}$$

Set $N = 2N_0 + 1$, and let $\hat{\imath}: ([0,1]^Z, \sigma^N) \to ([0,1]^{\mathbf{Z}}, \sigma)$ be the map

$$\hat{\imath}(\ldots, x_{-1}, x_0, x_1, \ldots) \mapsto (\ldots, x_{-N}, x_0, x_N, \ldots).$$

Since we assume (Y,T) to be imbedded in $([0,1]^{\mathbf{Z}}, \sigma)$ this map defines a map from (Y, T^N) to the infinite dimensional cube, and we will prove the image of Y contains the smaller infinite dimensional cube $H' \overset{\text{def}}{=} \left[\frac{1}{4}, \frac{3}{4}\right]^{\mathbf{Z}}$. Since it is easy to find a factor transformation from $[0,1]^{\mathbf{Z}}$ to itself such that the image of the smaller cube is $[0,1]^{\mathbf{Z}}$ this proves the theorem.

We want to see that $\hat{\imath}(Y) \supset H'$, that is that $\hat{\imath} \circ \hat{\phi}(H) \supset H'$. Since H is compact, so is $\hat{\imath} \circ \hat{\phi}(H)$. If we will show that for every point $h \in H'$ and every n there is a point $h' \in \hat{\imath} \circ \hat{\phi}(H)$ that agrees with h in the coordinates $-n, \ldots, n$ it will follow that h itself is in $\hat{\imath} \circ \hat{\phi}(H)$. So we need only look, for every n, at the projection π of $\hat{\imath} \circ \hat{\phi}(H)$ on that range of coordinates, and verify that they contain the projection of H'. We now have a function

$$g \overset{\text{def}}{=} \pi \circ \hat{\imath} \circ \hat{\phi}: H \to [0,1]^{2n+1}.$$

We will define a function $\psi: [0,1]^{2n+1} \to H$ such that $g \circ \psi: [0,1]^{2n+1} \to [0,1]^{2n+1}$ will satisfy the conditions of Lemma (2.2). Let $p^0, p^1 \in [0,1]^{\mathbf{Z}}$ such that $\phi(p^0) = 0$ and $\phi(p^1) = 1$. Set

$$p_k(t) = (1-t) \cdot p^0_k + t \cdot p^1_k \qquad \text{for } -\infty \le k \le \infty,$$

and define ψ as follows:

$$\begin{aligned}
&\psi(x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots x_n) \overset{\text{def}}{=} \\
&\quad (\ldots 0,0,0, p_{-N_0}(x_{-n}), p_{-N_0+1}(x_{-n}), \ldots p_{N_0}(x_{-n}), \\
&\qquad p_{-N_0}(x_{-n+1}), p_{-N_0+1}(x_{-n+1}), \ldots p_{N_0}(x_{-n+1}), \\
&\qquad \vdots \\
&\qquad p_{-N_0}(x_0), p_{-N_0+1}(x_0), \ldots p_0(x_0), \ldots p_{N_0}(x_0), \\
&\qquad \vdots \\
&\qquad p_{-N_0}(x_n), p_{-N_0+1}(x_n), \ldots p_{N_0}(x_n), 0,0,0, \ldots)
\end{aligned} \tag{2.1.2}$$

(the 0^{th} coordinate is $p_0(x_0)$).

Suppose $x_0 = 0$. Then $\psi(\dots, x_{-1}, 0, x_1, \dots)$ is identical to p^0 in the coordinates $-N_0, \dots, N_0$. Using (2.1.1) we see that

$$|\phi \circ \psi(\dots, x_{-1}, 0, x_1, \dots) - \phi(p^0)| < \tfrac{1}{10},$$

that is, $\phi \circ \psi(\dots, x_{-1}, 0, x_1, \dots) < \frac{1}{10}$. In this way we see that if $x_k = 0$ then the k^{th} coordinate of $g \circ \psi$ is smaller than $\frac{1}{10}$, and if $x_k = 1$ then the k^{th} coordinate is bigger than $\frac{9}{10}$ — i.e. $g \circ \psi$ satisfies the conditions of Lemma (2.2). Thus the image of this map contains the small cube $[\frac{1}{4}, \frac{3}{4}]^{2n+1}$. □

Conclusion (2.2) *Any factor of* $[0,1]^{\mathbf{Z}}$ *has infinite entropy.*

3. The existence of open sets with small boundary

If (X, T) is a dynamical system, we will denote by $\text{per}(X)$ the set of its periodic points, and by $\text{per}_k(X)$ the set of its periodic points with period $\leq k$. We shall denote the metric on X by $d(\cdot, \cdot)$; and to simplify notations we shall (without loss of generality) assume that for every $x, y \in X$, $d(x,y) < 1$. Also, in this and the next section every measure is assumed to be a normalized Borel measure. We will denote the set of all T-invariant measures of (X, T) by $\mathcal{M}_T$.

We wish to show that every open subset of X can be approximated (in a rather strong sense) by an open set such that the set of the non-periodic points on its boundary will be small, in the sense introduced in [7]:

Definition (3.1) *A set $E \subset X$ will be called (N, n)-disjoint if the intersection of any n sets from the collection*

$$E,\ T(E),\ T^2(E),\ T^3(E), \dots,\ T^{N-1}(E)$$

is empty.

Definition (3.2) *A set $E \subset X$ will be called T-**small** (or simply **small** if T is understood) if for every $\epsilon > 0$ there is an N such that E is $(N, \lfloor \epsilon N \rfloor)$-disjoint.*

Note that if a (measurable) set E is (N, n)-disjoint then no point of X can be covered by more than n sets from the collection $\{T^i(E)\}_{i=0}^{N-1}$ and so, for any $\mu \in \mathcal{M}_T$,

$$\mu(E) = \frac{1}{N} \sum \mu(T^i(E)) \leq \frac{n}{N} \mu(X) = \frac{n}{N}.$$

In particular, if a set E is small then for any such measure μ, the measure of E is zero. For closed sets the converse also holds, but we will not use this fact. For details see [7]. Clearly, a subset of a small set is also small; it is also very easy to see that the union of two small sets is small.

In this section we will prove the following theorem:

Theorem (3.3) *Let (X, T) be a finite dimensional dynamical system. Then for every open set $U \subset X$, and any open $V \supset \partial U \setminus \mathrm{per}(X)$, there is an open U', $U \subset U' \subset U \cup V$, such that $\partial U'$ is the union of a small set and a subset of* $\mathrm{per}(X)$.

We will see in the next section that the fact that any open set can be approximated by an open set with its boundary satisfying the conditions above is sufficient to ensure the existence of factors of the system (X, T) with arbitrary entropy.

We will prove this theorem using ideas from [5]. The following definition is also taken from [5]:

Definition (3.4) *A collection B of subsets of an n-dimensional space X will be said to be in* **general position** *if for every finite sub-collection $S \subset B$ with m elements,* $\dim \cap S \leq \max(-1, n-m)$.

(Recall that by definition $\dim M = -1$ iff $M = \emptyset$.) The motivation for this definition is that given a collection of $n-1$ dimensional subsets of an n dimensional space then generically any two will have intersection with dimension $\leq n-2$, etc. In the rest of this section, we will take n to be $\dim(X)$. Notice that if the collection $\{T^k E\}_{k=0}^{r-1}$ is in general position then in particular, since the intersection of any $n+1$ sets from this collection is empty, E is $(r, n+1)$-disjoint.

We will use the following standard results in dimension theory valid for every separable metric space M. For proofs of these results see [1].

- D1 ("The Subspace Theorem") If $A \subset B$ then $\dim A \leq \dim B$.
- D2 ("The Countable Closed Sum Theorem") Let $\{B_i\}$ be a countable collection of closed sets, with $\dim B_i \leq k$ for every i. Then $\dim \bigcup B_i \leq k$.
- D3 Let E be a zero dimensional subset of the space M. Then for every $x \in M$ and every open neighbourhood U of x there is a $U' \subset U$ with $x \in U'$ such that $\partial U' \cap E = \emptyset$.
- D4 If $\dim M > -1$ there is a zero dimensional subset E of M which is the union of a countable number of closed sets in M and such that $\dim M \setminus E = \dim M - 1$.

Also, the following result is a consequence of D1 and D2:

- D5 Any finite or countable union of $F_\sigma k$-dimensional sets is an F_σ k-dimensional set.

To see D5 is true note that each F_σ set is a countable union of closed sets that by D1 have dimension $\leq k$, and so the union of the original F_σ sets is a countable union of closed at most k-dimensional sets. By D2 this union has dimension $\leq k$.

Let $P_k = \mathrm{per}_k(X)$, and $P = \mathrm{per}(X)$.

The important step towards proving the main result of this section is the following lemma, which is a modification of a lemma proved by J. Kulesza in [5]:

Lemma (3.5) *Let (X,T) be an n dimensional dynamical system, and let $U \subset X$ be an open set. Then for every k and every open set $V \supset \partial U \setminus P$, there exists an open set U', $U \subset U' \subset U \cup V$ with $\partial U' \subset \partial U \cup V$ such that*

$$\{\partial U' \setminus P, T(\partial U') \setminus P, \dots, T^{k-1}(\partial U') \setminus P\}$$

is in general position.

(Notice that for any U', if $\partial U'$ contains a periodic point with a short period the more natural collection $\{\partial U', \dots, T^{k-1}(\partial U')\}$ will *not* be in general position.)

To prove Lemma (3.5) we will need the following simple result:

Lemma (3.6) *Let U be an open set, and let $E \subset X$ be zero dimensional. Then for any open $V \supset \mathrm{cl}(U)$, there exists an open set $U' \supset \mathrm{cl}(U)$, with $U' \subset V$ and such that $\partial U' \cap E = \emptyset$.*

Proof By D3 for every $x \in \partial U$ there is neighbourhood U_x inside V whose boundary does not intersect E. By compactness of ∂U a finite number of these, say $U_{x_1}, \dots, U_{x_r}$, suffice for covering ∂U. Set $U' = U \cup \bigcup_{i=1}^{r} U_{x_r}$.

Clearly $U' \subset V$ and $\mathrm{cl}(U) \subset U'$. Also, since the union is finite, every point x in $\partial U'$ is in ∂U or in some ∂U_{x_i}. The first possibility is impossible since the U_{x_j}'s cover ∂U and so every point of ∂U is an internal point of U'. Thus x is in ∂U_{x_i} for some i, and since this set has empty intersection with E, $x \notin E$. Thus $\partial U' \cap E = \emptyset$.
□

Proof of Lemma (3.5)

We will prove the lemma by induction on k. If $k = 0$ we can simply take for U' the set U itself.

Now, suppose that the lemma is true for some $k \geq 0$. We will prove it is true for $k+1$. Without loss of generality we can assume

$$V \cap P_{k+1} = \emptyset$$

for otherwise we can take $V' = V \setminus P_{k+1}$ that also satisfies the conditions of this lemma and if the lemma holds for V' it also holds for V. Define further for any integer r,

$$V_r = \left\{x : d(x, X \setminus V) \geq \frac{1}{r}\right\};$$

clearly each V_r is compact and $V = \bigcup_{r=0}^{\infty} V_r$. Using the induction hypothesis we know that there exists an open A_0 with $U \subset A_0 \subset U \cup V$ such that $\partial A_0 \subset \partial U \cup V$

and such that

$$\{\partial A_0 \setminus P,\, T(\partial A_0) \setminus P, \dots, T^{k-1}(\partial A_0) \setminus P\}$$

is in general position.

For every $x \in \partial A_0 \cap V$, there exists an $\eta_x > 0$ such that $\operatorname{cl} B(x, \eta_x) \subset V$ and such that the sets $B(x, \eta_x)$, $TB(x, \eta_x)$, $\dots$, $T^k B(x, \eta_x)$ are disjoint (since $x \notin P_{k+1}$). Let $B_x \stackrel{\text{def}}{=} B(x, \eta_x/2)$ and $\hat{B}_x \stackrel{\text{def}}{=} B(x, \eta_x)$. The B_x's form a cover of $\partial A_0 \cap V$, and we can find a countable sub-cover B_i such that

$$\lim_{i \to \infty} \operatorname{diam}(B_i) = 0, \tag{3.7.1}$$

and such that for any V_l there exists an r_l such that

$$V_l \cap \bigcap_{i=r_l}^{\infty} \hat{B}_i = \emptyset.$$

Indeed, for every integer r define the compact set C_r as

$$C_r \stackrel{\text{def}}{=} \left\{x \in \partial A_0 : \tfrac{1}{r} \geq d(x, X \setminus V) \geq \tfrac{1}{r+1}\right\}.$$

We can construct a sub-cover with the required properties by taking for every r a finite number of sets of the form B_x with $x \in C_r$ that cover C_r. Since $\partial A_0 \cap V = \bigcup_{r=1}^{\infty} C_r$ we have indeed a subcover of $\partial A_0 \cap V$. Furthermore, for any $x \in C_r$ the diameter of B_x is $\leq 1/r$, and so (3.7.1) holds. Since $d(V_l, C_r) > 1/r$ for r large enough, we see that the second condition imposed on the above sub-cover also holds, and so the existence of a sub-cover B_i with the required properties is established.

We will construct recursively, starting with A_0, a sequence of open sets $\{A_i\}_{i=0}^{\infty}$ such that

P1 $A_i \subset A_0 \cup \bigcup_{j=0}^{\infty} B_j$,

P2 $A_i \supset A_{i-1}$,

P3 $\partial A_i \subset \partial U \cup V$,

P4 $A_i \setminus \hat{B}_i = A_{i-1} \setminus \hat{B}_i$

P5 $\mathcal{A}_i \stackrel{\text{def}}{=} \left\{\partial A_i \setminus P,\, T(\partial A_i) \setminus P, \dots, T^{k-1}(\partial A_i) \setminus P,\, T^k\left(\partial A_i \cap \bigcup_{j=1}^{i} B_j\right)\right\}$ is in general position.

(It is easy to see that A_0 satisfies all these conditions for $i = 0$.)

Given this sequence we will define U' as $\bigcup_{i=0}^{\infty} A_i$. By P1, and since $\bigcup_{j=0}^{\infty} B_j \subset V$, we see that

$$U' \subset A_0 \cup V \subset (U \cup V) \cup V = U \cup V.$$

By P2 we see that $U \subset U'$. We now claim that $\partial U' \subset \partial U \cup V$. Indeed, let x be a boundary point of U'. Then either it is in ∂A_i for some i or it is the limit of a sequence $(x_l)_{l=1}^{\infty}$ with $x_l \in A_{n_l} \setminus A_{n_l-1}$ with n_l strictly increasing. Consequently, since by P4 $A_{n_l} \setminus A_{n_l-1} \subset \hat{B}_{n_l}$,

$$x \in \bigcap_{l=0}^{\infty} \mathrm{cl}\left(\bigcup_{l=r}^{\infty} \hat{B}_{n_l}\right) \subset \partial A_0$$

where the last inequality follows from the fact that the centers of the $\hat{B}_i$'s are in ∂A_0 and their radii tend to zero. Thus we see that

$$\partial U' \subset \partial A_0 \cup \bigcup_{i=0}^{\infty} \partial A_i \subset \partial U \cup V,$$

where the second inclusion is a consequence of P3.

It remains to be verified that

$$\{\partial U' \setminus P,\, T(\partial U') \setminus P, \dots, T^{k-1}(\partial U') \setminus P,\, T^k(\partial U') \setminus P\} \tag{3.7.2}$$

is in general position. First note that (for a given l) $\partial U' \cap V_l = \partial A_r \cap V_l$ if r is large enough. For according to P4, $A_r \setminus \bigcup_{i=r+1}^{\infty} \hat{B}_i = A_{r'} \setminus \bigcup_{i=r+1}^{\infty} \hat{B}_i$ for any $r' > r$, and so $U' \setminus \bigcup_{i=r+1}^{\infty} \hat{B}_i = A_r \setminus \bigcup_{i=r+1}^{\infty} \hat{B}_i$. According to the way the B_i's were chosen, there is an r such that for every $i > r$, $\hat{B}_i \cap V_{l+1} = \emptyset$, and so $U' \cap V_{l+1} = A_r \cap V_{l+1}$. Since $V_l \subset \mathrm{int}(V_{l+1})$ we see that indeed $(\partial U') \cap V_l = (\partial A_r) \cap V_l$. We can take r large enough so that $V_l \subset \bigcup_{i=1}^{r} B_i$. Now we know that

$$\left\{\partial A_r \setminus P,\, T(\partial A_r) \setminus P, \dots, T^{k-1}(\partial A_r) \setminus P,\, T^k\left(\partial A_r \cap \textstyle\bigcup_{j=1}^{r} B_j\right)\right\}$$

is in general positions and the same is true if we replace each set of the above collection by a smaller set, and so

$$\begin{aligned} &\{\partial A_r \cap V_l \setminus P,\, T(\partial A_r \cap V_l) \setminus P, \dots, T^{k-1}(\partial A_r \cap V_l) \setminus P,\, T^k(\partial A_r \cap V_l) \setminus P\} \\ &= \{\partial U' \cap V_l \setminus P,\, T(\partial U' \cap V_l) \setminus P, \dots, T^{k-1}(\partial U' \cap V_l) \setminus P,\, T^k(\partial U' \cap V_l) \setminus P\} \end{aligned}$$

is in general position. Since this is true for every l, and since $T^i(\partial U' \cap V_l) \setminus P$ is closed in $X \setminus P$, then an immediate consequence of the Countable Closed Sum Theorem (D2) is that the collection

$$\{\partial U' \cap V \setminus P, T(\partial U' \cap V) \setminus P, \ldots, T^{k-1}(\partial U' \cap V) \setminus P, T^k(\partial U' \cap V) \setminus P\}$$

is in general position. We have already seen that $\partial U' \subset \partial U \cup V$; since $\partial U \setminus P \subset V$ we see that $\partial U' \subset P \cup V$ and so the collection above is the same as the collection (3.7.2). Thus it only remains to be seen how to construct A_{i+1} using A_i.

For every finite sub-collection $S \subset \mathcal{A}_i$ with $\dim(\cap S) > -1$ there is according to D4 a zero-dimensional set $E_S \subset \cap S$, F_σ in $\cap S$, such that $\dim(\cap S \setminus E_S) = \dim(\cap S) - 1$ (for $S = \emptyset$, $\cap S$ is undefined; for convenience we set it to be X). E_S is in fact σ-compact, for $\cap S$ is σ-compact and an F_σ subset of a σ-compact space is σ-compact. Set

$$E = \bigcup_{\substack{S \subset \mathcal{A}_i \\ 0 \le j \le k}} T^{-j} E_S.$$

Since T is a homeomorphism the image of a zero dimensional σ-compact set is zero dimensional and σ-compact. Thus E is a finite union of σ-compact zero dimensional sets, and so according to D5, E itself is both σ-compact and zero dimensional. Using Lemma (3.6), there is an open set $W \supset \operatorname{cl}(B_{i+1} \cap A_i)$ such that $\operatorname{cl}(W) \subset \hat{B}_{i+1} \cap \left(U \cup \bigcup_{j=0}^{\infty} B_j\right) \cap V$, with $\partial W \cap E = \emptyset$. We claim that $A_{i+1} \overset{\text{def}}{=} A_i \cup W$ satisfies P1 – P5. That A_{i+1} satisfies P1 – P4 is trivial, and thus it only remains to be shown that P5 is satisfied.

Suppose that P5 is not satisfied, i.e. there exists a collection $S \subset \mathcal{A}_{i+1}$ of cardinality m such that $\dim \cap S > \max(-1, n-m)$. Let $S = \{S_1, S_2, \ldots, S_m\}$, where each S_l is a distinct element of $\mathcal{A}_{i+1}$, i.e. either $T^{j_l}(\partial A_{i+1}) \setminus P$ for some $j_l < k$ or else $T^k\left(\partial A_{i+1} \cap \bigcup_{j=1}^{i+1} B_j\right) \setminus P$ (in which case we set $j_l = k$). Notice that since the S_l's are distinct, $j_l \neq j_{l'}$ for $l \neq l'$.

Now $\partial A_{i+1} \subset (\partial A_i \setminus W) \cup \partial W$ and so for any l such that $j_l < k$,

$$S_l \subset (T^{j_l}(\partial A_i) \setminus P) \cup T^{j_l}(\partial W).$$

A similar result holds for $j_l = k$:

$$\begin{aligned} S_l &= T^r\left(\partial A_{i+1} \cap \bigcup_{j=1}^{i+1} B_j\right) \setminus P \\ &\subset \left(T^r\left((\partial A_i \setminus W) \cap \bigcup_{j=1}^{i+1} B_j\right) \cup T^r(\partial W)\right) \setminus P \end{aligned}$$

$$= \left(T^r\left((\partial A_i \setminus W) \cap \bigcup_{j=1}^{i} B_j\right) \cup T^r(\partial W)\right) \setminus P$$

$$\subset \left(T^r\left(\partial A_i \cap \bigcup_{j=1}^{i} B_j\right) \setminus P\right) \cup T^r(\partial W),$$

where the second equality holds since $\mathrm{cl}\,(B_{i+1} \cap A_i) \subset W$. Combining the two cases, we see that each S_l is a subset of the union of two sets: S_l^0, the element of $\mathcal{A}_i$ corresponding to S_l, and $S_l^1 \overset{\text{def}}{=} T^{j_l}(\partial W)$. Thus $\bigcap S$ is a subset of the finite union

$$\bigcap S \subset \bigcup_{\alpha \in \{0,1\}^m} \left(\bigcap_{l=1}^{m} S_l^{\alpha_l}\right)$$

and, since $\dim(\bigcap S) > \max(-1, n-m)$, at least one of these intersections must have dimension larger than $\max(-1, n-m)$. But since $\mathrm{cl}(W) \subset \hat{B}_{i+1}$ and the $T^j\hat{B}_{i+1}$'s for $j = 0,\dots,k$ are pairwise disjoint, any α with more than two α_l's 1 has $\bigcap_{l=1}^{m} S_l^{\alpha_l} = \emptyset$. For $\alpha = (0,\dots 0)$, each $S_l^{\alpha_l}$ is a (distinct) element of $\mathcal{A}_i$, and since this collection is in general position we see that $\dim \bigcap_{l=1}^{m} S_l^0 \leq \max(-1, n-m)$. We haven't assumed anything on the order of the S_l's yet, so without loss of generality we can assume that

$$\dim(S_1^1 \cup \bigcup_{l=2}^{m} S_l^0) > \max(-1, n-m).$$

But this too is impossible — to see this, let $\hat{S} = \{S_2^0, \dots, S_m^0\}$. Now $\hat{S} \subset \mathcal{A}_i$ so we know that

$$\dim\left(\bigcap \hat{S} \setminus E_{\hat{S}}\right) \leq \dim\left(\bigcap \hat{S}\right) - 1 \leq \min(-1, n-m).$$

By construction, ∂W is disjoint from E and thus also from $T^{-j_1}E_{\hat{S}} \subset E$, so that $S_1^1 = T^{j_1}(\partial W) \subset X \setminus E_{\hat{S}}$. Thus

$$S_1^1 \cap \bigcap_{l=2}^{m} S_l^0 \subset \bigcap \hat{S} \setminus E_{\hat{S}},$$

a set of dimension $\leq \min(-1, n-m)$ — a contradiction. □

Lemma (3.8) *Suppose that A is closed and that $A \setminus P$ is (k,m)-disjoint. Then there is an open set $V \supset A \setminus P$ such that V is also (k,m)-disjoint.*

Proof Set $A' \overset{\text{def}}{=} A \setminus P$.

Take any m sets from the collection $\{T^i(A')\}_{i=0}^{k-1}$, say $A_1, \ldots, A_m$, where, for every i,

$$A_i = T^{n_i}(A') = T^{n_i}(A) \setminus P.$$

Note that $\mathrm{cl}(A_i) \subset T^{n_i}(A) \subset A_i \cup P$, and so

$$A_1 \cap \bigcap_{i=2}^{m} \mathrm{cl}(A_i) \subset A_1 \cap \bigcap_{i=2}^{m} (A_i \cup P) = \bigcap_{i=1}^{m} A_i \cup (A_1 \cap P) = \emptyset, \tag{3.8.1}$$

where the last equality is true since $A' = A \setminus P$ is (k, m)-disjoint, and since $A_1 \cap P = \emptyset$.

Define for every $1 \le j \le k$ the open set

$$V_j = \left\{ x : d(x, T^j(A')) < \min_{\substack{S \subset \{1,\ldots,k\} \\ |S|=m-1, j \notin S}} \max_{r \in S} d(x, T^r(A')) \right\}.$$

From (3.8.1) we see that for every $S \subset \{1, \ldots, k\}$ with $|S| = m - 1$ and $j \notin S$, and every $x \in T^j(A')$, $\max_{r \in S} d(x, T^r(A')) > 0$ since $x \notin \bigcap_{r \in S} \mathrm{cl}\, T^r(A)$. It follows that every $x \in T^j(A')$ is in V_j, and thus for every j, $T^j(A') \subset V_j$.

Now take any m sets from the collection $\{V_j\}_{j=1}^{k}$, say $V_{j_1}, \ldots, V_{j_m}$, and let $S = \{j_2, \ldots, j_m\}$. Then

$$\bigcap_{l=1}^{m} V_{j_l} \subset \left\{ x : d(x, T^{j_1}(A')) < \max_{r \in S} d(x, T^r(A')) \right\}$$
$$\cap \bigcap_{r \in S} \left\{ x : d(x, T^r(A')) < d(x, T^{j_1}(A')) \right\} = \emptyset.$$

Define $V = \bigcap_{j=0}^{k} T^{-j}(V_j)$. Since $T^j(A') \subset V_j$ for every j we see that $A' = A \setminus P \subset V$. And since for every j, $T^j(V) \subset V_j$, the equation above shows that V is (k, m)-disjoint. □

We are now ready to prove Theorem (3.3). For the convenience of the reader we will restate it:

Theorem (3.3) *Let (X, T) be a finite dimensional dynamical system. Then for every open set $U \subset X$, and any open $V \supset \partial U \setminus P$, there is an open U', $U \subset U' \subset U \cup V$, such that $\partial U'$ is the union of a small set and a subset of P.*

Proof We will construct recursively two sequences of open sets $\{U_i\}_{i=0}^{\infty}$ and $\{V_i\}_{i=0}^{\infty}$, starting from $U_0 = U$ and $V_0 = V$ such that the following conditions hold:

C1 $\quad \partial U_k \setminus P \subset V_k$,

C2 $\quad U_{k+1} \subset U_k \cup V_k$, C3 $\quad \mathrm{cl}(V_{k+1}) \setminus P \subset V_k$,

C4 $\quad V_k$ is $(k, n+1)$-disjoint,

C5 $\quad V_{k+1} \subset V_k$ and $U_{k+1} \supset U_k$.

Suppose that U_i, V_i have already been defined. Define $U' = \bigcup_{i=0}^{\infty} U_i$. Using C2 and C5 we see that for every r,

$$U_{k+r} \cup V_k \subset U_{k+r-1} \cup V_{k+r-1} \cup V_k = U_{k+r-1} \cup V_k \subset \cdots \subset U_k \cup V_k,$$

and thus for every k, $U_k \subset U' \subset U_k \cup V_k$. Taking $k = 0$ we see that $U \subset U' \subset U \cup V$. Furthermore, using this, C1 and C2 we see that for every k

$$\begin{aligned} \partial U' \setminus P &\subset (\mathrm{cl}(U_{k+1} \cup V_{k+1}) \setminus U_{k+1}) \setminus P \\ &\subset (\partial U_{k+1} \setminus P) \cup (\mathrm{cl}(V_{k+1}) \setminus P) \\ &= \mathrm{cl}(V_{k+1}) \setminus P \subset V_k. \end{aligned}$$

Since V_k is $(k, n+1)$-disjoint the same is true for $\partial U' \setminus P$. Since n is fixed ($= \dim(X)$) and k arbitrary, we see that $\partial U' \setminus P$ is T-small.

Thus it only remains to show how to build such sequences of sets. Clearly U_0 and V_0 satisfy C1 and since any set is $(0, n+1)$-disjoint also C4. Suppose $U_0, \ldots, U_k$ and $V_0, \ldots, V_k$ have already been defined, satisfying the conditions imposed on these sets by C1 – C5.

From C1 we know that $\partial U_k \setminus P \subset V_k$, and so using Lemma (3.5) we can find a set U_{k+1} such that $U_k \subset U_{k+1} \subset U_k \cup V_k$, $\partial U_{k+1} \setminus P \subset V_k$, and such that $\partial U_{k+1} \setminus P$ is $(k, n+1)$-disjoint. Using the previous lemma we can find a $(k, n+1)$-disjoint open set $V'_{k+1} \supset \partial U_{k+1} \setminus P$. Define $W_k \subset V_k$ as

$$W_k = \{x : d(x, \partial U_{k+1}) < d(x, X \setminus V_k)\},$$

and set

$$V_{k+1} = V'_{k+1} \cap W_k.$$

Clearly U_{k+1} and V_{k+1} satisfy C2, C4 and C5. By definition $\partial U_{k+1} \setminus P \subset V'_{k+1}$. For every $x \in \partial U_{k+1} \setminus P$, $d(x, U_{k+1}) = 0$ but $d(x, X \setminus V_k) > 0$ (since $x \in V_k$). Thus $\partial U_{k+1} \setminus P \subset W_k$ and thus C1 holds. To prove C3 holds, it clearly suffices to show that $\mathrm{cl}(W_k) \setminus P \subset V_k$. So suppose $x \in \mathrm{cl}(W_k) \setminus V_k$. Then

$$d(x, \partial U_{k+1}) \leq d(x, X \setminus V_k) = 0,$$

and so $x \in \partial U_{k+1} \setminus V_k \subset P$. Since this is true for any $x \in \mathrm{cl}(W_k) \setminus V_k$ we see that indeed $\mathrm{cl}(W_k) \setminus P \subset V_k$. □

Note that we have in fact proved that for any k the set $\partial U' \setminus P$ is $(k, n+1)$-disjoint, which is a much stronger statement than merely saying that $\partial U' \setminus P$ is T-small. We will also need the following corollary to Theorem (3.3) and Lemma (3.8):

Corollary (3.10) *Suppose E is closed and $E \setminus P$ is (k,m)-disjoint, with $V \supset E$ an open set. Then there exists an open set U', $V \supset U' \supset E \setminus P$ such that U' is (k,m)-disjoint and such that $\partial U' \setminus P$ is small.*

Proof According to Lemma (3.8) there is an open, (k,m)-disjoint $V' \supset E \setminus P$. Without loss of generality $V' \subset V$ for otherwise we can replace V' by $V' \cap V$. Define the open set W as

$$W = \{x : d(x,E) < d(x, X \setminus V')\}.$$

As in the proof of Theorem (3.3), $\mathrm{cl}(W) \subset V' \cup P$. Thus we can apply Theorem (3.3) to find an open set U' with $W \subset U' \subset V'$ and such that $\partial U' \setminus P$ is small. Since $U' \subset V'$, U' is also (k,m)-disjoint. □

4. The construction of factors with arbitrary entropy

In this section we will prove that if the conclusion of Theorem (3.3) holds (and thus its Corollary (3.10)) for the dynamical system (X,T), then we can find for every $\xi, \xi' \in X$ and any $\eta > 0$ a factor (Y,S) with entropy $\leq \eta$ and such that the factor transformation maps ξ and ξ' into distinct points in Y. From this we will deduce that if (Z,R) is any factor of the system (X,T) with $h_{\mathrm{top}}(Z) < h_{\mathrm{top}}(X)$, then for any $h \in (h_{\mathrm{top}}(Z), h_{\mathrm{top}}(X))$, we can find a factor (Y,S) of (X,T) with topological entropy h, and such that if $\phi : X \to Z$ and $\psi : X \to Y$ are the respective factor transformations then $\phi \circ \psi^{-1}$ is well defined, and is a factor transformation $Y \to Z$.

We first need to see how the information given on the system (X,T) by (3.3) can help to construct factors with small entropy.

For any measurable set A we can construct in a natural way a (not necessarily continuous) map $m_A : X \to \{0,1\}^{\mathbf{Z}}$ as follows:

$$m_A : x \mapsto \left(\ldots, 1_A \circ T^{-1}(x), 1_A(x), 1_A \circ T(x), \ldots \right).$$

Define the set $\mathcal{S}_A \stackrel{\mathrm{def}}{=} \mathrm{cl}\ m_A(X) \subset \{0,1\}^{\mathbf{Z}}$. This set is shift invariant — and so together with the shift σ it is a dynamical system. We will define for any set A the entropy of A as

$$h(A) \stackrel{\mathrm{def}}{=} h_{\mathrm{top}}(\mathcal{S}_A).$$

Proposition (4.1) *For any two (measurable) sets A and B,*

$$h(A \cup B) \leq h(A) + h(B).$$

Proof Take any $x \in X$. Note that we can calculate $m_{A\cup B}(x)$ from $m_A(x)$ and $m_B(x)$, since

$$1_{A\cup B}(T^n x) = \max\big(1_A(T^n x), 1_B(T^n x)\big).$$

Consider the (continuous) map

$$f\colon (\ldots, a_{-1}, a_0, a_1, \ldots) \times (\ldots, b_{-1}, b_0, b_1, \ldots) \longmapsto (\ldots, \max(a_{-1}, b_{-1}), \max(a_0, b_0), \max(a_1, b_1), \ldots)$$

from $\mathcal{S}_A \times \mathcal{S}_B$ to $\{0,1\}^{\mathbf{Z}}$. We claim that $\mathcal{S}_{A\cup B} \subset f(\mathcal{S}_A \times \mathcal{S}_B)$. Indeed, for any $x \in X$,

$$f\,(m_A(x), m_B(x)) = m_{A\cup B}(x),$$

and so $m_{A\cup B}(X) \subset f(\mathcal{S}_A \times \mathcal{S}_B)$. Since f is continuous, $f(\mathcal{S}_A \times \mathcal{S}_B)$ is closed (as the image of a compact set) and so contains cl $m_{A\cup B} = \mathcal{S}_{A\cup B}$.

Thus $\mathcal{S}_{A\cup B}$ is a subsystem of a factor of $\mathcal{S}_A \times \mathcal{S}_B$, and so we have:

$$h(A\cup B) = h_{\text{top}}(\mathcal{S}_{A\cup B}) \leq h_{\text{top}}\left(f(\mathcal{S}_A \times \mathcal{S}_B)\right) \leq h_{\text{top}}(\mathcal{S}_A \times \mathcal{S}_B) = h(A) + h(B).$$

□

Lemma (4.2) *Let E be a closed set. If $E \setminus \mathrm{per}(X)$ is small then for any $V \supset E$, and any $\epsilon > 0$, there is an open set $V \supset U \supset E$ such that:*

(a) $h(U) < \epsilon$;

(b) $\partial U \setminus \mathrm{per}(X)$ *is small.*

Proof Fix some sequence of integers $N_0, N_1, \ldots$. We will use this sequence to construct an open U, and show that if the integers N_i are large enough this set has the required properties.

Applying Corollary (3.10), we can find an open set U_0, $V \supset U_0 \supset E \setminus P$ such that $\partial U_0 \setminus \mathrm{per}(X)$ is small and such that U_0 is $(N, \lfloor N/N_0 \rfloor)$-disjoint for some N and the given N_0. Using Theorem (3.3), there is a V', $V \supset V' \supset E$ such that $\partial V' \setminus P$ is small. For every k, let $P^k = \{x : T^k(x) = x\}$, and set

$$U_k \stackrel{\text{def}}{=} \bigcap_{i=0}^{N_k} T^{ki}(V').$$

Clearly $U_k \supset P_k \cap E$. Thus $\{U_k\}_{k=0}^{\infty}$ form an open cover of the compact set E, and so a finite number of these, say $U_0, \ldots, U_r$, suffice to cover E. Define $U \stackrel{\text{def}}{=} \bigcup_{i=0}^{r} U_i$.

Note that for any $k \geq 1$,

$$\partial(U_k \setminus P) \subset \bigcup_{i=0}^{N_k} T^{ki}(\partial V' \setminus P),$$

and so, since the union of a finite number of small sets is small, for every k the set $\partial U_k \setminus P$ is small. Thus $\partial U \setminus P$, being a subset of $\bigcup_{i=0}^{r}(\partial U_k \setminus P)$, is small. Also it is clear that $E \subset U \subset V$.

It remains to be shown that if the sequence N_i increases fast enough, then $h(U) < \epsilon$. Since by the previous proposition, $h(U) \leq \sum_{i=0}^{r} h(U_i)$, this lemma will be proved if we show that if N_k is large enough then $h(U_k) < \epsilon 2^{-k-1}$.

If $\mathcal{S} \subset \{0,1\}^{Z}$ is a shift invariant set, denote by $\mathcal{W}_n(\mathcal{S})$ the set

$$\mathcal{W}_n(\mathcal{S}) \stackrel{\text{def}}{=} \{(x_1, x_2, \ldots, x_n) : (\ldots, x_{-1}, x_0, x_1, \ldots) \in \mathcal{S}\}.$$

It is well known (and easy to deduce from the definition of topological entropy) that if $\mathcal{S} \subset \{0,1\}^{\mathbf{Z}}$, the entropy of the system $(\mathcal{S}, \sigma)$ is

$$h_{\text{top}}(\mathcal{S}) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{W}_n(\mathcal{S})| = \inf \frac{1}{n} \log |\mathcal{W}_n(\mathcal{S})|,$$

where the second equality is true since for any n and k, $|\mathcal{W}_{kn}(\mathcal{S})| \leq |\mathcal{W}_n(\mathcal{S})|^k$.

If some n-tuple is the first n coordinates of some point in $\mathcal{S}_{U_0}$, it is also the first n coordinates of a point of $m_{U_0}(X)$ (which is dense in $\mathcal{S}_{U_0}$). Since U_0 is $(N, \lfloor N/N_0 \rfloor)$-disjoint, for no $x \in X$ can $m_{U_0}(x)$ have more than N/N_0 ones in the first N coordinates. Thus

$$h(U_0) \leq \frac{1}{N} \log |\mathcal{W}_N(\mathcal{S}_{U_0})| \leq \frac{1}{N} \log \left(\sum_{i=0}^{\lfloor N/N_0 \rfloor} \binom{N}{i} \right)$$

which can be made arbitrarily small by taking N_0 to be very large.

We will now consider what n-tuples can appear in $\mathcal{W}_n(\mathcal{S}_{U_k})$ for $k > 0$. Recall that $U_k = \bigcap_{i=0}^{N_k} T^{ki}(V')$. Suppose now that (for some $x \in X$) $1_{U_k}(x) = 1$ but $1_{U_k}(T^{-k}(x)) = 0$. If this is true then this x must be in $T^{ki}(V')$ for $i = 0, \ldots, N_k$, but *not in* $T^{k(N_k+1)}(V')$. Thus we see that this x is not in $T^{ki}(U_k)$ (or, equivalently, $1_{U_k}(T^{-ki}(x)) = 0$) for $i = 1, \ldots, N_k$. Take $n = kN_k$, and consider for any n-tuple

$$(a_1, a_2, \ldots, a_{kN_k}) \in \mathcal{W}_{kN_k}(\mathcal{S}_{U_k})$$

the k N_k-tuples

$$(a_i, a_{i+k}, \ldots, a_{i+k(N_k-1)}) \qquad \text{for } i = 1, \ldots, N_k.$$

We know that if for some j, $a_j = 1$ and $a_{j-k} = 0$, then for any l such that $1 \le j - lk < j$ we know that $a_{j-lk} = 0$. Similarly, if for some x, $1_{U_k}(x) = 1$ but $1_{U_k}(T^k(x)) = 0$, then for every $1 \le i \le N_k$, $1_{U_k}(T^{ki}(x)) = 0$. Thus if for some j, $a_j = 1$ but $a_{j+k} = 0$, then for any l such that $j < j + kl \le kN_k$, $a_{j+lk} = 0$.

Thus the sequence $(a_i, a_{i+k}, \ldots, a_{i+k(N_k-1)})$ must be of the form

$$(0, 0, \ldots, 0, 1, 1, \ldots 1, 0, 0, \ldots, 0).$$

The number of N_k-tuples of this form is less than N_k^2, and so

$$|\mathcal{W}_{kN_k}(\mathcal{S}_{U_k})| < N_k^{2k}.$$

Thus

$$h(U_k) \le \frac{1}{kN_k} \log |\mathcal{W}_{kN_k}(\mathcal{S}_{U_k})| \le \frac{1}{kN_k} \log(N_k^{2k}) = \frac{2}{N_k} \log(N_k),$$

and so we see that indeed if N_k is large enough, $h(U_k)$ will be smaller than $2^{-k-1}\epsilon$.
□

(4.3) The construction

Let ξ, ξ' be distinct points of X, and $\epsilon > 0$. We will now construct a factor imbedded in $([0,1]^{\mathbf{Z}}, \sigma)$ and a (continuous) factor map $\hat{\phi}: X \to [0,1]^{\mathbf{Z}}$ such that $\hat{\phi}(\xi) \neq \hat{\phi}(\xi')$. In the next subsection we will show that the entropy of this factor is less than ϵ.

We will first find a countable collection $\mathcal{A}$ of open sets such that

A1 For any $U \in \mathcal{A}$ and any $\delta > 0$ there is a $V \in \mathcal{A}$ such that $\partial U \subset V \subset B(\partial U, \delta)$.

A2 $\sum_{U \in \mathcal{A}} h(U) < \epsilon$.

A3 There is a $U \in \mathcal{A}$ such that $\xi \in U$ but $\xi' \notin \mathrm{cl}(U)$.

This collection $\mathcal{A}$ will be the union of a monotone increasing sequence of collections

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots.$$

To construct these collections first note that for any $x \in X$, $\{x\} \setminus \mathrm{per}(X)$ is small, and thus according to Lemma (4.2) there is an open set U such that $\xi \in U$, $\xi' \notin \mathrm{cl}(U)$, the boundary of U is a union of a small set and a set of periodic points, and $h(U) < \epsilon/2$. Define $\mathcal{A}_1 = \{U\}$.

Suppose $\mathcal{A}_n$ has already been defined. According to Lemma (4.2) we can find for any $U \in \mathcal{A}_n$ a set $V_{U,n}$ such that

(a) $\partial U \subset V_{U,n} \subset B(\partial U, 1/n)$,

(b) $\partial V_{U,n}$ is a union of a small set and a set of periodic points,

(c) $h(V_{U,n}) < \left(2^{n+1} \cdot |\mathcal{A}_n|\right)^{-1} \epsilon$,

and define $\mathcal{A}_{n+1}$ as

$$\mathcal{A}_{n+1} \stackrel{\text{def}}{=} \{V_{U,n} : U \in \mathcal{U}\} \cup \mathcal{A}_n.$$

It is easy to see that $\mathcal{A} \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \mathcal{A}_n$ has all the required properties.

Define

$$\mathcal{A}_T = \{T^n U : n \in \mathbf{Z} \text{ and } U \in \mathcal{A}\}.$$

It is easy to see that $\mathcal{A}_T$ still satisfies properties A1 and A3. Indeed, $\mathcal{A}_T$ clearly satisfies the following three properties:

B1 For any $U \in \mathcal{A}_T$ and any $\delta > 0$ there is a $V \in \mathcal{A}_T$ such that $\partial U \subset V \subset B(\partial U, \delta)$.

B2 There is a $U \in \mathcal{A}_T$ such that $\xi \in U$ but $\xi' \notin \text{cl}(U)$.

B3 $U \in \mathcal{A}_T$ iff $T(U) \in \mathcal{A}_T$.

We will now define a (closed) equivalence relation $\sim$ on X: we will say that $x \sim y$ iff for every $U \in \mathcal{A}_T$, $1_U(x) = 1_U(y)$. To show $\sim$ is closed we must show that if $x \not\sim y$ there are open neighbourhoods $V_x \ni x$, $V_y \ni y$ such that for every $x' \in V_x$ and any $y' \in V_y$, $x' \not\sim y'$.

Without loss of generality we can assume that for some $U \in \mathcal{A}_T$, $x \in U$ but $y \notin U$. There are two cases:

A If $y \notin \partial U$, there is a neigbourhood $V_y \ni y$ such that $V_y \cap U = \emptyset$ and we can take as V_x the set U. Since all points of V_y are not in U, no point of V_y can be equivalent to a point of V_x.

B If $y \in \partial U$ then there is an n such that $d(x, \partial U) > 1/n$. According to the properties of the collection $\mathcal{A}_T$ there is a set $U' \in \mathcal{A}_T$ such that $\partial U \subset U' \subset B(\partial U, 1/n)$, and thus $y \in U'$, $x \notin \text{cl}(U')$, and Case A is applicable.

Using the well known construction of quotient space, $X/\sim$ is a compact metric space, and let $\pi: X \to X/\sim$ denote the projection map from X to $Y \stackrel{\text{def}}{=} X/\sim$. Recall that if $d(\cdot,\cdot)$ is a metric on X, the metric on $X/\sim$ is defined by

$$\rho(x,y) = \delta(\pi^{-1}x, \pi^{-1}y),$$

where $\delta(\cdot,\cdot)$ is the Hausdorff metric on the closed subsets of X, that is

$$\delta(A,A') = \inf\{\delta > 0 : A \subset B(A',\delta) \text{ and } A' \subset B(A,\delta)\}.$$

We can define a homeomorphism $S: Y \to Y$ by specifying that for any $x \in X$,

$$S \circ \pi(x) = \pi \circ T(x).$$

This map is well defined since $x \sim y$ iff $T(x) \sim T(y)$, and is easily seen to be continuous and to have a continuous inverse. Thus we have constructed a factor (Y, S) of (X, T).

(4.4) Estimating the entropy

Recall the following definition of topological entropy:

Let (Z, R) be a dynamical system and let $d(\cdot\,\cdot)$ be a metric on Z. A set E is said to be *n,δ-separated* if for every two distinct x and y in E, there is a $0 \leq k < n$ such that $d(T^k x, T^k y) > \delta$. Let $S(Z; n, \delta)$ (or $S(n, \delta)$ if Z is understood) be the maximal cardinality of an n,δ-separated set. Set

$$h_\delta(Z) \overset{\text{def}}{=} \limsup \frac{1}{n} \log S(Z; n, \delta),$$

then the topological entropy of (Z, R) is

(4.4.1) $$h_{\text{top}}(Z) \overset{\text{def}}{=} \lim_{\delta \to 0} h_\delta(Z).$$

We wish to show that the entropy of Y is less than ϵ. We will do so by showing that Y is not only a factor of X but also of some dynamical system Z with $h_{\text{top}}(Z) < \epsilon$.

$\mathcal{A}$ is denumerable, and so we can present it as

$$\mathcal{A} = \{U_1, U_2, \ldots\}.$$

For any U_i we have the dynamical system $(\mathcal{S}_{U_i}, \sigma)$ with entropy $h(U_i)$. Thus $Z' = \prod_{i=1}^{\infty} \mathcal{S}_{U_i}$, together with the transformation $\sigma' = \prod_{i=1}^{\infty} \sigma$, is a dynamical system with entropy at most $\sum_{i=1}^{\infty} h(U_i) < \epsilon$.

We have the following natural (non-continuous) map ν from X to Z':

$$\nu: x \mapsto (\nu_{i,j})_{\substack{1 \leq i < \infty \\ -\infty < j < \infty}},$$

where $\nu_{i,j} = 1_{T^j U_i}(x)$ for every i, j. This map ν induces a map ν' from $Y = X/\!\sim$ to Z since if $x \sim y$ then $\nu(x) = \nu(y)$. Let $Z = \text{cl}\ \nu(X) \subset Z'$ and define the transformation $R: Z \to Z$ as $R = \sigma'|_Z$. Since (Z, R) is a subsystem of (Z', σ'), $h_{\text{top}}(Z) < \epsilon$.

To show that $h_{\text{top}}(Y) < \epsilon$ we wish to construct a continuous factor map $f: Z \to Y$. For any i, define $U_i^0 = X \setminus U_i$ and $U_i^1 = \text{cl}\ U_i$. Take any $\alpha = \nu(x) \in \nu(X)$. Then

$$x \in \bigcap_{\substack{1 \leq i < \infty \\ -\infty < j < \infty}} T^j(U_i^{\alpha_{i,j}}),$$

and so for any $\alpha \in Z = \text{cl}\ \nu(X)$,

$$U_{\alpha,N} \stackrel{\text{def}}{=} \bigcap_{\substack{1 \le i \le N \\ -N \le j \le N}} T^j(U_i^{\alpha_{i,j}}) \neq \emptyset,$$

and thus $U_\alpha \stackrel{\text{def}}{=} \bigcap_{N=1}^{\infty} U_{\alpha,N} \neq \emptyset$.

Proposition (4.5) diam $(\pi(U_{\alpha,N}))$ *tends to zero uniformly in N.*

Proof Suppose that the proposition is false, that is there exists a $\delta > 0$ and a sequence of $\alpha_n \in Z$, an increasing sequence N_n and two sequences $x_n, y_n \in U_{\alpha_n,N_n}$ such that

$$\rho(\pi(x_n), \pi(y_n)) > \delta.$$

Since Z is compact there is a subsequence of α_n (which w.l.o.g. we can assume to be α_n itself) that converges to a point of Z, say α. This means that there is a sequence $N'_n \to \infty$, $N'_n < N_n$ for every n, such that $(\alpha_n)_{i,j} = \alpha_{i,j}$ for $1 \le i \le N'_n$, $-N'_n \le j \le N'_n$. Thus $x_n, y_n \in U_{\alpha,N'_n}$.

X is compact, and so again we may assume that x_n and y_n converge to $x \in U_\alpha$ and $y \in U_\alpha$, respectively. Since $\rho(\pi(x), \pi(y)) \ge \delta$, $x \not\sim y$. This means that there is some $T^j U_i$ such that $x \in T^j U_i$ and $y \notin T^j U_i$. Without loss of generality we can assume that

$$x \in T^j U_i \qquad \text{and} \qquad y \notin \text{cl}(T^j U_i), \tag{4.5.1}$$

for otherwise there is an $U_{i'} \supset \partial U_i$ such that $x \notin \text{cl}(T^j U_{i'})$, and as in the proof that $\sim$ is closed we see that if we exchange the names of x and y and take $T^j U_{i'}$ instead of $T^j U_i$, (4.5.1) will be true.

But this is impossible, for if $\alpha_{i,j} = 0$ both x and y cannot be in $T^j U_i$; and if $\alpha_{i,j} = 1$ both x and y must be in $\text{cl}(T^j U_i)$. □

Note that in particular we have proved that $\pi(U_\alpha)$ consists of exactly one point for any $\alpha \in Z$. We will define $f(\alpha)$ as the single point of Y in $\pi(U_\alpha)$. Clearly, as a direct consequence of the definitions of f and S,

$$f \circ R(\alpha) = S \circ f(\alpha) \qquad \text{for every } \alpha \in Z.$$

f is onto Y since for any $x \in X$, $x \in U_{\nu(x)}$, and so for any $y \in Y$, $y \in \pi\, U_{\nu'(y)}$. f is also continuous — this is merely a restatement of the fact that diam $\pi(U_{\alpha,N}) \to 0$ uniformly in N. Thus f is a factor transformation from Z to Y and we are done.

We have in fact proved the following theorem:

Theorem (4.6) *Let (X,T) be a finite dimensional dynamical system. Then for every two points $x,x' \in X$ and every $\epsilon > 0$ there is a factor transformation ψ*

that maps the system (X,T) *onto a system* (Y,S) *with* $h_{\text{top}}(Y) < \epsilon$ *and such that* $\psi(x) \neq \psi(x')$.

In fact, it is not hard to see that if we make sure that the collection $\mathcal{A}$ from (4.3) contains arbitrarily small neighbourhoods of x then the resulting factor transformation ψ will have the additional property that

$$\psi(x) \neq \psi(y) \quad \text{for } any\ y \in X.$$

(4.8) Constructing intermediate factors

We will need a small lemma before proving our main result:

Lemma (4.9) *Let* $\{(Y_i, S_i)\}_{i=1}^{\infty}$ *be a sequence of factors of the system* (X,T), *and let* $\psi_i: X \to Y_i$, $i = 1, \dots,$ *be the associated factor transformations. Define for every* n *the factor* (Z_n, R_n) *imbedded in*

$$(Y_1 \times Y_2 \times \cdots \times Y_n, S_1 \times S_2 \times \cdots \times S_n)$$

as the image of the system X *under the factor mapping* $\psi_1 \times \psi_2 \times \cdots \times \psi_n$. *Similarly, define* (Z_∞, R_∞) *as the image of* X *under the infinite product* $\psi_1 \times \psi_2 \times \cdots$. *Then*

$$h_{\text{top}}(Z_\infty) = \lim_{n\to\infty} h_{\text{top}}(Z_n). \tag{4.9.1}$$

Proof We will first define for each $n \leq \infty$ a metric $d(\cdot, \cdot)$ on Z_n as

$$d(z, z') = \sum_{i=1}^{n} 2^{-i} d_{Y_i}(y_i, y_i'),$$

where $z = (y_1, \dots, y_n)$ and $z' = (y_1', \dots, y_n')$.

Since (Z_m, R_m) is a factor of (Z_n, R_n) for $m < n \leq \infty$ the limit on the right-hand side of equation (4.9.1) exists and is no bigger than the left-hand side. However, in view of the metrics defined on $\prod_{i=1}^{\infty} Y_i$ and $\prod_{i=1}^{n} Y_i$, for every ϵ there is an N and ϵ' such that

$$h_\epsilon(Z_\infty) \leq h_{\epsilon'}(Z_N). \tag{4.9.2}$$

This is true since for large enough N, the projection $\pi_N: Z_\infty \to Z_N$ does not change the distances too much, i.e.

$$|d(x,y) - d(\pi_N(x), \pi_N(y))| < \epsilon/4$$

for any $x, y \in Z_\infty = \prod_{i=1}^{\infty} Y_i$. Thus the projection of any n, ϵ-separated set to the first N coordinates will be a n, $\epsilon/2$-separated set in Z_N, and so we have (4.9.2).

Assume $h_{\mathrm{top}}(Z_\infty) < \infty$ (the case $h_{\mathrm{top}}(Z_\infty) = \infty$ is almost the same and is omitted). Since $h_{\mathrm{top}}(Z_\infty) = \lim_{\epsilon\to 0} h_\epsilon(Z_\infty)$, for any δ there exists an ϵ such that $h_{\mathrm{top}}(Z_\infty) < h_\epsilon(Z_\infty) + \delta$. Thus

$$h_{\mathrm{top}}(Z_\infty) - \delta < h_\epsilon(Z_\infty) \le h_{\epsilon'}(Z_N) \le h_{\mathrm{top}}(Z_N) \le \lim_{n\to\infty} h_{\mathrm{top}}(Z_n). \qquad \square$$

Theorem (4.10) *Let (X, T) be a finite dimensional dynamical system, and let (Y', S') be a factor of (X, T) such that $h_{\mathrm{top}}(Y') < h_{\mathrm{top}}(X)$ (in particular (Y', S') can be the trivial factor consisting of a single point). Then for any $h \in (h_{\mathrm{top}}(Y'), h_{\mathrm{top}}(X))$, we can find a factor (Y, S) of (X, T) with topological entropy h, and such that if $\phi: X \to Y'$ and $\psi: X \to Y$ are the respective factor transformations then $\phi \circ \psi^{-1}$ is well defined, and is a factor transformation $Y \to Y'$.*

Proof Take any $h_{\mathrm{top}}(Y') < h < h_{\mathrm{top}}(X)$. Take $\epsilon_1 = (h - h_{\mathrm{top}}(Y'))/2$. According to Theorem (4.6), for every two points $x \neq x'$ in X there is a factor $(Y^1_{x,x'}, S^1_{x,x'})$ of X with entropy $\le \epsilon_1$ such that x and x' are mapped to different points in $Y^1_{x,x'}$. The factor transformation is continuous and so there are open sets $U^1_{x,x'} \ni x$ and $V^1_{x,x'} \ni x'$ such that the factor transformation maps $U^1_{x,x'}$ and $V^1_{x,x'}$ into disjoint sets.

The sets $U^1_{x,x'} \times V^1_{x,x'}$ are an open cover of $X \times X \setminus \Delta$, where Δ is the diagonal in $X \times X$. A countable number of these $U^1_{x,x'} \times V^1_{x,x'}$ suffice to cover $X \times X \setminus \Delta$, say $U^1_i \times V^1_i$. Denote by (Y^1_i, S^1_i) the associate factors and by ψ^1_i $(i > 1)$ the factor transformations. Also set $(Y^1_1, S^1_1) = (Y', S')$ (the given factor of X), and $\psi^1_1 = \phi$.

Define using the (Y^1_i, S^1_i)'s the factors (Z^1_n, R^1_n) and (Z^1_∞, R^1_∞) as in the previous lemma. The $U^1_i \times V^1_i$'s cover $X \times X \setminus \Delta$ and so for every $x \neq x'$ in X there is an i such that $\psi^1_i(x) \neq \psi^1_i(x')$. Thus the factor map $X \to Z^1_\infty$ is one-on-one; by definition it is onto. Since X is compact we see that Z^1_∞ is homeomorphic to X, and this homeomorphism is in fact an isomorphism between these two dynamical systems. Consequently, we see that $h_{\mathrm{top}}(Z^1_\infty) = h_{\mathrm{top}}(X) > h$. Using Lemma (4.9) we see that already for some finite N, the entropy $h_{\mathrm{top}}(Z^1_N) > h$. Note that for any N we have

$$h_{\mathrm{top}}(Z^1_{N+1}) \le h_{\mathrm{top}}(Z^1_N) + h_{\mathrm{top}}(Y^1_{N+1}) < h_{\mathrm{top}}(Z^1_N) + \epsilon_1,$$

since we can imbed (Z^1_{N+1}, R^1_{N+1}) as a dynamical system in $(Z^1_N \times Y^1_{N+1}, R^1_N \times S^1_{N+1})$. If we take N_1 to be the smallest integer such that $h_{\mathrm{top}}(Z^1_N) > h$, we see that

$$h - \epsilon_1 < h_{\mathrm{top}}\left(\psi^1_1 \times \psi^1_2 \times \cdots \times \psi^1_{N_1 - 1}(X)\right) \le h.$$

(Note that since $h_{\mathrm{top}}(Y') + \epsilon_1 < h$, the integer N_1 must be greater than 1.)

We now take $\epsilon_2 = \epsilon_1/2$ and find exactly as for ϵ_1 a sequence of factors (Y_i^2, T_i^2) and factor transformations ψ_i^2 such that for every i, $h_{\mathrm{top}}(Y_i^2) < \epsilon_2$, and so that for every $x \neq x'$ there is an i such that $\psi_i^2(x) \neq \psi_i^2(x')$. For the same reasons as in the preceding paragraph

$$\psi_1^1 \times \cdots \times \psi_{N_1-1}^1 \times \psi_1^2 \times \psi_2^2 \times \cdots (X) \cong X$$

and so there is an N_2 such that

$$h - \epsilon_2 < h_{\mathrm{top}}\left(\psi_1^1 \times \cdots \times \psi_{N_1-1}^1 \times \psi_1^2 \times \cdots \times \psi_{N_2-1}^2(X)\right) \leq h.$$

Proceeding in this way and using (4.9) we see that

$$h_{\mathrm{top}}\left(\psi_1^1 \times \cdots \times \psi_{N_1-1}^1 \times \psi_1^2 \times \cdots \times \psi_{N_2-1}^2 \times \psi_1^3 \times \cdots \times \psi_{N_3-1}^3 \times \cdots (X)\right) = h,$$

and so, if we define ψ to be

$$\psi \stackrel{\mathrm{def}}{=} \psi_1^1 \times \cdots \times \psi_{N_1-1}^1 \times \psi_1^2 \times \cdots \times \psi_{N_2-1}^2 \times \psi_1^3 \times \cdots \times \psi_{N_3-1}^3 \times \cdots,$$

$Y \stackrel{\mathrm{def}}{=} \psi(X)$ is the required factor. Indeed, since $\psi_1^1 = \phi$ the map $\phi \circ \psi^{-1} : Y \to Y'$ is simply a map that projects elements of Y (imbedded in $Y' \times Y_2^1 \cdots$) according to the first coordinate and so is obviously well defined and continuous. It is also onto Y' since ϕ is. □

5. A construction of a minimal dynamical system with no small factors

In Section 2 we used the fact that each coordinate of a point in $[0,1]^{\mathbf{Z}}$ can be varied continuously from 0 to 1 without having to change any other coordinate. Obviously there can't be any minimal system with this property, but we will build a minimal system with enough flexibility that we will be able to use reasoning similar to that in Section 2 to prove this system has no finite entropy factors. Our system will be imbedded in $[0,1]^{\mathbf{Z}}$, and we will use the following metric on this space:

$$d(x,y) \stackrel{\mathrm{def}}{=} \sum_{-\infty<k<\infty} 2^{-|k|}|x_k - y_k|.$$

We will define inductively a monotone decreasing sequence of subsystems (i.e. closed and shift invariant subset) $[0,1]^{\mathbf{Z}} \supset X_0 \supset X_1 \supset \cdots \supset X_n \cdots$ such that:

a. No X_n is empty.

b. For any two points x, $y \in X_n$ there is a k such that $d(y, \sigma^k x) < a_n$, where $(a_k)_{k=0}^{\infty}$ is a sequence tending to zero as $k \to \infty$.

Lemma (5.1) *If X_n satisfy conditions* **a** *and* **b** *above then $X \stackrel{\text{def}}{=} \bigcap_{n=0}^{\infty} X_n$ is minimal.*

Proof Since $[0,1]^{\mathbf{Z}}$ is compact $\bigcap X_n$ is nonempty. Let $x \in X$. We must show that $\overline{\{\sigma^k x\}} = X$. Take any $y \in X$. For every n, both x and y are in X_n; thus there is a k such that $d(y, \sigma^k x) < a_n$. Since $a_n \to 0$ this proves that y is in the closure of $\{\sigma^k x\}$. □

The following terminology will be convenient for the construction of the sequence X_n, and in proving X has no small factors. We will look at finite sequences of real numbers in the interval [0,1] as *words* from the (infinite) alphabet [0, 1] (the *letters*). By an infinite word we simply mean an element of $[0,1]^{\mathbf{Z}}$. If w is a word we will designate its length by $l(w)$. If $x \in [0,1]^{\mathbf{Z}}$, a and b in $\mathbf{Z} \cup \{-\infty, \infty\}$ define $x|_a^b = (x_a, x_{a+1}, \dots x_{b-1})$. We will also use the notation $w|_a^b$ for words, where the first letter of a word has index 1 (if $a < 1$ or $b > l(w)$ we set $w|_a^b \stackrel{\text{def}}{=} w|_{\max(a,1)}^{\min(b,l(w)+1)}$).

A word w' is said to be a *substring* of a (finite or infinite) word w (or $w' \preceq w$) if there exist integers t_1,t_2 such that $w' = w|_{t_1}^{t_2}$. If $X \subset [0,1]^{\mathbf{Z}}$ we define the words of X or $\mathcal{W}(X)$ to be:

$$\mathcal{W}(X) = \{w \mid \exists x \in X \quad w \preceq x\},$$

$\mathcal{W}_l(X)$ to be the words of X of length l, $\mathcal{W}_{\leq l}(X)$ those of length $\leq l$. And finally, if w,w' are two words then $w \cdot w'$ is their concatenation.

Let w_0 be the length 1 word '0', $t_0 = 100$. Set

$$X_0 = \left\{ x \in [0,1]^{\mathbf{Z}} \mid \forall t \quad w_0 \preceq x|_t^{t+t_0} \right\}.$$

For every k set $L_k = \{0, \frac{1}{2^k}, \dots, 1 - \frac{1}{2^k}\}^{\mathbf{Z}} \subset [0,1]^{\mathbf{Z}}$, with $L_\infty = \bigcup_{k=0}^{\infty} L_k$. We will continue inductively to define X_k,w_k and t_k, with the following properties:

1. $w_k \in \mathcal{W}(X_{k-1} \cap L_k)$.
2. Every word $w \in \mathcal{W}_{t_{k-1}}(X_{k-1} \cap L_k)$ is a substring of w_k.
3. $t_k = 50 \cdot 2^k l(w_k)$.
4. $X_k = \{x \in X_{k-1} \mid \forall t \quad w_k \preceq x|_t^{t+t_k}\}$.

If we will define how to get w_k with the above properties given X_{k-1} we will be done with the definition. Set $W = \mathcal{W}_{3t_{k-1}}(X_{k-1} \cap L_k)$. W is obviously a finite set. By property 4, for $k-1$, each $w \in W$ has a w_{k-1} as a substring of its first (and also last) t_{k-1} letters. For every such w remove at most t_{k-1} letters from its beginning and its end until we have a word that begins and ends with the substring w_{k-1}, and let W' be the set of all these smaller words. Take w_k to be the concatenation of all the $w' \in W'$.

Since each $w' \in \mathcal{W}(X_{k-1})$, and since they all begin and end with a w_{k-1}, $w_k \in \mathcal{W}(X_{k-1})$ and so $w_k \in \mathcal{W}(X_{k-1} \cap L_k)$ which is property 1. Also, since we have not tampered with the middle t_{k-1} letters of each $w \in W$ we have property 2. Properties 3 and 4 are merely definitions for t_k and X_k respectively.

In terms of the w_k and t_k we see that

$$X_n = \left\{ x \in [0,1]^{\mathbf{Z}} \mid \forall t, k \leq n \quad w_k \preceq x|_t^{t+t_k} \right\}. \tag{5.1.1}$$

X is the same but without the restriction that k must be $\leq n$.

Lemma (5.2) *The X_k defined above satisfy the conditions of Lemma* (5.1). *Thus $X = \bigcap_{k=0}^{\infty} X_k$ is minimal.*

Proof Since w_k is a word of X_{k-1}, that is, it has "enough" $w_{k'}$ substrings for $k' < k$, and it begins and ends with a w_{k-1}, the two sided infinite concatenation $\cdots w_k \cdot w_k \cdot w_k \cdots$ is in X_k, so it is clear that X_k is a non-empty closed shift invariant set. Thus condition **a** of Lemma (5.1) is satisfied.

To prove that condition **b** is satisfied let x be any point of X_k. Define a word $x' \in \mathcal{W}(X_{k-1} \cap L_k)$ as follows :

$$x'_l = \frac{\lfloor 2^k x_l \rfloor}{2^k} \qquad \text{for } -\tfrac{t_{k-1}}{2} \leq l < \tfrac{t_{k-1}}{2}$$

(it is a word of X_{k-1} since all the words $w_{k'}$, $k' < k$ are composed of letters of the form $\frac{\text{integer}}{2^k}$). Now according to property 2, x' is a substring of w_k and so of any $y \in X_k$. So we see that for any $y \in X_k$ there is an m such that for $-\frac{t_{k-1}}{2} \leq l < \frac{t_{k-1}}{2}$ the l^{th} coordinate of x is different from that of $\sigma^m y$ by less than 2^{-k}. But this means (since $t_{k-1} \gg k$) that there exists an m such that $d(x, \sigma^m y) < 2^{-k+2}$ — which is condition **b**. □

Since every w_n begins and ends with a w_{n-1}, there is a one sided word (with its first letter having index 1) w_∞, such that $w_\infty|_1^{l(w_n)+1} = w_n$ for every n, and another one sided word (with its last letter having index 0) $\bar{w}_\infty$ such that for every n, $\bar{w}_\infty|_{-l(w_n)+1}^{1} = w_n$.

We will say a word w is *legal* if for every n, $w_n \cdot w \cdot w_n \in \mathcal{W}(X)$, or since X is closed this is equivalent to $\bar{w}_\infty \cdot w \cdot w_\infty \in X$. This definition was chosen (in preference to the more natural definition that w be in $\mathcal{W}(X)$) so that we would have an easy way to extend any legal word to an element of X. It is easy to see that a word w is legal iff for every n

$$w_n \preceq (w_n \cdot w \cdot w_n)|_t^{t+t_n} \qquad \text{for every } 1 < t \leq l(w) + 2l(w_n) - t_n. \tag{5.2.1}$$

The fact, that for legal w equation (5.2.1) is satisfied for every n, is immediate from the definitions. But if (5.2.1) is satisfied then it is easy to see that $\bar{w}_\infty \cdot w \cdot w_\infty$ satisfies the conditions for being an element of X stated in (5.1.1).

Notice that for large enough n, $t_n > l(w) + 2l(w_n)$ since by definition $t_n \geq 100l(w_n)$, and so (5.2.1) is satisfied vacuously. It is easy to see that for every n the word w_n is legal; and every word in $\mathcal{W}(X \cap L_\infty)$ is a substring of w_n for some n, so that every such word is a substring of a legal one.

The following lemma supplies the necessary degrees of freedom we need to apply the methods of Section 2, and is the important step in proving our system does not have any small non-trivial factors:

Lemma (5.3) *If v_0 and v_1 are both legal words of common length l, then there exists a continuous path $v(t)$, $0 \leq t \leq 1$ between them consisting of legal words (a legal path).*

To prove this lemma we must look more closely at legal words. Say w is a legal word. Some of its letters we can change at will without touching any substring equal to a w_k — indeed in general letters may even fail to be of the form $\frac{\text{integer}}{2^k}$ and so by the definition of X given in terms of the t_k and w_k, (5.1.1), the changed words will all be legal. Other letters however can be crucial and changing them will destroy an essential appearance of a w_k substring. Even those crucial letters have different roles: some appear as part of an essential w_0 (i.e. if they are changed, there will be a t such that $w_0 \not\prec \bar{w}_\infty \cdot w \cdot w_\infty|_t^{t+t_0}$), others as part of an essential w_1, etc. Notice that if the word w is shorter than t_k there will be no essential w_{k+1}'s. Changing an essential letter (which is a thing we might need to do to pass from the word v_1 to v_2) involves constructing alternative copies of the relevant w_k which is harder for larger k's. To designate such a *pattern* of words with some of their letters fixed, and some ranging over all values, independently of one another, we will use a word in the extended alphabet $[0, 1] \cup \{*\}$, where a '$*$' should be interpreted as a letter that can be replaced with any value. To each pattern p of length l, we will designate the set of words that fit it by $\langle p \rangle$. The intersection of two such sets for two patterns p_1 and p_2 of common length l is either an empty set or a set associated to another pattern of the same length; if this pattern is p we will say that $p = p_1 \wedge p_2$. Thus the following definition, which will play an important role in proving Lemma (5.3), is natural:

Definition (5.4) *A* **structure** *of a word w is a sequence of patterns $(p_i)_{i=0}^\infty$ (i.e. a sequence of words in the extended alphabet $[0, 1] \cup \{*\}$) such that:*

1. *w is in $\langle p_k \rangle$ for every k.*

2. *For every k, $\langle p_k \rangle \subset \langle p_{k+1} \rangle$.*

3. *Setting*

$$X_{\geq n} \stackrel{\text{def}}{=} \{ x \in [0,1]^{\mathbf{Z}} \mid \forall t, k \geq n \quad w_k \preceq x|_t^{t+t_k} \}$$

every word v in $\langle p_n \rangle$ satisfies $w_k \cdot v \cdot w_k \in \mathcal{W}(X_{\geq n})$ for every k (or, equivalently, $\bar{w}_\infty \cdot v \cdot w_\infty \in X_{\geq n}$; we will call such words v legal words for $X_{\geq n}$).

The structure is **minimal** *if there is no other structure of w, say $(p'_i)_{i=0}^\infty$, such that for every i, $\langle p'_i \rangle \supset \langle p_i \rangle$.*

Remarks Notice that if we have a structure $(p_i)_{i=0}^\infty$ for a word w of length $< t_k$, then $(p'_i)_{i=0}^\infty$, where $p'_i = p_i$ for $i \leq k$ and $p'_i = * * \cdots *$ otherwise, is also a structure. In particular, if $l(w) < t_k$ then for any minimal structure $(p_i)_{i=0}^\infty$ of w, for every $k' > k$, $p_{k'} = * * \cdots *$. Another thing worth noting is that every legal word w has a structure. Indeed the sequence of patterns $p_i = w$ is a structure, for both the first and second conditions are trivial since all the p_is are the same, and the third follows for every n from the legality of w.

We will need the following lemma, which shows that many patterns satisfying condition 3 in (5.4) for some n can be completed to form a structure.

Lemma (5.5) *Let p be a pattern satisfying property 3 of Definition* (5.4) *for some n. Suppose in addition that for every t such that the* t^{th} *letter of p is not '$*$' there are t' and t'', $t' \leq t \leq t''$ such that $(\bar{w}_\infty \cdot p \cdot w_\infty)|_{t'}^{t''} = w_{n'}$ with $n' \geq n$ (fix the indexing of this infinite word by taking the last letter of $\bar{w}_\infty$ as its* 0^{th} *coordinate). Then $\langle p \rangle$ contains a legal word.*

Proof By induction on n. For $n = 0$ Definition (5.4) says every word in $\langle p \rangle$ is legal so the lemma obviously holds.

If $n > 0$ we will show that there is a pattern p' such that $\langle p' \rangle \subset \langle p \rangle$ and p' satisfies the conditions of this lemma for $n - 1$.

We get the pattern p' from p by taking any $l(w_{n-1})$ consecutive '$*$'s of p and replacing them by w_{n-1} until there are no $l(w_{n-1})$ consecutive '$*$'s in p'. This new pattern clearly satisfies the second condition of this lemma, namely that every non '$*$' of p' is part of a $w_{n'}$ substring for $n' \geq n - 1$. We already know that for every $w \in \langle p' \rangle$, the infinite word $\bar{w}_\infty \cdot w \cdot w_\infty$ (again the last letter of $\bar{w}_\infty$ is the 0^{th} coordinate of the infinite word) is in $X_{\geq n}$, and wish to show that it is in $X_{\geq n-1}$. Thus we only need to show that for every t,

$$w_{n-1} \preceq (\bar{w}_\infty \cdot w \cdot w_\infty)\,\big|_t^{t+t_{n-1}}\,. \tag{5.5.1}$$

Since $\bar{w}_\infty$ ends with w_{n-1}, and any finite substring of it is a substring of some $w_{n'}$, (5.5.1) is true for $t \leq -l(w_{n-1})$ and in the same way we can see that (5.5.1) is true for $t \geq l(w) - t_{n-1} + l(w_{n-1})$. Suppose there is t', $-l(w_{n-1}) < t' < l(w) - t_{n-1} + l(w_{n-1})$ that does not satisfy (5.5.1) for some word $w \in \langle p' \rangle$. Changing the indexation we see there is a t', $0 < t' < l(w) - t_{n-1} + 2l(w_{n-1})$ and a $w \in \langle p' \rangle$, such that w_{n-1} is not a substring of $(w_{n-1} \cdot w \cdot w_{n-1})|_{t'}^{t'+t_{n-1}}$. Such a w exists iff w_{n-1} is not a substring of the pattern $(w_{n-1} \cdot p' \cdot w_{n-1})|_{t'}^{t'+t_{n-1}}$ (patterns are simply words in an extended alphabet, and so we can use the word terminology for patterns as well). So we see that (5.5.1) is equivalent to having

$$w_{n-1} \preceq (w_{n-1} \cdot p' \cdot w_{n-1})|_{t}^{t+t_{n-1}} \tag{5.5.2}$$

for every $1 < t \leq l(w) - t_{n-1} + 2l(w_{n-1})$.

Now assume again that there is a t' that does not satisfy (5.5.2). Since in p' there are no more than $l(w_{n-1}) - 1$ consecutive '$*$'s, there is a non '$*$' letter in the (smaller) substring $(w_{n-1} \cdot p' \cdot w_{n-1})\,|_{t'+l(w_{n-1})}^{t'+t_{n-1}-l(w_{n-1})}$ which is the substring $p'\,|_{t'}^{t'+t_{n-1}-2l(w_{n-1})}$. We know it is part of a $w_{n'}$ substring of $\hat{w}_\infty \cdot p' \cdot w_\infty$ for $n' \geq n-1$. If it is part of a w_{n-1} then we get a contradiction to t' being 'bad'. Suppose, then, that this letter is a part of a $w_{n'}$ for $n' \geq n$. If this $w_{n'}$ contains all the letters of $w_{n-1} \cdot p' \cdot w_{n-1}$ in the range t' to $t' + t_{n-1} - 1$, then since $w_{n'} \in \mathcal{W}(X)$ these letter must contain a w_{n-1} substring — a contradiction. So this $w_{n'}$ substring must have either its end or its beginning in this range of letters. Assume that its end is in this range — the other case is exactly the same. We know that a letter in the range $t' + l(w_{n-1})$ to $t' + t_{n-1} - 1 - l(w_{n-1})$ is part of this $w_{n'}$, so the last $l(w_{n-1})$ letters of this $w_{n'}$ are a substring of $(w_{n-1} \cdot p' \cdot w_{n-1})|_{t'}^{t'+t_{n-1}}$. Since $w_{n'}$ ends with a w_{n-1}, we again get a contradiction.

Thus the pattern p' satisfies the conditions of this lemma for $n-1$. By the induction hypothesis we know that $\langle p' \rangle$ must contain a legal word; and since $\langle p \rangle \supset \langle p' \rangle$ so does $\langle p \rangle$. □

Corollary (5.6) *Any pattern p satisfying the conditions of Lemma* (5.5) *for some n, can be completed to form a structure whose n^{th} element is p.*

Proof Let w be a legal word in $\langle p \rangle$; then

$$p_i = \begin{cases} w, & \text{if } i < n \\ p, & \text{otherwise} \end{cases}$$

is a structure for the word w. □

We return to Lemma (5.3). Let $(p_i^0)_{i=0}^{\infty}$ and $(p_i^1)_{i=0}^{\infty}$ be structures for the two legal words v_0 and v_1 respectively. Also define k_0 to be the smallest integer such that $t_{k_0} > l(v_0)$.

Now suppose p_0^0 and p_0^1 are the same. This means that there is a pattern which both v_0 and v_1 satisfy, such that any word satisfying this pattern is legal. So simply vary every letter different in v_0 and v_1 continuously from its value in v_0 to that in v_1 and we get the required legal path. On the other hand, from the remark following Definition (5.4), if $(p_i^0)_{i=0}^\infty$ and $(p_i^1)_{i=0}^\infty$ are minimal structures, $p_{k_0}^0$ and $p_{k_0}^1$ are always the same, namely the trivial pattern '$**\cdots*$'.

We will prove by induction on r the following lemma.

Lemma (5.3)′ *Suppose v_0 and v_1 are both legal words of common length l, with structures $(p_i^0)_{i=0}^\infty$ and $(p_i^1)_{i=0}^\infty$ respectively. If $p_r^0 = p_r^1$ for some r, then there exists a legal path between them.*

Remark Since we can choose any structures we like for v_0 and v_1, we can choose minimal structures, which we know satisfy $p_r^0 = p_r^1$ for r large enough. So (5.3)′ does imply (5.3).

Proof of the Lemma Note that we have proved the case $r = 0$. So suppose it is true for $r' = r - 1$; we must show the lemma is true also for r. Let k_0 be the smallest k such that $l < t_{k_0}$.

(5.8) We proceed to show that the structures $(p_k^0)_{k=0}^\infty$ and $(p_k^1)_{k=0}^\infty$ can be assumed to be minimal. As we have already seen we can assume that for $k > k_0$, p_k^0 and p_k^1 are '$**\cdots*$'. Also we can assume that for $k \geq r$ we have $p_k^0 = p_k^1$ — in fact we can take both of these equal to p_r^0. Now we can start at $k = k_0$ and find a pattern p_k' that satisfies all the properties it needs to in Definition (5.4), and in addition $\langle p_k^0\rangle \subset \langle p_k'\rangle$, such that for every other pattern q satisfying these conditions, $\langle p_k'\rangle \not\subset \langle q\rangle$. (There is such a pattern, because $\langle p_k'\rangle \subset \langle q\rangle$ simply means that some of the letters $\in [0,1]$ in p_k' were replaced by '$*$'s. So after a *finite* number of stages we must find the pattern p_k' we wanted.) After we have defined p_k', we can define in the same way, but for $k' = k-1$, a pattern p_{k-1}' and continue inductively. The resulting sequence of patterns together with the all '$*$' patterns for large k form a *minimal* structure for v_0.

Performing the same process for v_1 we can construct a minimal structure p_k'' for v_1 satisfying $p_k'' = p_k'$ for $k \geq r$, for in finding p_k' for $k \geq r$ we only used p_k^0 for $k \geq r$ and these are the same as p_k^1 for these ks. In particular $p_r'' = p_r'$, which is exactly the condition imposed on the original structures by the assumptions of Lemma (5.3)′. From now on we will assume that $(p_k^0)_{k=0}^\infty$ and $(p_k^1)_{k=0}^\infty$ are minimal.

(5.9) Now suppose we find a pattern ξ, such that

(i) ξ satisfies property 3 of Definition (5.4) for $n = r - 1$,

(ii) any non '$*$' letter of ξ is part of a w_k substring of $\bar{w}_\infty \cdot \xi \cdot w_\infty$ ($k \geq r-1$),

(iii) $\langle\xi\rangle \cap \langle p_{r-1}^0\rangle \neq \emptyset$ and $\langle\xi\rangle \cap \langle p_{r-1}^1\rangle \neq \emptyset$,

(iv) $\langle\xi\rangle \subset \langle p_r^0\rangle \quad (= \langle p_r^1\rangle)$.

Assume further that we can find legal words v_0' in $\langle \xi \wedge p_{r-1}^0 \rangle$ and v_1' in $\langle \xi \wedge p_{r-1}^1 \rangle$. We can easily find structures $(\xi_k^0)_{k=0}^{\infty}$ and $(\xi_k^1)_{k=0}^{\infty}$ for v_0' and v_1' respectively, such that $\xi = \xi_{r-1}^0 = \xi_{r-1}^1$ (simply take $\xi_k^0 = v_0'$ for $k < r-1$ and $\xi_k^0 = \xi$ for $k \geq r-1$ and the same also applies to ξ_k^1). So according to the induction hypothesis v_0' and v_1' can be connected with a legal path. From exactly the same reasons, completing p_{r-1}^0 to structures for v_0 and v_0' we see they can be connected, and the same is true for v_1 and v_1'. Combining these three legal paths we get a legal path from v_0 to v_1.

We can see that there is a legal word in $\langle \xi \wedge p_{r-1}^0 \rangle$ using Lemma (5.5). For assumption (i) on ξ says any word in $\langle \xi \rangle$ is legal for $X_{\geq r-1}$ and so the same is true for the smaller set $\langle \xi \wedge p_{r-1}^0 \rangle$, which is the first part of the conditions of Lemma (5.5). In addition p_{r-1}^0 as the $r-1^{\text{th}}$ element of a minimal structure also satisfies assumption (ii), and so every non '$*$' letter of $\xi \wedge p_{r-1}^0$ being a non '$*$' letter in either ξ or p_{r-1}^0 is part of a w_k substring in the concatenation of this pattern with $\bar{w}_\infty$ and w_∞ and so is part of a w_k in the concatenation of $\xi \wedge p_{r-1}^0$ with these two one sided infinite words, which is the second part of the conditions of that lemma. The same is of course true for the pattern $\xi \wedge p_{r-1}^1$, so we see that the existence of v_0' and v_1' is a consequence of the conditions set on ξ.

(5.10) We will now construct ξ by starting with $\xi_0' \stackrel{\text{def}}{=} p_r^0$, and modifying it until at last we get a pattern that will satisfy the conditions on ξ, in a way very similar to the way we got the pattern p' from the pattern p in the proof of Lemma (5.5). Suppose we have defined already ξ_k'. There are two cases:

A: There is a $1 \leq t \leq l - l(w_{r-1}) + 1$ such that

$$\xi_k' \Big|_t^{t+l(w_{r-1})} = p_{r-1}^0 \Big|_t^{t+l(w_{r-1})} = ** \cdots * = p_{r-1}^1 \Big|_t^{t+l(w_{r-1})},$$

in which case we get ξ_{k+1}' by replacing the $l(w_{r-1})$ '$*$'s starting at place t with w_{r-1}.

B: There is no such t, in which case we take ξ to be ξ_k'.

Case A can be applied only a finite number of times. So eventually we do get to case B and thus ξ is defined. ξ clearly satisfies the assumptions (iii) and (iv). Also, since we started with p_r^0 which is an r^{th} element of a minimal structure and replaced '$** \cdots *$' substrings with w_{r-1}, assumption (ii) also holds.

To show that assumption (i) holds, we need to show that for every word w in $\langle \xi \rangle$,

$$\bar{w}_\infty \cdot w \cdot w_\infty \in X_{\geq r-1}.$$

Since we know that $\langle \xi \rangle \subset \langle p_r^0 \rangle$, a set of legal words for $X_{\geq r}$, then exactly as in the proof of condition (5.5.2) in Lemma (5.5) it remains to be shown that for every

$1 \le t \le l + 2l(w_{r-1}) - t_{r-1}$,

$$(5.10.1) \qquad w_{r-1} \preceq (w_{r-1} \cdot \xi \cdot w_{r-1}) \big|_t^{t+t_{r-1}} .$$

If there is a t in the range $0 < t \le l + 2l(w_{r-1}) - t_{r-1}$ that does not satisfy (5.10.1) then $(w_{r-1} \cdot \xi \cdot w_{r-1}) \big|_t^{t+t_{r-1}}$ is a pattern with t_{r-1} letters but no w_{r-1} substrings. We want to show that this contradicts the fact that ξ satisfies case B, and we start by showing (using arguments very similar to those in the proof of (5.5)) that

$$(5.10.2) \qquad (w_{r-1} \cdot p_r^0 \cdot w_{r-1}) \big|_{t+l(w_{r-1})}^{t+t_{r-1}-l(w_{r-1})} = * * \cdots * .$$

Since $(p_k^0)_{k=0}^{\infty}$ is minimal every non '$*$' letter in p_r^0 must be part of a w_k substring for $k \ge r$. If all the letters between t and $t + t_{r-1}$ are part of this w_k then, since $w_k \in \mathcal{W}(X)$, there is in this range of letters a w_{r-1} substring, which is a contradiction. We know w_k begins and ends with w_{r-1}, so if the letters $t, \ldots, t + t_{r-1} - 1$ of $(w_{r-1} \cdot p_r^0 \cdot w_{r-1})$ are not part of the w_k, the first or last letters of this w_k are in this range of letters; in order not to get a w_{k-1} substring (and a contradiction to the existence of a 'bad' t) we see that the letters of p_r^0, in the range that interests us, that can appear as part of any w_k for $k \ge r$ are at most the first and last $l(w_{k-1}) - 1$ letters. Thus the central letters between $t + l(w_{r-1})$ and $t + t_{r-1} - 1 - l(w_{r-1})$ are '$*$'s.

Both p_{r-1}^0 and p_{r-1}^1 can be derived from p_r^0 by replacing '$*$'s with letters in $[0, 1]$. If we look at the letters from $t + l(w_{r-1})$ to $t + t_{r-1} - l(w_{r-1})$ in p_{r-1}^0 we claim that they cannot contain more than two complete appearances of w_{r-1}. For if they contain three appearances of w_{r-1} the middle one is redundant — any t_{r-1} consecutive letters that contain the middle w_{r-1} contains either the first or last w_{r-1} — and so we can replace by '$*$' all the letters that compose the middle w_{r-1} and are not part of the other two w_{r-1}s contradicting minimality. So the letters $t + 2l(w_{r-1})$ to $t + t_{r-1} - 2l(w_{r-1})$ cannot contain more than two blocks of non '$*$' letters, each of length $\le l(w_{r-1})$ (the two blocks can merge together to form a larger block but no bigger than twice this size). The same is true for p_{r-1}^1, and so since $t_{r-1} \ge 100 l(w_{r-1})$ there must be in the range $t + 2l(w_{r-1})$ to $t + t_{r-1} - 2l(w_{r-1})$ a subrange from t' to $t' + 3l(w_{r-1})$ such that in both p_{r-1}^0 and p_{r-1}^1 the letters in this range are only '$*$'. We know however that the letters of ξ in this range do not have a w_{r-1} substring. Recall that ξ was derived from p_r^0 by changing substrings of $l(w_{r-1})$ '$*$'s by w_{r-1}. Since the letters t' to $t' + 3l(w_{r-1})$ do not contain any full copy of w_{r-1}, and since they were in p_r^0 all '$*$', we see that the middle third of these letters must remain all '$*$'s. Thus at last we see that ξ has an all $*$ substring satisfying the conditions of case A — a contradiction.

Since we have seen in the previous subsection that the existence of a ξ satisfying assumptions (i)–(iv) implies the existence of a legal path between v_0 and v_1 we are done. □

We need one more simple lemma:

Lemma (5.11) *If $v_1, \ldots, v_n$ are all legal words of common length $l \le t_k - 2l(w_k)$, and $n(l + l(w_k)) \le t_{k+1} - 2l(w_{k+1})$, then $v_1 \cdot w_k \cdot v_2 \cdots w_k \cdot v_n$ is a legal word.*

The proof of this lemma is very similar to the proof of Lemma (5.5) and the last subsection of the proof of Lemma (5.3)$'$, and involves no new ideas, so it will be omitted.

At last we are ready to prove:

Theorem (5.12) *Every factor of the system $(X, \sigma|_X)$ has infinite topological entropy.*

Proof Recall the notations and the definition of entropy given at the beginning of subsection (4.3).

Again, as in Theorem (2.1), we may consider only factors of the form $(Y, \sigma|_Y)$, Y a subset of $[0,1]^{\mathbf{Z}}$, with a factor map $\hat{\phi}$ generated from a continuous function $\phi: X \to [0,1]$ (with both 0 and 1 in its image) as in (1.2.1).

There is an N_0 such that for any two points $x, y \in X$, if $x|_{-N_0}^{N_0} = y|_{-N_0}^{N_0}$ then $|\phi(x) - \phi(y)| < \frac{1}{20}$. Since the points of $X \cap L_\infty$ are dense in X, we can find x^0 and $x^1 \in X \cap L_\infty$ such that $\phi(x^0) < \frac{1}{20}$, $\phi(x^1) > \frac{19}{20}$.

Let $v_0' = x^0|_{-N_0}^{N_0}$, and in the same way define the word v_1' using x^1. Both v_0' and v_1' are of course in $\mathcal{W}(X \cap L_\infty)$, and so both of them are substrings of $w_{n'}$ for an n' large enough. Suppose that v_i' begins at letter s_i of $w_{n'}$ for $i = 0, 1$. Set for $i = 0, 1$

$$p_i = \underbrace{* * \cdots *}_{s_{1-i} \text{ times}} \cdot w_{n'} \cdot \underbrace{* * \cdots *}_{s_i \text{ times}}.$$

Both p_0 and p_1 satisfy the conditions of Lemma (5.5) for $n = n'$, and so there are legal words $v_0 \in \langle p_0 \rangle$ and $v_1 \in \langle p_1 \rangle$ of common length $l \stackrel{\text{def}}{=} l(w_{n'}) + s_0 + s_1$. These words satisfy $v_i|_{s_0+s_1}^{s_0+s_1+2N_0} = v_i'$ for $i = 0, 1$ and so we see there is (an easily computable) s such that for any $x \in X$,

$$x|_{-s}^{-s+l-1} = v_i \implies \begin{cases} \phi(x) < 1/10, & \text{if } i = 0, \\ \phi(x) > 9/10, & \text{if } i = 1. \end{cases}$$

According to Lemma (5.3), there exists a legal path, which we will denote by $v(t)$, $0 \le t \le 1$, such that $v(0) = v_0$ and $v(1) = v_1$.

Using Lemma (5.11), and since we know $l < t_{n'} - 2l(w_{n'})$, for every $s_1, \dots, s_{r_1} \in [0,1]$ with

$$r_1 = \left\lfloor \frac{t_{n'+1} - 2l(w_{n'+1})}{t_{n'}} \right\rfloor$$

the word

$$\eta_1(s_1, \dots, s_{r_1}) \overset{\text{def}}{=} v(s_1) \cdot w_{n_0} \cdot v(s_2) \cdots w_{n_0} \cdot v(s_{r_1})$$

is legal. Let $\tau_1, \dots, \tau_{r_1}$ be the indices of the first letter of $v(s_1), \dots, v(s_{r_1})$ respectively. By definition of legality, $\bar{w}_\infty \cdot \eta_1(s_1, \dots, s_{r_1}) \cdot w_\infty$ (the 0^{th} coordinate is the last letter of $\bar{w}_\infty$) is in X. If we take the projection of

$$\hat{\phi}(\bar{w}_\infty \cdot \eta_1(s_1, \dots, s_{r_1}) \cdot w_\infty)$$

to the coordinates $\tau_1 + s, \dots, \tau_{r_1} + s$ we get a function from $[0,1]^{r_1}$ to $[0,1]^{r_1}$ satisfying the conditions of Lemma (2.2). Thus the whole cube $[\frac{1}{4}, \frac{3}{4}]^{r_1}$ is in its image. Since there are m^{r_1} points that differ from each other in at least one coordinate by more that $1/2m$ in this cube, we see that $s(Y; t_{n'+1}, 1/2m) \geq s(Y; \tau_{r_1} + s, 1/2m) \geq m^{r_1}$. We can continue in this way one more step. Set

$$r_2 = \left\lfloor \frac{t_{n'+2} - 2l(w_{n'+2})}{t_{n'+1}} \right\rfloor,$$

and let

$$\begin{aligned}\eta_2(s_1, s_2, \dots, s_{r_2 r_1}) \overset{\text{def}}{=} \eta_1(s_1, \dots, s_{r_1}) \cdot w_{n'+1} \cdot \eta_1(s_{r_1+1}, \dots, s_{2r_1}) \cdots \\ \cdot w_{n'+1} \cdot \eta_1(s_{(r_2-1)r_1+1}, \dots, s_{r_2 r_1}).\end{aligned}$$

And in the same way as before show that $s(Y; t_{n'+2}, \frac{1}{2m}) \geq m^{r_2 r_1}$. We can proceed further in this way to show that if

$$r_i = \left\lfloor \frac{t_{n'+i} - 2l(w_{n'+i})}{t_{n'+i-1}} \right\rfloor$$

then

$$s(Y; t_{n'+i}, \frac{1}{2m}) \geq m^{r_i r_{i-1} \cdots r_1}.$$

Now since

$$t_{n'+i-1} \leq l(w_{n'+i}) = \frac{t_{n'+i}}{50 \cdot 2^{n'+i}},$$

$$\begin{aligned} r_i &\geq \left(1 - \frac{2}{50 \cdot 2^{n'+i}}\right) \frac{t_{n'+i}}{t_{n'+i-1}} - 1 \\ &\geq \left(1 - \frac{1}{2^{n'+i}}\right) \frac{t_{n'+i}}{t_{n'+i-1}}, \end{aligned}$$

and so we have

$$
\begin{aligned}
\frac{1}{t_{n'+i}} \log s\Big(Y; t_{n'+i}, \frac{1}{2m}\Big) &\geq \frac{r_i r_{i-1} \cdots r_1}{t_{n'+i}} \log m \\
&\geq \frac{1}{t_{n'}} \log m \prod_{i=1}^{\infty} \Big(1 - \frac{1}{2^{n'+i}}\Big) \qquad (5.12.1) \\
&\geq c \log m
\end{aligned}
$$

for a positive constant $c > 0$ that does not depend on i and m, and $c > 0$ since the product above converges. But (5.12.1) exactly says that $h_{1/2m}(Y) \geq c \log m$ and thus $h_{\mathrm{top}}(Y) = \infty$. □

6. Two minimal systems with a common almost 1-1 extension but no common factors

In this section we present an application, due to S. Glasner, of the results of the previous section.

We will say a map $\phi: X \to Y$ is almost 1-1 if the set $Y_0 \subset Y$ of points y such that the set $\phi^{-1}(y)$ consists of exactly one point is dense. It is a well known result of elementary topology that the set Y_0 so defined is a G_δ set for any map ϕ, and so if ϕ is almost 1-1 then $|\phi^{-1}(y)| = 1$ on a dense G_δ subset of Y. We will say (Z, R) is an almost 1-1 extension of (X, T) if (X, T) is a factor of (Z, R) and the factor transformation $Z \to X$ is almost 1-1.

Let $(\bar{X}, \bar{T})$ denote the minimal system we have constructed in the previous section (or any other minimal system with no finite entropy factors). We wish to find an almost 1-1, zero dimensional extension of $\bar{X}$. The existence of such an extension is part of the folklore on almost 1-1 extensions (see [2]). However, there does not appear to be any published proof of this fact.

Lemma (6.1) *Let (X, T) be a minimal system, and assume X is infinite. Then X has a minimal, almost 1-1, zero dimensional extension (Z, R).*

Proof X is, by assumption, a compact metric space. Thus, since (X, T) is minimal and infinite, X has no isolated points. It is well known that under these circumstances there exists an almost 1-1 map ϕ from the cantor set $C = \{0, 1\}^{\mathbf{N}}$ onto X. Let X_0 be the set of points where $|\phi^{-1}(x)| = 1$. Recall that X_0 is a dense G_δ subset of X, by definition of ϕ.

The system Z we will construct will be a subsystem of $C^{\mathbf{Z}}$ with the shift operation σ. We first define

$$Z' = \left\{(\dots, c_{-1}, c_0, c_1, \dots) \in C^{\mathbf{Z}} : \forall i \in \mathbf{Z} \quad \phi(c_i) = T^i \phi(c_0)\right\}.$$

The map $\bar{\phi} : Z' \to X$, defined by

$$\bar{\phi}: (\dots, c_0, \dots) \mapsto \phi(c_0),$$

is a factor transformation $Z' \to X$.

Now, any dynamical system (and in particular Z') has a minimal subsystem. Indeed, any subsystem of Z' with no subsystems is minimal, and such subsystems exist by Zorn's Lemma. Let $(Z, \sigma|_Z)$ be a minimal subsystem of Z'.

$\bar{\phi}(Z)$ is a nonempty, closed and T invariant subset of X. Thus, since X is minimal $\bar{\phi}(Z) = X$ and Z is an extension of X. Clearly Z is zero dimensional, so we only need to prove that $\bar{\phi}$ is almost 1-1. Since X is minimal, we only need to prove $|\bar{\phi}^{-1}(x_0)| = 1$ for some point $x_0 \in X$.

Let

$$x_0 \in \bigcap_{i=-\infty}^{\infty} T^i(X_0).$$

This set is nonempty since it is the intersection of a countable number of dense G_δ sets. Suppose c and d are two distinct points of Z' such that

$$\bar{\phi}(c) = \bar{\phi}(d) = x_0.$$

Then for some i,

$$c_i \neq d_i \qquad \text{but} \qquad \phi(c_0) = \phi(d_0) = x_0$$

so that

$$\phi(c_i) = T^i\phi(c_0) = T^i(x_0) = T^i\phi(d_0) = \phi(d_i),$$

and so $T^i(x_0) \notin X_0$ — a contradiction. □

Let (Z, R) be a minimal, almost 1-1, zero dimensional extension of $\bar{X}$. It is not hard to modify slightly the proof of Theorem (4.10) and show that any finite dimensional dynamical system has a factor with any positive entropy we wish (smaller than the entropy of the original system) such that the factor transformation is almost 1-1. For the very special case of a zero dimensional minimal systems we can give a direct proof of this fact, due to B. Weiss:

Theorem (6.2) *Any zero dimensional dynamical system (Z, R) has, for any $\eta > 0$, a factor (Y, S) with entropy $h_{\text{top}}(Y) < \eta$ such that the factor transformation is almost 1-1.*

Proof Take any $z \in Z$. Since Z is minimal it suffices to show that the factor transformation $Z \to Y$ is 1-1 at z. Let d be the metric on Z.

Pick some large integer N. Let

$$\delta' = \tfrac{1}{2} \min_{i=1,\dots,N} d(z, T^i z). \tag{6.2.1}$$

Choose any $\delta > 0$ small enough so that for any x_1 and x_2, and any $-N \leq n \leq N$, if $d(x_1, x_2) < \delta$ then $d(T^n x_1, T^n x_2) < \delta'$.

Since Z is zero dimensional, there is some clopen set U such that

$$z \in U \subset B(z, \delta).$$

Notice that for any $0 < k \leq N$,

$$U \cap T^k(U) = \emptyset.$$

For assume $u \in U \cap T^k(U)$. Then $d(u, z) < \delta$ and $d(T^{-k}u, z) < \delta$. Thus we see that $d(u, T^k z) < \delta'$ and so

$$d(z, T^k z) < 2\delta',$$

in contradiction to (6.2.1).

Define the map $\phi : Z \to \{0,1\}^{\mathbf{Z}}$ by

$$\phi : z \mapsto (\dots 1_U(T^{-1}z), 1_U(z), 1_U(Tz) \dots).$$

Since U is clopen, ϕ is continuous. Any infinite word of 0's and 1's in $\phi(X)$ must have at least N zero's between every two occurrences of 1's, and so it is immediate that by taking N large enough the entropy of the factor $\phi(Z)$ can be made as small as we wish. Notice also that if $d(x, z) > \delta$ then $\phi(x) \neq \phi(z)$.

To summarize, we have so far seen that for every ϵ, $\delta > 0$, there exists a factor $Y_{\epsilon,\delta}$ of Z such that $h_{\text{top}}(Y_{\epsilon,\delta}) < \epsilon$, and such that if $d(x, z) > \delta$ then x and z are mapped into distinct points of $Y_{\epsilon,\delta}$.

For every $n > 0$, let $Y_n = Y_{2^{-n}\eta, n^{-1}}$, and let ϕ_n be the associated factor transformations. Define $\phi_\infty : Z \to \prod_{n=1}^{\infty} Y_n$ by

$$\phi_\infty : x \mapsto (\phi_1(x), \phi_2(x), \dots),$$

and set $Y = \phi_\infty(Z)$.

Clearly,

$$h_{\text{top}}(Y) \leq \sum_{n=1}^{\infty} h_{\text{top}}(Y_n) < \eta,$$

and if $x \neq z$ then, for every n large enough, $\phi_n(x) \neq \phi_n(z)$ and so $\phi_\infty(x) \neq \phi_\infty(z)$. □

Thus Z has a finite entropy factor (Y, S) with an almost 1-1 factor transformation. Every factor of $(\bar{X}, \bar{T})$ has infinite entropy, every factor of (Y, S) finite entropy. So these two minimal systems have a common almost 1-1 extension but no common factors, answering the question posed by H. Furstenberg.

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