# ON ELLIPTIC MONGE-AMPÈRE EQUATIONS AND WEYL'S EMBEDDING PROBLEM

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## Introduction.

In this paper we shall deal with Monge-Ampère equations of the following kind:

$$(1) \qquad F \equiv A(x, y, z, p, q)r + 2B(x, y, z, p, q)s + C(x, y, z, p, q)t + (rt - s^2) - E(x, y, z, p, q) = 0$$

$$(p = z_x, \quad q = z_y; \quad r = z_{xx}, \quad s = z_{xy}, \quad t = z_{yy}).$$

We assume that the equation is elliptic for the given solution $z = z(x, y)$; i. e., the inequality

$$(2) \qquad \Delta = AC - B^2 + E > 0$$

holds.

Equations of this type frequently occur in problems of differential geometry in the large. In this connection the problems of Minkowski and Weyl [1] deserve particular attention, since they stimulated many researches on this subject. They lead to the following Monge-Ampère equations which in terms of local parameters $(u, v)$ take the form:

$$(3) \qquad rt - s^2 = f(u, v) > 0,$$

$$(4) \qquad \left(\rho_{uu} - \left\{{}^{1}{}_{1}{}^{1}\right\}\rho_u - \left\{{}^{1}{}_{2}{}^{1}\right\}\rho_v\right)\left(\rho_{vv} - \left\{{}^{2}{}_{1}{}^{2}\right\}\rho_u - \left\{{}^{2}{}_{2}{}^{2}\right\}\rho_v\right) - \left(\rho_{uv} - \left\{{}^{1}{}_{1}{}^{2}\right\}\rho_u - \left\{{}^{1}{}_{2}{}^{2}\right\}\rho_v\right)^2 = K(u, v)\left[EG - F^2 - (G\rho_u^2 - 2F\rho_u\rho_v + E\rho_v^2)\right].$$

In the last equation, which is due to Darboux [7], $\left\{{}^{1}{}_{1}{}^{1}\right\}, \dots, \left\{{}^{2}{}_{2}{}^{2}\right\}$ are the Christoffel symbols with respect to the line element

$$(5) \qquad ds^2 = E(u, v)\,du^2 + 2F(u, v)\,du\,dv + G(u, v)\,dv^2.$$

Furthermore $K(u, v)$ is the Gauss curvature of $ds^2$, which is assumed to be positive, and $\rho(u, v)$ stands for any component of the position vector

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1. For a detailed account on the history of these problems see Nirenberg [25]. A. D. Alexandrow's work on Weyl's embedding problem is described by Efimow [9] and Pogorelow [27]. For a discussion of Weyl's problem see also Wintner [32].

(6) $$X(u, v) = (x(u, v), y(u, v), z(u, v))$$

of the surface under consideration.

In solving the problems of Minkowski and Weyl by a continuity method [2] one is confronted with the problem of obtaining suitable bounds for the second and higher order derivatives of the solutions of (3) and (4) in any compact subset of the parameter domain (so-called a priori estimates). In order to get a precise formulation of this question let us impose the following conditions on the coefficients of the Monge-Ampère equation (1) and the solution $z = z(x, y)$:

(i) The function $z(x, y)$ is of class $C^{3+\varkappa}$ $(0 < \varkappa < 1)$ in a domain $\Omega$ of the $xy$-plane and satisfies (1), where the coefficients

$$A = A(x, y, z, p, q), \ldots, E = E(x, y, z, p, q)$$

are of class $C^2$ in an open set of the 5-dimensional space containing the hypersurface:

(7) $$\Sigma = \{(x, y, z(x, y),\ p(x, y),\ q(x, y)),\ (x, y) \in \Omega\}.$$

(ii) The coefficients $A, \ldots, E$ satisfy for $(x, y, z, p, q) \in \Sigma$ the inequalities:

(8) $$|A|, \ldots, |E| \leqq \alpha_0,$$

(9) $$\Delta = AC - B^2 + E \geqq \alpha_0^{-1},$$

(10) $$|A_x|, \ldots, |E_q| \leqq \alpha_1,$$

(11) $$|A_{xx}|, \ldots, |E_{qq}| \leqq \alpha_2,$$

where $\alpha_0, \alpha_1, \alpha_2$ are finite positive constants.

(iii) For the solution $z(x, y)$ of (1) we have the estimates:

(12) $$|z(x, y)| \leqq \gamma_0,$$

(13) $$|p(x, y)|,\ |q(x, y)| \leqq \gamma_1,$$

where $(x, y) \in \Omega$ and $\gamma_0, \gamma_1$ are finite positive numbers.

Now let $d$ be a positive number and $\Omega_d$ the set of all points in $\Omega$ whose distance from the boundary of $\Omega$ exceeds $d$. Then the main problem, with which we are concerned in the present paper, consists in establishing inequalities of the form

(14) $$|r|, |s|, |t| \leqq k_0(\alpha_0, \alpha_1, \alpha_2, \gamma_0, \gamma_1, d) < \infty,$$

and

2. See Weyl [31], Nirenberg [25], and Lewy [18], [19].

$$(15) \qquad |r_x|, \dots, |t_y| \leqq k_1(\alpha_0, \alpha_1, \alpha_2, \gamma_0, \gamma_1, d) < \infty,$$

valid for $(x, y) \in \Omega_d$. (3)

The first decisive approach to this problem was made by H. Lewy [16], [17] in the case of analytic Monge-Ampère equations. In [16] he studies the analogous problem for the more general Monge-Ampère equation

$$(16) \qquad Ar + 2Bs + Ct + D(rt - s^2) = E$$

under the additional assumption that the coefficients $A, \dots, E$ are analytic in a complex neighborhood of $\Sigma$, and a bound for the moduli of the second order derivatives of the solution is known, say $\gamma_2$. From these data he derives estimates for the third and higher order derivatives of $z(x, y)$. In [17] he carries the analysis still further by removing the condition

$$(17) \qquad |r|, |s|, |t| < \gamma_2, (x, y) \in \Omega,$$

and thus proves a general theorem of compactness for a certain class of Monge-Ampère equations under the assumption that the first derivatives of the solutions are uniformly bounded. However he does not give an explicit estimate for their second derivatives. As a principal tool for establishing his results he makes extensive use of the characteristic equations associated with (1) and (16), which furnish the analytic continuation of the original solutions into the complex domain. He also observes in [17] that an a priori limitation of the second derivatives is impossible for the general Monge-Ampère equation (1). (4)

Applying his results to the equation (3) and an equation of the Darboux Type (4) (5) he solves the problem of Minkowski and Weyl in the case of analytic data (see [18] and [19]).

Since the papers of H. Lewy much effort has been concentrated on the question of whether the condition of analyticity in the equations (1) and (16) can be dropped. The first decisive step in this direction was made

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3. No assumptions are made regarding the boundary values of $z$. Estimates for the derivatives of $z(x, y)$, which are valid in the closure of $\Omega$ and depend on the smoothness of boundary values and the boundary curve of $\Omega$, were given by Bernstein [3], Leray [15], and Pogorelow [28].

4. See the example on p. 35 of this paper. This phenomenon is remarkable in view of the fact that the characteristic equations of (1) do not involve the second derivatives of the solution $z(x, y)$.

5. See the remark at the end of this paper.

by Nirenberg [24] and Pogorelow [27], [28] [6] who considered the general elliptic equation in two variables:

$$F(x, y, z, p, q, r, s, t) = 0, \quad F_r F_t - \frac{1}{4} F_s^2 > 0. \tag{18}$$

So far as the estimation of the third derivatives is concerned, the sharpest result is due to Nirenberg, and is as follows: Let $z = z(x, y)$ be of class $C^2$ and a solution of (18), where the function $F$ has continuous first derivatives in an open set of the 8-dimensional space containing the hypersurface

$$T = \{(x, y, z(x, y),\ p(x, y),\ q(x, y),\ r(x, y),\ s(x, y),\ t(x, y)),\ (x, y) \in \Omega\} \tag{19}$$

and satisfies for $(x, y, z, p, q, r, s, t) \in T$ the inequalities

$$|F_x|, \ldots, |F_t| \leqq \alpha < \infty, \tag{20}$$

$$F_r F_t - \frac{1}{4} F_s^2 \geqq \beta > 0. \tag{21}$$

Furthermore, let the estimates (13) and (17) hold. Then the second derivatives of $z(x, y)$ satisfy for $(x, y) \in \Omega_d$ a Hölder condition whose coefficient and exponent depend only on $\alpha, \beta, \gamma_1, \gamma_2$, and $d$. [7]

This theorem immediately gives us a partial answer to our main problem, for on applying the well-known Schauder estimates for linear equations [8] it follows that under the hypotheses (i)—(iii) an inequality of the type (15) is established once we can prove (14). [9] Furthermore on account of a differentiability theorem of E. Hopf [13] the assumption that $z(x, y)$ be of class $C^{3+\varkappa}$ $(0 < \varkappa < 1)$ is implied by the fact that $z(x, y)$ has continuous first and second derivatives. [10]

The methods of Nirenberg and Pogorelow are essentially different in character. Nirenberg's results, which are based on the earlier work of Morrey [23], can be regarded as theorems on linear elliptic equations with

6. For an English summary of part of Pogorelow's results see [26].

7. See Nirenberg [24], Theorem I.

8. See [29], Theorem 1. Simplified proofs have been given by Miranda [22], Chapter V, and Douglis-Nirenberg [8].

9. The corresponding result of Pogorelow (see [27], Appendix 2) is somewhat weaker in the sense that it furnishes estimates for the third order derivatives of the solutions of (18) depending also on the bounds for the third order derivatives of $F$ on $T$.

10. See Nirenberg [24], Theorem 3.

bounded coefficients and thus yield information concerning the differentiability of the solutions of (18). On the other hand Pogorelow makes strong use of the non-linear character of the equations under consideration and thus derives even bounds for the second derivatives of the Monge-Ampère equations (3) and (4). Completing the earlier work of A. D. Alexandrow on the Weyl embedding problem he considers in [27], Chapter IV, § 4, a convex cap with a one-to-one projection on the plane $z=0$ and a planar boundary; i. e., a vector $X(u,v)=(x(u,v), y(u,v), z(u,v))$ defined for $(u,v)\in\Omega$ such that $z(u,v)\neq 0$, $x_u y_v - x_v y_u \neq 0$ for $(u,v)\in\Omega$ and $z=0$ on the boundary of $\Omega$. Applying his specific method of auxiliary functions to the Darboux equation (4) he derives appropriate estimates for the quantities $|X_{uu}|, \ldots, |X_{vv}|$ in terms of the line element $ds^2$ of the surface $X=X(u,v)$. His result is contained in Theorem 5 of the present paper. Furthermore, with the aid of a geometrical theorem on convex surfaces, due to A. D. Alexandrow [2], he proves in [28], § 2, the existence of a finite positive number $k=k(\alpha_0, \alpha_1, \alpha_2, \gamma_0, d)$ such that the second derivatives of the solution of (3) satisfy the inequality

$$|r|, |s|, |t| \leqq k(\alpha_0, \alpha_1, \alpha_2, \gamma_0, d) \tag{22}$$

for $(u,v)\in\Omega_d$. In this connection mention should be made of the well-known inequalities of Weyl [31] and Miranda [21].(11) However, since they are established for the solutions of the corresponding differential equations defined on the closed unit sphere, they do not yield interior estimates for the associated Monge-Ampère equations (3) and (4) in the sense described above.

The method adopted in the present paper for estimating the solutions of the Monge-Ampère equation (1) can be regarded as a further development of the ideas of H. Lewy and is thus different from the methods of Nirenberg and Pogorelow. Now since the equation (1) ceases to be analytic, we can no longer set up the corresponding characteristic equations in the complex domain, as Lewy has done, but must restrict ourselves to real variables. Let $z=z(x,y)$ be a solution of (1) and let the hypothesis (i)—(iii) be

11. For a proof of these inequalities see also Nirenberg [25] § 10 and § 16. An estimate analogous to that of Miranda has been derived by Pogorelow [28] in § 5 of this paper. A thorough treatment of the inequalities of Weyl and Miranda and their corresponding identities has recently been given by Wintner [33].

satisfied. Then on account of the uniformization theorem we can introduce real parameters $(u, v)$ in the large, such that the equation

$$\varphi \equiv F_t dx^2 - F_s dx\, dy + F_r dy^2 = \lambda (du^2 + dv^2) \tag{23}$$

holds identically in $dx, dy$. More precisely, we can determine two real-valued functions $x(u, v)$ and $y(u, v)$, such that the following conditions are satisfied:

(i′) $x = x(u, v)$ and $y = y(u, v)$ map the unit disc $u^2 + v^2 \leqq 1$ homeomorphically onto the disc

$$\Gamma = \{(x - x_0)^2 + (y - y_0)^2 \leqq R^2\} \in \Omega \tag{24}$$

such that $x(0, 0) = x_0$ and $y(0, 0) = y_0$.

(ii′) For $u^2 + v^2 < 1$ the functions $x(u, v)$ and $y(u, v)$ belong to $C^2$. Furthermore, we have $\frac{\partial (x, y)}{\partial (u, v)} \neq 0$ and the representations:

$$\frac{A+t}{\sqrt{\Delta}} = \frac{x_u^2 + x_v^2}{x_u y_v - x_v y_u}, \quad \frac{B-s}{\sqrt{\Delta}} = \frac{x_u y_u + x_v y_v}{x_u y_v - x_v y_u}, \quad \frac{C+r}{\sqrt{\Delta}} = \frac{y_u^2 + y_v^2}{x_u y_v - x_v y_u}. \tag{25}$$

Obviously $\gamma = u + iv$ and $\gamma^* = u - iv$ are characteristic parameters with respect to the equation (1), and our method corresponds to that of solving the Cauchy problem for non-linear hyperbolic equations.[12] By some formal computations one can derive from (25) the following differential equations:

$$\begin{aligned} x_{uu} + x_{vv} = {} & h_1(x, y, z, p, q)(x_u^2 + x_v^2) + h_2(x, y, z, p, q)(x_u y_u + x_v y_v) \\ & + h_3(x, y, z, p, q)(y_u^2 + y_v^2) + h_4(x, y, z, p, q)(x_u y_v - x_v y_u), \end{aligned} \tag{26}$$

$$\begin{aligned} y_{uu} + y_{vv} = {} & \tilde{h}_1(x, y, z, p, q)(x_u^2 + x_v^2) + \tilde{h}_2(x, y, z, p, q)(x_u y_u + x_v y_v) \\ & + \tilde{h}_3(x, y, z, p, q)(y_u^2 + y_v^2) + \tilde{h}_4(x, y, z, p, q)(x_u y_v - x_v y_u), \end{aligned} \tag{26′}$$

where $h_1, \ldots, \tilde{h}_4$ can be expressed in terms of the coefficients of (1) and their first derivatives (see Lemma 3). Furthermore, on account of (i′) and (ii′) it is easy to establish the following inequality for the Dirichlet integral of the mapping functions (see Lemma 4):

$$\iint\limits_{u^2+v^2<1} (x_u^2 + x_v^2 + y_u^2 + y_v^2)\, du\, dv \leqq N(\alpha_0, \gamma_1, R) < \infty. \tag{27}$$

Suppose now that we wish to estimate the second and higher order

12. See Courant-Hilbert [6], Chapter V, especially pp. 344—345, also Hadamard [10], Appendix III.

derivatives of the solution at the point $(x_0, y_0)$. Then from the representations (25) it follows that this can be done provided that we succeed in determining upper bounds for the derivatives of the functions $x(u, v)$ and $y(u, v)$ and a positive lower bound for the modulus of the Jacobian of the mapping $(u, v) \to (x, y)$. Now since the mapping functions are solutions of the system (26), (26′) and satisfy hypothesis (i′) together with the inequality (27) we can apply our earlier results [12] and thus establish the following propositions:

Theorem 1. Let the hypotheses (i), (ii) and (iii) together with the inequality (17) be satisfied. Furthermore, let $\nu$ be a real number in the interval $0<\nu<1$. Then we can determine three finite positive numbers

$$\vartheta_0 = \vartheta_0(\alpha_0, \alpha_1, \gamma_1, \gamma_2, d, \nu),$$
$$\vartheta_1 = \vartheta_1(\alpha_0, \alpha_1, \alpha_2, \gamma_1, \gamma_2, d),$$
$$\vartheta_2 = \vartheta_2(\alpha_0, \alpha_1, \alpha_2, \gamma_1, \gamma_2, d, \nu)$$

such that the following estimates hold:

$$(28)\qquad \begin{aligned} &|r(x_1, y_1) - r(x_0, y_0)| \leqq \vartheta_0[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2} \\ &\qquad\cdot \\ &\qquad\cdot \\ &\qquad\cdot \\ &|t(x_1, y_1) - t(x_0, y_0)| \leqq \vartheta_0[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2} \end{aligned}$$

for $(x_0, y_0) \in \Omega_d$ and $(x_1, y_1) \in \Omega_d$,

$$(29)\qquad |r_x|, \ldots, |t_y| \leqq \vartheta_1 \quad \text{for} \quad (x, y) \in \Omega_d,$$

and

$$(30)\qquad \begin{aligned} &|r_x(x_1, y_1) - r_x(x_0, y_0)| \leqq \vartheta_2[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2} \\ &\qquad\cdot \\ &\qquad\cdot \\ &\qquad\cdot \\ &|t_y(x_1, y_1) - t_y(x_0, y_0)| \leqq \vartheta_2[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2} \end{aligned}$$

for $(x_0, y_0) \in \Omega_d$, $(x_1, y_1) \in \Omega_d$.

Theorem 2. Let the hypotheses (i)—(iii) be satisfied. Furthermore, let the coefficients $A, B, C$ be independent of $p$ and $q$. Then we have for $(x, y) \in \Omega_d$ estimates of the form:

$$(31)\qquad |r|, |s|, |t| \leqq \Theta(\alpha_0, \alpha_1, \gamma_1, d) < \infty.$$

Theorem 3. Let the hypotheses (i)—(iii) be satisfied. Furthermore, let the coefficients $h_1, \ldots, \tilde{h}_4$ occuring in the characteristic equations (26), (26′) be independent of $z$, $p$ and $q$ and satisfy for $(x, y) \in \Omega$ the inequalities

$$|h_1(x, y)|, \ldots, |\tilde{h}_4(x, y)| \leqq \beta_0, \tag{32}$$

$$\left|\frac{\partial h_1}{\partial x}\right|, \ldots, \left|\frac{\partial \tilde{h}_4}{\partial y}\right| \leqq \beta_1, \tag{33}$$

where $\beta_0$ and $\beta_1$ are finite positive constants. Then we have for $(x, y) \in \Omega_d$ the estimates

$$|r|, |s|, |t| \leqq \Lambda(\alpha_0, \beta_0, \beta_1, \gamma_1, d), \tag{34}$$

where $\Lambda$ is a finite positive constant.

As mentioned above, the estimates (29) and (30) of Theorem 1 are also consequences of Nirenberg's results. Regarding the a priori limitation of the second derivatives carried out in Theorem 2, it is remarkable that the bound $\alpha_2$ for the moduli of the second derivatives of the coefficients $A, \ldots, E$ does not enter into the inequality (31), in contrast to the estimates of Pogorelow [28] and Miranda [21]. This fact enables us to weaken the differentiability conditions in certain existence theorems (for instance, in Theorem 4 of Nirenberg [25]). Theorem 3 constitutes a generalization of one of Lewy's principal results (see [17], p. 373) to the case where the coefficients $A, \ldots, E$; $h_1, \ldots, \tilde{h}_4$ are no longer assumed to be analytic. If we apply Theorem 1 and 3 to the Darboux equation (4) we obtain the following esimates for a convex surface in terms of its metric $ds^2$:

Theorem 5. Hypotheses:

(i) The vector $X(u, v) = (x(u, v), y(u, v), z(u, v))$ is of class $C^{3+\varkappa}$ $(0 < \varkappa < 1)$ in a domain $\Omega$ of the $uv$-plane and satisfies the equations

$$X_u^2 = E, \quad X_u X_v = F, \quad X_v^2 = G. \tag{35}$$

(ii) The functions $E(u, v)$, $F(u, v)$ and $G(u, v)$ are of class $C^4$ and satisfy for $(u, v) \in \Omega$ the inequalities

$$\begin{aligned} &|E|, \ldots, |G| \leqq \alpha, \\ &|E_u|, \ldots, |G_v| \leqq \alpha, \\ &|E_{uu}|, \ldots, |G_{vv}| \leqq \alpha, \\ &|E_{uuu}|, \ldots, |G_{vvv}| \leqq \alpha, \\ &|E_{uuuu}|, \ldots, |G_{vvvv}| \leqq \alpha, \end{aligned} \tag{36}$$

(37) $$EG - F^2 \geqq \alpha^{-1},$$

and

(38) $$K(u, v) \geqq \alpha^{-1},$$

where $K(u,v)$ is the Gauss curvature of the line element $ds^2 = E\,du^2 + 2F du\,dv + G dv^2$, and $\alpha$ is a fixed positive constant.

(iii) There exists a fixed unit vector $Z$, such that the inequality

(39) $$|(Z, \bar{X})| \geqq \gamma > 0$$

holds for $(u,v) \in \Omega$, where $\bar{X}$ represents the unit normal of the surface $X = X(u,v)$.

Conclusion: Let $\nu$ be a real number in the interval $0 < \nu < 1$. Then there exist three positive numbers

$\tau_0 = \tau_0(\alpha, \gamma, d)$, $\tau_1 = \tau_1(\alpha, \gamma, d)$ and $\tau_2 = \tau_2(\alpha, \gamma, d, \nu)$

such that the following estimates hold:

(40) $$|X_{uu}|, \ldots, |X_{vv}| \leqq \tau_0$$

(41) $$|X_{uuu}|, \ldots, |X_{vvv}| \leqq \tau_1$$

for $(u, v) \in \Omega_d$,

and

(42)
$$\begin{aligned} &|X_{uuu}(u_1, v_1) - X_{uuu}(u_0, v_0)| \leqq \tau_2 [(u_1 - u_0)^2 + (v_1 - v_0)^2]^{\nu/2} \\ &\quad \vdots \\ &|X_{vvv}(u_1, v_1) - X_{vvv}(u_0, v_0)| \leqq \tau_2 [(u_1 - u_0)^2 + (v_1 - v_0)^2]^{\nu/2} \end{aligned}$$

for $(u_0, v_0) \in \Omega_d$ and $(u_1, v_1) \in \Omega_d$.[12a]

In the case where the line element $ds^2$ is invariantly defined on the unit sphere, we can eliminate the condition (39) and thus obtain the estimates which are required for an analytical treatment of Weyl's embedding problem (see Theorem 6). Apart from some occasional modifications the procedure for establishing the inequalities in quesion corresponds to that of H. Lewy [18]. Since no detailed presentation of Lewy's method is available, it has been felt desirable to give a thorough treatment of the characteristic equations associated with (1) and especially with the Darboux equation (4). Incidentally we thus re-establish a classical result of Darboux [7]. This is done in sections 2.1 and 3.1. Furthermore, for the sake of comple-

12a. In the corresponding theorem of Pogorelow quoted above $\Omega_d$ is replaced by the subset of points in $\Omega$ where $|(Z, X)| > d$, and it is assumed that $(Z, X)$ vanishes for $(u, v) \in \dot{\Omega}$.

teness, a proof of the fundamental theorem of conformal mapping with respect to a Riemannian metric is given in § 1, which makes the paper independent of others. The principal results, part of which have already been published in [11], are contained in Sections 2.2 and 3.2.

## § 1. CONFORMAL MAPPING WITH RESPECT TO A RIEMANNIAN METRIC

### 1.1. The Beltrami Equations.

Consider a pair of functions $u(x, y)$ and $v(x, y)$, which are of class $C'$ in a domain $\Omega$ of the $xy$-plane and satisfy the Beltrami system

$$\left\{ \begin{aligned} u_x &= \frac{-bv_x + av_y}{\sqrt{ac - b^2}} \\ u_y &= \frac{-cv_x + bv_y}{\sqrt{ac - b^2}} \,. \end{aligned} \right. \tag{1.1.1}$$

Here $a(x, y)$, $b(x, y)$, $c(x, y)$ denote three continuous functions in $\Omega$ such that $ac - b^2 > 0$. If we furthermore assume that

$$v_x^2 + v_y^2 > 0 \tag{1.1.2}$$

holds, then we have

$$a(u_x v_y - u_y v_x) = a \frac{cv_x^2 - 2bv_x v_y + av_y^2}{\sqrt{ac - b^2}} > 0 \,, \tag{1.1.3}$$

and

$$\varphi = adx^2 + 2bdx\,dy + cdy^2 = \frac{\sqrt{ac - b^2}}{u_x v_y - u_y v_x} (du^2 + dv^2) \tag{1.1.4}$$

identically in $dx$, $dy$.

From (1.1.3) and (1.1.4) it follows that to each point $(x_0, y_0) \in \Omega$ there exists a neighborhood which is mapped conformally with respect to the metric $\varphi$ onto a domain of the $uv$-plane. If $u^*(x, y)$ and $v^*(x, y)$ is another pair of functions satisfying the above conditions, then we have

$$(du^*)^2 + (dv^*)^2 = \lambda (du^2 + dv^2) \,, \tag{1.1.5}$$

and

$$\frac{\partial(u^*, v^*)}{\partial(u, v)} > 0 \,, \tag{1.1.5'}$$

hence

$$u^* + iv^* = \psi(u + iv) \,, \tag{1.1.6}$$

where $\psi$ is an ordinary analytic function with non-vanishing derivative.

For later purposes it is necessary to consider the inverse functions $x(u, v)$ and $y(u, v)$. In contrast to $u(x, y)$ and $v(x, y)$ they satisfy a simple non-linear system of differential equations of the type considered in [12]. In order to derive these equations we first observe that (1.1.1) can also be written in the form:

$$(1.1.7) \qquad \begin{cases} v_x = \dfrac{bu_x - au_y}{\sqrt{ac - b^2}} \\[2ex] v_y = \dfrac{cu_x - bu_y}{\sqrt{ac - b^2}} . \end{cases}$$

Now we have

$$(1.1.8) \qquad u_x = \delta y_v , \quad u_y = -\delta x_v ; \quad v_x = -\delta y_u , \quad v_y = \delta x_u ,$$

where $\delta = \dfrac{\partial(u, v)}{\partial(x, y)}$. Inserting this into (1.1.1) and (1.1.7) we obtain

$$(1.1.9) \qquad \begin{cases} x_u = \dfrac{bx_v + cy_v}{\sqrt{ac - b^2}} \\[2ex] x_v = -\dfrac{bx_u + cy_u}{\sqrt{ac - b^2}} \end{cases}$$

and

$$(1.1.9') \qquad \begin{cases} y_u = -\dfrac{ax_v + by_v}{\sqrt{ac - b^2}} \\[2ex] y_v = \dfrac{ax_u + by_u}{\sqrt{ac - b^2}} \end{cases}$$

To proceed further we make the additional assumption that the coefficients $a(x, y)$, $b(x, y)$ and $c(x, y)$ are of class $C^1$ and the mapping functions $x(u, v)$ and $y(u, v)$ belong to $C^2$. Differentiating (1.1.9) and (1.1.9′) with respect to $u$ and $v$ we obtain the following differential equations:

$$(1.1.10) \quad \Delta x = \left[\frac{\partial}{\partial x}\left(\frac{c}{\sqrt{ac - b^2}}\right) - \frac{\partial}{\partial y}\left(\frac{b}{\sqrt{ac - b^2}}\right)\right](x_u y_v - x_v y_u),$$

$$(1.1.10') \quad \Delta y = \left[\frac{\partial}{\partial y}\left(\frac{a}{\sqrt{ac - b^2}}\right) - \frac{\partial}{\partial x}\left(\frac{b}{\sqrt{ac - b^2}}\right)\right](x_u y_v - x_v y_u).$$

Furthermore, from (1.1.9) we derive the representations:

$$
(1.1.11) \qquad
\begin{aligned}
\frac{a}{\sqrt{ac-b^2}} &= \frac{y_u^2+y_v^2}{x_u y_v - x_v y_u}, \\
\frac{b}{\sqrt{ac-b^2}} &= -\frac{x_u y_u + x_v y_v}{x_u y_v - x_v y_u}, \\
\frac{c}{\sqrt{ac-b^2}} &= \frac{x_u^2+x_v^2}{x_u y_v - x_v y_u},
\end{aligned}
$$

which are in fact equivalent to (1.1.1).

## 1.2. Existence Proof.

We shall now show that it is possible to map a circular disc, whose closure is contained in $\Omega$, conformally with respect to the metric $\varphi$ onto the unit disc. In the case of an analytic metric $\varphi$ this is an immediate consequence of the theorem of Cauchy-Kowalewski and the uniformization theorem. In order to extend this result to the case of a non-analytic metric $\varphi$, we shall approximate $\varphi$ by a sequence of analytic metrics $\varphi_n$ and then apply the results obtained in [12].[13] It is thus possible to avoid the classical theorems of Korn [14] and Lichtenstein [20] on the existence of isothermal parameters in the small. Another proof, based on a different method, is due to Ahlfors [1].

**Lemma 1.** *Let the functions $a(x,y)$, $b(x,y)$ and $c(x,y)$ be analytic in $\Omega$ (i.e., regular power series in the vicinity of each point in $\Omega$) and satisfy the inequality $ac-b^2>0$. Then to each point $(x_0,y_0)\in\Omega$ there exists a pair of functions $u=u(x,y)$ and $v=v(x,y)$ which are analytic in a neighborhood of $(x_0,y_0)$ and satisfy the equations (1.1.1). Furthermore, we have*

$$
a(x_0,y_0)\,(u_x v_y - u_y v_x)_{\substack{x=x_0\\ y=y_0}} > 0 .
$$

**Proof.** Let $u(x,y)$ and $v(x,y)$ be a pair of analytic solutions of (1.1.1). Then by elimination of $u$ we obtain the following differential equation for $v$:

---

13. In the case of a metric $\varphi$ with bounded coefficients a similar approximation process has been carried out by Morrey using his results on quasi-conformal mappings (see [23], Theorem 3).

$$(1.2.1) \qquad \frac{\partial}{\partial x}\left(\frac{cv_x - bv_y}{\sqrt{ac-b^2}}\right) + \frac{\partial}{\partial y}\left(\frac{-bv_x + av_y}{\sqrt{ac-b^2}}\right) = 0 .$$

Conversely, if $v(x, y)$ is an analytic solution of (1.2.1) in a neighborhood of $(x_0, y_0)$, such that $(v_x^2 + v_y^2)_{\substack{x=x_0\\y=y_0}} > 0$, and if we set

$$(1.2.2) \qquad u(x, y) = -\int_{(x_0, y_0)}^{(x, y)} \left(\frac{bv_x - av_y}{\sqrt{ac-b^2}}\,dx + \frac{cv_x - bv_y}{\sqrt{ac-b^2}}\,dy\right),$$

where the path of integration is the straight segment joining the points $(x_0, y_0)$ and $(x, y)$, then the function $u(x, y)$ is uniquely defined in a neighborhood of $(x_0, y_0)$, and the equations (1.1.1) are satisfied. Furthermore, according to (1.1.3), we have

$$a(x_0, y_0)\,(u_x v_y - u_y v_y)_{\substack{x=x_0\\y=y_0}} > 0 .$$

If we write (1.2.1) in the form

$$(1.2.3) \qquad v_{xx} = \frac{2b}{c} v_{xy} - \frac{a}{c} v_{yy}$$
$$+ \frac{\sqrt{ac-b^2}}{c}\left[-\frac{\partial}{\partial x}\left(\frac{c}{\sqrt{ac-b^2}}\right) + \frac{\partial}{\partial y}\left(\frac{b}{\sqrt{ac-b^2}}\right)\right] v_x$$
$$+ \frac{\sqrt{ac-b^2}}{c}\left[\frac{\partial}{\partial x}\left(\frac{b}{\sqrt{ac-b^2}}\right) - \frac{\partial}{\partial y}\left(\frac{a}{\sqrt{ac-b^2}}\right)\right] v_y$$

and prescribe the initial conditions

$$(1.2.4) \qquad v(x_0, y) = 0, \quad v_x(x_0, y) = 1,$$

then the assertion of the lemma follows from the theorem of Cauchy-Kowalewski.[14]

Lemma 2. *Let the functions* $a(x, y)$, $b(x, y)$ *and* $c(x, y)$ *be of class* $C^{1+\varkappa}$ $(0<\varkappa<1)$[15] *in a domain* $\Omega$ *and satisfy the inequality* $ac-b^2>0$. *Furthermore let the disc*

$$D = \{(x-x_0)^2 + (y-y_0)^2 \leqq R^2\}$$

*be contained in* $\Omega$.

*Then there exists a pair of functions* $x(u, v)$ *and* $y(u, v)$ *with the following properties:*

---

14. See, for instance, Hadamard [10], pp. 14—19.

15. A function is said to be of class $C^{k+\nu}$ in $\Omega$, $(k=0, 1, \ldots; 0 \leqq \nu < 1)$ if it has continuous derivatives up to the $k^{th}$ order, which satisfy Hölder conditions of exponent $\nu$ in any compact subset of $\Omega$.

(i) $x=x(u, v)$ and $y=y(u, v)$ map the unit disc $u^2+v^2 \leqq 1$ homeomorphically onto the disc $D$, such that $x(0, 0)=x_0$ and $y(0, 0)=y_0$.

(ii) For $u^2+v^2<1$ the functions $x(u, v)$ and $y(u, v)$ are of class $C^2$ and satisfy the differential equations:

$$(1.2.5) \qquad \Delta x = \left[\frac{\partial}{\partial x}\left(\frac{c}{\sqrt{ac-b^2}}\right) - \frac{\partial}{\partial y}\left(\frac{b}{\sqrt{ac-b^2}}\right)\right](x_u y_v - x_v y_u),$$

$$(1.2.5') \qquad \Delta y = \left[\frac{\partial}{\partial y}\left(\frac{a}{\sqrt{ac-b^2}}\right) - \frac{\partial}{\partial x}\left(\frac{b}{\sqrt{ac-b^2}}\right)\right](x_u y_v - x_v y_u).$$

(iii) For $u^2+v^2<1$ we have $x_u y_v - x_v y_u \neq 0$, and

$$(1.2.6) \qquad \begin{aligned} \frac{a}{\sqrt{ac-b^2}} &= \frac{y_u^2+y_v^2}{x_u y_v - x_v y_u}, \\ \frac{b}{\sqrt{ac-b^2}} &= -\frac{x_u y_u + x_v y_v}{x_u y_v - x_v y_u}, \\ \frac{c}{\sqrt{ac-b^2}} &= \frac{x_u^2+x_v^2}{x_u y_v - x_v y_u}. \end{aligned}$$

Proof. (I) We first prove the assertion under the additional assumption that the functions $a(x, y)$, $b(x, y)$ and $c(x, y)$ are analytic in the disc $D$. According to Lemma 1 we can determine to each point $(x^*, y^*) \in D$ a positive number $\varepsilon$ and a pair of functions $u^*(x, y)$, $v^*(x, y)$ with the following properties:

(i′) The functions $u^* = u^*(x, y)$, $v^* = v^*(x, y)$ map the disc

$$K = \{(x-x^*)^2 + (y-y^*)^2 < \varepsilon^2\}$$

topologically onto a domain in the $u^* v^*$-plane.

(ii′) $u(x, y)$ and $v(x, y)$ are analytic in $K$ and satisfy the Beltrami system (1.1.1).

(iii′) For $(x, y) \in K$ we have $a(x, y) \cdot \dfrac{\partial(u^*, v^*)}{\partial(x, y)} > 0$.

Furthermore, if $(x^*-x_0)^2 + (y^*-y_0)^2 = R^2$, we can choose the functions $u(x, y)$, $v(x, y)$ such that

(i″) the circular arc

$$\sigma = K \cap \{(x-x_0)^2 + (y-y_0)^2 = R^2\}$$

is mapped onto a segment of the real axis $v^* = 0$.

Introducing the system of local parameters $(u^*, v^*)$ the disc $D$ thus becomes a simply connected finite Riemann surface $S$. From the uniformization principle for finite Riemann surfaces [16] it follows that $S$ can be mapped conformally onto the unit disc $u^2 + v^2 < 1$. The mapping functions $u = u(x, y)$ and $v = v(x, y)$ are analytic for $(x, y) \in D$ and satisfy the conditions (1.1.3) and (1.1.4). Furthermore, we have $u(x_0, y_0) = v(x_0, y_0) = 0$. Hence they are solutions of the Beltrami system (1.1.1). The statements (ii) and (iii) of the lemma then follow from (1.1.10), (1.1.10′) and (1.1.11).

(II) We shall now remove the restriction that the functions $a(x, y)$, $b(x, y)$ and $c(x, y)$ be analytic in $D$. According to the Weierstrass approximation theorem there exist sequences of polynomials

$$\{a_n(x, y)\}, \quad \{b_n(x, y)\}, \quad \{c_n(x, y)\},$$

such that the relations

$$\left.\begin{array}{l} a_n(x, y) \to a(x, y), \ldots, c_n(x, y) \to c(x, y); \\ \dfrac{\partial a_n}{\partial x} \to \dfrac{\partial a}{\partial x}, \ldots, \dfrac{\partial c_n}{\partial y} \to \dfrac{\partial c}{\partial y} \end{array}\right\} n \to \infty \tag{1.2.7}$$

hold uniformly in $D$. Consequently there exist two finite positive constants $c_1$ and $c_2$ such that for $(x, y) \in D$ and $n \geqq n_0$ the inequalities

$$|a_n(x, y)| + \ldots + |c_n(x, y)| + \left|\frac{\partial a_n}{\partial x}\right| + \ldots + \left|\frac{\partial c_n}{\partial y}\right| \leqq c_1, \tag{1.2.8}$$

and

$$a_n(x, y)\, c_n(x, y) - b_n(x, y)^2 \geqq c_2 \tag{1.2.9}$$

are satisfied. Let $u_n(x, y)$ and $v_n(x, y)$ be the mapping functions defined in (I) and $x_n(u, v)$, $y_n(u, v)$ be the inverse functions satisfying the conditions (i), (ii) and (iii), where the functions $a$, $b$, $c$ have to be replaced by $a_n$, $b_n$, $c_n$. From (1.2.6) we conclude

$$\frac{a_n + c_n}{\sqrt{a_n c_n - b_n^2}} = \frac{\left(\dfrac{\partial x_n}{\partial u}\right)^2 + \left(\dfrac{\partial x_n}{\partial v}\right)^2 + \left(\dfrac{\partial y_n}{\partial u}\right)^2 + \left(\dfrac{\partial y_n}{\partial v}\right)^2}{\dfrac{\partial(x_n, y_n)}{\partial(x, y)}} \tag{1.2.10}$$

$$= \frac{\left(\dfrac{\partial u_n}{\partial x}\right)^2 + \left(\dfrac{\partial u_n}{\partial v}\right)^2 + \left(\dfrac{\partial v_n}{\partial x}\right)^2 + \left(\dfrac{\partial v_n}{\partial y}\right)^2}{\dfrac{\partial(u_n, v_n)}{\partial(x, y)}}$$

---

16. See, for instance, Schiffer-Spencer [30], § 2.10.

Hence by (1.2.8) and (1.2.9) we have for $n \geqq n_0$ the estimates:

$$(1.2.11)\quad \left(\frac{\partial x_n}{\partial u}\right)^2 + \left(\frac{\partial x_n}{\partial v}\right)^2 + \left(\frac{\partial y_n}{\partial u}\right)^2 + \left(\frac{\partial y_n}{\partial v}\right)^2 \leqq \frac{c_1}{\sqrt{c_2}} \left| \frac{\partial (x_n, y_n)}{\partial (u, v)} \right|,$$

and

$$(1.2.11')\quad \left(\frac{\partial u_n}{\partial x}\right)^2 + \left(\frac{\partial u_n}{\partial y}\right)^2 + \left(\frac{\partial v_n}{\partial x}\right)^2 + \left(\frac{\partial v_n}{\partial y}\right)^2 \leqq \frac{c_1}{\sqrt{c_2}} \left| \frac{\partial (u_n, v_n)}{\partial (x, y)} \right|.$$

On integrating (1.2.11) and (1.2.11′) we obtain the inequalities

$$(1.2.12)\quad \iint\limits_{u^2+v^2<1} \left[\left(\frac{\partial x_n}{\partial u}\right)^2 + \left(\frac{\partial x_n}{\partial v}\right)^2 + \left(\frac{\partial y_n}{\partial u}\right)^2 + \left(\frac{\partial y_n}{\partial v}\right)^2\right] du\, dv$$

$$\leqq \frac{c_1}{\sqrt{c_2}} \iint\limits_{u^2+v^2<1} \left| \frac{\partial (x_n, y_n)}{\partial (u, v)} \right| du\, dv = \frac{c_1}{\sqrt{c_2}} \iint\limits_{D} dx\, dy = \frac{\pi c_1 R^2}{\sqrt{c_2}},$$

and

$$(1.2.12')\quad \iint\limits_{D} \left[\left(\frac{\partial u_n}{\partial x}\right)^2 + \left(\frac{\partial u_n}{\partial y}\right)^2 + \left(\frac{\partial v_n}{\partial x}\right)^2 + \left(\frac{\partial v_n}{\partial y}\right)^2\right] dx\, dy$$

$$\leqq \frac{c_1}{\sqrt{c_2}} \iint\limits_{D} \left| \frac{\partial (u_n, v_n)}{\partial (u, v)} \right| dx\, dy = \frac{c_1}{\sqrt{c_2}} \iint\limits_{u^2+v^2<1} du\, dv = \frac{c_1 \pi}{\sqrt{c_2}}.$$

According to [12], Theorem 9, (i) and (ii) together with the inequalities (1.2.8), (1.2.9) and (1.2.12) imply estimates of the following kind:

$$(1.2.13)\quad \left|\frac{\partial x_n}{\partial u}\right|, \ldots, \left|\frac{\partial y_n}{\partial v}\right| \leqq c_3(R, \rho) < \infty,$$

$$(u^2 + v^2 \leqq \rho^2 < 1, \quad n \geqq n_0),$$

and

$$(1.2.14)\quad | x_{nu}(u_1, v_1) - x_{nu}(u_2, v_2) | \leqq c_4(R, \rho, \nu) [(u_1-u_2)^2 + (v_1-v_2)^2]^{\nu/2}$$

$$\vdots$$

$$| y_{nv}(u_1, v_1) - y_{nv}(u_2, v_2) | \leqq c_4(R, \rho, \nu) [(u_1-u_2)^2 + (v_1-v_2)^2]^{\nu/2},$$

where $c_4(R, \rho, \nu) < \infty$, $n \geqq n_0$, and

$$u_1^2 + v_1^2 \leqq \rho^2 < 1, \quad u_2^2 + v_2^2 \leqq \rho^2 < 1, \quad 0 < \nu < 1.$$

Furthermore, from [12], Theorem 7, we conclude that an estimate of the form

$$(1.2.15)\quad \left(\frac{\partial x_n}{\partial u}\right)^2 + \left(\frac{\partial x_n}{\partial v}\right)^2 + \left(\frac{\partial y_n}{\partial u}\right)^2 + \left(\frac{\partial y_n}{\partial v}\right)^2 \geqq c_5(R . \rho) > 0$$

holds for $u^2+v^2 \leqq \rho^2 < 1$, and $n \geqq n_0$. Using (1.2.11) we obtain the inequality

$$\left| \frac{\partial(x_n, y_n)}{\partial(u, v)} \right| \geqq \frac{\sqrt{c_2} \cdot c_5(R, \rho)}{c_1} > 0, \quad (n \geqq n_0). \tag{1.2.16}$$

From (1.2.7), (1.2.13) and (1.2.14) we infer that there exists a subsequence $\{n_k\}$ of positive integers such that the relations

$$\left.\begin{array}{l} x_{n_k}(u, v) \to x(u, v) \\ y_{n_k}(u, v) \to y(u, v) \end{array}\right\} \quad k \to \infty, \tag{1.2.17}$$

$$\left.\begin{array}{c} \dfrac{\partial x_{n_k}}{\partial u} \to \dfrac{\partial x}{\partial u} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial y_{n_k}}{\partial v} \to \dfrac{\partial y}{\partial v} \end{array}\right\} \quad k \to \infty, \tag{1.2.18}$$

and

$$\left.\begin{array}{l} \Delta x_{n_k} \to \left[\dfrac{\partial}{\partial x}\left(\dfrac{c}{\sqrt{ac-b^2}}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{b}{\sqrt{ac-b^2}}\right)\right](x_u y_v - x_v y_u) \\ \Delta y_{n_k} \to \left[\dfrac{\partial}{\partial y}\left(\dfrac{a}{\sqrt{ac-b^2}}\right) - \dfrac{\partial}{\partial x}\left(\dfrac{b}{\sqrt{ac-b^2}}\right)\right](x_u y_v - x_v y_u) \end{array}\right\} k \to \infty \tag{1.2.19}$$

hold uniformly in every closed disc $u^2+v^2 \leqq \rho^2 < 1$. Since by (1.2.14) the functions $x(u, v)$ and $y(u, v)$ are of class $C^{1+\nu}$ $(0<\nu<1)$ for $u^2+v^2<1$, the limit functions $\lim_{k\to\infty} \Delta x_{n_k}$ and $\lim_{k\to\infty} \Delta y_{n_k}$ belong to $C^{0+\varkappa}$; hence by [12], Lemma 2, $x(u, v)$ and $y(u, v)$ are of class $C^{2+\tilde{\varkappa}}$ $(0<\tilde{\varkappa}<1)$ for $u^2+v^2<1$ and satisy (1.2.5) and (1.2.5'). Furthermore, from (1.2.16) we conclude that for $u^2+v^2 \leqq \rho^2 < 1$ the inequality

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| \geqq \frac{\sqrt{c_2} \cdot c_5(R, \rho)}{c_1} > 0 \tag{1.2.20}$$

holds, hence the functions $x(u, v)$ and $y(u, v)$ satisfy the equations (1.2.6). According to a classical result of Lebesgue (see [12], Lemma 16) statement (i) together with the inequalities (1.2.12) and (1.2.12') implies that the functions $x_n(u, v)$, $y_n(u, v)$ and their inverses $u_n(x, y)$, $v_n(x, y)$, $(n \geqq n_0)$ are equicontinuous for $u^2+v^2 \leqq 1$ and $(x-x_0)^2+(y-y_0) \leqq R^2$, respectively. Therefore the functions $x(u, v)$ and $y(u, v)$ are continuous for $u^2+v^2 \leqq 1$ and map the disc $u^2+v^2 \leqq 1$ in a one-to-one way onto the disc $(x-x_0)^2+(y-y_0)^2 \leqq R^2$ such that $x(0, 0)=x_0$ and $y(0, 0)=y_0$. This completes the proof of Lemma 2.

## § 2. ELLIPTIC MONGE-AMPÈRE EQUATIONS

### 2.1. Characteristic Parameters.

Let $z=z(x, y)$ be a real-valued function, which is of class $C^2$ in a domain $\Omega$ of the $xy$-plane and satisfies the partial differential equation

$$F(x, y, z, p, q, r, s, t)=0 \tag{2.1.1}$$

$$(p=z_x,\ q=z_y;\ r=z_{xx},\ s=z_{xy},\ t=z_{yy}).$$

In order that (2.1.1) has a meaning we assume that $F$ is a real-valued function of class $C^1$ in an open set of the 8-dimensional space

$$(x, y, z, p, q, r, s, t)$$

containing the hypersurface

$$T=\{(x, y,\ z(x, y),\ p(x, y),\ q(x, y),\ r(x, y),\ s(x, y),\ t(x, y)),\ (x, y)\in\Omega\}.$$

Now the equation $F=0$ is called elliptic, if for

$$(x, y, z, p, q, r, s, t)\in T$$

the inequality

$$D=F_r F_t-\frac{1}{4}F_s^2>0 \tag{2.1.2}$$

holds. (2.1.2) implies that the characteristic differential form

$$\varphi=F_t\,dx^2-F_s\,dx\,dy+F_r\,dy^2 \tag{2.1.3}$$

is (positive or negative) definite. If we furthermore assume that the functions $z(x, y)$ and $F$ be sufficiently regular the results of § 1 show that we can introduce new independent variables $u=u(x, y)$ and $v=v(x, y)$ in the large such that

$$\varphi=\lambda(x, y)\,(du^2+dv^2), \tag{2.1.4}$$

and

$$F_t\cdot(u_x v_y-u_y v_x)>0 \tag{2.1.4'}$$

hold. In accordance with the general theory of hyperbolic equations in two variables [17] the quantities $x, y, z, p, q, r, s, t$, considered as functions of $u$ and $v$, satisfy certain non-linear elliptic systems of the type considered in [12], § 3. Although not strictly logical, we shall call the parameters $(u, v)$

17. See Courant-Hilbert [6], Chapter V, also Hadamard [10], Appendix III.

introduced by (2.1.4) and (2.1.4'), characteristic with respect to the equation $F = 0$.

In the following investigations we shall be concerned with the Monge-Ampère equation

$$F \equiv Ar + 2Bs + Ct + rt - s^2 - E = 0 \tag{2.1.5}$$
$$(A = A(x, y, z, p, q), \dots, E = E(x, y, z, p, q)).$$

For any solution of (2.1.5) we have

$$\begin{aligned} D = F_r F_t - \frac{1}{4} F_s^2 &= (A + t)(C + r) - (B - s)^2 \\ &= AC - B^2 + Ar + 2Bs + Ct + rt - s^2 \\ &= AC - B^2 + E. \end{aligned} \tag{2.1.6}$$

Ellipticity of the equation (2.1.5) therefore means that $AC - B^2 + E > 0$. The main object of this section consists in setting up the differential equations satisfied by the functions $x(u, v)$ and $y(u, v)$, where $(u, v)$ are characteristic parameters with respect to the equation (2.1.5). This procedure will be carried out in the next lemma, which is derived from Lemma 2 by formal computations.

Lemma 3. *Let $z = z(x, y)$ belong to $C^{3+\varkappa}$ $(0 < \varkappa < 1)$ and satisfy the elliptic Monge-Ampère equation*

$$\begin{aligned} Ar + 2Bs + Ct + rt - s^2 &= E, \\ D = AC - B^2 + E &> 0 \end{aligned} \tag{2.1.7}$$

*in a domain $\Omega$ of the $xy$-plane. Furthermore, let the coefficients $A = A(x, y, z, p, q), \dots, E = E(x, y, z, p, q)$ be of class $C^2$ in an open set of the 5-dimensional $(x, y, z, p, q)$-space containing the hypersurface*

$$\Sigma = \{(x, y, z(x, y), p(x, y), q(x, y)), (x, y) \in \Omega\},$$

*and let the circular disc $(x - x_0)^2 + (y - y_0)^2 \leq R^2$ belong to $\Omega$. Then there exists a pair of functions $x(u, v)$ and $y(u, v)$ with the following properties:*

(i) *$x = x(u, v)$ and $y = y(u, v)$ map the unit disc $u^2 + v^2 \leq 1$ homeomorphically onto the disc $(x - x_0)^2 + (y - y_0)^2 \leq R$ such that $x(0, 0) = x_0$ and $y(0, 0) = y_0$.*

(ii) *For $u^2 + v^2 < 1$ the functions $x(u, v)$ and $y(u, v)$ are of class $C^2$ and satisfy the differential equations*

$$(2.1.8)\quad \Delta x = h_1 (x_u^2 + x_v^2) + h_2 (x_u y_u + x_v y_v) + h_3 (y_u^2 + y_v^2) + h_4 (x_u y_v - x_v y_u),$$

$$(2.1.8')\quad \Delta y = \tilde{h}_1 (x_u^2 + x_v^2) + \tilde{h}_2 (x_u y_u + x_v y_v) + \tilde{h}_3 (y_u^2 + y_v^2) + \tilde{h}_4 (x_u y_v - x_v y_u),$$

where the functions $h_1, \ldots, \tilde{h}_4$ are defined by the expressions

$$(2.1.9)\quad \begin{aligned} h_1 &= B_q - \frac{1}{2D}(D_x + D_z p - D_p C + D_q B) \\ h_2 &= -A_q - B_p - \frac{1}{2D}(D_y + D_z q + D_p B - D_q A) \\ h_3 &= A_p \\ h_4 &= \frac{1}{\sqrt{D}}\left(A_x + B_y + A_z p + B_z q - A_p C + (A_q + B_p) B - B_q A - \frac{1}{2} D_p\right), \end{aligned}$$

and

$$(2.1.9')\quad \begin{aligned} \tilde{h}_1 &= C_q \\ \tilde{h}_2 &= -B_q - C_p - \frac{1}{2D}(D_x + D_z p - D_p C + D_q B) \\ \tilde{h}_3 &= B_p - \frac{1}{2D}(D_y + D_z q + D_p B - D_q A) \\ \tilde{h}_4 &= \frac{1}{\sqrt{D}}\left(C_y + B_x + C_z q + B_z p - B_p C + (B_q + C_p) B - C_q A - \frac{1}{2} D_q\right). \end{aligned}$$

(iii) For $u^2 + v^2 < 1$ we have $x_u y_v - x_v y_u \neq 0$ and the representations

$$(2.1.10)\quad \frac{A+t}{\sqrt{D}} = \frac{x_u^2 + x_v^2}{x_u y_v - x_v y_u}, \quad \frac{B-s}{\sqrt{D}} = \frac{x_u y_u + x_v y_v}{x_u y_v - x_v y_u}, \quad \frac{C+r}{\sqrt{D}} = \frac{y_u^2 + y_v^2}{x_u y_v - x_v y_u}.$$

**Proof.** The characteristic form associated with the equation (2.1.7) is

$$(2.1.11)\quad \varphi = (C + r)\, dx^2 - 2(B - s)\, dx\, dy + (A + t)\, dy^2,$$

and by (2.1.6) we have

$$(2.1.12)\quad (C + r)(A + t) - (B - s)^2 = D > 0$$

for $(x, y) \in \Omega$. Since by hypothesis the coefficients of $\varphi$ are of class $C^{1+\varkappa}$ $(0 < \varkappa < 1)$ in $\Omega$, the statements (i) and (iii) are immediate consequences of Lemma 2, (i) and (iii). Furthermore, from (1.2.5) and (1.2.5′) we obtain the differential equations

$$(2.1.13) \qquad \Delta x = \left[\frac{\partial}{\partial x}\left(\frac{A+t}{\sqrt{D}}\right) + \frac{\partial}{\partial y}\left(\frac{B-s}{\sqrt{D}}\right)\right](x_u y_v - x_v y_u),$$

and

$$(2.1.13') \qquad \Delta y = \left[\frac{\partial}{\partial y}\left(\frac{C+r}{\sqrt{D}}\right) + \frac{\partial}{\partial x}\left(\frac{B-s}{\sqrt{D}}\right)\right](x_u y_v - x_v y_u).$$

We shall now transform the right-hand sides of (2.1.13) and (2.1.13′) into expressions which contain the quantities $x, y, z, p, q;\ x_u, \dots, y_v$ only.

First of all, we have

$$(2.1.14) \qquad \frac{\partial}{\partial x}\left(\frac{A+t}{\sqrt{D}}\right) + \frac{\partial}{\partial y}\left(\frac{B-s}{\sqrt{D}}\right)$$

$$= \frac{1}{\sqrt{D}}(A_x + A_z p + A_p r + A_q s + B_y + B_z q + B_p s + B_q t)$$

$$- \frac{A+t}{2D\sqrt{D}}(D_x + D_z p + D_p r + D_q s) - \frac{B-s}{2D\sqrt{D}}(D_y + D_z q + D_p s + D_q t)$$

$$= \frac{1}{\sqrt{D}}[(A_x + A_z p + B_y + B_z q - A_p C + B(A_q + B_p) - B_q A) + A_p(C+r)$$

$$- (A_q + B_p)(B-s) + B_q(A+t)]$$

$$- \frac{A+t}{2D\sqrt{D}}[(D_x + D_z p - D_p C + D_q B) + D_p(C+r) - D_q(B-s)]$$

$$- \frac{B-s}{2D\sqrt{D}}[(D_y + D_z q + D_p B - D_q A) - D_p(B-s) + D_q(A+t)]$$

$$= \frac{1}{\sqrt{D}}[A_x + A_z p + B_y + B_z q - A_p C + (A_q + B_p)B - B_q A]$$

$$+ \frac{A+t}{\sqrt{D}}\left[B_q - \frac{1}{2D}(D_x + D_z p - D_p C + D_q B)\right]$$

$$+ \frac{B-s}{\sqrt{D}}\left[-(A_q + B_p) - \frac{1}{2D}(D_y + D_z q + D_p B - D_q A)\right]$$

$$+ \frac{C+r}{\sqrt{D}} \cdot A_p - \frac{D_p}{2D\sqrt{D}}[(A+t)(C+r) - (B-s)^2].$$

Using (2.1.12) we obtain

$$(2.1.15) \qquad \frac{\partial}{\partial x}\left(\frac{A+t}{\sqrt{D}}\right) + \frac{\partial}{\partial y}\left(\frac{B-s}{\sqrt{D}}\right) = h_1\frac{A+t}{\sqrt{D}} + h_2\frac{B-s}{\sqrt{D}} + h_3\frac{C+r}{\sqrt{D}} + h_4,$$

where the coefficients $h_1, \dots, h_4$ are defined by the equations (2.1.9). In the same way we derive the equations

$$(2.1.16)\qquad \frac{\partial}{\partial y}\left(\frac{C+r}{\sqrt{D}}\right)+\frac{\partial}{\partial x}\left(\frac{B-s}{\sqrt{D}}\right)$$

$$= \frac{1}{\sqrt{D}}(C_y+C_zq+C_ps+C_qt+B_x+B_zp+B_pr+B_qs)$$

$$- \frac{C+r}{2D\sqrt{D}}(D_y+D_zq+D_ps+D_qt)-\frac{B-s}{2D\sqrt{D}}(D_x+D_zp+D_pr+D_qs)$$

$$= \frac{1}{\sqrt{D}}[(C_y+C_zq+B_x+B_zp+(C_p+B_q)B-C_qA-B_pC)$$

$$-(C_p+B_q)(B-s)+C_q(A+t)+B_p(C+r)]$$

$$- \frac{C+r}{2D\sqrt{D}}[(D_y+D_zq+D_pB-D_qA)-D_p(B-s)+D_q(A+t)]$$

$$- \frac{B-s}{2D\sqrt{D}}[(D_x+D_zp-D_pC+D_qB)+D_p(C+r)-D_q(B-s)]$$

$$= \frac{1}{\sqrt{D}}[(C_y+C_zq+B_x+B_zp+(C_p+B_q)B-C_qA-B_pC)]$$

$$+ \frac{A+t}{\sqrt{D}}\cdot C_q+\frac{B-s}{\sqrt{D}}[-C_p-B_q-\frac{1}{2D}(D_x+D_zp-D_pC+D_qB)]$$

$$+ \frac{C+r}{\sqrt{D}}[B_p-\frac{1}{2D}(D_y+D_zq+D_pB-D_qA)]$$

$$- \frac{D_q}{2D\sqrt{D}}[(A+t)(C+r)-(B-s)^2]\,.$$

With the aid of (2.1.12) it follows that

$$(2.1.17)\quad \frac{\partial}{\partial y}\left(\frac{C+r}{\sqrt{D}}\right)+\frac{\partial}{\partial x}\left(\frac{B-s}{\sqrt{D}}\right)=\tilde{h}_1\cdot\frac{A+t}{\sqrt{D}}+\tilde{h}_2\cdot\frac{B-s}{\sqrt{D}}+\tilde{h}_3\cdot\frac{C+r}{\sqrt{D}}+\tilde{h}_4\,,$$

where the coefficients $\tilde{h}_1, \ldots, \tilde{h}_4$ are given by the formulas (2.1.9′). If we now insert the expressions (2.1.15) and (2.1.17) into the equations (2.1.13) and (2.1.13′) and use (2.1.10) we arrive at the differential equations (2.1.8) and (2.1.8′). Lemma 3 is thus proved.

## 2.2. Interior Estimates.

The essential feature of the preceding lemma lies in the fact that the second order derivatives of the solutions of (2.1.7) are given by the simple formulas (2.1.10), whereas they do not enter into the differential equations (2.1.8) and (2.1.8′). This enables us to establish a priori estimates for the second and higher order derivatives of the solutions of certain Monge-Ampère equations in terms of bounds for the first order derivatives. The proofs of these statements will depend on the results obtained in [12], § 3. In order to apply these we shall first derive from Lemma 3 a simple inequality satisfied by the functions $x(u, v)$ and $y(u, v)$:

Lemma 4. Let the function $z(x, y)$ satisfy the hypotheses of Lemma 3. Furthermore, let the inequalities

$$\left.\begin{aligned} &|A| \leqq \alpha, \ldots, |E| \leqq \alpha; \\ &D = AC - B^2 + E \geqq \alpha^{-1} \end{aligned}\right\} (x, y, z, p, q) \in \Sigma, \quad 0 < \alpha < \infty, \tag{2.2.1}$$

and

$$|p(x, y)|, \quad |q(x, y)| \leqq \gamma < \infty, \quad (x, y) \in \Omega, \tag{2.2.2}$$

be satisfied. Then we have the estimate

$$\iint\limits_{u^2+v^2<1} (x_u^2 + x_v^2 + y_u^2 + y_v^2)\, du\, dv \leqq 2\pi\, \alpha^{3/2} R^2 + 4\pi\, \alpha^{1/2} \gamma R, \tag{2.2.3}$$

where $x(u, v)$ and $y(u, v)$ are defined by Lemma 3.

Proof. Let $\rho$ be a real number in the interval $0 < \rho < 1$. Then from (i), Lemma 3, it follows that there exists a real number $R_0$ satisfying the inequality $0 < R_0 < R$ such that the image domain $K^*$ of the circular disc

$$K = \{(x - x_0)^2 + (y - y_0)^2 \leqq R_0^2\}$$

under the mapping $(x, y) \rightarrow (u, v)$ covers the disc $u^2 + v^2 \leqq \rho^2$. For abbreviation let $K'$ denote the circle $K' = \{(x - x_0)^2 + (y - y_0)^2 = R_0^2\}$. Now on account of (2.1.10), we have

$$\frac{A + C + r + t}{\sqrt{D}} = \frac{x_u^2 + x_v^2 + y_u^2 + y_v^2}{x_u y_v - x_v y_u} \neq 0 \tag{2.2.4}$$

for $u^2 + v^2 < 1$. Hence with the aid of (2.2.1) and (2.2.2) we obtain the estimate

$$\begin{aligned} \iint\limits_{u^2+v^2\leqq\rho^2} (x_u^2+x_v^2+y_u^2+y_v^2)\, du\, dv &\leqq \iint\limits_{K^*} (x_u^2+x_v^2+y_u^2+y_v^2)\, du\, dv \\ &= \iint\limits_{K} \frac{|A + C + r + t|}{\sqrt{D}}\, dx\, dy \leqq \alpha^{1/2} \left| \iint\limits_{K} (A+C+r+t)\, dx\, dy \right| \\ &= \alpha^{1/2} \left| \iint\limits_{K} (A+C)\, dx\, dy + \oint\limits_{K'} (-q\,dx + p\,dy) \right| \\ &\leqq \alpha^{1/2} (2\pi\, \alpha R_0^2 + 4\pi\, \gamma R_0) \leqq 2\pi\, \alpha^{3/2} R^2 + 4\pi\, \alpha^{1/2} \gamma R. \end{aligned} \tag{2.2.5}$$

If we now let $\rho$ tend to 1, we arrive at the desired inequality (2.2.3).

Theorem 1.

Hypotheses:

(i) The function $z = z(x, y)$ is of class $C^{3+\varkappa}$ $(0 < \varkappa < 1)$ in a domain $\Omega$ of the $xy$-plane and satisfies the Monge-Ampère equation

$$Ar + 2Bs + Ct + rt - s^2 = E, \tag{2.2.6}$$

where the coefficients $A = A(x, y, z, p, q), \ldots, E = E(x, y, z, p, q)$ are of class $C^2$ in an open set of the 5-dimensional space containing the hypersurface

$$\Sigma = \{(x, y, z(x, y),\ p(x, y)\ q(x, y)),\ (x, y) \in \Omega\}.$$

(ii) The coefficients $A, \ldots, E$ satisfy for $(x, y, z, p, q) \in \Sigma$ the inequalities

$$|A|, \ldots, |E| \leqq \alpha_0, \tag{2.2.7}$$

$$D = AC - B^2 + E \geqq \alpha_0^{-1}, \tag{2.2.8}$$

$$|A_x|, \ldots, |E_q| \leqq \alpha_1, \tag{2.2.9}$$

$$|A_{xx}|, \ldots, |E_{qq}| \leqq \alpha_2, \tag{2.2.10}$$

where $\alpha_0, \alpha_1$ and $\alpha_2$ are finite positive constants.

(iii) For the function $z(x, y)$ we have the estimates

$$|p(x, y)| \leqq \gamma_1, \quad |q(x, y)| \leqq \gamma_1 \tag{2.2.11}$$

and

$$|r(x, y)| \leqq \gamma_2, \quad |s(x, y)| \leqq \gamma_2, \quad |t(x, y)| \leqq \gamma_2, \tag{2.2.12}$$

where $(x, y) \in \Omega$ and $\gamma_1, \gamma_2$ are finite positive constants.

Conclusion:

Let $\Omega_d$ denote the set of all points of $\Omega$, whose distance from the boundary exceeds a positive number $d$. Furthermore, let $\nu$ be a real number in the interval $0 < \nu < 1$. Then the function $z(x, y)$ satisfies the inequalities

(2.2.13)

$$|r(x_1, y_1) - r(x_0, y_0)| \leqq \vartheta_0(\alpha_0, \alpha_1, \gamma_1, \gamma_2, d, \nu)\,[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2}$$

$$\vdots$$

$$|t(x_1, y_1) - t(x_0, y_0)| \leqq \vartheta_0(\alpha_0, \alpha_1, \gamma_1, \gamma_2, d, \nu)\,[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2}$$

for $(x_1, y_1) \in \Omega_d$, $(x_0, y_0) \in \Omega_d$,

(2.2.14) $$|r_x|, \ldots, |t_y| \leqq \vartheta_1(\alpha_0, \alpha_1, \alpha_2, \gamma_1, \gamma_2, d)$$

for $(x, y) \in \Omega_d$, and

(2.2.15)
$$|r_x(x_1, y_1) - r_x(x_0, y_0)| \leqq \vartheta_2(\alpha_0, \alpha_1, \alpha_2, \gamma_1, \gamma_2, d, \nu)\,[(x_1-x_0)^2 + (y_1-y_0)^2]^{\nu/2}$$
$$\vdots$$
$$|t_y(x_1, y_1) - t_y(x_0, y_0)| \leqq \vartheta_2(\alpha_0, \alpha_1, \alpha_2, \gamma_1, \gamma_2, d, \nu)\,[(x_1-x_0)^2 + (y_1-y_0)^2]^{\nu/2}$$

for $(x_1, y_1) \in \Omega_d$, $(x_0, y_0) \in \Omega_d$, where $\vartheta_0, \vartheta_1$ and $\vartheta_2$ are finite positive constants.

Proof: Let $(x_0, y_0)$ be an arbitrary point in $\Omega_d$. According to Lemma 3 there exists a homeomorphism $x = x(u, v)$, $y = y(u, v)$ of the disc $u^2 + v^2 \leqq 1$ onto the disc $(x-x_0)^2 + (y-y_0)^2 \leqq d^2$ such that $x(0, 0) = x_0$, $y(0, 0) = y_0$, and the equations (2.1.8), (2.1.8′), (2.1.10) are satisfied. Now let

(2.2.16) $$X(u, v) = \frac{x(u, v) - x_0}{d}, \quad Y(u, v) = \frac{y(u, v) - y_0}{d},$$

and

(2.2.17) $$H_1(X, Y) = dh_1, \ldots, \tilde{H}_4(X, Y) = d\tilde{h}_4,$$

where the quantities $h_1, \ldots, \tilde{h}_4$ are defined by the formulas (2.1.9) and (2.1.9′). Then the functions $X(u, v)$ and $Y(u, v)$ are of class $C^2$ for $u^2 + v^2 < 1$ and map the disc $u^2 + v^2 \leqq 1$ homeomorphically onto the disc $X^2 + Y^2 \leqq 1$ such that $X(0, 0) = Y(0, 0) = 0$ and the differential equations

(2.2.18) $$\Delta X = H_1(X, Y)(X_u^2 + X_v^2) + H_2(X, Y)(X_u Y_u + X_v Y_v) + H_3(X, Y)(Y_u^2 + Y_v^2) + H_4(X, Y)(X_u Y_v - X_v Y_u),$$

(2.2.18′) $$\Delta Y = \tilde{H}_1(X, Y)(X_u^2 + X_v^2) + \tilde{H}_2(X, Y)(X_u Y_u + X_v Y_v) + \tilde{H}_3(X, Y)(Y_u^2 + Y_v^2) + \tilde{H}_4(X, Y)(X_u Y_v - X_v Y_u)$$

hold. Furthermore, the coefficients $H_1(X, Y), \ldots, \tilde{H}_4(X, Y)$ are of class $C^{(1)}$ for $X^2 + Y^2 < 1$ and satisfy inequalities of the form

(2.2.19) $$|H_1(X, Y)| \leqq d\mu_0(\alpha_0, \alpha_1, \gamma_1),$$
$$\vdots$$
$$|\tilde{H}_4(X, Y)| \leqq d\mu_0(\alpha_0, \alpha_1, \gamma_1),$$

and

$$(2.2.20) \qquad \left|\frac{\partial H_1}{\partial X}\right| \leqq d^2 \mu_1 (\alpha_0, \alpha_1, \alpha_2, \gamma_1, \gamma_2)$$

$$\vdots$$

$$\left|\frac{\partial \tilde{H}_4}{\partial Y}\right| \leqq d^2 \mu_1 (\alpha_0, \alpha_1, \alpha_2, \gamma_1, \gamma_2) \quad \text{for} \quad X^2 + Y^2 < 1,$$

where $\mu_0$ and $\mu_1$ are finite positive constants. As to the mapping functions $X(u, v)$ and $Y(u, v)$, we derive from Lemma 4 the estimate

$$(2.2.21) \qquad \iint\limits_{u^2+v^2<1} (X_u^2+X_v^2+Y_u^2+Y_v^2)\, du\, dv \leqq 2\pi\alpha_0^{3/2} + 4\pi\alpha_0^{1/2} \gamma_1 d^{-1}.$$

Furthermore, we have $\dfrac{\partial (X, Y)}{\partial (u, v)} \neq 0$.

Now applying Theorem 9 and Lemma 1 of [12], we obtain inequalities of the form:

$$(2.2.22) \qquad |X_u|, \ldots, |Y_v| \leqq \mu_2 (\alpha_0, \alpha_1 . \gamma_1, d, \rho) < \infty$$
$$(u^2 + v^2 \leqq \rho^2 < 1),$$

(2.2.23)

$$|X_u(u_1, v_1) - X_u(u_0, v_0)| \leqq \mu_3 (\alpha_0, \alpha_1, \gamma_1, d, \rho, \nu) [(u_1-u_0)^2 + (v_1-v_0)^2]^{\nu/2}$$

$$\vdots$$

$$|Y_v(u_1, v_1) - Y_v(u_0, v_0)| \leqq \mu_3 (\alpha_0, \alpha_1, \gamma_1, d, \rho, \nu) [(u_1-u_0)^2 + (v_1-v_0)^2]^{\nu/2}$$
$$(u_1^2 + v_1^2 \leqq \rho^2 < 1, \quad u_0^2 + v_0^2 \leqq \rho^2 < 1, \quad 0 < \nu < 1, \quad \mu_3 < \infty);$$

$$(2.2.24) \qquad |X_{uu}|, \ldots, |Y_{vv}| \leqq \mu_4 (\alpha_0, \alpha_1, \alpha_2, \gamma_1, \gamma_2, d, \rho) < \infty,$$
$$(u^2 + v^2 \leqq \rho^2 < 1)$$

and

(2.2.25)

$$|X_{uu}(u_1, v_1) - X_{uu}(u_0, v_0)| \leqq \mu_5(\alpha_0, \alpha_1, \alpha_2, \gamma_1, \gamma_2, d, \rho, \nu) [(u_1-u_0)^2 + (v_1-v_0)^2]^{\nu/2},$$

$$\vdots$$

$$|Y_{vv}(u_1, v_1) - Y_{vv}(u_0, v_0)| \leqq \mu_5 (\alpha_0, \alpha_1, \alpha_2, \gamma_1, \gamma_2, d, \rho, \nu) [(u_1-u_0)^2 + (v_1-v_0)^2]^{\nu/2},$$
$$(u_1^2 + v_1^2 \leqq \rho^2 < 1, \quad u_0^2 + v_0^2 \leqq \rho^2 < 1, \quad 0 < \nu < 1, \quad \mu_5 < \infty).$$

Furthermore, from [12], Theorem 7, we derive an estimate of the following kind:

$$X_u^2 + X_v^2 + Y_u^2 + Y_v^2 \geqq \mu_6(\alpha_0, \alpha_1, \gamma_1, d, \rho) > 0, \quad (u^2 + v^2 \leqq \rho^2 < 1), \tag{2.2.26}$$

and by (2.2.4) we have

$$\frac{X_u^2 + X_v^2 + Y_u^2 + Y_v^2}{|X_u Y_v - X_v Y_u|} = \frac{|A + C + r + t|}{\sqrt{D}} \leqq \frac{|A| + |C| + |r| + |t|}{\sqrt{D}} \leqq 2\alpha_0^{1/2}(\alpha_0 + \gamma_2) < \infty . \tag{2.2.27}$$

Combining (2.2.26) and (2.2.27) we obtain

$$|X_u Y_v - X_v Y_u| \geqq \frac{\mu_6(\alpha_0, \alpha_1, \gamma_1, d, \rho)}{2\alpha_0^{1/2}(\alpha_0 + \gamma_2)} > 0, \tag{2.2.28}$$

where $u^2 + v^2 \leqq \rho^2 < 1$. Now let $(x_1, y_1)$ be a point in $\Omega$ such that $(x_1 - x_0)^2 + (y_1 - y_0)^2 < \frac{1}{4} d^2$, and put

$$X_1 = \frac{x_1 - x_0}{d}, \quad Y_1 = \frac{y_1 - y_0}{d}. \tag{2.2.29}$$

If $(u_1, v_1)$ is the image point of $(X_1, Y_1)$ under the mapping $(X, Y) \to (u, v)$, then we have

$$u_1 = \int_{(0,0)}^{(X_1, Y_1)} \left(\frac{\partial u}{\partial X} dX + \frac{\partial u}{\partial Y} dY\right), \tag{2.2.30}$$

$$v_1 = \int_{(0,0)}^{(X_1, Y_1)} \left(\frac{\partial v}{\partial X} dX + \frac{\partial v}{\partial Y} dY\right), \tag{2.2.30'}$$

where the path of integration is the straight segment joining the points $(0, 0)$ and $(X, Y)$. From (2.2.30) and (2.2.30′) we conclude

$$(u_1^2 + v_1^2)^{1/2} \leqq (X_1^2 + Y_1^2)^{1/2} \max_{X^2+Y^2 \leqq 1/4} \frac{(X_u^2 + X_v^2 + Y_u^2 + Y_v^2)^{1/2}}{|X_u Y_v - X_v Y_u|}. \tag{2.2.31}$$

Now since $X = X(u, v)$, $Y = Y(u, v)$ is a homeomorphism of the disc $u^2+v^2 \leqq 1$ onto the disc $X^2+Y^2 \leqq 1$, such that $X(0, 0) = Y(0, 0) = 0$, and (2.2.21) holds, it follows from [12], Lemma 16, that we can determine a positive number $\rho = \rho(\alpha_0, \gamma_1, d) < 1$ such that the image domain of the disc $X^2 + Y^2 \leqq \frac{1}{4}$ under the mapping $(X, Y) \to (u, v)$ is contained in the disc $u^2 + v^2 \leqq \rho^2$. Hence by combining (2.2.22), (2.2.28), (2.2.29) and (2.2.31) we obtain the estimates

$$(2.2.32) \qquad (u_1^2+v_1^2)^{1/2} \leqq \rho(\alpha_0, \gamma_1, d) < 1,$$

and

$$(2.2.32') \qquad (u_1^2+v_1^2)^{1/2} \leqq \mu_7(\alpha_0, \alpha_1, \gamma_1, \gamma_2, d)\,[(x_1-x_0)^2+(y_1-y_0)^2]^{1/2},$$

where $\mu_7$ denotes a finite positive constant.

The proof of the lemma is completed by using the equations (2.1.10) (see Lemma 3). First of all we get

$$(2.2.33) \qquad r = -C + \sqrt{D}\cdot\frac{Y_u^2+Y_v^2}{X_uY_v-X_vY_u},$$

$$s = B - \sqrt{D}\cdot\frac{X_uY_u+X_vY_v}{X_uY_v-X_vY_u},$$

$$t = -A + \sqrt{D}\cdot\frac{X_u^2+X_v^2}{X_uY_v-X_vY_u}.$$

Differentiating these equations with respect to $x$ and $y$ we obtain

$$(2.2.34) \qquad r_x = -(C_x + C_zp + C_pr + C_qs)$$

$$+\frac{1}{2\sqrt{D}}(D_x+D_zp+D_pr+D_qs)\cdot\frac{Y_u^2+Y_v^2}{X_uY_v-X_vY_u}$$

$$+\frac{\sqrt{D}}{d}\left[\frac{\partial}{\partial u}\left(\frac{Y_u^2+Y_v^2}{X_uY_v-X_vY_u}\right)\frac{Y_v}{X_uY_v-X_vY_u}-\frac{\partial}{\partial v}\left(\frac{Y_u^2+Y_v^2}{X_uY_v-X_vY_u}\right)\frac{Y_u}{X_uY_v-X_vY_u}\right],$$

and similar representations for the functions $r_y, \ldots, t_y$. From the equations (2.2.33) and (2.2.34) the inequalities (2.2.13)—(2.2.15) are derived successively by making use of the estimates (2.2.22)—(2.2.25), (2.2.28), (2.2.32) and (2.2.32'). The theorem is thus proved.

Our next object consists in exhibiting certain classes of Monge-Ampère equations for which interior estimates of the second order derivatives can be obtained. Two cases of such equations will be considered. Both of them allow applications to problems of differential geometry in the large.

Theorem 2.

Hypotheses:

(i) The function $z = z(x,y)$ is of class $C^{3+\varkappa}$ $(0<\varkappa<1)$ in a domain $\Omega$ of the $xy$-plane and satisfies the Monge-Ampère equation

$$(2.2.35) \qquad A(x,y,z)r + 2B(x,y,z)s + C(x,y,z)t + (rt-s^2) = E(x,y,z,p,q),$$

where the coefficients $A, B, C, E$ are of class $C^{(2)}$ in an open set of the 5-dimensional space containing the hypersurface

$$\Sigma = \{(x, y, z(x, y),\ p(x, y),\ q(x, y)),\ (x, y) \in \Omega\}.$$

(ii) The coefficients $A, \dots, E$ satisfy for $(x, y, z, p, q) \in \Sigma$ the inequalities

$$|A|, \dots, |E| \leqq \alpha_0, \tag{2.2.36}$$

$$D = AC - B^2 + E \geqq \alpha_0^{-1}, \tag{2.2.37}$$

and

$$|A_x|, \dots, |E_q| \leqq \alpha_1, \tag{2.2.38}$$

where $\alpha_0$ and $\alpha_1$ are finite positive constants.

(iii) For the solution of (2.2.35) we have the estimates

$$|p(x, y)|,\ |q(x, y)| \leqq \gamma_1, \tag{2.2.39}$$

where $(x, y) \in \Omega$, and $\gamma_1$ is a finite positive constant.

Conclusion:

Let $\Omega_d$ denote the set of all points of $\Omega$, whose distance from the boundary exceeds a positive number $d$. Then the second derivatives of the function $z(x, y)$ satisfy for $(x, y) \in \Omega_d$ the inequalities

$$|r|, |s|, |t| \leqq \Theta(\alpha_0, \alpha_1, \gamma_1, d), \tag{2.2.40}$$

where $\Theta(\alpha_0, \alpha_1, \gamma_1, d)$ is a finite positive constant.

Proof: Since the coefficients $A, B, C$ of the equation (2.2.35) do not depend on $p$ and $q$, the associated functions $h_1, \dots, \tilde{h}_4$, defined by the formulas (2.1.9) and (2.1.9′) fulfill the relations

$$h_1 = \tilde{h}_2,\ \ h_2 = \tilde{h}_3 \ \text{ and } \ h_3 = \tilde{h}_1 = 0: \tag{2.2.41}$$

Retaining the notations introduced in the proof of the preceding theorem it follows that the mapping $(u, v) \to (X, Y)$ satisfies all the hypotheses of [12], Theorem 10 with

$$M = d\mu_0(\alpha_0, \alpha_1, \gamma_1), \tag{2.2.42}$$

and

$$N = 2\pi\alpha_0^{3/2} + 4\pi\alpha_0^{1/2}\gamma_1 d^{-1}. \tag{2.2.43}$$

Hence we conclude that the inequality

$$(|X_u Y_v - X_v Y_u|)_{u=v=0} \geqq \lambda_4(M, N, 0) > 0 \tag{2.2.44}$$

holds, where $\lambda_4(M, N, \rho)$ is defined in [12], Theorem 10.

Applying now (2.2.22) and (2.2.44) we derive from (2.2.33) the following estimates:

$$\left.\begin{array}{l}|r(x_0, y_0)| \\ |s(x_0, y_0)| \\ |t(x_0, y_0)|\end{array}\right\} \leqq \alpha_0 + \sqrt{\alpha_0^2 + \alpha_0} \cdot \frac{2\mu_2(\alpha_0, \alpha_1, \gamma_1, d, 0)^2}{\lambda_4(M, N, 0)}. \tag{2.2.45}$$

If we now set

$$\Theta(\alpha_0, \alpha_1, \gamma_1, d) = \alpha_0 + \sqrt{\alpha_0^2 + \alpha_0} \cdot \frac{2\mu_2(\alpha_0, \alpha_1, \gamma_1, d, 0)^2}{\lambda_4(M, N, 0)}, \tag{2.2.46}$$

and remember that $(x_0, y_0)$ is an arbitrary point in $\Omega_d$, we arrive at the conclusion of the theorem.

Theorem 3.[18]

Hypotheses:

(i) The function $z(x,y)$ is of class $C^{3+\varkappa}$ $(0<\varkappa<1)$ in a domain $\Omega$ of the $xy$-plane and satisfies the Monge-Ampère equation

$$Ar + 2Bs + Ct + rt - s^2 = E, \tag{2.2.47}$$

where the coefficients $A = A(x, y, z, p, q), \ldots, E = E(x, y, z, p, q)$ belong to $C^2$ in an open set of the 5-dimensional space containing the hypersurface

$$\Sigma = \{(x, y, z(x,y),\ p(x,y),\ q(x,y)),\ (x,y) \in \Omega\}.$$

(ii) The coefficients $A, \ldots, E$ satisfy for $(x, y, z, p, q) \in \Sigma$ the inequalities

$$|A|, \ldots, |E| \leqq \alpha_0, \tag{2.2.48}$$

and

$$D = AC - B^2 + E \geqq \alpha_0^{-1}, \tag{2.2.49}$$

where $\alpha_0$ is a finite positive constant.

(iii) The functions $x(u,v)$ and $y(u,v)$ defined in Lemma 3 satisfy the partial differential equations

$$\Delta x = h_1(x,y)(x_u^2 + x_v^2) + h_2(x,y)(x_u y_u + x_v y_v) + h_3(x,y)(y_u^2 + y_v^2) + h_4(x,y)\frac{\partial(x,y)}{\partial(u,v)}, \tag{2.2.50}$$

18. See Theorem 4 of [11], where an analogous result is proved.

$$(2.2.50') \qquad \Delta y = \tilde{h}_1(x,y)(x_u^2 + x_v^2) + \tilde{h}_2(x,y)(x_u y_u + x_v y_v) + \tilde{h}_3(x,y)(y_u^2 + y_v^2) + \tilde{h}_4(x,y)\frac{\partial(x,y)}{\partial(u,v)}.$$

Here the coefficients $h_1(x,y), \ldots, \tilde{h}_4(x,y)$ belong to $C^1$ for $(x,y) \in \Omega$, and we have the inequalities:

$$(2.2.51) \qquad |h_1(x,y)|, \ldots, |\tilde{h}_4(x,y)| \leqq \beta_0,$$

$$(2.2.52) \qquad \left|\frac{\partial h_1}{\partial x}\right|, \ldots, \left|\frac{\partial \tilde{h}_4}{\partial y}\right| \leqq \beta_1,$$

where $\beta_0$ and $\beta_1$ are fixed positive constants.

(iv) For the solution $z$ of (2.2.47) we have the inequalities

$$(2.2.53) \qquad |p(x,y)|, \quad |q(x,y)| \leqq \gamma_1,$$

where $(x,y) \in \Omega$ and $\gamma_1$ denotes a finite positive number.

Conclusion:

For $(x,y) \in \Omega_d$ the estimates

$$(2.2.54) \qquad |r|, |s|, |t| \leqq \Lambda(\alpha_0, \beta_0, \beta_1, \gamma_1, d)$$

hold, where $\Lambda$ is a finite positive constant.

Proof: We again use the same notations as in the proof of Theorem 1. Let the functions $X(u,v)$, $Y(u,v)$; $H_1(X,Y), \ldots, \tilde{H}_4(X,Y)$ be defined by the equations (2.2.16) and (2.2.17), respectively, where now $h_1, \ldots, \tilde{h}_4$ are the coefficients occurring in the equations (2.2.50) and (2.2.50′), Then from the inequalities (2.2.21), (2.2.51) and (2.2.52) it follows that the mapping $(u,v) \to (X,Y)$ satisfies all the hypotheses of [12], Theorem 11 with

$$(2.2.55) \qquad M_0 = d\beta_0, \quad M_1 = d^2\beta_1,$$

and

$$(2.2.56) \qquad N = 2\pi\alpha_0^{3/2} + 4\pi\alpha_0^{1/2}\gamma_1 d^{-1}.$$

Hence we have the estimate

$$(2.2.57) \qquad (|X_u Y_v - X_v Y_u|)_{u=v=0} \geqq \lambda_5(M_0, M_1, N, 0) > 0,$$

where $\lambda_5(M_0, M_1, N, 0)$ is defined in [12], Theorem 11. Furthermore, from [12], Theorem 9, we conclude that estimates of the form

$$(2.2.58) \qquad |X_u(0,0)|, \ldots, |Y_v(0,0)| \leqq \mu_8(\alpha_0, \beta_0, \gamma_1, d)$$

hold, where $\mu_8(\alpha_0, \beta_0, \gamma_1, d)$ is a finite positive number. If we now combine the inequalities (2.2.57) and (2.2.58) with the representations (2.2.33) we obtain the estimates

$$\left.\begin{array}{l} |r(x_0, y_0)| \\ |s(x_0, y_0)| \\ |t(x_0, y_0)| \end{array}\right\} \leqq \alpha_0 + \sqrt{\alpha_0^2 + \alpha_0} \cdot \frac{2\mu_8(\alpha_0, \beta_0, \gamma_1, d)^2}{\lambda_5(M_0, M_1, N, 0)}. \tag{2.2.59}$$

Putting

$$\Lambda(\alpha_0, \beta_0, \beta_1, \gamma_1, d) = \alpha_0 + \sqrt{\alpha_0^2 + \alpha_0} \cdot \frac{2\mu_8(\alpha_0, \beta_0, \gamma_1, d)^2}{\lambda_5(M_0, M_1, N, 0)}, \tag{2.2.60}$$

and remembering that $(x_0, y_0)$ is an arbitrary point of $\Omega_d$, we arrive at the conclusion of our theorem.

As an application of the results hitherto obtained, we shall treat the case where the coefficients of the equation (2.1.5) are independent of $z, p$, and $q$. It then becomes possible to estimate $|p|$ and $|q|$ in terms of bounds for $|z|$ and state:

Theorem 4.[19] Hypotheses:

(i) The function $z = z(x, y)$ belongs to $C^{3+\varkappa}$ $(0 < \varkappa < 1)$ for $(x, y) \in \Omega$ and satisfies the Monge-Ampère equation

$$A(x, y) r + 2B(x, y) s + C(x, y) t + rt - s^2 = E(x, y), \tag{2.2.61}$$

where the coefficients $A, \dots, E$ are of class $C^2$ for $(x, y) \in \Omega$.

(ii) For $(x, y) \in \Omega$ we have the inequalities

$$|A|, \dots, |E| \leqq \alpha_0, \tag{2.2.62}$$

$$D = AC - B^2 + E \geqq \alpha_0^{-1}, \tag{2.2.63}$$

$$|A_x|, \dots, |E_y| \leqq \alpha_1, \tag{2.2.64}$$

$$|A_{xx}|, \dots, |E_{yy}| \leqq \alpha_2, \tag{2.2.65}$$

where $\alpha_0, \alpha_1, \alpha_2$ are finite positive constants.

(iii) The function $z(x, y)$ satisfies for $(x, y) \in \Omega$ the inequality

$$|z(x, y)| \leqq \gamma_0 < \infty. \tag{2.2.66}$$

Conclusion: We have the estimates

$$|p|, |q| \leqq \eta_1(\alpha_0, \gamma_0, d) \quad \text{for} \quad (x, y) \in \Omega_d, \tag{2.2.67}$$

$$|r|, |s|, |t| \leqq \eta_2(\alpha_0, \alpha_1, \gamma_0, d) \quad \text{for} \quad (x, y) \in \Omega_d, \text{[20]} \tag{2.2.68}$$

---

19. This is an improvement of Theorem 4' of [11]. In the case where the coefficients $A(x, y), \dots, E(x, y)$ are analytic, an analogous result was previously established by H. Lewy (see [17], Theorem 2').

20. This contains the inequality (22) of Pogorelow [28].

(2.2.69)

$$|r(x_1, y_1) - r(x_0, y_0)| \leqq \eta_3(\alpha_0, \alpha_1, \gamma_0, d, \nu)[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2}$$

$$\vdots$$

$$|t(x_1, y_1) - t(x_0, y_0)| \leqq \eta_3(\alpha_0, \alpha_1, \gamma_0, d, \nu)[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2}$$

for $(x_0, y_0) \in \Omega_d$, $(x_1, y_1) \in \Omega_d$, and $0 < \nu < 1$,

(2.2.70) $|r_x|, \ldots, |t_y| \leqq \eta_4(\alpha_0, \alpha_1, \alpha_2, \gamma_0, d)$ for $(x, y) \in \Omega_d$,

and

(2.2.71)

$$|r_x(x_1, y_1) - r_x(x_0, y_0)| \leqq \eta_5(\alpha_0, \alpha_1, \alpha_2, \gamma_0, d, \nu)[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2}$$

$$\vdots$$

$$|t_y(x_1, y_1) - t_y(x_0, y_0)| \leqq \eta_5(\alpha_0, \alpha_1, \alpha_2, \gamma_0, d, \nu)[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2}$$

for $(x_0, y_0) \in \Omega_d$, $(x_1, y_1) \in \Omega_d$, and $0 < \nu < 1$, where $\eta_1, \ldots, \eta_5$ are finite positive constants.

Proof: We shall first establish (2.2.67). First of all, from (2.1.6) and (2.2.63) we conclude that the functions $A + t$ and $C + r$ have the same sign in $\Omega$. Performing the substitutions $A \to -A$, $B \to -B$, $C \to -C$, $E \to E$, $z \to -z$, we may suppose that the inequalities

(2.2.72) $$A + t > 0, \quad C + r > 0$$

hold throughout $\Omega$, without changing either hypotheses or conclusion of the theorem. Now let $(x_0, y_0)$ be an arbitrary point of $\Omega_d$ and put

(2.2.73) $$Z(x, y) = z(x, y) + \frac{\alpha_0}{2}[(x - x_0)^2 + (y - y_0)^2].$$

On account of (2.2.62) and (2.2.72) we have $Z_{xx} \geqq 0$ and $Z_{yy} \geqq 0$. Consequently the functions $Z_x$ and $Z_y$ are monotonic in $x$ and $y$, respectively. There exist real numbers $x', x'', y', y''$ satisfying the inequalities

(2.2.74) $$x_0 - d \leqq x' \leqq x_0, \quad x_0 \leqq x'' \leqq x_0 + d,$$

and

(2.2.74') $$y_0 - d \leqq y' \leqq y_0, \quad y_0 \leqq y'' \leqq y_0 + d,$$

such that the equations

$$\frac{Z(x_0, y_0) - Z(x_0 - d, y_0)}{d} = Z_x(x', y_0), \tag{2.2.75}$$

$$\frac{Z(x_0 + d, y_0) - Z(x_0, y_0)}{d} = Z_x(x'', y_0), \tag{2.2.75'}$$

$$\frac{Z(x_0, y_0) - Z(x_0, y_0 - d)}{d} = Z_y(x_0, y'), \tag{2.2.76}$$

and

$$\frac{Z(x_0, y_0 + d) - Z(x_0, y_0)}{d} = Z_y(x_0, y'') \tag{2.2.76'}$$

hold. From the above facts we conclude

$$\begin{aligned} |z_x(x_0, y_0)| = |Z_x(x_0, y_0)| &\leqq \operatorname{Max}\{|Z_x(x', y_0)|, |Z_x(x'', y_0)|\} \\ &\leqq \frac{2}{d} \operatorname*{Max}_{|x-x_0|\leqq d} |Z(x, y_0)| \leqq \frac{2}{d}\left(\gamma_0 + \frac{\alpha_0}{2} d^2\right), \end{aligned} \tag{2.2.77}$$

and

$$\begin{aligned} |z_y(x_0, y_0)| = |Z_y(x_0, y_0)| &\leqq \operatorname{Max}\{|Z_y(x_0, y')|, |Z_y(x_0, y'')|\} \\ &\leqq \frac{2}{d} \operatorname*{Max}_{|y-y_0|\leqq d} |Z(x_0, y)| \leqq \frac{2}{d}\left(\gamma_0 + \frac{\alpha_0}{2} d^2\right). \end{aligned} \tag{2.2.78}$$

Putting

$$\eta_1 = \frac{2}{d}\left(\gamma_0 + \frac{\alpha_0}{2} d^2\right) \tag{2.2.79}$$

we arrive at the estimates (2.2.67).

Now consider the function $z(x, y)$ in the disc

$$\tilde{\Omega} = \{(x - x_0)^2 + (y - y_0)^2 < \tfrac{1}{4} d^2\}.$$

In virtue of (2.2.67) we have for $(x, y) \in \tilde{\Omega}$ the inequalities

$$|p(x, y)|, \ |q(x, y)| \leqq \eta_1(\alpha_0, \gamma_0, \tfrac{1}{2} d) = \gamma_1. \tag{2.2.80}$$

Now applying Theorem 2 we obtain

$$\left.\begin{array}{l} |r(x_0, y_0)| \\ |s(x_0, y_0)| \\ |t(x_0, y_0)| \end{array}\right\} \leqq \Theta(\alpha_0, \alpha_1, \gamma_1, \tfrac{1}{2} d). \tag{2.2.81}$$

If we set

$$\eta_2(\alpha_0, \alpha_1, \gamma_0, d) = \Theta(\alpha_0, \alpha_1, \gamma_1, \tfrac{1}{2} d) \tag{2.2.82}$$

and remember that $(x_0, y_0)$ is an arbitrary point in $\Omega_d$, the inequalities (2.2.68) are established.

Let us now put

$$\gamma_2 = \eta_2(\alpha_0, \alpha_1, \gamma_0, \tfrac{1}{2} d) \tag{2.2.83}$$

and identify $\Omega$ with $\tilde{\Omega}$. Then all the hypotheses of Theorem 1 are satisfied. Hence from (2.2.13) we conclude that for

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 \leqq \frac{d^2}{16}$$

and $0 < \nu < 1$ the inequalities

(2.2.84)

$$\left.\begin{array}{l} |r(x_1, y_1) - r(x_0, y_0)| \\ \quad \cdot \\ \quad \cdot \\ \quad \cdot \\ |t(x_1, y_1) - t(x_0, y_0)| \end{array}\right\} \leqq \vartheta_0(\alpha_0, \alpha_1, \gamma_1, \gamma_2, \frac{d}{4}, \nu)\,[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{\nu/2}$$

hold. If we set

(2.2.85)

$$\eta_3(\alpha_0, \alpha_1, \gamma_0, d, \nu) = \operatorname{Max}\left\{\vartheta_0(\alpha_0, \alpha_1, \gamma_1, \gamma_2, \frac{d}{4}, \nu),\ \frac{2\eta_2(\alpha_0, \alpha_1, \gamma_0, d)}{\left(\frac{d}{4}\right)^{\nu}}\right\}$$

we arrive at the estimates (2.2.69).

Finally, by a repetition of the preceding arguments, we successively derive the inequalities (2.2.70) and (2.2.71) from (2.2.14) and (2.2.15). Theorem 4 is thus proved.

The following example, which is essentially due to H. Lewy [17], shows that an a priori limitation of the second order derivatives of the solutions of the Monge-Ampère equation (2.1.5) in terms of bounds for the first order derivatives is in general impossible. Let $f(x)$ be the real solution of the ordinary differential equation

$$f'(x')^3 + \varepsilon f'(x) = x, \quad f(0) = 0, \tag{2.2.86}$$

where $\varepsilon$ is a positive parameter. Obviously the function $f(x)$ thus defined is analytic in the vicinity of each point of the real axis. Furthermore, for $|x| < 1$ we have

$$|f'(x)| < 1, \tag{2.2.87}$$

and

$$f''(x) = \frac{1}{3f'(x)^2 + \varepsilon}. \tag{2.2.87'}$$

From this it follows that the function

$$z(x, y) = f(x) + \frac{y^2}{2} \tag{2.2.88}$$

is a solution of the Monge-Ampère equation (2.1.5) with

$$A = -1 + 3p^2 + \varepsilon, \quad B = C = 0, \quad E = 1. \tag{2.2.89}$$

For $p < 1$, $q < 1$, and $0 < \varepsilon < 1$ we obviously have estimates of the form

$$|A|, \ldots, |E_{qq}| \leqq \alpha, \tag{2.2.90}$$

and

$$D = AC - B^2 + E \geqq \alpha^{-1}, \tag{2.2.90'}$$

where $\alpha$ is a fixed positive number. Furthermore, from (2.2.87) we conclude that the inequalities

$$|p(x, y)|, \quad |q(x, y)| < 1 \tag{2.2.91}$$

hold for $x^2 + y^2 < 1$. On the other hand, by (2.2.87') we have

$$r(0, 0) = \frac{1}{\varepsilon}, \tag{2.2.92}$$

which tends to $+\infty$ for $\varepsilon \to 0$.

## § 3. THE WEYL EMBEDDING PROBLEM.

### 3.1. Some Classical Results in Differential Geometry.

As an application of the results obtained in the preceding section we shall treat the following non-linear system of differential equations:

$$\begin{aligned} x_u^2 + y_u^2 + z_u^2 &= E(u, v), \\ x_u x_v + y_u y_v + z_u z_v &= F(u, v), \\ x_v^2 + y_v^2 + z_v^2 &= G(u, v). \end{aligned} \tag{3.1.1}$$

This system, which can also be written in the more compact form

$$dx^2 + dy^2 + dz^2 = Edu^2 + 2Fdu\,dv + Gdv^2, \tag{3.1.2}$$

occurs in the problem of realizing a given line element

$$ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2 \tag{3.1.3}$$

by a surface $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ in the three dimensional space. Unless otherwise indicated we shall make the following assumptions concerning the functions $x, y, z; E, F, G$.

(i) $x, y, z; E, F, G$ are real and of class $C^2$ in a domain $\Omega$ of the $uv$-plane.

(ii) The discriminant $EG - F^2$ of the differential form $ds^2$ is positive.

(iii) The Gauss curvature $K(u, v)$ of the line element $ds^2$ is everywhere positive.

Our main object consists in deriving certain a priori bounds for the derivatives of the coordinate functions $x(u, v)$, $y(u, v)$ and $z(u, v)$, which are relevant in the discussion of the Weyl embedding problem. This procedure will be carried out in the next section. As a preliminary step we shall first derive an equation of Monge-Ampère type satisfied by $x(u, v)$, $y(u, v)$ and $z(u, v)$ and then study its characteristic equations (see Lemma 5 and 6). All the results proved in this section are due to Darboux [7].

To proceed further we shall make use of some of the standard notations and results in differential geometry. We shall denote vectors in three-space by capital letters:

$$X = (x, y, z), \quad Y = (x', y', z'), \dots,$$

and put

(3.1.4) $$(X, Y) = xx' + yy' + zz' \quad \text{(inner product)},$$

(3.1.5) $$XY = (yz' - y'z, \; zx' - z'x, \; xy' - x'y) \quad \text{(cross product)}.$$

Furthermore, we set $|X| = (X, X)^{1/2}$, and occasionally we shall write $XY$ and $X^2$ in place of $(X, Y)$ and $(X, X)$. With these abbreviations the unit normal $\overline{X}$ of the surface $X = X(u, v) = (x(u, v), \; y(u, v), \; z(u, v))$ is given by the formula

(3.1.6) $$\overline{X} = \frac{X_u \times X_v}{\sqrt{EG - F^2}},$$

and the system (3.1.1) takes the form

(3.1.7) $$E = X_u^2, \quad F = X_u X_v, \quad G = X_v^2.$$

Furthermore, the second fundamental quantities $L, M, N$ of the surface $X = X(u, v)$ are defined by the equations

(3.1.8) $$L = \overline{X} X_{uu}, \quad M = \overline{X} X_{uv}, \quad N = \overline{X} X_{vv}.$$

With these notations the Gauss and Weingarten equations take the form

(3.1.9)
$$\begin{aligned} X_{uu} &= \left\{ {}^{1}{}_{1}{}^{1} \right\} X_u + \left\{ {}^{1}{}_{2}{}^{1} \right\} X_v + L\overline{X}, \\ X_{uv} &= \left\{ {}^{1}{}_{1}{}^{2} \right\} X_u + \left\{ {}^{1}{}_{2}{}^{2} \right\} X_v + M\overline{X}, \\ X_{vv} &= \left\{ {}^{2}{}_{1}{}^{2} \right\} X_u + \left\{ {}^{2}{}_{2}{}^{2} \right\} X_v + N\overline{X}, \end{aligned}$$

and

$$(3.1.10)\qquad \begin{aligned} -\overline{X}_u &= L_1^1 X_u + L_1^2 X_v \\ -\overline{X}_v &= L_2^1 X_u + L_2^2 X_v , \end{aligned}$$

where the quantities $\left\{{}^{1}{}_{1}{}^{1}\right\}, \ldots, L_2^2$ are given by the expressions

$$(3.1.11)\qquad \begin{aligned} \left\{{}^{1}{}_{1}{}^{1}\right\} &= \frac{GE_u + FE_v - 2FF_u}{2\,(EG-F^2)}, \\ \left\{{}^{1}{}_{2}{}^{1}\right\} &= \frac{2EF_u - EE_v - FE_u}{2\,(EG-F^2)}, \\ \left\{{}^{1}{}_{1}{}^{2}\right\} &= \frac{GE_v - FG_u}{2\,(EG-F^2)}, \\ \left\{{}^{1}{}_{2}{}^{2}\right\} &= \frac{EG_u - FE_v}{2\,(EG-F^2)}, \\ \left\{{}^{2}{}_{1}{}^{2}\right\} &= \frac{2GF_v - GG_u - FG_v}{2\,(EG-F^2)}, \\ \left\{{}^{2}{}_{2}{}^{2}\right\} &= \frac{EG_v - 2FF_v + FG_u}{2\,(EG-F^2)}, \end{aligned}$$

and

$$(3.1.12)\qquad \begin{aligned} L_1^1 &= \frac{GL-FM}{EG-F^2}, \quad L_1^2 = \frac{EM-FL}{EG-F^2}, \\ L_2^1 &= \frac{GM-FN}{EG-F^2}, \quad L_2^2 = \frac{EN-FM}{EG-F^2}. \end{aligned}$$

The Gauss theorema egregium and the Codazzi equations, which are the integrability conditions for the combined system (3.1.9) and (3.1.10), can be written as follows:

$$(3.3.13)\qquad \frac{LN-M^2}{EG-F^2} = K$$

$$= \frac{1}{(EG-F^2)^2}\left\{ \begin{vmatrix} E, & F, & F_v - \frac{1}{2}G_u \\ F, & G, & \frac{1}{2}G_v \\ \frac{1}{2}E_u, & F_u - \frac{1}{2}E_v, & F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} \end{vmatrix} - \begin{vmatrix} E, & F, & \frac{1}{2}E_v \\ F, & G, & \frac{1}{2}G_u \\ \frac{1}{2}E_v, & \frac{1}{2}G_u, & 0 \end{vmatrix} \right\},$$

$$(3.1.14)\qquad \begin{aligned} L_v - M_u &= \left\{{}^{1}{}_{1}{}^{2}\right\} L + \left(\left\{{}^{1}{}_{2}{}^{2}\right\} - \left\{{}^{1}{}_{1}{}^{1}\right\}\right) M - \left\{{}^{1}{}_{2}{}^{1}\right\} N \\ M_v - N_u &= \left\{{}^{2}{}_{1}{}^{2}\right\} L + \left(\left\{{}^{2}{}_{2}{}^{2}\right\} - \left\{{}^{1}{}_{1}{}^{2}\right\}\right) M - \left\{{}^{1}{}_{2}{}^{2}\right\} N . \end{aligned}$$

In deriving the Codazzi equations (3.1.14) we have to assume that $X(u, v)$ is of class $C^3$ for $(u, v) \in \Omega$.

With the aid of these results it is easy to establish the following two lemmas:

Lemma 5.[21] Let $Z$ be a fixed unit vector such that $(Z, \overline{X}) \neq 0$. Then the function $\rho(u, v) = (Z, X)$ satisfies the Monge-Ampère equation

(3.1.15) $$T \equiv (\rho_{uu} - \left\{{}^{1}{}_{1}{}^{1}\right\}\rho_u - \left\{{}^{1}{}_{2}{}^{1}\right\}\rho_v)(\rho_{vv} - \left\{{}^{2}{}_{1}{}^{2}\right\}\rho_u - \left\{{}^{2}{}_{2}{}^{2}\right\}\rho_v) - (\rho_{uv} - \left\{{}^{1}{}_{1}{}^{2}\right\}\rho_u - \left\{{}^{1}{}_{2}{}^{2}\right\}\rho_v)^2 - K(u, v)[EG - F^2 - (G\rho_u^2 - 2F\rho_u\rho_v + E\rho_v^2)] = 0 .$$

We have

(3.1.16) $$T_{\rho_{uu}} T_{\rho_{vv}} - \tfrac{1}{4} T^2_{\rho_{uv}} = K(u, v)(EG - F^2)(Z, \overline{X})^2 .$$

Proof: From the Gauss equations (3.1.9) we conclude

(3.1.17) $$\begin{aligned} \rho_{uu} - \left\{{}^{1}{}_{1}{}^{1}\right\}\rho_u - \left\{{}^{1}{}_{2}{}^{1}\right\}\rho_v &= L(Z, \overline{X}) \\ \rho_{uv} - \left\{{}^{1}{}_{1}{}^{2}\right\}\rho_u - \left\{{}^{1}{}_{2}{}^{2}\right\}\rho_v &= M(Z, \overline{X}) \\ \rho_{vv} - \left\{{}^{2}{}_{1}{}^{2}\right\}\rho_u - \left\{{}^{2}{}_{2}{}^{2}\right\}\rho_v &= N(Z, \overline{X}) . \end{aligned}$$

Hence with the aid of (3.1.13) we obtain

(3.1.18) $$(\rho_{uu} - \left\{{}^{1}{}_{1}{}^{1}\right\}\rho_u - \left\{{}^{1}{}_{2}{}^{1}\right\}\rho_v)(\rho_{vv} - \left\{{}^{2}{}_{1}{}^{2}\right\}\rho_u - \left\{{}^{2}{}_{2}{}^{2}\right\}\rho_v) - (\rho_{uv} - \left\{{}^{1}{}_{1}{}^{2}\right\}\rho_u - \left\{{}^{1}{}_{2}{}^{2}\right\}\rho_v)^2 = K(u, v)(EG - F^2)(Z, \overline{X})^2 .$$

Now we have

(3.1.19) $$\begin{aligned} (EG - F^2)(Z, \overline{X})^2 &= (Z, X_u \times X_v)^2 \\ &= \begin{vmatrix} Z^2, & ZX_u, & ZX_v \\ X_u Z, & X_u^2, & X_u X_v \\ X_v Z, & X_v X_u, & X_v^2 \end{vmatrix} = \begin{vmatrix} 1 & \rho_u & \rho_v \\ \rho_u & E & F \\ \rho_v & F & G \end{vmatrix} \\ &= EG - F^2 - (G\rho_u^2 - 2F\rho_u\rho_v + E\rho_v^2) . \end{aligned}$$

(3.1.18) together with (3.1.19) imply (3.1.15) and (3.1.16), which proves the lemma.

Lemma 6. Let $X(u, v)$ and the coefficients $E(u, v)$, $F(u, v)$, $G(u, v)$ belong to $C^{3+\varkappa}$ $(0 < \varkappa < 1)$ and $C^4$ for $(u, v) \in \Omega$, respectively. Furthermore, let the independent variables $(u, v)$ be considered as functions of the parameters $(\alpha, \beta)$ which are characteristic with respect to the Monge-Ampère equation (3.1.15), in the manner described in

21. See Darboux [7], § 703, also Bianchi [4], p. 115.

Lemma 3.[22] Then the functions $u = u(\alpha, \beta)$ and $v = v(\alpha, \beta)$ satisfy the following system of partial differential equations:[23]

$$(3.1.20)\qquad \Delta u + \left(\left\{{}^{1}{}_{1}{}^{1}\right\} + \frac{K_u}{2K}\right)(u_\alpha^2 + u_\beta^2) + \left(2\left\{{}^{1}{}_{1}{}^{2}\right\} + \frac{K_v}{2K}\right)(u_\alpha v_\alpha + u_\beta v_\beta) + \left\{{}^{2}{}_{1}{}^{2}\right\}(v_\alpha^2 + v_\beta^2) = 0\,,$$

$$(3.1.20')\qquad \Delta v + \left\{{}^{1}{}_{2}{}^{1}\right\}(u_\alpha^2 + u_\beta^2) + \left(2\left\{{}^{1}{}_{2}{}^{2}\right\} + \frac{K_u}{2K}\right)(u_\alpha v_\alpha + u_\beta v_\beta) + \left(\left\{{}^{2}{}_{2}{}^{2}\right\} + \frac{K_v}{2K}\right)(v_\alpha^2 + v_\beta^2) = 0\,.$$

Proof: From (3.1.17) we conclude that the characteristic form associated with the Darboux equation (3.1.15) is

$$(3.1.21)\qquad \varphi = (Z, \overline{X})(L du^2 + 2M\, du\, dv + N dv^2)\,.$$

Hence on account of Lemma 2 and 3 it is evident that the functions $u(\alpha, \beta)$ and $v(\alpha, \beta)$ satisfy the equations

$$(3.1.22)\qquad \Delta u = \varepsilon\left[\frac{\partial}{\partial u}\left(\frac{N}{\sqrt{LN - M^2}}\right) - \frac{\partial}{\partial v}\left(\frac{M}{\sqrt{LN - M^2}}\right)\right](u_\alpha v_\beta - u_\beta v_\alpha)\,,$$

$$(3.1.22')\qquad \Delta v = \varepsilon\left[\frac{\partial}{\partial v}\left(\frac{L}{\sqrt{LN - M^2}}\right) - \frac{\partial}{\partial u}\left(\frac{M}{\sqrt{LN - M^2}}\right)\right](u_\alpha v_\beta - u_\beta v_\alpha)\,,$$

and

$$(3.1.23)\qquad \begin{aligned} \frac{\varepsilon L}{\sqrt{LN - M^2}} &= \frac{v_\alpha^2 + v_\beta^2}{u_\alpha v_\beta - u_\beta v_\alpha} \\ \frac{\varepsilon M}{\sqrt{LN - M^2}} &= -\frac{u_\alpha v_\alpha + u_\beta v_\beta}{u_\alpha v_\beta - u_\beta v_\alpha} \\ \frac{\varepsilon N}{\sqrt{LN - M^2}} &= \frac{u_\alpha^2 + u_\beta^2}{u_\alpha v_\beta - u_\beta v_\alpha}\,, \quad \text{where } \varepsilon = \pm 1\,. \end{aligned}$$

The idea of the proof consists in eliminating the unknown quantities $L, M, N$ from the differential equations (3.1.22), (3.1.22′) by making use

22. On account of the differentiability hypotheses concerning $X$; $E, F, G$ the conditions of Lemma 3 are all satisfied.

23. See Darboux [7], § 725, also Bianchi [4], § 112. The Darboux equations are obtained from (3.1.20) and (3.1.20′) by the formal substitutions:

$$\gamma = \alpha + i\beta\,, \quad \overline{\gamma} = \alpha - i\beta\,.$$

of the Gauss theorema egregium, the Codazzi equations, and the representations (3.1.23). Let us put for abbreviation

$$(3.1.24) \quad P(u,v) = \varepsilon\left[\frac{\partial}{\partial u}\left(\frac{N}{\sqrt{LN-M^2}}\right) - \frac{\partial}{\partial v}\left(\frac{M}{\sqrt{LN-M^2}}\right)\right],$$

and

$$(3.1.24') \quad Q(u,v) = \varepsilon\left[\frac{\partial}{\partial v}\left(\frac{L}{\sqrt{LN-M^2}}\right) - \frac{\partial}{\partial u}\left(\frac{M}{\sqrt{LN-M^2}}\right)\right].$$

Then on account of the Gauss formula (3.1.13) we have

$$(3.1.25) \quad \varepsilon P(u,v) = \frac{N_u - M_v}{\sqrt{LN-M^2}} - \frac{N}{2\sqrt{LN-M^2}}\left(\frac{\partial \log K}{\partial u} + \frac{\partial \log(EG-F^2)}{\partial u}\right) + \frac{M}{2\sqrt{LN-M^2}}\left(\frac{\partial \log K}{\partial v} + \frac{\partial \log(EG-F^2)}{\partial v}\right),$$

and

$$(3.1.25') \quad \varepsilon Q(u,v) = \frac{L_v - M_u}{\sqrt{LN-M^2}} - \frac{L}{2\sqrt{LN-M^2}}\left(\frac{\partial \log K}{\partial v} + \frac{\partial \log(EG-F^2)}{\partial v}\right) + \frac{M}{2\sqrt{LN-M^2}}\left(\frac{\partial \log K}{\partial u} + \frac{\partial \log(EG-F^2)}{\partial u}\right).$$

Using the Codazzi equations (3.1.14) we obtain

$$\begin{aligned}(3.1.26) \quad \varepsilon P(u,v) &= -\left\{{}^{2}{}_{1}{}^{2}\right\}\frac{L}{\sqrt{LN-M^2}} \\ &+ \left(\left\{{}^{1}{}_{1}{}^{2}\right\} - \left\{{}^{2}{}_{2}{}^{2}\right\} + \frac{1}{2}\frac{\partial \log(EG-F^2)}{\partial v} + \frac{1}{2}\frac{\partial \log K}{\partial v}\right)\frac{M}{\sqrt{LN-M^2}} \\ &+ \left(\left\{{}^{1}{}_{2}{}^{2}\right\} - \frac{1}{2}\frac{\partial \log(EG-F^2)}{\partial u} - \frac{1}{2}\frac{\partial \log K}{\partial u}\right)\frac{N}{\sqrt{LN-M^2}} \\ &= -\left\{{}^{2}{}_{1}{}^{2}\right\}\frac{L}{\sqrt{LN-M^2}} + \left(2\left\{{}^{1}{}_{1}{}^{2}\right\} + \frac{K_v}{2K}\right)\frac{M}{\sqrt{LN-M^2}} \\ &- \left(\left\{{}^{1}{}_{1}{}^{1}\right\} + \frac{K_u}{2K}\right)\frac{N}{\sqrt{LN-M^2}},\end{aligned}$$

and

$$\begin{aligned}(3.1.26') \quad \varepsilon Q(u,v) &= \left(\left\{{}^{1}{}_{1}{}^{2}\right\} - \frac{1}{2}\frac{\partial \log(EG-F^2)}{\partial v} - \frac{1}{2}\frac{\partial \log K}{\partial v}\right)\frac{L}{\sqrt{LN-M^2}} \\ &+ \left(\left\{{}^{1}{}_{2}{}^{2}\right\} - \left\{{}^{1}{}_{1}{}^{1}\right\} + \frac{1}{2}\frac{\partial \log(EG-F^2)}{\partial u} + \frac{1}{2}\frac{\partial \log K}{\partial u}\right)\frac{M}{\sqrt{LN-M^2}} \\ &- \left\{{}^{1}{}_{2}{}^{1}\right\}\frac{N}{\sqrt{LN-M^2}}\end{aligned}$$

$$= -\left(\left\{{}^{2}{}_{2}{}^{2}\right\} + \frac{K_v}{2K}\right)\frac{L}{\sqrt{LN-M^2}} + \left(2\left\{{}^{1}{}_{2}{}^{2}\right\} + \frac{K_u}{2K}\right)\frac{M}{\sqrt{LN-M^2}} - \left\{{}^{1}{}_{2}{}^{1}\right\}\frac{N}{\sqrt{LN-M^2}}.$$

If we now express the functions $\frac{L}{\sqrt{LN-M^2}}$, $\frac{M}{\sqrt{LN-M^2}}$ and $\frac{N}{\sqrt{LN-M^2}}$ in terms of $u_\alpha, \dots, v_\beta$ by means of the relations (3.1.23) and then substitute the resulting expressions for $P$ and $Q$ into the equations (3.1.22) and (3.1.22′), we arrive at the conclusion of the lemma.

### 3.2. A Priori Estimates.

The preceding two lemmas show that the Darboux equation (3.1.15) associated with the given line element $ds^2$ belongs to the type of Monge-Ampère equations considered in Theorem 3. This fact enables us to establish a priori estimates for the second and higher order derivatives of the vector $X(u, v)$ in terms of the coefficients of the metric $ds^2$. We shall first treat the case where the surface $S: X = X(u, v)$ has a boundary (Theorem 5) and then pass to the more special situation where $S$ is closed (Theorem 6). In the latter case, which occurs in the discussion of Weyl's problem, the line element is defined in terms of local parameters on the unit sphere.

Theorem 5.

Hypotheses:

(i) The vector $X(u,v) = (x(u,v), y(u,v), z(u,v))$ is of class $C^{3+\varkappa}$ $(0<\varkappa<1)$ in a domain $\Omega$ of the $uv$-plane and satisfies the equations

$$X_u^2 = E, \quad X_u X_v = F, \quad X_v^2 = G. \tag{3.2.1}$$

(ii) The functions $E(u,v)$, $F(u,v)$ and $G(u,v)$ are of class $C^4$ and satisfy for $(u,v)\in\Omega$ the inequalities

$$\begin{aligned} &|E|, |F|, |G| \leqq \alpha, \\ &|E_u|, \dots, |G_v| \leqq \alpha, \\ &|E_{uu}|, \dots, |G_{vv}| \leqq \alpha, \\ &|E_{uuu}|, \dots, |G_{vvv}| \leqq \alpha, \\ &|E_{uuuu}|, \dots, |G_{vvvv}| \leqq \alpha, \end{aligned} \tag{3.2.2}$$

$$EG - F^2 \geqq \alpha^{-1}, \tag{3.2.3}$$

and

$$K(u, v) \geqq \alpha^{-1}, \tag{3.2.4}$$

where $K(u,v)$ is the Gauss curvature of the line element $ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2$, and $\alpha$ is a fixed positive constant.

(iii) There exists a fixed unit vector $Z$, such that the inequality

$$(3.2.5) \qquad |(Z, \overline{X})| \geqq \gamma > 0$$

holds for $(u,v) \in \Omega$.

Conclusion:

Let $\Omega_\varepsilon$ be the set of all points of $\Omega$ whose distance from the boundary of $\Omega$ exceeds $\varepsilon > 0$, and $\nu$ a real number in the interval $0 < \nu < 1$. Then we have the estimates

$$(3.2.6) \qquad |X_{uu}|, \ldots, |X_{vv}| \leqq \tau_0(\alpha, \gamma, \varepsilon) < \infty \quad \text{for} \quad (u,v) \in \Omega_\varepsilon,$$

$$(3.2.7) \qquad |X_{uuu}|, \ldots, |X_{vvv}| \leqq \tau_1(\alpha, \gamma, \varepsilon) < \infty \quad \text{for} \quad (u,v) \in \Omega_\varepsilon,$$

and

(3.2.8)

$$|X_{uuu}(u_1, v_1) - X_{uuu}(u_0, v_0)| \leqq \tau_2(\alpha, \gamma, \varepsilon, \nu)\,[(u_1 - u_0)^2 + (v_1 - v_0)^2]^{\nu/2}$$

$$\vdots$$

$$|X_{vvv}(u_1, v_1) - X_{vvv}(u_0, v_0)| \leqq \tau_2(\alpha, \gamma, \varepsilon, \nu)\,[(u_1 - u_0)^2 + (v_1 - v_0)^2]^{\nu/2}$$

for $(u_0, v_0) \in \Omega_\varepsilon$ and $(u_1, v_1) \in \Omega_\varepsilon$, where $\tau_2(\alpha, \gamma, \varepsilon, \nu)$ is a fixed positive number.

**Proof**: The Darboux equation (3.1.15) satisfied by the function $\rho(u,v) = (Z, X(u,v))$ can be written in the form

$$(3.2.9) \qquad a\rho_{uu} + 2b\rho_{uv} + c\rho_{vv} + (\rho_{uu}\rho_{vv} - \rho_{uv}^2) = e,$$

where the coefficients $a, b, c, e$ are given by the expressions

$$(3.2.10) \qquad \begin{aligned} a &= -\left\{{}^{2}{}_{1}{}^{2}\right\}\rho_u - \left\{{}^{2}{}_{2}{}^{2}\right\}\rho_v, \\ b &= \left\{{}^{1}{}_{1}{}^{2}\right\}\rho_u + \left\{{}^{1}{}_{2}{}^{2}\right\}\rho_v, \\ c &= -\left\{{}^{1}{}_{1}{}^{1}\right\}\rho_u - \left\{{}^{1}{}_{2}{}^{1}\right\}\rho_v, \\ e &= (\left\{{}^{1}{}_{1}{}^{2}\right\}\rho_u + \left\{{}^{1}{}_{2}{}^{2}\right\}\rho_v)^2 \\ &\quad - (\left\{{}^{1}{}_{1}{}^{1}\right\}\rho_u + \left\{{}^{1}{}_{2}{}^{1}\right\}\rho_v)(\left\{{}^{2}{}_{1}{}^{2}\right\}\rho_u + \left\{{}^{2}{}_{2}{}^{2}\right\}\rho_v) \\ &\quad + K(u,v)\,[EG - F^2 - (G\rho_u^2 - 2F\rho_u\rho_v + E\rho_v^2)]. \end{aligned}$$

Furthermore, on account of (3.1.16) we have

$$(3.2.11) \qquad d = ac - b^2 + e = K(u,v)(EG - F^2)(Z, \overline{X})^2.$$

Now since for $(u,v) \in \Omega$ the estimates

$$|\rho_u| \leqq |X_u| \leqq \alpha^{1/2}, \tag{3.2.12}$$

and

$$|\rho_v| \leqq |X_v| \leqq \alpha^{1/2} \tag{3.2.12'}$$

hold, we conclude from the hypotheses of the theorem that the coefficients $a, b, c, e$ considered as functions of the variables $u, v, \rho, \rho_u, \rho_v$ satisfy for

$$(u, v, \rho, \rho_u, \rho_v) \in \Sigma = \{(u, v, \rho(u, v), \ \rho_u(u, v), \ \rho_v(u, v)), \ (u, v) \in \Omega\}$$

the inequalities

$$|a|, \dots, |e| \leqq \alpha_0(\alpha) < \infty, \tag{3.2.13}$$

$$|a_u|, \dots, |e_{\rho_v}| \leqq \alpha_1(\alpha) < \infty, \tag{3.2.14}$$

$$|a_{uu}|, \dots, |e_{\rho_v \rho_v}| \leqq \alpha_2(\alpha) < \infty, \tag{3.2.15}$$

and

$$d = ac - b^2 + e \geqq \alpha^{-2} \gamma^2 > 0. \tag{3.2.16}$$

From these facts and Lemma 6 it follows that the Darboux equation (3.2.9) satisfies all the conditions of Theorem 3 with

$$\begin{aligned} \alpha_0 &= \alpha_0(\alpha) + \alpha^2 \gamma^{-2}, \\ \beta_0 &= \beta_0(\alpha) < \infty, \\ \beta_1 &= \beta_1(\alpha) < \infty, \\ \gamma_1 &= \alpha^{1/2}. \end{aligned} \tag{3.2.17}$$

Hence from (2.2.54) we deduce estimates of the form

$$|\rho_{uu}|, |\rho_{uv}|, |\rho_{vv}| \leqq \tau'_0(\alpha, \gamma, \varepsilon) < \infty, \tag{3.2.18}$$

where $(u, v)$ is an arbitrary point in $\Omega_\varepsilon$ and $\tau'_0(\alpha, \gamma, \varepsilon)$ is a fixed positive number. Now applying Theorem 1 we obtain the inequalities

$$|\rho_{uuu}|, \dots, |\rho_{vvv}| \leqq \tau'_1(\alpha, \gamma, \varepsilon) < \infty \tag{3.2.19}$$

for $(u, v) \in \Omega_\varepsilon$, and

(3.2.20)

$$|\rho_{uuu}(u_1, v_1) - \rho_{uuu}(u_0, v_0)| \leqq \tau'_2(\alpha, \gamma, \varepsilon, \nu) \, [(u_1 - u_0)^2 + (v_1 - v_0)^2]^{\nu/2}$$

$$\vdots$$

$$|\rho_{vvv}(u_1, v_1) - \rho_{vvv}(u_0, v_0)| \leqq \tau'_2(\alpha, \gamma, \varepsilon, \nu) \, [(u_1 - u_0)^2 + (v_1 - v_0)^2]^{\nu/2}$$

where $(u_0, v_0) \in \Omega_\varepsilon$, $(u_1, v_1) \in \Omega_\varepsilon$, $0 < \nu < 1$, and $\tau'_2(\alpha, \gamma, \varepsilon, \nu)$ is a fixed constant.

The proof of the theorem is now completed by taking account of the systems of equations:

$$(3.2.21) \qquad \begin{cases} ZX_{uu} = \rho_{uu} \\ X_u X_{uu} = \frac{1}{2} E_u \\ X_v X_{uu} = F_u - \frac{1}{2} E_v , \end{cases}$$

$$(3.2.21') \qquad \begin{cases} ZX_{uv} = \rho_{uv} \\ X_u X_{uv} = \frac{1}{2} E_v \\ X_v X_{uv} = \frac{1}{2} G_u , \end{cases}$$

and

$$(3.2.21'') \qquad \begin{cases} ZX_{vv} = \rho_{vv} \\ X_u X_{vv} = F_v - \frac{1}{2} G_u \\ X_v X_{vv} = \frac{1}{2} G_v \end{cases}$$

and the fact that the determinant $|(Z , X_u \times X_v)|$ is $\geqq \alpha^{-1/2} \gamma > 0$.

We shall now apply the preceding theorem in order to establish the a priori estimates required for an analytical treatment of the Weyl embedding problem.[24] To fix the ideas we consider two overlapping regions $\Sigma_1$ and $\Sigma_2$ on the unit sphere $\Sigma$ which are bounded by parallels of latitude, such that $\Sigma_1$ contains the north pole and $\Sigma_2$ contains the south pole, and introduce a parameter system in each. The parameter system $(u_1 , v_1)$ for $\Sigma_1$ is obtained by stereographic projection of $\Sigma$ from the south pole onto the plane tangent to $\Sigma$ at the north pole, and similarly the parameter system $(u_2 , v_2)$ is constructed by stereographic projection of $\Sigma$ from the north pole onto the plane tangent to $\Sigma$ at the south pole. Evidently the parameter domains $\Omega_1$ and $\Omega_2$, which correspond to $\Sigma_1$ and $\Sigma_2$, are circular discs. Every point of $\Sigma$ lying in the intersection of $\Sigma_1$ and $\Sigma_2$ is represented by two points $(u_1 , v_1)$ and $(u_2 , v_2)$ lying in $\Omega_1$ and $\Omega_2$, respectively. Furthermore, $(u_1 , v_1)$ and $(u_2 , v_2)$ are connected by an analytic transformation having a non-vanishing jacobian. Consider now a vector-valued function $X = (x , y , z)$ on $\Sigma$. In each of the parameter domains $\Omega_i$, $X$ becomes an ordinary function $X_i (u_i , v_i)$, $(i = 1, 2)$, and we have

$$(3.2.22) \qquad X_1 (u_1 , v_1) = X_2 (u_2 , v_2)$$

for any pairs $(u_1 , v_1)$ and $(u_2 , v_2)$ representing the same point on $\Sigma$. $X$ is said to be of class $C^{k+\nu}$ $(k = 0, 1, \dots ;\ 0 \leqq \nu < 1)$ on $\Sigma$, if the functions

24. See Nirenberg [25], in particular § 10 and § 11.

$X_i(u_i, v_i)$ $(i = 1, 2)$ belong to $C^{k+\nu}$ in $\Omega_i$. In the same way a quadratic differential form (3.2.23):

$$ds^2 = \begin{cases} E_1(u_1, v_1)\,du_1^2 + 2F_1(u_1, v_1)\,du_1 dv_1 + G_1(u_1, v_1)\,dv_1^2, & (u_1, v_1) \in \Omega_1, \\ E_2(u_2, v_2)\,du_2^2 + 2F_2(u_2, v_2)\,du_2 dv_2 + G_2(u_2, v_2)\,dv_2^2, & (u_2, v_2) \in \Omega_2, \end{cases}$$

is said to be of class $C^{k+\nu}$, if all the coefficients $E_1(u_1, v_1), \ldots, G_2(u_2, v_2)$ belong to $C^{k+\nu}$ in the respective parameter domains. In order that $ds^2$ be invariantly defined on $\Sigma$ we have to assume that the equation

$$(3.2.24) \qquad \begin{aligned} &E_1(u_1, v_1)\,du_1^2 + 2F_1(u_1, v_1)\,du_1\,dv_1 + G_1(u_1, v_1)\,dv_1^2 \\ &= E_2(u_2, v_2)\,du_2^2 + 2F_2(u_2, v_2)\,du_2\,dv_2 + G_2(u_2, v_2)\,dv_2^2 \end{aligned}$$

holds for any pairs $(u_1, v_1)$ and $(u_2, v_2)$ representing the same point on $\Sigma$. Furthermore, we shall require that the differential form $ds^2$ be positive definite throughout $\Sigma$. Our object consists in deriving a priori estimates for the solutions of the equation

$$(3.2.25) \qquad (dX)^2 = ds^2,$$

where $ds^2$ is a given line element of positive Gauss curvature $K$, defined on $\Sigma$. Geometrically speaking, the equation (3.2.25) expresses the fact that the line element $ds^2$ is realized by a closed convex surface in the three-dimensional space (Weyl's embedding problem).

It is convenient to state the next theorem in the language of functional analysis, as Nirenberg [25] has done. For this purpose we shall introduce certain norms for functions $X$ and quadratic differential forms $ds^2$ defined on $\Sigma$. Let us put

$$(3.2.26) \qquad \begin{aligned} \|X\|_{k+\nu} = &\underset{(u_1, v_1) \in \Omega_1}{\text{l. u. b.}} |X_1| + \underset{(u_2, v_2) \in \Omega_2}{\text{l. u. b.}} |X_2| + \ldots \\ &+ \underset{(u_1, v_1) \in \Omega_1}{\text{l. u. b.}} \left| \frac{\partial^k X_1}{\partial u_1^k} \right| + \ldots + \underset{(u_2, v_2) \in \Omega_2}{\text{l. u. b.}} \left| \frac{\partial^k X_2}{\partial v_2^k} \right| \\ &+ \underset{\substack{(u_1, v_1) \in \Omega_1 \\ (u'_1, v'_1) \in \Omega_1 \\ (u_1, v_1) \neq (u'_1, v'_1)}}{\text{l. u. b.}} \frac{\left| \dfrac{\partial^k X_1}{\partial u_1^k}(u_1, v_1) - \dfrac{\partial^k X_1}{\partial u_1^k}(u'_1, v'_1) \right|}{[(u_1 - u'_1)^2 + (v_1 - v'_1)^2]^{\nu/2}} + \ldots \\ &+ \underset{\substack{(u_2, v_2) \in \Omega_2 \\ (u'_2, v'_2) \in \Omega_2 \\ (u_2, v_2) \neq (u'_2, v'_2)}}{\text{l. u. b.}} \frac{\left| \dfrac{\partial^k X_2}{\partial v_2^k}(u_2, v_2) - \dfrac{\partial^k X_2}{\partial v_2^k}(u'_2, v'_2) \right|}{[(u_2 - u'_2)^2 + (v_2 - v'_2)^2]^{\nu/2}} \end{aligned}$$

for $X \in C^{k+\nu}$ $(k=0, 1, \dots;\ 0 \leqq \nu < 1)$, and similarly

(3.2.26′) $$\| ds^2 \|_{k+\nu} = \underset{(u_1, v_1) \in \Omega_1}{\text{l.u.b.}} |E_1| + \dots + \underset{(u_2, v_2) \in \Omega_2}{\text{l.u.b.}} |G_2| + \dots$$
$$+ \underset{(u_1, v_1) \in \Omega_1}{\text{l.u.b.}} \left| \frac{\partial^k E_1}{\partial u_1^k} \right| + \dots + \underset{(u_2, v_2) \in \Omega_2}{\text{l.u.b.}} \left| \frac{\partial^k G_2}{\partial v_2^k} \right|$$
$$+ \underset{\substack{(u_1, v_1) \in \Omega_1 \\ (u'_1, v'_1) \in \Omega_1 \\ (u_1, v_1) \neq (u'_1, v'_1)}}{\text{l.u.b.}} \frac{\left| \dfrac{\partial^k E_1}{\partial u_1^k}(u_1, v_1) - \dfrac{\partial^k E_1}{\partial u_1^k}(u'_1, v'_1) \right|}{[(u_1 - u'_1)^2 + (v_1 - v'_1)^2]^{\nu/2}} + \dots$$
$$+ \underset{\substack{(u_2, v_2) \in \Omega_2 \\ (u'_2, v'_2) \in \Omega_2 \\ (u_2, v_2) \neq (u'_2, v'_2)}}{\text{l.u.b.}} \frac{\left| \dfrac{\partial^k G_2}{\partial v_2^k}(u_2, v_2) - \dfrac{\partial^k G_2}{\partial v_2^k}(u'_2, v'_2) \right|}{[(u_2 - u'_2)^2 + (v_2 - v'_2)^2]^{\nu/2}}$$

for $ds^2 \in C^{k+\nu}$ $(k=0, 1;\ 0 \leqq \nu < 1)$. Then the following theorem holds:

Theorem 6. Let $X$ be of class $C^{3+\varkappa}$ $(0<\varkappa<1)$ on $\Sigma$ and satisfy the equation

(3.2.27) $$(dX)^2 = ds^2,$$

where $ds^2$ is a given line element of class $C^4$ on $\Sigma$. Furthermore, let the following inequalities be satisfied:

(3.2.28) $$\| ds^2 \|_4 \leqq \alpha,$$

(3.2.29) $$E_i G_i - F_i^2 \geqq \alpha^{-1}, \qquad (i = 1, 2),$$

(3.2.30) $$K \geqq \alpha^{-1},$$

where $K$ is the Gauss curvature of $ds^2$ and $\alpha$ is a finite positive constant. Then there exists a fixed vector $Y$ such that we have for $0<\nu<1$ an estimate of the form:

(3.2.31) $$\| X - Y \|_{3+\nu} \leqq \omega(\alpha, \nu) < \infty.$$

In order to prove the above theorem, use is made of the following lemma on closed convex surfaces, which is an immediate consequence of the results established in Blaschke [5], § 25 and § 26.

Lemma 7. Let $S$ be a closed regular surface of class $C^2$. Furthermore, let the Gauss curvature $K$ at each point $X$ of $S$ satisfy the inequality

(3.2.32) $$0<\beta_1 \leqq K \leqq \beta_2 < \infty .$$

Then there exists a fixed vector $Y$ such that at each point $X \in S$ the estimates

(3.2.33) $$|X - Y| \leqq \sigma(\beta_1) < \infty ,$$

and

(3.2.34) $$|(X - Y , \overline{X})| \geqq \sigma^*(\beta_1 , \beta_2) > 0$$

hold, where $\overline{X}$ is the unit normal on $S$.

Proof of Theorem 6.

From (3.2.28) and (3.2.29) it follows that the Gauss curvature $K$ at each point of the surface $S$ realizing $ds^2$ satisfies an inequality of the form :

(3.2.35) $$K \leqq \beta(\alpha) < \infty .$$

If we combine this with (3.2.30) and then apply Lemma 7, we infer the existence of a fixed vector $Y$ such that at each point $X \in S$ the estimates

(3.2.36) $$|X - Y| \leqq \sigma^+(\alpha) < \infty ,$$

and

(3.2.37) $$|(X - Y , \overline{X})| \geqq \sigma^-(\alpha) > 0$$

hold with

(3.2.38) $$\sigma^+(\alpha) = \sigma(\alpha^{-1}) ,$$

and

(3.2.39) $$\sigma^-(\alpha) = \sigma^*(\alpha^{-1} , \beta(\alpha)) .$$

Now let $\Omega$ be one of the parameter domains $\Omega_i$ $(i = 1, 2)$, and write $u , v$; $X(u , v)$; $E(u , v)$, $F(u , v)$, $G(u , v)$ in place of $u_i , v_i$; $X_i(u_i , v_i)$; $E_i(u_i , v_i)$, $F_i(u_i , v_i)$ and $G_i(u_i , v_i)$. Furthermore, let $\overline{\Omega}$ be any closed subdomain of $\Omega$ whose distance from the boundary of $\Omega$ exceeds a positive number $\varepsilon$. Then from (3.2.26) and the construction of the parameter systems $(u_i , v_i)$ it is evident that our theorem is proved once we can establish estimates of the following kind:

(3.2.40) $$|X_{uu}|, \ldots , |X_{vv}| \leqq \omega_0(\alpha , \varepsilon) < \infty,$$

(3.2.41) $$|X_{uuu}|, \ldots , |X_{vvv}| \leqq \omega_1(\alpha , \varepsilon) < \infty \quad \text{for} \quad (u , v) \in \overline{\Omega} ,$$

and

(3.2.42) $$|X_{uuu}(u , v) - X_{uuu}(u' , v')| \leqq \omega_2(\alpha , \varepsilon , \nu)\,[(u-u')^2 + (v-v')^2]^{\nu/2}$$

$$|X_{vvv}(u , v) - X_{vvv}(u' , v')| \leqq \omega_2(\alpha , \varepsilon , \nu)\,[(u-u')^2 + (v-v')^2]^{\nu/2}$$

for $(u,v) \in \bar{\Omega}$, $(u',v') \in \bar{\Omega}$, and $0 < \nu < 1$, where $\omega_2(\alpha, \varepsilon, \nu) < \infty$.

In order to prove these inequalities, we first consider the disc $\Gamma = \{(u-u_0)^2 + (v-v_0)^2 < \delta^2\}$, where $(u_0, v_0)$ is an arbitrary point in $\bar{\Omega}$, and

$$(3.2.43) \qquad \delta = \delta(\alpha, \varepsilon) = \operatorname{Min}\left(\varepsilon, \frac{\sigma^-(\alpha)}{2\alpha^{1/2}}\right).$$

Obviously $\Gamma$ is contained in $\Omega$ and by (3.2.28) we have

$$(3.2.44) \qquad |X(u,v) - X(u_0,v_0)| \leqq \tfrac{1}{2}\sigma^-(\alpha) \quad \text{for} \quad (u,v) \in \Gamma.$$

Combining this with (3.2.37) we obtain

$$(3.2.45) \qquad |(X(u_0,v_0) - Y, \bar{X}(u,v))| \geqq \tfrac{1}{2}\sigma^-(\alpha) \quad \text{for} \quad (u,v) \in \Gamma.$$

If we now put

$$(3.2.46) \qquad Z = \frac{X(u_0,v_0) - Y}{|X(u_0,v_0) - Y|},$$

then from (3.2.45) and (3.2.36) we conclude that for $(u, v) \in \Gamma$ the inequality

$$(3.2.47) \qquad |(Z, \bar{X})| \geqq \frac{\sigma^-(\alpha)}{2\sigma^+(\alpha)} > 0$$

holds. Hence Theorem 5 becomes applicable and yields the inequalities:

$$(3.2.48) \qquad |X_{uu}(u_0,v_0)|, \ldots, |X_{vv}(u_0,v_0)| \leqq \tau_0\left(\alpha, \frac{\sigma^-(\alpha)}{2\sigma^+(\alpha)}, \delta(\alpha,\varepsilon)\right);$$

$$(3.2.49) \qquad |X_{uuu}(u_0,v_0)|, \ldots, |X_{vvv}(u_0,v_0)| \leqq \tau_1\left(\alpha, \frac{\sigma^-(\alpha)}{2\sigma^+(\alpha)}, \delta(\alpha,\varepsilon)\right),$$

and

$$(3.2.50) \qquad |X_{uuu}(u_1, v_1) - X_{uuu}(u_0, v_0)| \leqq \tau_2\left(\alpha, \frac{\sigma^-(\alpha)}{2\sigma^+(\alpha)}, \frac{1}{2}\delta(\alpha,\varepsilon), \nu\right) \cdot [(u_1-u_0)^2 + (v_1-v_0)^2]^{\nu/2}.$$

$$X_{vvv}(u_1, v_1) - X_{vvv}(u_0, v_0)| \leqq \tau_2\left(\alpha, \frac{\sigma^-(\alpha)}{2\sigma^+(\alpha)}, \frac{1}{2}\delta(\alpha,\varepsilon), \nu\right) [(u_1-u_0)^2 + (\ _1 - v_0)^2]^{\nu/2}$$

for

$$(u_1-u_0)^2 + (v_1-v_0)^2 \leqq \tfrac{1}{4}\delta(\alpha, \varepsilon)^2.$$

If we set

(3.2.51) $$\omega_k(\alpha, \varepsilon) = \tau_k\left(\alpha, \frac{\sigma^-(\alpha)}{2\sigma^+(\alpha)}, \delta(\alpha, \varepsilon)\right), \quad (k=0, 1),$$

and

(3.2.52)

$$\omega_2(\alpha, \varepsilon, \nu) = \operatorname{Max}\left\{\tau_2\left(\alpha, \frac{\sigma^-(\alpha)}{2\sigma^+(\alpha)}, \frac{1}{2}\delta(\alpha, \varepsilon), \nu\right), \frac{2\tau_1\left(\alpha, \frac{\sigma^-(\alpha)}{2\sigma^+(\alpha)}, \delta(\alpha, \varepsilon)\right)}{(\frac{1}{2}\delta(\alpha, \varepsilon))^\nu}\right\}$$

we arrive at the inequalities (3.2.40) — (3.2.42), which proves our theorem.

## Concluding Remarks.

The proof of Theorem 6 can be somewhat simplified by using the auxiliary function $\rho(u, v) = \frac{1}{2}(x^2 + y^2 + z^2)$ instead of the coordinates, as Lewy [18] and Nirenberg [25] have done. Since the characteristic form of the corresponding Monge-Ampère equation (25) is proportional to the second fundamental form of the surface $X = X(u, v)$, Lemma 6, hence Theorem 3, can be applied and yield a priori estimates of the type (3.2.31). Furthermore, it follows from the differentiability theorem of Nirenberg-Hopf, mentioned in the introduction, that in hypothesis (i) of Theorems 1—6 the assumption

$$z(x, y) \in C^{3+\varkappa} \quad (\text{or } X(u, v) \in C^{3+\varkappa})$$

can be replaced by the weaker requirement that the respective solutions belong to $C^2$ in their domain of existence.

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25. See Darboux [7], § 707, also Bianchi [4], pp. 116. A simple derivation of this equation is given by Nirenberg [25], § 3.

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(Added in proof)

In his recent note "On the regularity of convex surfaces with a regular metric in spaces of constant curvature" (in Russian), Doklady Akad. Nauk. SSSR (1958), vol. 122, No. 2, pp. 186—187, A. V. Pogorelow extends some of his previous results to the case where the embedding space of the line element has constant curvature. Applying Lemma 3 it can be shown that the Monge-Ampère equation, on which his considerations are based, satisfies the hypotheses of Theorem 3 of the present paper.