ISRAEL JOURNAL OF MATHEMATICS, Vol. 62, No. 2, 1988

FIXED POINT VARIETIES ON AFFINE FLAG MANIFOLDS

BY

D. KAZHDAN[a] AND G. LUSZTIG[b]

[a]*Department of Mathematics, Harvard University, Cambridge, MA 02138, USA; and*
[b]*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

ABSTRACT

We study the space of Iwahori subalgebras containing a given element of a semisimple Lie algebra over $\mathbf{C}((\varepsilon))$. We also define and study a map from nilpotent orbits in a semisimple Lie algebra over $\mathbf{C}$ to conjugacy classes in the Weyl group.

TABLE OF CONTENTS

§0. Introduction

Let G be a semisimple, simply connected algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. We denote by $\mathscr{B}$ the variety of Borel subalgebras of $\mathfrak{g}$. For any nilpotent element $N_0 \in \mathfrak{g}$, we consider the closed subvariety $\mathscr{B}_{N_0}$ of $\mathscr{B}$ consisting

[†] Both authors were supported in part by the National Science Foundation.
Received June 7, 1987 and in revised form December 31, 1987

of all Borel subalgebras containing N_0. The geometry of the varieties $\mathscr{B}_{N_0}$ has been investigated by Springer, Steinberg, Spaltenstein and others; it has interesting applications to representation theory.

In this paper we shall begin the study of the affine analogue of this situation. Let $F = \mathbf{C}((\varepsilon))$ be the field of formal power series, $A = \mathbf{C}[[\varepsilon]]$ the ring of integers of F, $\hat{G} = G(F)$, $\mathfrak{g}_F = \mathfrak{g} \otimes_{\mathbf{C}} F$, $\hat{\mathscr{B}}$ the set of Iwahori subalgebras of $\mathfrak{g}_F$. It is known that $\hat{\mathscr{B}}$ is an infinite dimensional algebraic variety over $\mathbf{C}$; more precisely, it is an increasing union of ordinary projective algebraic varieties over $\mathbf{C}$.

For any $N \in \mathfrak{g}_F$ we consider the subset $\hat{\mathscr{B}}_N$ of $\hat{\mathscr{B}}$ consisting of all Iwahori subalgebras containing N. We shall restrict ourselves to the case where N is *topologically nilpotent* in the sense that $\mathrm{ad}(N)^r \to 0$ in $\mathrm{End}\, \mathfrak{g}_F$ for $r \to \infty$. (We then say that N is a nil-element.) This condition implies in particular that $\hat{\mathscr{B}}_N$ is non-empty. We show that $\hat{\mathscr{B}}_N$ is infinite dimensional unless N is regular semisimple (Lemma 2.5). If N is regular semisimple, then $\hat{\mathscr{B}}_N$ is a locally finite union of ordinary irreducible projective algebraic varieties over $\mathbf{C}$, all of the same dimension; moreover, there is a free abelian group Λ_N of finite rank which acts on $\hat{\mathscr{B}}_N$ without fixed points and $\Lambda_N \backslash \hat{\mathscr{B}}_N$ is an algebraic variety (§3, §4). We say that N is *elliptic* if its centralizer in $\hat{G}$ is an anisotropic torus; when N is elliptic we have $\Lambda_N = \{e\}$, so that $\hat{\mathscr{B}}_N$ is an algebraic variety (it has finitely many irreducible components).

Let X be the set of all $\hat{G}$-conjugates of $\mathfrak{g}_A = \mathfrak{g} \otimes A \subset \mathfrak{g}_F$. This is just like $\hat{\mathscr{B}}$ an increasing union of projective algebraic varieties over $\mathbf{C}$ and there is a canonical $\hat{G}$-invariant map $p : \hat{\mathscr{B}} \to X$. We denote by X_N the set of subalgebras in X which contain N. Then p maps $\hat{\mathscr{B}}_N$ onto X_N. In §4 we show that if N is a regular semisimple nil-element of $\mathfrak{g}_F$ then there exists some irreducible component of $\hat{\mathscr{B}}_N$ which has the same dimension (over $\mathbf{C}$) as its image under p. It follows that $\dim \hat{\mathscr{B}}_N = \dim X_N$. This implies that N is $\hat{G}$-conjugate to an element of $\mathfrak{g}_A$ whose image in $\mathfrak{g}_A/\varepsilon\mathfrak{g}_A$ is regular nilpotent.

Let $\mathfrak{h}_N$ be the unique Cartan subalgebra of $\mathfrak{g}_F$ containing N. The $\hat{G}$-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_F$ are parameterized by conjugacy classes in the Weyl group (see §1) hence to $\mathfrak{h}_N$ corresponds an element $\sigma(N)$ in the Weyl group of $\mathfrak{h}_N$, well defined up to conjugacy. It is known that the field $\bar{F} = \bigcup_{n \geqq 1} \mathbf{C}((\varepsilon^{1/n}))$ is an algebraic closure of F. Let $v : \bar{F} \to Q \cup \infty$ be the valuation defined by $v(a_0\varepsilon^{m/n} + \text{higher powers of } \varepsilon) = m/n$, for $a_0 \in \mathbf{C}^*$ and $v(0) = \infty$. We conjecture that for any regular semisimple nil element $N \in \mathfrak{g}_F$ we have

$$2 \dim \hat{\mathscr{B}}_N = v(\det_F(\mathrm{ad}\, N : \mathfrak{g}_F/\mathfrak{h}_N \to \mathfrak{g}_F/\mathfrak{h}_N)) - \mathrm{rank}_F(\sigma(N) - 1 : \mathfrak{h}_N \to \mathfrak{h}_N).$$

The analysis in §5 reduces the verification of this conjecture to the case where N is elliptic.

Now let N_0 be a nilpotent element in $\mathfrak{g}$. If $Y \in \mathfrak{g}_A$, then $N = N_0 + \varepsilon Y$ is a nil-element of $\mathfrak{g}_F$ and we show that for "almost all" choices of Y, N is a regular semisimple nil-element of $\mathfrak{g}_F$, the conjugacy class of $\sigma(N)$ in the Weyl group is independent of Y, and that $\dim \hat{\mathscr{B}}_N = \dim \mathscr{B}_{N_0}$. (In fact $\mathscr{B}_{N_0}$ is naturally imbedded in $\hat{\mathscr{B}}_N$ as a union of irreducible components of $\hat{\mathscr{B}}_N$.) This gives a map $N_0 \to \sigma(N)$ from the set of nilpotent orbits in $\mathfrak{g}$ to the set of conjugacy classes in the Weyl group. We show that this map takes "distinguished" ([BC]) nilpotent orbits to Weyl group elements without eigenvalue 1. We also compute explicitly this map in a number of cases and show that it restricts to a map defined in [CE] on a certain subset of the set of nilpotent orbits. Our map is bijective in type A and seems to be injective (but not bijective) for other types.

We hope that the study of $\hat{\mathscr{B}}_N$ will be useful in the character theory of semisimple groups over p-adic fields; in §10 we state some results in this direction.

We wish to thank N. Spaltenstein for some very useful comments.

§1. Conjugacy classes of Cartan subalgebras and conjugacy classes in the Weyl group

Let $F = \mathbf{C}((\varepsilon))$ be the field of formal power series, and $\bar{F}$ its algebraic closure. It is well known that $\bar{F}$ is the union of the subfields $F_n = F(\varepsilon^{1/n})$. In particular, (1) the Galois group $\Gamma = \mathrm{Gal}(\bar{F}/F)$ is (non-canonically) isomorphic to the profinite completion $\hat{\mathbf{Z}}$ of $\mathbf{Z}$, and (2) $\dim F = 1$, namely F is a field of cohomological dimension 1. Therefore (see [S1], p. 170), for any torus T over F we have $H^1(\Gamma, T(\bar{F})) = 0$. Let N be an algebraic F-group such that its connected component is a torus T. Denote by W the quotient group.

LEMMA 1. *The map* $H^1(\Gamma, N(\bar{F})) \xrightarrow{p} H^1(\Gamma, W)$ *is an isomorphism.*

PROOF. Since $\dim F = 1$, it follows from Corollary to Proposition I.41 in [S2] that p is surjective. Fix $\beta \in H^1(\Gamma, W)$, and denote by T^β a form of T over F which corresponds to the action of Γ on $T(\bar{F})$ given by

$$\gamma : t \to \beta(\gamma) t^\gamma \beta(\gamma)^{-1} \qquad \text{for } \gamma \in \Gamma, \quad t \in T(\bar{F}).$$

It follows from [S2], I.5.5, that $p^{-1}(\beta)$ is a quotient of $H^1(\Gamma, T^\beta(\bar{F}))$. As noted above, the last group is trivial. Hence Lemma 1 is proved.

LEMMA 2. *Let G be a reductive algebraic group over* $\mathbf{C}$, $\mathfrak{g}$ *the Lie algebra of* G, $\hat{G} = G(F)$, $\mathfrak{g}_F = \mathfrak{g} \otimes_{\mathbf{C}} F$. *Then the set of conjugacy classes of Cartan subalgebras in* $\mathfrak{g}_F$ *is in one-to-one correspondence with the set of conjugacy classes in the Weyl group* W *of* $\mathfrak{g}$.

PROOF. Let $C \subset \mathfrak{g}$ be a (split) Cartan subalgebra, and N the normalizer of C in G. It is well known that the set of conjugacy classes of maximal tori in $\hat{G}$ is in one-to-one correspondence with the set $H^1(\Gamma, N(\bar{F}))$. By Lemma 1, the map $p : H^1(\Gamma, N(\bar{F})) \to H^1(\Gamma, W)$ is an isomorphism. Now choose an isomorphism $\phi : \Gamma \xrightarrow{\sim} \hat{\mathbf{Z}}$. Since Γ acts trivially on W, $H^1(\Gamma, W)$ is isomorphic to the quotient of $\mathrm{Hom}(\hat{\mathbf{Z}}, W)$ under conjugation by W. Since $\mathrm{Hom}(\hat{\mathbf{Z}}, W) = W$ this quotient is the set of conjugacy classes of W. It is well known [(Sp 3]) that for any w in W and integer n prime to the order of W, the element w^n is conjugate to w in W. Therefore the isomorphism between $H^1(\Gamma, W)$ and the set of conjugacy classes of W does not depend on a choice of an isomorphism $\phi : \Gamma \xrightarrow{\sim} \hat{\mathbf{Z}}$. Lemma 2 is proved.

DEFINITION. A Cartan subalgebra which corresponds to a conjugacy class γ in W by the correspondence of the proof of Lemma 2 is said to be of *type* γ.

For any conjugacy class γ in W, we denote by $\mathfrak{g}_F^\gamma$ the subset of regular semisimple elements N in $\mathfrak{g}_F$ such that the centralizer $\hat{\mathbf{Z}}_N$ of N in $\mathfrak{g}_F$ is a Cartan subalgebra of type γ.

Let T be a torus over F. Put $X_*(T) = \mathrm{Hom}_F(G_m, T)$ and $\bar{X}_*(T) = \mathrm{Hom}_{\bar{F}}(G_m, T)$, where G_m is the multiplicative group. It is clear that (1) $\bar{X}_*(T)$ is isomorphic to $\mathbf{Z}^d$, where $d = \dim_F T$; (2) Γ acts on $\bar{X}_*(T)$, and $X_*(T)$ is the subgroup of invariant elements. We say that T *is elliptic* if $X_*(T) = \langle e \rangle$. We denote by Λ_T (or Λ) the subgroup consisting of all $x(\varepsilon)$ in T where x ranges over $X_*(T)$.

§2. $\hat{G}/K$ as a union of algebraic varieties

Let F be the field $\mathbf{C}((\varepsilon))$, A the ring $\mathbf{C}[[\varepsilon]]$ of integers in F, and $v : F^* \to \mathbf{Z}$ the valuation with $v(\varepsilon) = 1$. Let G be a reductive $\mathbf{C}$-group, $\mathfrak{g}$ its Lie algebra, $\hat{G} = G(F)$, $K = G(A)$, $\mathfrak{g}_F = \mathfrak{g} \otimes_{\mathbf{C}} F$, $\mathfrak{g}_A = \mathfrak{g} \otimes_{\mathbf{C}} A$ and $p : \mathfrak{g}_A \to \mathfrak{g}$ the natural projection. We say that an element $N \in \mathfrak{g}_F$ is *nil* if $(\operatorname{ad} N)^r \to 0$ for $r \to \infty$.

LEMMA 1. (a) *The element* $N \in \hat{\mathfrak{g}}_F$ *is nil if and only if there exists* $g \in \hat{G}$ *such that* $\operatorname{Ad}(g)N \in \mathfrak{g}_A$ *and* $p(\operatorname{Ad}(g)N)$ *is nilpotent.*

(b) *If* N *is nil and* $\operatorname{Ad}(g)N \subset \mathfrak{g}_A$ *for some* $g \in \hat{G}$, *then* $p(\operatorname{Ad}(g)N)$ *is nilpotent.*

PROOF. Suppose that $\operatorname{Ad}(g)N \in \mathfrak{g}_A$ and $p(\operatorname{Ad}(g)N)$ is nilpotent. To show that N is nil it suffices to show that $N_1 \overset{\text{def}}{=} \operatorname{Ad}(g)N$ is nil. Since $p(N_1)$ is nilpotent there exists a positive integer r such that $[\operatorname{ad} p(N_1)]^r = 0$. Then we have $(\operatorname{ad} N_1)^r \mathfrak{g}_A \subset \varepsilon \mathfrak{g}_A$. It is clear now that $(\operatorname{ad} N_1)^r \to 0$ for $r \to \infty$.

Suppose that N is nil. Then there exists $l \in \mathbf{Z}^+$ such that $(\operatorname{ad} N)^l \mathfrak{g}_A \subset \varepsilon \mathfrak{g}_A$. Let $L = \bigcup_{r=0}^{l-1} (\operatorname{ad} N)^r L_0$. Then L is a lattice in $\mathfrak{g}_F$ and $(\operatorname{ad} N)^l L \subset \varepsilon L$. Therefore the series for

$$\exp N \overset{\text{def}}{=} \sum_{r=0}^{\infty} \frac{(\operatorname{ad} N)^r}{r!}$$

is convergent, and $\operatorname{Ad}(\exp N)L = L$. By [BT] there exists g in $\hat{G}$ such that $g(\exp N)g^{-1} \in K$. Then $\operatorname{Ad}(g)N \in \mathfrak{g}_A$ and it is clear that $p(\operatorname{Ad}(g)N)$ is nilpotent. Lemma 1 is proved.

The quotient space $X = \hat{G}/K$ has a natural structure of an infinite-dimensional algebraic variety. More precisely, X is an increasing union of finite-dimensional projective varieties $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$, where the variety X_n can be described as follows (see [L]). As a set, X_n is the set of x in $\hat{G}/K$ such that $\operatorname{Ad} x(\mathfrak{g}_A) \subset \varepsilon^{-n}\mathfrak{g}_A$. Let Gr_n be the Grassmanian of the subspaces in $\varepsilon^{-n}\mathfrak{g}_A/\varepsilon^n\mathfrak{g}_A$ whose dimension is equal to the dimension of $\mathfrak{g}_A/\varepsilon^n\mathfrak{g}_A$. Consider the map $\phi : X_n \to \mathrm{Gr}_n$ given by

$$\phi(x) = \operatorname{Ad}(x)\mathfrak{g}_A/\varepsilon^n\mathfrak{g}_A \subset \varepsilon^{-n}\mathfrak{g}_A/\varepsilon^n\mathfrak{g}_A.$$

It is clear that ϕ is an imbedding. It is shown in [L] that the image $\phi(X_n)$ of X_n is an algebraic subvariety in Gr_n. It is easy to see that the induced algebraic structures on X_n are compatible with the imbeddings $X_n \to X_m$ for $n < m$.

The group $\hat{G}$ acts naturally on X. Analogously, any N in $\mathfrak{g}_F$ defines a vector field on X and we denote by $X_N \subset X$ the set of zeroes of this vector field. It is clear that X_N is the set of x in $\hat{G}/K$ with $N \in \operatorname{Ad}(x)\mathfrak{g}_A$.

Let $p : K \to G(\mathbf{C})$ be the natural projection, $U \subset G(\mathbf{C})$ a maximal unipotent subgroup with Lie algebra $\mathfrak{u} \subset \mathfrak{g}$, $R = p^{-1}(U) \subset K$, $\mathfrak{r} = p^{-1}(\mathfrak{u}) \subset \mathfrak{g}_A$. The quotient space $X' = \hat{G}/R$ has the structure of an infinite-dimensional algebraic

G-variety over $\mathbf{C}$ such that the natural map $\pi: X' \to X$ is an algebraic morphism with fibers isomorphic to $G(\mathbf{C})/U$. (Namely, the map $\pi: X'_n \overset{\text{def}}{=} \pi^{-1}(X_n) \to X_n$ is algebraic for all n.) Let $\tilde{F}$ be a finite extension of F, $\tilde{\Gamma} = \mathrm{Gal}(\tilde{F}/F)$ ($\simeq \mathbf{Z}/d\mathbf{Z}$, $d = [\tilde{F}: F]$), $\tilde{G} = G(\tilde{F})$, and $\tilde{R} \subset \tilde{K}$ be the subgroups of $\tilde{G}$ analogous to R, K. As before, we define infinite-dimensional algebraic varieties $\tilde{X}$, $\tilde{X}'$ over $\mathbf{C}$ such that $X = \tilde{G}/\tilde{K}$, $\tilde{X}' = \tilde{G}/\tilde{R}$. The Galois group $\tilde{\Gamma}$ acts naturally on $\tilde{X}$, $\tilde{X}'$.

Lemma 2. *We have* $(\tilde{X}'(\mathbf{C}))^{\tilde{\Gamma}} = X'(\mathbf{C})$.

Proof. Fix $\tilde{x}$ in $(\tilde{G}/\tilde{R})^{\tilde{\Gamma}}$. Let $\tilde{g}$ be a representative in $\tilde{G}$. Consider the map $c: \tilde{\Gamma} \to \tilde{U}$ given by $c(\gamma) = \tilde{g}^{\gamma}\tilde{g}^{-1}$. It is clear that c is a one-cocycle on $\tilde{\Gamma}$ with values in $\tilde{R}$. Since $\tilde{R}$ is a pro-unipotent group, c is a coboundary. Lemma 2 is proved.

Remark. $(\tilde{X}(\mathbf{C}))^{\tilde{\Gamma}} \neq X(\mathbf{C})$.

Now $\hat{G}$ acts on X', and any N in $\mathfrak{g}_F$ defines a vector field on X'. We denote by X'_N the subvariety of zeroes of this vector field. It is clear that as a set, X'_N is the set of x' in $\hat{G}/R$ with N in $\mathrm{Ad}(x')\mathfrak{r}$.

Lemma 3. *For any nil element N in $\mathfrak{g}_F$, the map $\pi: X'_N \to X_N$ is surjective.*

Proof. This follows from Lemma 1.

We assume from now on (except in Lemma 6) that N is a regular semisimple nil element in $\mathfrak{g}_F$. Let T_N be the centralizer of N in $\hat{G}$. It is clear that T_N is a maximal torus in $\hat{G}$ and T_N acts naturally on X_N. Let Λ_N be the subgroup of T_N defined at the end of §1.

Proposition 1. *There exists a positive integer n such that*

$$X_N = \Lambda_N(X_N \cap X_n) \quad \text{and} \quad X'_N = \Lambda_N(X'_N \cap X'_n).$$

Proof. First we consider the case when N is a split element. In this case T_N is a split torus and we can assume that it is defined over $\mathbf{C}$. We denote this torus by T. It is clear that $T(F) = \Lambda T(A)$ and $T(A) \subset K$. Fix a Borel subgroup $B = TU$ in G. Define $V_N = \{u \in U(F) \mid \mathrm{Ad}(u)N \in \mathfrak{g}_A\}$ and $Y_N = V_N K/K \subset X_N$. It is clear that $T(A)$ normalizes V_N and stabilizes Y_N.

Let $\mathrm{Ad}: \hat{G} \to \mathrm{Aut}\,\mathfrak{g}_F$ be the adjoint representation of $\hat{G}$ and $\rho = \Lambda^{\dim \mathfrak{g} - \mathrm{rank}\,\mathfrak{g}}(\mathrm{Ad})$ its exterior power, $d\rho$ the corresponding representation of $\mathfrak{g}_F$.

Define the functions $\Delta : \mathfrak{g}_F \to F$ and $\delta : \mathfrak{g}_F \to \mathbf{Z} \cup -\infty$ by $\Delta(N) = \operatorname{tr} d\rho(N)$, $\delta(N) = \iota(\Delta(N))$. It is clear that $\Delta(N) \neq 0$ if and only if N is a regular semisimple element. For any $\delta \in \mathbf{Z}^+$ put $\mathfrak{g}_A^\delta = \{N \in \mathfrak{g}_A \mid \delta(N) = \delta\}$.

Lemma 4. *For any $\delta \in \mathbf{Z}^+$ there exists $n(\delta) \in \mathbf{Z}^+$ such that $Y_N \subset X_n$ for all $N \in \mathfrak{g}_F^\delta$.*

Proof. Let $\alpha_1, \ldots, \alpha_M$ be the set of positive roots of (G, B). We can consider α_i as functions $\alpha_i : C \to F$ on the split Cartan subalgebra $C \subset \mathfrak{g}_F$ corresponding to $T \subset \hat{G}$. Define $\tilde{\alpha}_i = v \circ \alpha_i : C \to \mathbf{Z}$. It is clear that $\tilde{\alpha}_i(N) \geqq 0$ and $\Sigma_{i=1}^{r} \tilde{\alpha}_i(N) = 2\delta$. Therefore $\tilde{\alpha}_i(N) \leqq 2\delta$ for all i, $1 \leqq i \leqq M$. It is clear then that $Y_N \subset X_n$ for $n = \delta^M$. Lemma 4 is proved.

Lemma 5. *For any x in X_N there exists λ in Λ_N such that λx lies in Y_N.*

Proof. Fix x in X_N. Since $\hat{G}$ is equal to $T(F)U(F)K$, we can choose a representative of x in $\hat{G}$ of the form tu, $t \in T(F)$, $u \in U(F)$. Since $x \in X_N$ we have $\operatorname{Ad}(tu)N \in \mathfrak{g}_A$. But $\operatorname{Ad}(tu)N = \operatorname{Ad}(u)N$. Therefore $u \in V_N$. Choose $\lambda \in \Lambda_N$ such that $\lambda t \in T(A)$. Then we have $\lambda x \in Y_N$. Lemma 5 is proved.

By Lemma 4, there exists $n \in \mathbf{Z}^+$ such that $Y_N \subset X_n$. Therefore $X_N = \Lambda_N(X_N \cap X_n)$. It follows from Lemma 3 that $X'_N = \Lambda_N(X'_N \cap X'_n)$.

Now let N be any regular semisimple element. Let $\tilde{F} = F(\varepsilon^{1/d})$ be an extension over which the torus T_N splits. Put $\tilde{G} = G(\tilde{F})$. Let $\tilde{X}$ be defined as above. Then $\Lambda_N = (\tilde{\Lambda}_N)^\Gamma$ and $X'_N = (\tilde{X}'_N)^\Gamma$. Let $\Lambda = \Lambda_N$, $\tilde{\Lambda} = \tilde{\Lambda}_N$. We have $\tilde{X}_N = \tilde{\Lambda}\tilde{Y}_N$. Take $n = \delta(N)^M$. Then $\tilde{Y}_N \subset \tilde{X}_n$. Let $S = \{\tilde{\lambda} \in \tilde{\Lambda} \mid \tilde{\lambda}\tilde{Y}_N \cap \tilde{Y}_N \neq \varnothing\}$. It is clear that the set S is finite. Let $\gamma \in \Gamma$ be a generator and $\tilde{S} = \{\tilde{\lambda} \in \tilde{\Lambda} \mid \tilde{\lambda}^\gamma \tilde{\lambda}^{-1} \in S\}$. Since $\tilde{\Lambda}^\gamma = \Lambda$ there exists a finite set $D \subset \tilde{\Lambda}$ such that $\tilde{S} = \Lambda D$. Let $\tilde{n} \in \mathbf{Z}^+$ be such that $D\tilde{X}_n \subset \tilde{X}_{\tilde{n}}$. Now it is easy to show that $X'_N \subset \Lambda(X'_N \cap X'_{\tilde{n}})$. Indeed, fix $x \in X_N$. We can find $\tilde{\lambda} \in \tilde{\Lambda}$ and $\tilde{y} \in \tilde{Y}_N \subset \tilde{X}$ such that $x = \tilde{\lambda}\tilde{y}$. But then $(\tilde{\lambda}\tilde{y})^\gamma = \tilde{\lambda}\tilde{y}$. That is, $\tilde{\lambda}^{-1}\tilde{\lambda}^\gamma \tilde{y}^\gamma = \tilde{y}$. Therefore $\tilde{\lambda}^{-1}\tilde{\lambda}^\gamma \in S$ and $\tilde{\lambda} \in \tilde{S}$. Therefore we can find $\lambda \in \Lambda$ such that $\lambda^{-1}\tilde{\lambda} \in D$. Then $\lambda^{-1}x = (\lambda^{-1}\tilde{\lambda})\tilde{y} \in \tilde{X}_{\tilde{n}} \cap X = X_{\tilde{n}}$. Therefore $X_N = \Lambda(X_N \cap X_{\tilde{n}})$. It is clear that $X'_N = \Lambda(X'_N \cap X'_{\tilde{n}})$. Proposition 1 is proved.

Proposition 1′. *For any $\delta \in \mathbf{Z}^+$ there exists $n(\delta) \in \mathbf{Z}^+$ such that for any $N \in \mathfrak{g}_A^\delta$ we have*

$$X'_N = \Lambda_N(X'_N \cap X'_{n(\delta)}).$$

Proof. The proof is the same as that of Proposition 1.

COROLLARY. *Let N be a regular semisimple nil element in $\mathfrak{g}_A$. Then* $\dim X_N < \infty$.

The converse statement is also true. Indeed, we have:

LEMMA 6. *Let $N \in \mathfrak{g}_A$ be a nil element which is not regular semisimple. Then* $\dim X_N = \infty$.

PROOF. Since N is not a regular semisimple element, the centralizer Z_N of N in $\hat{G}$ contains a nontrivial one-parameter unipotent subgroup $V(t) \subset \hat{G}$, $t \in F$. Let $x_0 \in X(\mathbf{C})$ be the point corresponding to $K/K \in \hat{G}/K$. Then $x_0 \in X_N$. Therefore $V(t)x_0 \in X_N$ for all $t \in F$. Let $Y_n = \{C(t)x_0 \mid t \in \varepsilon^{-n}A\}$. It is clear that Y_n is an algebraic subvariety in X and $\dim Y_n \to \infty$ for $n \to \infty$. Lemma 6 is proved.

For any $l \in \mathbf{Z}^+$ we denote by $K_l \subset K$ the congruence subgroup mod ε^l.

PROPOSITION 2. *For any $\delta \in \mathbf{Z}^+$ there exists $l, n \in \mathbf{Z}^+$ such that for any $N \in \mathfrak{g}_A^\delta$, and any $N' \in N + \varepsilon^l \mathfrak{g}_A$, we have*

$$X'_{N'} = \Lambda_{N'}(X'_N \cap X'_n).$$

PROOF. We start with a series of simple lemmas.

LEMMA 7. *For any $n \in \mathbf{Z}^+$ there exists $l(n) \in \mathbf{Z}^+$ such that K_l acts trivially on X_n.*

PROOF. Clear.

LEMMA 8. *For any $\delta \in \mathbf{Z}^+$ there exists $l_1(\delta) \in \mathbf{Z}^+$ such that for any $N \in \mathfrak{g}_A^\delta$ and any $N' \in (N + \varepsilon^{l_1(\delta)}\mathfrak{g}_A) \cap \hat{Z}_N$, we have $X'_{N'} = X'_N$; here $\hat{Z}_N$ is the centralizer of N in $\mathfrak{g}_F$.*

PROOF. As follows from the proof of Lemma 4 there exists $l_1(\delta) \in \mathbf{Z}^+$ such that for any split $N \in \mathfrak{g}_A^\delta$ and $N' \in (N + \varepsilon^{l_1(\delta)}\mathfrak{g}_A) \cap \hat{Z}_N$, we have $V_N = V_{N'}$. Then $Y_N = Y_{N'}$. Define $Y'_N = \pi^{-1}(Y_N) \cap X'_N$ and $Y'_{N'} = \pi^{-1}(Y_{N'}) \cap X'_{N'}$. We have

$$X'_{N'} = \Lambda_N(Y'_{N'}) = \Lambda_N(Y_N) = X_N.$$

If N is not split we proceed as in the proof of Proposition 1. We find that for any $N \in \mathfrak{g}_A^\delta$ and $N' \in (N + \varepsilon^{l_1(\sigma)}\mathfrak{g}_A) \cap Z_N$, we have

$$X'_{N'} = (\tilde{X}'_{N'})^\Gamma = (\tilde{X}'_N)^\Gamma = X'_N.$$

Lemma 8 is proved.

LEMMA 9. *For any δ and any $l \in \mathbf{Z}^+$ there exists $l_2(\delta, l) \in \mathbf{Z}^+$ such that for any $N \in \mathfrak{g}_A^\delta$ and any $N' \in N + \varepsilon^{l_2(\delta,l)}\mathfrak{g}_A$ there exists $k \in K_l$ such that* $\mathrm{Ad}(k)N' \in \hat{Z}_N$.

PROOF. Clear.

Now we can prove Proposition 2. By Proposition 1′ there exists $n \in \mathbf{Z}^+$ such that for any $N \in \mathfrak{g}_A^\delta$, $X'_N = \Lambda_N(X'_N \cap X'_n)$. Let $l = l_2(\delta, \max(l(n), l_1(\delta)))$. By the definition of $l_2(\delta, l)$, for any $N \in \mathfrak{g}_A^\delta$ and $N' \in N + \varepsilon^l \mathfrak{g}_A$ we can find $k \in K_{l(n)}$ such that $N'' \stackrel{\mathrm{def}}{=} \mathrm{Ad}(k)N' \in \hat{Z}_N$. It is clear that $N'' \in (N + \varepsilon^{l(n)}\mathfrak{g}_A) \cap \hat{Z}_N$. Therefore $X'_{N''} = X'_N$ and

$$\begin{aligned} X'_{N'} = k^{-1}X'_{N''} = k^{-1}X'_N = k^{-1}\Lambda_N(X'_N \cap X'_n) &= \Lambda_{N'}(k^{-1}(X'_N \cap X'_n)) \\ &= \Lambda_{N'}(X'_N \cap X'_n). \end{aligned}$$

Proposition 2 is proved.

§3. Finite dimensionality of $\hat{\mathscr{B}}_N$

From now on we assume that G is simply connected.

Let $\mathscr{B}$ be the variety of Borel subalgebras in $\mathfrak{g}$. Fix $\mathfrak{b}_0 \in \mathscr{B}(\mathbf{C})$ and denote by $\hat{\mathfrak{b}}_0$ the preimage of $\mathfrak{b}_0$ under the natural projection $\mathfrak{g}_A \to \mathfrak{g}$. By an *Iwahori subalgebra* we mean a subalgebra of $\mathfrak{g}_F$ which is conjugate to $\hat{\mathfrak{b}}_0$. Let $B \subset G$ be the subgroup corresponding to $\mathfrak{b}_0$ and $B_0 \subset K$ be the preimage of $B \subset G$ under the projection $p: K \to G$. It is well known that the set $\hat{\mathscr{B}}$ of all Iwahori subalgebras of $\mathfrak{g}_F$ is isomorphic to $\hat{G}/\hat{B}_0$. The inclusion $\hat{\mathscr{B}}_0 \to K$ induces the map $p: \hat{\mathscr{B}} \to X$ with fiber $K/\hat{B}_0 \simeq \mathscr{B}$. It is easy to define a structure of an infinite-dimensional algebraic $\hat{G}$-variety on $\hat{\mathscr{B}}$ such that for any $n \geqq 0$ in $\mathbf{Z}$, $\tilde{X}_n \stackrel{\mathrm{def}}{=} p^{-1}(X_n)$ is an algebraic subvariety of $\hat{\mathscr{B}}$, and the map $\tilde{X}_n \to X_n$ is an algebraic morphism with fibers isomorphic to $\mathscr{B}$. It is known that $\tilde{X}_n$ is a projective variety.

Let Σ (resp. $\hat{\Sigma}$) be the set of vertices of the Dynkin diagram (resp. extended Dynkin diagram) for G. We identify $\hat{\Sigma}$ with the set of simple reflections $\hat{S}$ in the affine Weyl group $\hat{W}$ of G, by $\alpha \to s_\alpha$ (see [B]). We denote by $l(w)$ the length of $w \in \hat{W}$, and by $\leqq$ the standard partial order on $\hat{W}$.

LEMMA 1.

(a) *For any $w \in \hat{W}$ there exists a constructible subvariety $\hat{\mathscr{B}}_w < \hat{\mathscr{B}}$ such that $\hat{\mathscr{B}}_w = \hat{B}_0 w \hat{\mathfrak{b}}_0 \subset \hat{\mathscr{B}} = \hat{G}/\hat{B}_0$.*

(b) *$\hat{\mathscr{B}}_w$ is isomorphic to the affine $l(w)$-dimensional space $A^{l(w)}$.*

(c) *Let* $\bar{\hat{\mathcal{B}}}_w$ *be the closure of* $\hat{\mathcal{B}}_w$ *in* $\hat{\mathcal{B}}$. *Then* $\bar{\hat{\mathcal{B}}}_w = \bigcup \hat{\mathcal{B}}_{w'}$, *where the union is over all* w' *in* $\hat{W}$ *with* $w' \leqq w$.

(d) *We have* $\hat{\mathcal{B}} = \bigcup \hat{\mathcal{B}}_w$, *where the union ranges over all* w *in* $\hat{W}$.

PROOF. Follows from [Ka].

COROLLARY. *For any pair of Iwahori subalgebras* $\hat{\mathfrak{b}}'$ *and* $\hat{\mathfrak{b}}''$ *in* $\hat{\mathcal{B}}$ *there exists* g *in* $\hat{G}$ *and* $\hat{w} \in \hat{W}$ *such that* $g\hat{\mathfrak{b}}' = \hat{\mathfrak{b}}_0$ *and* $g\hat{\mathfrak{b}}'' = \hat{w}\hat{\mathfrak{b}}_0$. *The element* $\hat{w}$ *is uniquely determined.*

We say in this case that $\hat{\mathfrak{b}}'$ and $\hat{\mathfrak{b}}''$ are in the relative position $\hat{w}$ and write $\hat{\mathfrak{b}}' \overset{\hat{w}}{—} \hat{\mathfrak{b}}''$.

For any $\alpha \in \hat{\Sigma}$ the subset $\hat{P}_{0,\alpha} \overset{\text{def}}{=} \hat{B}_0 \cup \hat{B}_0 s_\alpha \hat{B}_0 \subset \hat{G}$ is a subgroup and $\hat{\mathfrak{p}}_{0,\alpha} = \hat{\mathfrak{b}}_0 + s_\alpha \cdot \hat{\mathfrak{b}}_0$ is a subalgebra of $\mathfrak{g}_F$. A conjugate of $\hat{\mathfrak{p}}_{0,\alpha}$ is said to be a parabolic subalgebra of type α. The set $\hat{P}_\alpha$ of parabolic subalgebras of type α may be identified naturally with $\hat{G}/\hat{P}_\alpha$; it has a natural structure of complex algebraic $\hat{G}$-variety such that the natural map $\pi_\alpha : \hat{\mathcal{B}} \to \hat{\mathcal{P}}_\alpha$ is a $\mathbf{P}^1$-fibration. The fibers of π_α are called *lines of type* α. We denote by $\mathcal{E}_\alpha$ the line bundle on $\hat{\mathcal{B}}$ such that $\mathcal{E}_\alpha(\hat{\mathfrak{b}})$ is the cotangent space to the fiber of π_α through $\hat{\mathfrak{b}}$, for every $\hat{\mathfrak{b}}$ in $\hat{\mathcal{B}}$.

For any regular semisimple N in $\mathfrak{g}_F$, we denote by $\hat{\mathcal{B}}_N$ the subvariety in $\hat{\mathfrak{b}}$ of zeros of the corresponding vector field.

PROPOSITION 1.

(a) $\hat{\mathcal{B}}_N$ *is a locally finite union of finite-dimensional algebraic subvarieties in* $\hat{\mathcal{B}}$.

(b) Λ_N *acts without fixed points on* $\hat{\mathcal{B}}_N$.

Let C_N *be the set of irreducible components of* $\hat{\mathcal{B}}_N$.

(c) *For any* Z *in* C_N, *the stationary subgroup of* Z *in* Λ_N *is trivial.*

(d) C_N *is a finite union of* Λ_N*-orbits.*

PROOF. Follows from the proof of Proposition 2.1.

COROLLARY 1. *There exists an algebraic variety* $\mathcal{B}_N$ *and a map* $\hat{\mathcal{B}}_N \to \mathcal{B}_N$ *such that the fibers are* Λ_N*-orbits.*

PROOF. Follows from Theorem 3.2 in [DG].

COROLLARY 2. *If* N *is elliptic then* $\hat{\mathcal{B}}_N$ *is an algebraic variety.*

§4. Irreducible components of $\hat{\mathscr{B}}_N$

Fix $\alpha \in \hat{\Sigma}$. Let $\mathscr{E}_\alpha^N$ be the restriction to $\hat{\mathscr{B}}_N$ of the line bundle $\mathscr{E}_\alpha$ on $\hat{\mathscr{B}}$ defined in §3. For any $\hat{\mathfrak{b}} \in \hat{\mathscr{B}}_N$ let $\mathbf{P}_\alpha^1(\hat{\mathfrak{b}})$ be the fiber of π_α through $\hat{\mathfrak{b}}$. Since $\exp N$ preserves $\hat{B}$ it maps $\mathbf{P}_\alpha^1(\hat{\mathfrak{b}})$ into itself. Therefore N defines a vector field $\lambda^N(\hat{\mathfrak{b}})$ on $\mathbf{P}_\alpha^1(\hat{\mathfrak{b}})$ which is zero at the point $\hat{B}$ of $\mathbf{P}_\alpha^1(\hat{\mathfrak{b}})$. Since N is nil, λ^N has a zero of second order at $\hat{\mathfrak{b}}$. Therefore $\lambda^N(\hat{\mathfrak{b}})$ defines an element $s_\alpha^N(\hat{\mathfrak{b}})$ in the cotangent space to $\mathbf{P}_\alpha^1(\hat{\mathfrak{b}})$ at $\hat{\mathfrak{b}}$. This defines a section s_α^N of $\mathscr{E}_\alpha^N$. It is clear that $s_\alpha^N(\hat{\mathfrak{b}}) = 0$ if and only if $\lambda^N(\hat{\mathfrak{b}}) \equiv 0$; in this case $\mathbf{P}_\alpha^1(\hat{\mathfrak{b}})$ lies in $\hat{\mathscr{B}}_N$.

Remark. We could define the line bundle $\mathscr{E}_\alpha$ by $\mathscr{E}_\alpha(\hat{\mathfrak{b}}) = \mathfrak{n}/\mathfrak{n}_\alpha$, where $\hat{\mathfrak{p}}_\alpha$ is the unique parabolic subgroup of type α containing $\hat{\mathfrak{b}}$, and $\mathfrak{n}$, $\mathfrak{n}_\alpha$ are the nilradicals of $\hat{\mathfrak{b}}$ and $\hat{\mathfrak{p}}_\alpha$. Then $s_\alpha^N(\hat{\mathfrak{b}})$ is the image of N in $\mathfrak{n}/\mathfrak{n}_\alpha$.

Proposition 1. *Let $N \in \mathfrak{g}_F$ be a regular semisimple nil element. Then all irreducible components of $\hat{\mathscr{B}}_N$ have the same dimension.*

Remark. This Proposition is due to Spaltenstein [Spa] in finite dimensions; our proof is inspired by Spaltenstein's.

Proof. Let d be the maximum possible dimension of an irreducible component of $\hat{\mathscr{B}}_N$. (See Proposition 3.1(a).)

Lemma 1. *Let Y be an irreducible component of dimension d of $\hat{\mathscr{B}}_N$ and let $\mathscr{L} \subset \hat{\mathscr{B}}$ be a line of type α such that $\mathscr{L} \subset \hat{\mathscr{B}}_N$ and $\mathscr{L} \cap Y \neq \varnothing$. Then there exists an irreducible component Y' of $\hat{\mathscr{B}}_N$ such that $\mathscr{L} \subset Y'$ and* $\dim Y' = d$.

Proof of Lemma. If $\mathscr{L} \subset Y$, there is nothing to prove, so we can assume that $\mathscr{L} \not\subset Y$. Let $Z \subset Y$ be the set of zeroes of the restriction of s_α^N to Y. Then, clearly

$$Z = \{\hat{\mathfrak{b}} \in Y \mid \pi_\alpha^{-1}(\pi_\alpha(\hat{\mathfrak{b}})) \subset \hat{\mathscr{B}}_N\}.$$

By Krull's theorem, we have either $Z = Y$ or $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_n$, where the Z_i are irreducible subvarieties of Y of dimension $d - 1$.

Assume first that $Z = Y$. Then $\tilde{Y} = \pi_\alpha^{-1}(\pi_\alpha Y)$ is contained in $\hat{\mathscr{B}}_N$, and it is irreducible. Moreover, it strictly contains Y, since $\mathscr{L} \subset \tilde{Y}$, $\mathscr{L} \cap Y \neq \varnothing$, and $\mathscr{L} \not\subset Y$. Hence $\dim \tilde{Y} > d$. This contradicts the definition of d. Assume next that $Z \neq Y$. Let $\tilde{Y}_i = \pi_\alpha^{-1}(\pi_\alpha Y_i)$ $(1 \leqq i \leqq n)$. For some i, we have $\mathscr{L} \subset \tilde{Y}_i$. Then again $\tilde{Y}_i$ is contained in $\hat{\mathscr{B}}_N$, it is irreducible, and it strictly contains Z_i. Hence $\dim \tilde{Y}_i > \dim Z_i = d - 1$ and $\dim \tilde{Y}_i = d$. From the definition of d, it follows that $\tilde{Y}_i$ is an irreducible component of $\hat{\mathscr{B}}_N$. It contains $\mathscr{L}$, hence the lemma is proved.

LEMMA 2. *If $\hat{\mathfrak{b}}$ and $\hat{\mathfrak{b}}'$ lie in $\hat{\mathscr{B}}_N$, then there exist lines $\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_n$ of type $\alpha_1, \alpha_2, \dots, \alpha_n$ in $\hat{\mathscr{B}}$, such that*

$$\hat{\mathfrak{b}} \in \mathscr{L}_1,\ \hat{\mathfrak{b}}' \in \mathscr{L}_n,\ \mathscr{L}_1 \cap \mathscr{L}_2 \neq \varnothing,\ \mathscr{L}_2 \cap \mathscr{L}_3 \neq \varnothing,\ \cdots,\ \mathscr{L}_{n-1} \cap \mathscr{L}_n \neq \varnothing.$$

In particular, $\hat{\mathscr{B}}_N$ is connected.

(This is due to Tits in finite dimensions, the proof in our case is the same as that of Tits.)

PROOF. Let w in W be the relative position of $\hat{\mathfrak{b}}$ and $\hat{\mathfrak{b}}'$. Let $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n}$ be a reduced expression for w, where the α_i are simple affine roots. There is a unique sequence $\hat{\mathfrak{b}}_0, \hat{\mathfrak{b}}_1, \dots, \hat{\mathfrak{b}}_n$ in $\hat{\mathscr{B}}$, such that $\hat{\mathfrak{b}}_0 = \hat{\mathfrak{b}}$, $\hat{\mathfrak{b}}_n = \hat{\mathfrak{b}}'$, and $\hat{\mathfrak{b}}_{i-1}, \hat{\mathfrak{b}}_i$ are in relative position s_{α_i} for $1 \leqq i \leqq n$. Let $x = \exp N$. Then the sequence ${}^x\hat{\mathfrak{b}}_0, {}^x\hat{\mathfrak{b}}_1, \dots, {}^x\hat{\mathfrak{b}}_n$ has the same properties, hence it coincides with $\hat{\mathfrak{b}}_0, \hat{\mathfrak{b}}_1, \dots, \hat{\mathfrak{b}}_n$. Thus ${}^x\hat{\mathfrak{b}}_i = \hat{\mathfrak{b}}_i$ $(0 \leqq i \leqq n)$. Let $\mathscr{L}_i$ be the line of type α_i such that $\mathscr{L}_i$ contains $\hat{\mathfrak{b}}_{i-1}$ and $\hat{\mathfrak{b}}_i$. Clearly $\mathscr{L}_{i-1} \cap \mathscr{L}_i \neq \varnothing$ for $1 \leqq i \leqq n$. It is easy to see that for any line $\mathscr{L}$ of type α we have either $\mathscr{L} \subset \hat{\mathscr{B}}_N$ or $\mathscr{L} \cap \hat{\mathscr{B}}_N =$ point, or $\mathscr{L} \cap \hat{\mathscr{B}}_N =$ empty. For $\mathscr{L}_i$, the intersection $\mathscr{L}_i \cap \hat{\mathscr{B}}_N$ contains at least two points $\hat{\mathfrak{b}}_{i-1}$ and $\hat{\mathfrak{b}}_i$; it follows that $\mathscr{L}_i \subset \hat{\mathscr{B}}_N$. The lemma is proved.

PROOF OF PROPOSITION. Let Y be an irreducible component of $\hat{\mathscr{B}}_N$, $\hat{\mathfrak{b}}' \in Y$ a point of Y not contained in any other irreducible component of $\hat{\mathscr{B}}_N$, and Y_0 an irreducible component of $\hat{\mathscr{B}}_N$ of dimension d. Fix $\hat{\mathfrak{b}}$ in Y_0. Let $\mathscr{L}_1, \dots, \mathscr{L}_N$ be as in Lemma 3.

Applying Lemma 1 to Y_0 and $\mathscr{L}_1$, we see that $\mathscr{L}_1 \subset Y_1$, where Y_1 is an irreducible component of dimension d of $\hat{\mathscr{B}}_N$. Applying Lemma 1 to Y_1 and $\mathscr{L}_2$, we see that $\mathscr{L}_2 \subset Y_2$ where Y_2 is an irreducible component of dimension d of $\hat{\mathscr{B}}_N$.

Continuing in this way we find a sequence $Y_1, \dots, Y_n$ of irreducible components of dimension d of $\hat{\mathscr{B}}_N$ such that $\mathscr{L}_i \subset Y_i$ for $1 \leqq i \leqq n$. Since $\hat{\mathfrak{b}}'$ lies in $\mathscr{L}_n$ we have that $\hat{\mathfrak{b}}'$ lies in Y_n. From the definition of $\hat{\mathfrak{b}}'$ it follows that $Y_n = Y$. Hence $\dim Y = d$, as required.

REMARK. The same proof shows that for $N \in \mathfrak{g}_F$ nil, not necessarily regular semisimple, the following holds: if $\hat{\mathscr{B}}_N$ contains some irrreducible projective variety of dimension n, then for any $\hat{\mathfrak{b}} \in \hat{\mathscr{B}}_N$ there exists an irreducible projective variety of dimension n, contained in $\hat{\mathscr{B}}_N$ and containing $\hat{\mathfrak{b}}$.

Let N be a regular semisimple nil element in $\hat{\mathfrak{g}}$ and Y an irreducible component of $\hat{\mathscr{B}}_N$. We denote by $\tau(Y)$ the set of all α in $\hat{\Sigma}$ such that $Y = \pi_\alpha^{-1}(\pi_\alpha(Y))$.

LEMMA 3. *If* $\dim Y > 0$, *then* $\tau(Y) \neq \varnothing$.

PROOF. Suppose that $\tau(Y) = \varnothing$. Then we can find y in Y such that y does not lie in the closure of $\hat{\mathscr{B}}_N - Y$ and $\pi_\alpha^{-1}(\pi_\alpha(y)) \cap \hat{\mathscr{B}}_N = \{y\}$ for all α in $\hat{\Sigma}$. If $\dim Y > 0$ then we can find y_1 in $\hat{\mathscr{B}}_N$ with $y_1 \neq y$. By Lemma 2, there exists an α in $\hat{\Sigma}$ and a line $\mathscr{L}$ of type α through y in $\hat{\mathscr{B}}_N$. Since y is not in the closure of $\hat{\mathscr{B}}_N - Y$, we have $\mathscr{L} \subset Y$. This contradiction proves Lemma 3.

(An analogous result holds for the varieties $\mathscr{B}_N$ in finite dimensions (Spaltenstein).)

THEOREM 1. *Let N be a nil element in $\mathfrak{g}_F$ such that* $\dim \hat{\mathscr{B}}_N > 0$. *Then there exists an irreducible component Y of $\hat{\mathscr{B}}_N$ such that $\tau(Y) = \alpha_0$, where* $\{\alpha_0\} = \hat{\Sigma} - \Sigma$.

PROOF. We start the proof with a study of the action of the affine Weyl group on the homology of $\hat{\mathscr{B}}_N$.

It is well known (see [Ka]) that $\hat{W}$ acts naturally on the homology and cohomology groups of $\hat{\mathscr{B}}$. We describe in detail the action of $\hat{W}$ on $H_2(\hat{\mathscr{B}}, \mathbf{C})$. Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. Let $\mathfrak{h}'$ be the dual space. Let M_1 be the space of affine functions on $\mathfrak{h}'$. Then the group $\hat{W}$ acts naturally on the space M_1.

LEMMA 4. *The action of $\hat{W}$ on $H_2(\hat{\mathscr{B}}, \mathbf{C})$ is isomorphic to the action on M_1.*

PROOF. See [Ka] p. 206.

LEMMA 5. *Let $V \subset \mathrm{Sym}^k(M_1)$ be a $\hat{W}$-invariant subspace, and k a positive integer. Then $V^{\hat{W}} \neq \{0\}$.*

PROOF. Let $\Lambda \subset \hat{W}$ be the translation part of $\hat{W}$ (see [B]). We may identify $\mathrm{Sym}^k M_1$ (as a $\hat{W}$-module) with the space of polynomials of degree $\leqq k$ on $\mathfrak{h}'$. It is clear that any Λ-invariant nontrivial subspace in $\mathrm{Sym}^k M_1$ contains the function $\mathbf{1}$ of $\mathrm{Sym}^k M_1$, which is $\hat{W}$-invariant. Lemma 5 is proved.

Let $p_k : H_{2k}(\hat{\mathscr{B}}, \mathbf{C}) \to \mathrm{Sym}^k H_2(\hat{\mathscr{B}}, \mathbf{C})$ be the projection $\gamma \to p_k(\gamma)$ defined by the condition $\langle \mu^k, p_k(\gamma)\rangle = \langle \mu^k, \gamma \rangle$ for all μ in $H^2(\hat{\mathscr{B}}, \mathbf{C})$ (= dual of $H_2(\hat{\mathscr{B}}, \mathbf{C})$).

LEMMA 6. *Let Y be an irreducible algebraic variety of dimension k in $\hat{\mathscr{B}}$. Let γ_Y be the corresponding homology class in $H_{2k}(\hat{\mathscr{B}})$. Then $p_k(\gamma_Y) \neq 0$.*

PROOF. We can find a positive integer n such that Y is contained in $\tilde{X}_n$ (see the beginning of §3). It follows from the Bruhat decomposition (Lemma 3.1) that the map $H^2(\hat{\mathscr{B}}, \mathbf{C}) \to H^2(\tilde{X}_n)$ is an isomorphism (if n is sufficiently large). Since $\tilde{X}_n$ is projective we can imbed $\tilde{X}_n$ in a projective space. Let $\tilde{\mu}$ be the class of a hyperplane section in $H^2(\tilde{X}_n, \mathbf{C})$, and μ the corresponding class in $\mathrm{H}^2(\hat{\mathscr{B}}, \mathbf{C})$. It is clear that $\langle \mu^k, \gamma_Y \rangle \neq 0$. Therefore $p_k(\gamma_Y) \neq 0$. Lemma 6 is proved.

Let $N \subset \mathfrak{g}_F$, be a regular semisimple nil element. Put $d_N \stackrel{\text{def}}{=} \dim \hat{\mathscr{B}}_N$. We denote by $\hat{V}_N$ the image of $H_{2d_N}(\hat{\mathscr{B}}_N)$ in $H_{2d_N}(\hat{\mathscr{B}})$.

LEMMA 7. *$\hat{V}_N$ is a $\hat{W}$-invariant subspace in $H_{2d_N}(\hat{\mathscr{B}})$.*

PROOF. Follows from [K–L, 2.2].

LEMMA 8. *We have $\hat{V}_N^W \neq \{0\}$.*

PROOF. Let V_N be the image of $\hat{V}_N$ in $\mathrm{Sym}^{d_N}(H_2(\hat{\mathscr{B}}, \mathbf{C}))$ under the projection p_{d_N}. It follows from Lemma 6 that $V_N \neq \{0\}$ and therefore (by Lemma 5) that $V_N^W \neq \{0\}$. Since W is a finite group we conclude that $\hat{V}_N^W \neq \{0\}$.

Let $\rho: \hat{W} \to \mathrm{Aut}\, H_{2d_N}(\hat{\mathscr{B}})$ be the natural action of $\hat{W}$ on $H_{2d_N}(\hat{\mathscr{B}})$. Put $T_\alpha \stackrel{\text{def}}{=} \mathrm{Id} + \rho(s_\alpha)$ for any α in $\hat{\Sigma}$. For any irreducible component Y of $\hat{\mathscr{B}}_N$ we denote by $[Y] \subset H_{2d_N}(\hat{\mathscr{B}})$ the homology class represented by Y.

LEMMA 9. *For any α in $\tau(Y)$, we have $T_\alpha[Y] = 0$.*

PROOF. Follows from [Ka], p. 206.

Let T denote the sum of $\rho(w)$ over all w in W.

For any α in Σ let $W^{(\alpha)}$ be the set of w in W such that $l(ws_\alpha) > l(w)$. Put $Q_\alpha = \Sigma_{w \in W^{(\alpha)}} \rho(w)$. It is clear that $T = Q_\alpha \cdot T_\alpha$.

Now we can prove Theorem 1. Assume that for any irreducible component Y of $\hat{\mathscr{B}}_N$ we have $\tau(Y) \neq \{\alpha_0\}$. Since $\tau(Y) \neq \{\varnothing\}$ there exists α_Y in $\Sigma \cap \tau(Y)$. Then $T[Y] = Q_{\alpha_Y} T_{\alpha_Y}[Y] = 0$ for any irreducible component Y of $\hat{\mathscr{B}}_N$. Hence $T\hat{V}_N = 0$. But this contradicts Lemma 8. Theorem 1 is proved.

COROLLARY 1. *For any regular semisimple nil element N in $\mathfrak{g}_F$ there exists g in $\hat{G}$ such that* $\mathrm{Ad}(g)N$ *lies in $\mathfrak{g}_A$ and its reduction mod ε is a regular nilpotent element in $\mathfrak{g}$.*

PROOF. Let $Y \subset \hat{\mathscr{B}}_N$ be a component such that $\tau(Y) = \{\alpha_0\}$. Then there exist y in $Y(\mathbf{C})$ such that $\pi_\alpha^{-1}(\pi_\alpha(y)) \cap \hat{\mathscr{B}}_N = \{y\}$ for all α in Σ. Let $N_1 =$

$\mathrm{Ad}(y)N$. Then N_1 lies in $\mathfrak{g}_A$. Let $n_1 = \pi(N_1)$ be the reduction of N_1 mod ε. It is an element of $\mathfrak{g}$. Let $\mathscr{B}_{n_1}$ be the set of points in $\mathscr{B}$ fixed by $\exp n_1$. We have $\tau(\mathscr{B}_{n_1}) \subset \tau(\mathscr{B}_{N_1}) \cap \Sigma = \varnothing$. Therefore $\mathscr{B}_{n_1} = \{\text{point}\}$ and n_1 is a regular nilpotent element in $\mathfrak{g}$. Corollary 1 is proved.

COROLLARY 2. *For any regular semisimple nil element N we have*

$$\dim \hat{\mathscr{B}}_N = \dim X_N.$$

PROOF. We have a natural surjective map $\hat{\mathscr{B}}_N \xrightarrow{p} X_N$. Let Y be a component of $\hat{\mathscr{B}}_N$ such that $\tau(Y) = \{\alpha_0\}$. It follows from Corollary 1 that for generic y in Y, we have $p^{-1}(p(y)) = \{y\}$. Therefore $\dim Y \leqq \dim X_N$. On the other hand, it follows from Proposition 1 that $\dim \hat{\mathscr{B}}_N = \dim Y$. Since $\dim X_N \leqq \dim \hat{\mathscr{B}}_N$, we see that $\dim \hat{\mathscr{B}}_N = \dim X_N$. Corollary 2 is proved.

§5. $\hat{\mathscr{B}}_N$ for N in a Levi subalgebra

Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n} \subset \mathfrak{g}$ be a parabolic subalgebra, $P = MU \subset G$ the corresponding parabolic subgroup, $\hat{P} = P(F)$, $\hat{M} = M(F)$, $\hat{U} = U(F)$. Let X^M be the algebraic subvariety of X such that $X^M = \hat{M}/M(A) \subset \hat{G}/G(A) = X$. For any regular semisimple nil element N in $\mathfrak{m}_F \subset \mathfrak{g}_F$ we put $X_N^M = X^M \cap X_N$. It is clear that X_N^M is the set of m in $\hat{M}/\hat{M}(A)$ with $\mathrm{Ad}(m)N$ in $\mathfrak{m}_A$. Fix $x \in X$. Since $\hat{G} = \hat{P}G(A)$ we can find $p = mr$ in $\hat{P}$, where m lies in $\hat{M}$ and r in $\hat{U}$, such that $x = pG(A)$. Denote the element $mM(A)/M(A)$ of X^M by $q(x)$.

LEMMA 1. (a) *The element $q(x)$ is well-defined.*
(b) *$q(x) \in X_N^M$ for $x \in X_N$.*

PROOF. (a) is clear. To prove (b) we note that $\mathrm{Ad}(p^{-1})N \subset \mathfrak{g}_A$. But $\mathrm{Ad}(p^{-1})N = \mathrm{Ad}(r^{-1})(\mathrm{Ad}(m^{-1})N)$. Since $\mathrm{Ad}(m^{-1})N \subset \mathfrak{m}_F$, we have $\mathrm{Ad}(r^{-1})(\mathrm{Ad}(m^{-1})N) = \mathrm{Ad}(m^{-1})N + V$, where V lies in $\mathfrak{n}_F$. Therefore $\mathrm{Ad}(m^{-1})N$ lies in $(\mathfrak{g}_A + \mathfrak{n}_F) \cap \mathfrak{m}_F = \mathfrak{m}_A$. In other words, $q(x)$ is in X_N^M. Lemma 1 is proved.

Denote by q_N the restriction of q to X_N.

For any regular semisimple nil element N in the subalgebra $\mathfrak{m}_F$ of $\mathfrak{g}_F$ we denote by δ_N the determinant in F of the restriction of ad N on $\mathfrak{n}_F$. Let ν_N be the valuation of δ_N; it is a non-negative integer.

DEFINITION. An algebraic variety Z is called "almost an affine space" if either Z is an affine space or there exists a fibration of Z over an affine space such that all fibers are almost affine spaces of smaller dimension.

EXAMPLE. A two-dimensional algebraic variety Z is an almost affine space if and only if there exists a fibration $Z \to \mathbf{A}^1$ with fibers isomorphic to $\mathbf{A}^1$.

PROPOSITION 1. *The fibers of q_N are almost affine spaces of dimension ν_N.*

PROOF. Let Φ^+ be the set of positive roots of P with respect to the center of M, and Σ the set of roots in Φ^+ which are not sums of two or more roots in Φ^+. For any positive integer k we denote by Φ_k^+ the subset of roots in Φ^+ which are sums of at least k roots in Σ. Let $\mathcal{N}_k, \mathcal{V}_k \subset \mathcal{N}$ be the spans of the root subspaces corresponding to the roots in Φ_k^+ and $\Phi_k^+ - \Phi_{k+1}^+$, respectively. It is clear that $\mathcal{N}_k$ is a normal subalgebra in $\mathcal{N}$, and that $\mathcal{V}_k = \mathcal{N}_k/\mathcal{N}_{k+1}$. Let $U_k = \exp \mathcal{N}_k$ be the corresponding normal subgroup in $\hat{U}$. Fix y in $X_N^M(\mathbf{C})$, and a representative m of y in $\hat{M}$. Put $N_m = \mathrm{Ad}(m)N$; it is an element of $\mathcal{M}(A)$. Consider the map $\varphi_m^1 : \mathcal{V}_1(F) \to \mathcal{V}_1(F)$ defined by $\varphi_m^1(v) = [v, N_m]$. Put $Z_1 = (\mathcal{V}_1(A))/\phi_m^1(\mathcal{V}_1(A))$. It is clear that Z_1 has a natural structure of an affine space. Put $Z = q_N^{-1}(y)$. For any z in $Z \subset X_N(\mathbf{C})$ we can find a representative of z of the form mr_z, where r_z lies in $\hat{U} = U(F)$ and is uniquely determined up to a right multiplication by elements in $U(A)$. Let $\bar{r}_z$ be the image of r_z in $\mathcal{V}_1 = U_1/U_2$. Then $\varphi_m^1(\bar{r}_z) \in \mathcal{V}_1(A)$. Let $q_1(z)$ be the image of $\varphi_m^1(\bar{r}_z)$ in Z_1.

It is clear that the element $q_1(z)$ is well defined. $q_1 : Z \to Z_1$ is an algebraic fibration. Let $\varphi_m^2 : \mathcal{V}_2(F) \to \mathcal{V}_2(F)$ be the map given by $\varphi_m^2(v) = [v, N_m]$. Put $Z_2 = \mathcal{V}_2(A)/\varphi_m^2(\mathcal{V}_2(A))$. Fix z_1 in $Z_1(\mathbf{C})$, and an element r in $U(F)$, such that $\mathrm{Ad}(r)N_m - N_m$ is contained in $\mathcal{N}(A)$, and $\varphi_m^1(\bar{r}) = z_1$. For any z in $q_1^{-1}(z_1)$ we can find a representative of the form $mrr_z^{(2)}$, where $r_z^{(2)}$ is in $U_2(F)$, and $r_z^{(2)}$ is uniquely determined up to a right multiplication by an element in $U_2(A)$. Define the map q_2 from $Z^{(1)} \stackrel{\text{def}}{=} q_1^{-1}(Z_1)$ to Z_2 by letting $q_2(z)$ be the image of $\bar{r}_z^{(2)}$ in $\mathcal{N}_2(A)/\varphi_m^2(\mathcal{N}_2(A)) + \mathcal{N}_3(A)$, where $\bar{r}_z^2$ is the image of $r_z^{(2)}$ in $U_2/U_3 \approx \mathcal{V}_2$. It is clear that q_2 is a fibration. Repeating these arguments we obtain Proposition 1.

§6. Constructible subsets of $\mathfrak{g}_A$

Given a subset $\hat{Y}$ of $\mathfrak{g}_A$ we say that $\hat{Y}$ is *constructible* if there exists a positive integer $l = l(Y)$ and a subvariety $Y \subset \mathfrak{g}_A/\varepsilon^l\mathfrak{g}_A$ which is constructible in the Zariski topology, such that $\hat{Y} = p_l^{-1}(Y)$ where p_l is the natural projection $p_l : \mathfrak{g}_A \to \mathfrak{g}_A/\varepsilon^l\mathfrak{g}_A$. We say that $\hat{Y}$ is irreducible if Y is irreducible. We denote by $p_l^{l_1}$, $l_1 > l$ the natural projection $p_l^{l_1} : \mathfrak{g}_A/\varepsilon^{l_1}\mathfrak{g}_A \to \mathfrak{g}_A/\varepsilon^l\mathfrak{g}_A$.

We say that the subset $\hat{Y}_1$ of $\hat{Y}$ is open if there exists a positive integer

$l_1, l_1 > l$, and an open subset Y_1 of $(p_l^{l_1})^{-1}(Y)$, such that $\hat{Y}_1 = p_{l_1}^{-1}(Y_1)$. Let $\Delta : \mathfrak{g}_F \to F$, $\delta : \mathfrak{g}_F \to \mathbf{Z}^+ \cup \infty$ be the functions defined in §2.

For any constructible subset $\hat{Y}$ in $\mathfrak{g}_A$ we define $\delta(\hat{Y}) = \min_{y \in \hat{Y}} \delta(y)$. It is clear that $\delta(\hat{Y})$ is finite if $\hat{Y}$ is not empty. Put $\hat{Y}_0 = \{y \in \hat{Y} Z \delta(y) = \delta(\hat{Y})\}$.

LEMMA 1. *The set $\hat{Y}_0$ is open in $\hat{Y}$.*

PROOF. Put $l = \max(l(Y), \delta(\hat{Y}) + 1)$. It is clear that $\hat{Y}_0 = p_l^{-1}(Y_0)$, where $Y_0 = p_l(\hat{Y}_0)$ is an open subset in

$$(p_{l(Y)}^l)^{-1} Y \subset \mathfrak{g}_A / \varepsilon^l \mathfrak{g}_A.$$

Lemma 1 is proved.

COROLLARY. *For any constructible nonempty set $\hat{Y}$ there exists a nonempty open subset $\hat{Y}_0$ of $\hat{Y}$ which consists of regular semisimple elements.*

Let $\mathcal{N} \subset \mathfrak{g}$ be the subvariety of nilpotent elements and $\hat{\mathcal{N}}$ its preimage in $\mathfrak{g}_A$. Define a function d on $\hat{\mathcal{N}}$ by $d(N) = \dim \hat{\mathscr{B}}_N$. For any subset $\hat{Y} \subset \hat{\mathcal{N}}$ we define $d(\hat{Y}) = \min_{N \in \hat{Y}} d(N)$.

LEMMA 2. *For any constructible subset $\hat{Y}$ of $\hat{\mathcal{N}}$ ther exists a nonempty open subset $\hat{Y}_1$ of $\hat{Y}$ such that $d(y) = d(\hat{Y})$ for any y in $\hat{Y}_1$.*

PROOF. By Lemma 1 we may assume that $\delta(y) = \delta(\hat{Y})$ for all y in $\hat{Y}$. By Propositions 2.1′ and 2.2 there exist positive integers n and l such that for any N in $\hat{Y}$ we have $\dim \hat{\mathscr{B}}_N = \dim(\hat{\mathscr{B}}_N \cap \tilde{X}_n)$, where $\tilde{X}_n = \pi^{-1}(X_n)$ and $(\hat{\mathscr{B}}_N \cap \tilde{X}_n) = \hat{\mathscr{B}}_{N+z} \cap \tilde{X}_n$ for all z in $\varepsilon^l \mathfrak{g}_A$. We may assume that $\hat{Y} = p_l^{-1}(Y)$, where Y is a constructible subset of $\mathfrak{g}_A / \varepsilon^l \mathfrak{g}_A$. Define a function $D : Y \to \mathbf{Z}$ by $D(y) = \dim \hat{\mathscr{B}}_{N_y} \cap \tilde{X}_n$, where N_y is in $p_l^{-1}(y)$. Since D is constructible, Lemma 2 is proved.

Let $W_\sim$ be the set of conjugacy classes in W. For any c in $W_\sim$ denote by $\mathfrak{g}_F^c \subset \mathfrak{g}_F$, the subset of regular semisimple elements y such that the centralizer T_y of y in $\hat{G}$ is of type c (see Lemma 1.2). Fix a Cartain subalgebra $\mathfrak{h}_c$ of $\mathfrak{g}_F$ of type c.

PROPOSITION 1. *For any constructible subset $\hat{Y}$ of $\mathfrak{g}_A$ there exists a nonempty open subset $\hat{Y}_0$ of $\hat{Y}$ and c in $W_\sim$ such that $\hat{Y}_0$ is contained in $\mathfrak{g}_F^c$.*

PROOF. By Lemma 1 we may assume that $\delta(y) = \delta(= \delta(\hat{Y}))$ for all y in $\hat{Y}$. In particular, any element y in $\hat{Y}$ is regular semisimple. We can assume that $l(Y) > \delta$. For any $l \in \mathbf{Z}^+$ we define $K_l = \{k \in K \mid k \equiv 1 \bmod \varepsilon^l\}$.

LEMMA 3. *For any y in $\hat{Y}$ and z in $\varepsilon^{l(Y)}\mathfrak{g}_A$, there exist k in K_1 such that $k^{-1}(y+z)k$ lies in $\mathcal{T}_y$, where $\mathcal{T}_y$ is the centralizer of y in $\mathfrak{g}_F$.*

PROOF. Clear.

For any positive integer n we denote by $\hat{G}_n$ the preimage of X_n in $\hat{G}$ (see §2), and choose a positive integer l_n such that $g^{-1}K_{l_n}g$ is contained in $K_{l(Y)}$ for all g in $\hat{G}_n$. The set $\hat{G}_n/K_{l_n}$ has a natural structure of an algebraic variety. We define a subvariety $Z_{n,c}$ of $(\mathfrak{h}_c \cap \varepsilon^{-n}\mathfrak{g}_A/\varepsilon^{l(Y)}\mathfrak{g}_A) \times \hat{G}/K_{l_n}$ by $Z_{n,c} = \{(\bar{h}, x) \mid [x, \bar{h}] \in Y\}$. Let $Y_{n,c}$ be the image of $Z_{n,c}$ in Y under the morphism $(\bar{h}, g) \rightarrow [x, \bar{h}]$. It follows from a theorem of Chevalley that $Y_{n,c}$ is a constructible subset in Y. Since

$$Y = \bigcup_{\substack{c \in W_\sim \\ n \in \mathbf{Z}^+}} Y_{n,c},$$

it follows from Baire's theorem that there exists a conjugacy class c in $W_\sim$ and a positive integer n, such that $Y_{n,c}$ contains an open subset in Y. Proposition 1 is proved.

COROLLARY. *For any irreducible constructible subset $\hat{Y} \subset \mathfrak{g}_A$ there exists a unique conjugacy class $c = c(\hat{Y})$ in $W_\sim$ such that $\mathfrak{g}_F^c \cap \hat{Y}$ contains an open subset of $\hat{Y}$.*

§7. The sets Z_w

7.1. We fix $\mathfrak{b}_0 \in \hat{\mathcal{B}}$ with nil-radical $\mathfrak{n}_0$. For each $w \in \hat{W}$, let $\hat{\mathcal{B}}_{(w)} = \{\mathfrak{b} \in \mathcal{B} \mid \mathfrak{b}_0 \overset{w}{\text{---}} \mathfrak{b}\}$. (Here $\mathfrak{b}_0 \overset{w}{\text{---}} \mathfrak{b}$ means: $\mathfrak{b}_0$, $\mathfrak{b}$ are in relative position w.) For any $k \in \mathbf{N}$, let $\hat{\mathcal{B}}^k = \{\mathfrak{b} \in \hat{\mathcal{B}} \mid \mathfrak{b} \supset \varepsilon^k \mathfrak{n}_0\}$. Then $\hat{\mathcal{B}}^k$ is a projective subvariety of $\hat{\mathcal{B}}$; it is a union of finitely many $\hat{\mathcal{B}}_{(w)}$. Conversely, it is clear that any $\hat{\mathcal{B}}_{(w)}$ is contained in some $\hat{\mathcal{B}}^k$ for sufficiently large k.

Let $Z_w = \{(x, \mathfrak{b}) \in \mathfrak{n}_0 \times \hat{\mathcal{B}}_{(w)} \mid x \in \mathfrak{b}\}$.

If $k \in \mathbf{N}$ is such that $\hat{\mathcal{B}}_{(w)} \subset \hat{\mathcal{B}}^k$ we define

$$Z_w^k = \left\{(x, \mathfrak{b}) \in \frac{\mathfrak{n}_0}{\varepsilon^k \mathfrak{n}_0} \times \hat{\mathcal{B}}_{(w)} \mid \dot{x} \in \mathfrak{b}\right\}$$

where $\dot{x}$ is a representative for x in $\mathfrak{n}_0$. (The condition $\dot{x} \in \mathfrak{b}$ is independent of the choice of representative $\dot{x}$ since $\varepsilon^k \mathfrak{n}_0 \subset \mathfrak{b}$).

Then Z_w^k is an irreducible algebraic variety over $\mathbf{C}$ of dimension equal to

$\dim(\mathfrak{n}_0/\varepsilon^k\mathfrak{n}_0)$. (Indeed, the second projection $Z_w^k \to \hat{\mathscr{B}}_{(w)}$ is a vector bundle whose fibre at $\mathfrak{b} \in \hat{\mathscr{B}}_{(w)}$ is $(\mathfrak{n}_0 \cap \mathfrak{b})/\varepsilon^k\mathfrak{n}_0$, a vector space of dimension

$$\dim \frac{\mathfrak{n}_0}{\varepsilon^k\mathfrak{n}_0} - \dim \frac{\mathfrak{n}_0}{\mathfrak{n}_0 \cap \mathfrak{b}} = \dim \frac{\mathfrak{n}_0}{\varepsilon^k\mathfrak{n}_0} - l(w);$$

moreover $\hat{\mathscr{B}}_{(w)} \cong \mathbf{C}^{l(w)}$.)

Note that Z_w is the inverse image of Z_w^k under the natural map

$$\pi_k : \mathfrak{n}_0 \times \hat{\mathscr{B}}_{(w)} \to \frac{\mathfrak{n}_0}{\varepsilon^k\mathfrak{n}_0} \times \hat{\mathscr{B}}_{(w)}.$$

We shall denote by $\pi_k^{k+1} : Z_w^{k+1} \to Z_w^k$ the map induced by the natural map

$$\frac{\mathfrak{n}_0}{\varepsilon^{k+1}\mathfrak{n}_0} \times \hat{\mathscr{B}}_{(w)} \to \frac{\mathfrak{n}_0}{\varepsilon^k\mathfrak{n}_0} \times \hat{\mathscr{B}}_{(w)}.$$

Let $\bar{Z}_w^k$ be the Zariski closure of Z_w^k in $(\mathfrak{n}_0/\varepsilon^k\mathfrak{n}_0) \times \bar{\hat{\mathscr{B}}}_{(w)}$. We define $\bar{Z}_w$ to be the inverse image of $\bar{Z}_w^k$ under the natural map

$$\mathfrak{n}_0 \times \bar{\hat{\mathscr{B}}}_{(w)} \to \frac{\mathfrak{n}_0}{\varepsilon^k\mathfrak{n}_0} \times \bar{\hat{\mathscr{B}}}_{(w)}.$$

Then $\bar{Z}_w$ is independent of the choice of k. Indeed, $\bar{Z}_w^{k+1}$ is the inverse image of $\bar{Z}_w^k$ under the natural map

$$\frac{\mathfrak{n}_0}{\varepsilon^{k+1}\mathfrak{n}_0} \times \bar{\hat{\mathscr{B}}}_{(w)} \to \frac{\mathfrak{n}_0}{\varepsilon^k\mathfrak{n}_0} \times \bar{\hat{\mathscr{B}}}_{(w)}.$$

We have $Z_w \subset \bar{Z}_w \subset \{(x, \mathfrak{b}) \in \mathfrak{n}_0 \times \hat{\mathscr{B}} \mid x \in \mathfrak{b}\}$.

7.2. PROPOSITION. *Given $w \in \hat{W}$, the set $Z_w - [\bigcup_{y \in \hat{W}, y \neq w} \bar{Z}_y]$ is non-empty.*

PROOF. If k is such that $\hat{\mathscr{B}}_{(w)} \subset \hat{\mathscr{B}}^k$, we set $Y(k) = \{y \in \hat{W} \mid y \neq w, \hat{\mathscr{B}}_{(y)} \subset \hat{\mathscr{B}}^k\}$; this is a finite set. For each $y \in Y(k)$, Z_y^k (and $\bar{Z}_y^k$) is an irreducible subvariety of $(\mathfrak{n}_0/\varepsilon^k\mathfrak{n}_0) \times \hat{\mathscr{B}}$ of the same dimension ($= \dim(\mathfrak{n}_0/\varepsilon^k\mathfrak{n}_0)$ as Z_w^k. It follows that $Z_w^k \cap [\bigcup_{y \in Y(k)} \bar{Z}_y^k]$ is a closed subvariety of Z_w^k, distinct from Z_w^k; hence its complement Φ_w^k in Z_w^k is non-empty in Z_w^k.

It is clear that $\pi_k^{k+1}(\Phi_w^{k+1}) \subset \Phi_w^k$. Note that the maps $\pi_k^{k+1} : Z_w^{k+1} \to Z_w^k$ are open (they are vector bundle maps). We now fix k_0 such that $\hat{\mathscr{B}}_{(w)} \subset \hat{\mathscr{B}}^{k_0}$. For each $k \geqq k_0$, $\pi_{k_0}^{k_0+1} \circ \cdots \circ \pi_{k-2}^{k-1} \circ \pi_{k-1}^k(\Phi_w^k)$ is a non-empty open subset of $Z_w^{k_0}$, hence it is open dense; the intersection of this countable family of open dense subsets of $Z_w^{k_0}$ is non-empty by Baire's theorem applied to $Z_w^{k_0}$ with the standard locally compact topology. Let α_{k_0} be a point in this intersection.

We shall construct a sequence of points $\alpha_k \in \Phi_w^k$ $(k \geqq k_0)$ such that $\pi_k^{k+1}(\alpha_{k+1}) = \alpha_k$ for all $k \geqq k_0$. Assume that α_k is already defined for all k, $k_0 \leqq k < k_1$ where $k_1 \geqq k_0$, and it has the required property as far as it makes sense.

Consider the subset $R = (\pi_{k_1}^{k_1+1})^{-1}(\alpha_{k_1})$ of $Z_w^{k_1+1}$. This set is an algebraic variety isomorphic to $\mathfrak{n}_0/\varepsilon\mathfrak{n}_0$.

The sets $\pi_{k_1+1}^{k_1+2} \circ \cdots \circ \pi_{k'-2}^{k'-1}\pi_{k'-1}^{k'}(\Phi_w^{k'}) \cap R$ $(k' \geqq k_1 + 1)$ are non-empty, open subsets of R (by the properties of $\alpha_{k_0}, \ldots, \alpha_{k_1}$); the intersection of this countable family of open dense subsets of R is non-empty by Baire's theorem. Let α_{k_1+1} be a point in this intersection. Thus, we have an inductive construction of the sequence α_k $(k \geqq k_0)$ with the required properties.

Since $Z_w = \varprojlim_k Z_w^k$, there exists $\alpha \in Z_w$ such that $\pi_k \alpha = \alpha_k$ for all $k \geqq k_0$. We now show that for any $y \in \hat{W}$, $y \neq w$, we have $\alpha \notin \bar{Z}_y$. Assume that $\alpha \in \bar{Z}_y$. We can choose some $k \geqq k_0$ such that $\hat{\mathscr{B}}_y \subset \hat{\mathscr{B}}^k$. Then $\pi_k \alpha \in \bar{Z}_y^k$. But $\pi_k \alpha = \alpha_k \in \Phi_w^k$, hence $\pi_k \alpha$ is not contained in $\bar{Z}_y^k$, by the definition of Φ_w^k. This is a contradiction. The proposition is proved.

7.3. Proposition. *If* $(x, \mathfrak{b}) \in Z_w - (\bigcup_{y \in \hat{W}} \bar{Z}_y)$, *then* x *is a regular semisimple nil-element and* $\dim \hat{\mathscr{B}}_x \leqq l(w)$.

Proof. Consider an irreducible component X of the algebraic variety $\hat{\mathscr{B}}_w \cap \bar{\hat{\mathscr{B}}}_{(w')}$ $(w' \in \hat{W}, w' \geqq w)$ such that $\mathfrak{b} \in X$. There is a unique $y \in \hat{W}$ such that $\hat{\mathscr{B}}_{(y)} \cap X$ is open dense in X. We can find a sequence $\mathfrak{b}_i$ $(i \geqq 1)$ in $\hat{\mathscr{B}}_{(y)} \cap X$ such that $\lim_{i \to \infty} \mathfrak{b}_i = \mathfrak{b}$ in X. Then $\lim_{i \to \infty}(x, \mathfrak{b}_i) = (x, \mathfrak{b})$, hence $(x, \mathfrak{b}) \in \bar{Z}_y$. By our assumption on $(x, \mathfrak{b})$, we must then have $y = w$. Thus $\hat{\mathscr{B}}_{(w)} \cap X$ is open dense in X. Hence $X \subset \bar{\hat{\mathscr{B}}}_{(w)}$, and $\dim X \leqq l(w)$. Since w' was an arbitrary element of $\hat{W}$ such that $w' \geqq w$, the proposition follows from Lemma 2.5 and the remark preceding Lemma 4.3.

For any simple reflection $s \in W$, let $\mathfrak{n}_{0,s} = \mathfrak{b}_0 \cap \mathfrak{b}$ where $\mathfrak{b} \in \hat{\mathscr{B}}$ is such that $\mathfrak{b}_0 \overset{s}{\text{——}} \mathfrak{b}$. (Then $\mathfrak{n}_{0,s}$ is independent of the choice of $\mathfrak{b}$.) The following propositions (7.4, 7.5, 7.6) correspond to the study of regular and subregular nilpotent elements in semisimple Lie algebras.

7.4. Proposition. *The following conditions for* $x \in \mathfrak{n}_0$ *are equivalent*

(a) $(x, \mathfrak{b}_0) \in Z_e - \bigcup_{y \neq e} \bar{Z}_y$,

(b) $x \in \mathfrak{n}_0 - \mathfrak{n}_{0,s}$ *for any simple reflection* s,

(c) $\hat{\mathscr{B}}_x = \{\mathfrak{b}_0\}$.

PROOF. Assume that $x \in \mathfrak{n}_{0,s}$ for some s. Then $(x, \mathfrak{b}_0)$ is in $Z_e \cap \bar{Z}_s$. Thus (a)$\Rightarrow$(b).

Assume that $(x, \mathfrak{b}_0) \in \bar{Z}_y$ for some $y \neq e$. We can find a simple reflection such that $l(sy) < l(y)$. If $(x', \mathfrak{b}) \in Z_y$, there is a unique $\mathfrak{b}' \in \hat{\mathscr{B}}$ such that $\mathfrak{b}_0 \overset{s}{\text{---}} \mathfrak{b}'$, $\mathfrak{b}' \overset{sy}{\text{---}} \mathfrak{b}$ and $\mathfrak{b}'$ automatically contains x'. Hence $x' \in \mathfrak{n}_0 \cap \mathfrak{b}' = \mathfrak{n}_{0,s}$. It follows that for any $(x', \mathfrak{b}) \in \bar{Z}_y$ we have $x' \in \mathfrak{n}_{0,s}$. In particular this holds for $x' = x$. Thus (b)$\Rightarrow$(a).

If (a) holds, then x is a nil-element and $\hat{\mathscr{B}}_x$ is of dimension 0 (see Proposition 7.3); since $\hat{\mathscr{B}}_x$ is connected, it is $\{\mathfrak{b}_0\}$. Thus (a)$\Rightarrow$(c). If $x \in \mathfrak{n}_{0,s}$ then clearly $\hat{\mathscr{B}}_x$ contains a projective line (= the set of $\mathfrak{b} \in \hat{\mathscr{B}}$ such that $\mathfrak{b} \supset \mathfrak{n}_{0,s}$). Thus (c)$\Rightarrow$(b). The proposition is proved.

7.5. PROPOSITION. *Let s be a simple reflection in $\hat{W}$, and let*

$$(x, \mathfrak{b}) \in Z_s - \Bigl(\bigcup_{\substack{y \in \hat{W} \\ y \neq s}} \bar{Z}_y \Bigr).$$

Then x is a nil element and $\dim \hat{\mathscr{B}}_x = 1$.

PROOF. We have $\dim \hat{\mathscr{B}}_x \leqq 1$ by 7.3; on the other hand $\hat{\mathscr{B}}_x$ clearly contains the line of type s through $\mathfrak{b}$, hence $\dim \hat{\mathscr{B}}_x = 1$.

7.6. Now let x' be any nil-element such that $\dim \hat{\mathscr{B}}_{x'} = 1$. Then each irreducible component of $\hat{\mathscr{B}}_{x'}$ is a line of type s' for some simple reflection s'. (See Lemma 4.4.) The pattern of intersection of these lines can be described by a graph Γ, whose vertices are in 1–1 correspondence with the irreducible components of $\hat{\mathscr{B}}_{x'}$; two vertices are joined precisely when the corresponding components of $\hat{\mathscr{B}}_{x'}$ meet each other.

7.7. PROPOSITION. *In the set up of* 7.6, *Γ is an extended Dynkin graph of type*:

(a) *$\tilde{D}_n$ (resp. $\tilde{E}_n$) if $\mathfrak{g}$ is of type D_n (resp. E_n).*

(b) *$\tilde{D}_{2n}$ (resp. $\tilde{D}_{n+2}$) if $\mathfrak{g}$ is of type B_n (resp. C_n).*

(c) $\tilde{\mathrm{E}}_7$ if $\mathfrak{g}$ *is of type F_4.*

(d) *$\tilde{E}_6$ if $\mathfrak{g}$ is of type G_2.*

If $\mathfrak{g}$ is of type A_n $(n > 2)$ then Γ is the universal covering of the extended Dynkin graph of type $\tilde{A}_n$.

PROOF. Consider for example the case where $\mathfrak{g}$ if of type A_n $(n \geqq 3)$. The simple reflections in $\hat{W}$ can be labelled s_i $(i \in Z/(n+1))$ where $s_i \neq s_j$ commute if and only if $i-j\not\equiv\pm 1 \pmod{n+1}$. If X is a line of type s_i contained in $\hat{\mathscr{B}}_{x'}$ then X meets exactly one line X' of type s_{i+1} contained in $\hat{\mathscr{B}}_{x'}$, exactly one line X'' of type s_{i-1} contained in $\hat{\mathscr{B}}_{x'}$ and no lines of types $s_j, j\not\equiv i\pm 1 \pmod{n+1}$, contained in $\hat{\mathscr{B}}_{x'}$. This follows from the structure of the varieties $\mathscr{B}_x$ for subregular nilpotent elements in semisimple Lie algebras of type A_2 and $A_1\times A_1$. Moreover $X\cap X'\cap X''$ is empty: two lines of type s_{i+1}, s_{i-1} contained in $\hat{\mathscr{B}}_{x'}$ cannot intersect since $(i+1)-(i-1)\not\equiv\pm 1 \pmod{n+1}$ for $n\geqq 3$.

It follows that Γ must be a covering of the extended Dynkin graph of type $\tilde{A}_n$.

Now let $\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_n$ be a sequence of lines of type $s_{j+1}, s_{j+2}, \dots, s_{j+m}$ respectively, each contained in $\hat{\mathscr{B}}_{x'}$ such that $\mathscr{L}_1\cap\mathscr{L}_2\neq\varnothing$,

$$\mathscr{L}_2\cap\mathscr{L}_3\neq\varnothing, \dots, \mathscr{L}_{m-1}\cap\mathscr{L}_m\neq\varnothing.$$

(Here $j+1, j+2, \dots, j+m$ are taken modulo $n+1$.) Then $s_{j+1}s_{j+2}\cdots s_{j+m}$ has length m in $\hat{W}$. Since this element describes the relative position of some point in $\mathscr{L}_1$ with some point in $\mathscr{L}_m$, it follows that $\mathscr{L}_1\neq\mathscr{L}_m$. This shows that Γ is the universal covering of the extended Dynkin graph of type $\tilde{A}_n$. Analogous arguments apply for $\mathfrak{g}$ of type D_n or E_8; the argument is actually simpler, since the Dynkin graph is already simply connected. The arguments in the other cases are similar. (This proof is suggested by the analysis of subregular nilpotent elements in semisimple Lie algebras, see [St, 3.10].)

7.8. Let x' be as in 7.6. If $\mathfrak{g}$ is of type A_2 or A_1 then Γ is

either the infinite graph $\cdots - \circ - \circ - \circ - \cdots$ or the

finite graph $\circ - \circ$ (for A_1) or  (for A_2).

The infinite graph is realized by

$$x'=\begin{bmatrix}\varepsilon & 0\\ 0 & -\varepsilon\end{bmatrix}\in\mathfrak{sl}_2 \quad\text{and}\quad x'=\begin{bmatrix}0&0&1\\0&0&0\\ \varepsilon&0&0\end{bmatrix}\in\mathfrak{sl}_3.$$

The finite graphs are realized by

$$x'=\begin{bmatrix}0 & \varepsilon\\ \varepsilon^2 & 0\end{bmatrix}\in\mathfrak{sl}_2 \quad\text{and}\quad x'=\begin{bmatrix}0&0&1\\ \varepsilon&0&0\\ 0&\varepsilon&0\end{bmatrix}\in\mathfrak{sl}_3.$$

In the last case, $\hat{\mathscr{B}}_{x'}$ consists of three lines meeting in a single point.

7.9. The discussion in 7.1 can be carried out also in the case of a finite-dimensional reductive complex Lie algebra ${}^1\mathfrak{g}$. (See [St. 3].) We shall state briefly the results which we need. Let ${}^1\mathscr{B}$ be the variety of Borel subalgebras of ${}^1\mathfrak{g}$. Let ${}^1W$ be the Weyl group. We fix ${}^1\mathfrak{b}_0 \in {}^1\mathscr{B}$, and let ${}^1\mathscr{B}_w$ ($w \in {}^1W$) be the set of all ${}^1\mathfrak{b} \in {}^1\mathscr{B}$ such that ${}^1\mathfrak{b}_0$, ${}^1\mathfrak{b}$ are in relative position w. Let ${}^1\mathfrak{n}_0$ be the nil-radical of ${}^1\mathfrak{b}_0$. Let ${}^1Z_w = \{(x, {}^1\mathfrak{b}) \in {}^1\mathfrak{n}_0 \times {}^1\mathscr{B}_w \mid x \in {}^1\mathfrak{b}\}$. Let ${}^1\bar{Z}_w$ be the closure of ${}^1Z_w$ in ${}^1Z = \{(x, {}^1\mathfrak{b}) \in {}^1\mathfrak{n}_0 \times {}^1\mathscr{B} \mid x \in {}^1\mathfrak{b}\}$. The ${}^1\bar{Z}_w$ ($w \in {}^1W$) are the irreducible components of ${}^1Z$; note that ${}^1Z_w$ is isomorphic to ${}^1\mathfrak{n}_0$.

Let $\mathscr{C}_w$ be the unique nilpotent orbit in ${}^1\mathfrak{g}$ such that for any ${}^1\mathfrak{b}$, ${}^1\mathfrak{b}' \in {}^1\mathscr{B}$ in relative position w, the intersection ${}^1\mathfrak{b} \cap {}^1\mathfrak{b}' \cap \mathscr{C}_w$ is open dense in the nilpotent radical of ${}^1\mathfrak{b} \cap {}^1\mathfrak{b}'$.

(a) $w \to \mathscr{C}_w$ is a surjective map from ${}^1W$ to the set of nilpotent orbits in ${}^1\mathfrak{g}$. (See [St], [BC].)

Since the Z_w ($w \in {}^1W$) are distinct, irreducible of the same dimension, we have

(b) $Z_w - [\bigcup_{y \in {}^1W, y \neq w} \bar{Z}_y] \neq \varnothing$ ($w \in {}^1W$).

Moreover:

(c) If $(x, {}^1\mathfrak{b})$ is in the set of (b), then $x \in \mathscr{C}_w$.

Let ${}^1\mathscr{B}_x$ be the variety of all Borel subalgebras of $\mathfrak{g}$ containing x. Let X (resp. X') be an irreducible component of ${}^1\mathscr{B}_x$ containing ${}^1\mathfrak{b}_0$ (resp. ${}^1\mathfrak{b}$). Let $y \in {}^1W$ be such that a generic point of X and a generic point of x' are in relative position y. From [St] it follows that $x \in \mathscr{C}_y$. It is clear that we can find a sequence $\beta_i \in X$ and a sequence $\beta_i' \in X'$ ($i = 1, 2, \ldots$) such that $\lim_{i\to\infty} \beta_i = {}^1\mathfrak{b}_0$, $\lim_{i\to\infty} \beta_i' = {}^1\mathfrak{b}$, and (β_i, β_i') are in relative position y for all i. We fix $\beta^* \in {}^1\mathscr{B}$ opposed to ${}^1\mathfrak{b}_0$, let U^* be the unipotent radical of the corresponding Borel subgroup. We can assume that β_i is opposed to β^* for all i. Hence there is a well-defined sequence $u_i \in U^*$ such that $\beta_i = \mathrm{Ad}(u_i)\,{}^1\mathfrak{b}_0$. We have $\lim_{i\to\infty}(u_i) = 1$ since $\lim_{i\to\infty} \beta_i = {}^1\mathfrak{b}_0$. Consider the sequence $(x_i, {}^1\mathfrak{b}_i) = (\mathrm{Ad}(u_i)^{-1}x, \mathrm{Ad}(u_i)^{-1}(\beta_i'))$ in ${}^1Z$ ($i \geqq 1$). We have $\lim_{i\to\infty}(x_i, {}^1\mathfrak{b}_i) = (x, {}^1\mathfrak{b})$ (since $\lim_{i\to\infty} \beta_i' = {}^1\mathfrak{b}$ and $\lim_{i\to\infty} u_i = 1$). Note that $(x_1, {}^1\mathfrak{b}_i) \in {}^1Z_y$. It follows that $(x, {}^1\mathfrak{b}) \in {}^1\bar{Z}_y$. From our assumption on $(x, {}^1\mathfrak{b})$ it now follows that $y = w$. Since $x \in \mathscr{C}_y$, we have $x \in \mathscr{C}_w$, as desired.

§8. Induction of nilpotent classes

8.1. We wish to extend to $\mathfrak{g}_F$ some results of [LS] on induced nilpotent classes.

We fix a parahoric subalgebra $\mathfrak{p} \subset \mathfrak{g}_F$ with nil-radical $\mathfrak{n}$. Let P be the corresponding parahoric subgroup of $\hat{G}$. The quotient ${}^1\mathfrak{g}=\mathfrak{p}/\mathfrak{n}$ is a finite-dimensional, reductive complex Lie algebra. Let $\pi:\mathfrak{p}\rightarrow{}^1\mathfrak{g}$ be the natural projection. We shall use the notations of 7.9 for ${}^1\mathfrak{g}$. If x is a nilpotent element of ${}^1\mathfrak{g}$, then the variety ${}^1\mathscr{B}_x$ of Borel subalgebras of ${}^1\mathfrak{g}$ containing x may be identified, via ${}^1\mathfrak{b}\rightarrow\pi^{-1}({}^1\mathfrak{b})$, with the variety $\hat{\mathscr{B}}_{\mathfrak{p},x}$ of all Iwahori subgroups $\mathfrak{b}$ of $\mathfrak{g}_F$ such that $\mathfrak{b}\subset\mathfrak{p}$ and $x\in\pi(\mathfrak{b})$.

If $\tilde{x}$ is any element of $\mathfrak{p}$ such that $\pi(\tilde{x})=x$, then clearly $\hat{\mathscr{B}}_{\mathfrak{p},x}\subset\hat{\mathscr{B}}_{\tilde{x}}$. We seek nil-elements $\tilde{x}\in\pi^{-1}(x)\subset\mathfrak{p}$ such that dim $\hat{\mathscr{B}}_{\tilde{x}}$ is as small as possible.

8.2. Proposition. *Given a nilpotent element $x\in{}^1\mathfrak{g}=\mathfrak{p}/\mathfrak{n}$, there exists an open subset $\mathscr{U}$ of $\pi^{-1}(x)$ such that any $\tilde{x}\in\mathscr{U}$ is a nil-element satisfying* dim $\hat{\mathscr{B}}_{\tilde{x}}=$ dim ${}^1\mathscr{B}_x$. *(For such $\tilde{x}$, $\hat{\mathscr{B}}_{p,x}$ is a union of irreducible components of $\hat{\mathscr{B}}_{\tilde{x}}$.)*

Proof. Let $w\in{}^1W$ be such that $x\in\mathscr{C}_w$, relative to ${}^1\mathfrak{g}$ (see 7.9(a)). We imbed in a natural way ${}^1W$ as a parabolic subgroup of $\hat{W}$, hence we regard w as an element of $\hat{W}$.

Let $\mathfrak{b}_0$ be an Iwahori subalgebra of $\mathfrak{g}$ such that $\mathfrak{b}_0\subset\mathfrak{p}$ and let ${}^1\mathfrak{b}_0=\pi(\mathfrak{b}_0)$ be the corresponding Borel subalgebra of ${}^1\mathfrak{g}$.

Let $(x',\mathfrak{b})$ be an element of $Z_w-[\bigcup_{y\in\hat{W},y\neq w}\bar{Z}_y]$ (see Proposition 7.2), where Z_w, $\bar{Z}_y$ are defined relative to $\mathfrak{b}_0$. In particular, we have $x'\in\mathfrak{b}_0\cap\mathfrak{b}$ and $\mathfrak{b}_0\overset{w}{\text{---}}\mathfrak{b}$. Let $\tilde{x}=\pi(x')$, ${}^1\mathfrak{b}=\pi(\mathfrak{b})$. Let ${}^1X$ be an irreducible component of ${}^1\mathscr{B}_{\tilde{x}}$ containing ${}^1\mathfrak{b}$ and let X be the corresponding irreducible component of $\hat{\mathscr{B}}_{\mathfrak{p},\tilde{x}}$. We know that x' is a nil-element (see Proposition 7.3). Let X' be an irreducible component of $\hat{\mathscr{B}}_{x'}$ containing X. Let $X'_{\mathfrak{p}}=\{\mathfrak{b}'\in X'\mid\mathfrak{b}'\subset\mathfrak{p}\}$. Then $X\subset X'_{\mathfrak{p}}\subset X'$. Assume that $X'_{\mathfrak{p}}\subsetneqq X'$. Then $X'-X'_{\mathfrak{p}}$ is an open dense subset of X' hence there is a sequence $\mathfrak{b}_i\in X'-X'_{\mathfrak{p}}$ $(i\geqq 1)$ such that $\lim_{i\to\infty}\mathfrak{b}_i=\mathfrak{b}$. Passing to a subsequence, if necessary, we may assume that there exists $y\in\hat{W}$ such that $(x',\mathfrak{b}_i)\in Z_y$ for all i. It follows that $(x',\mathfrak{b})\in\bar{Z}_y$. By the definition of $(x',\mathfrak{b})$, we must have $y=w$, hence we have $(x',\mathfrak{b}_i)\in Z_w$ for all i. But then $\mathfrak{b}_0\overset{w}{\text{---}}\mathfrak{b}_i$ so that $\mathfrak{b}_i\subset\mathfrak{p}$ (since $w\in{}^1W$). This is a contradiction. Hence we have $X'_{\mathfrak{p}}=X'$. Thus X' is an irreducible component of $\hat{\mathscr{B}}_{\mathfrak{p},\tilde{x}}$ containing X, hence $X'=X$. It follows that dim $\hat{\mathscr{B}}_{x'}=$ dim ${}^1\mathscr{B}_{\tilde{x}}$. From the inclusion $(x',\mathfrak{b})\in Z_w-[\bigcup_{y\in\hat{W},y\neq w}\bar{Z}_y]$ it follows that

$$(\tilde{x},{}^1\mathfrak{b})\in{}^1Z_w-\left[\bigcup_{\substack{y\in\hat{W}\\ y\neq w}}{}^1\bar{Z}_y\right].$$

Using now 7.9(c) it follows that $\tilde{x}\in\mathscr{C}_w$. Hence $\tilde{x}$ is conjugate to x under an element in the adjoint group ${}^1G$ of ${}^1\mathfrak{g}$. Hence x' is conjugate under an element of P to a nil-element $\tilde{x} \in \mathfrak{p}$ such that $\pi(\tilde{x}) = x$. We have

$$\dim \hat{\mathscr{B}}_{\tilde{x}} = \dim \hat{\mathscr{B}}_{x'} = \dim {}^1\mathscr{B}_{\tilde{x}} = \dim {}^1\mathscr{B}_x.$$

Thus $\min_{\tilde{x} \in \pi^{-1}x} \dim \hat{\mathscr{B}}_{\tilde{x}} = \dim {}^1\mathscr{B}_x$. Using now Lemma 6.2, we see that there exists an open subset $\mathscr{U}$ of $\pi^{-1}(x)$ such that for all $\tilde{x} \in \mathscr{U}$ we have $\dim \hat{\mathscr{B}}_{\tilde{x}} = \dim {}^1\mathscr{B}_x$. The proposition is proved.

8.3. Proposition. *Let x, $\mathfrak{p}$, $\mathscr{U} \subset \pi^{-1}(x)$ be as in* 8.2; *we assume that $\mathfrak{p}$ is a maximal parahoric subalgebra and that x is not contained in any Levi subalgebra of a proper parabolic subalgebra of ${}^1\mathfrak{g}$. Then any element $\tilde{x} \in \mathscr{U}$ is elliptic.*

Proof. We use notations from the proof of Proposition 8.2. From our assumption it follows that ${}^1\mathfrak{g}$ is a semisimple Lie algebra over $\mathbf{C}$ and the centalizer $Z_{{}^1G}(x)$ has the property that the elements of finite order in it have bounded order.

Assume that $\tilde{x} \in \mathscr{U}$ is not elliptic. Then there exists an injective homomorphism $\phi : G_m \to G$ of F-algebraic groups such that $\phi(F^*)$ centralizes $\tilde{x}$; hence $\phi(\mathbf{C}^*)$ ($\subset \phi(F^*) \subset G(F)$) centralizes $\tilde{x}$. Now $\phi(\mathbf{C}^*)$ acts on $\hat{\mathscr{B}}_{\tilde{x}}$ and it leaves stable each irreducible component of $\hat{\mathscr{B}}_{\tilde{x}}$. Now $\hat{\mathscr{B}}_{\tilde{x}}$ has at least one irreducible component X' such that $\mathfrak{b} \in X' \Rightarrow \mathfrak{b} \subset \mathfrak{p}$; since $\phi(\mathbf{C}^*)$ leaves X' stable, it must leave $\mathfrak{p}$ stable. Hence $\phi(\mathbf{C}^*) \subset P$. Let $h : P \to {}^1G$ be the canonical homomorphism. Its kernel contains no elements of finite order > 1; hence the composition $h \circ \phi : \mathbf{C}^* \to {}^1G$ is injective at least on the group of roots of 1 in $\mathbf{C}^*$. The image of $h \circ \phi$ is contained in $Z_{{}^1G}(x)$. Thus $Z_{{}^1G}(x)$ contains elements of finite, arbitrarily large order. This contradiction proves the proposition.

8.4. The converse to the previous proposition is false. It may happen for ($\mathfrak{p} = \mathfrak{g}_A$) that x is a nilpotent element contained in a Levi subalgebra of a proper parabolic subalgebra of $\mathfrak{g}$, and that there is an open subset $\mathscr{U} \subset \pi^{-1}(x)$ such that any $\tilde{x} \in \mathscr{U}$ is elliptic. An example is obtained by taking x to be a subregular nilpotent element in $\mathfrak{sp}_4$. Another example, arising in $\mathfrak{sp}_6$, is given in the Appendix.

§9. A map from nilpotent orbits in $\mathfrak{g}$ to conjugacy classes in W

9.1. Let $\mathfrak{g}$, $\mathfrak{g}_F$, $\mathfrak{g}_A$ be as in §2. We shall denote $\mathfrak{n} = \varepsilon\mathfrak{g}_A$, so that $\mathfrak{g}_A = \mathfrak{g} \oplus \mathfrak{n}$. For any nilpotent element $N \in \mathfrak{g}$, we consider the subset $N + \mathfrak{n}$ of $\mathfrak{g}_A$. By the last Corollary in §6, there exists a non-empty open subset $\mathscr{U}$ of $N + \mathfrak{n}$ and a conjugacy class (σ) in W such that for any $x \in \mathscr{U}$, x is regular semisimple in a Cartan subalgebra of $\mathfrak{g}_F$ of type (σ). Since any two non-empty open sets in $N + \mathfrak{n}$ have non-empty intersection, it follows that (σ) is an invariant of N. It clearly depends only on the G-orbit of N. Thus, we have defined a map

$$\text{(a)} \qquad N \to (\sigma) = (\sigma_N)$$

from the set of nilpotent orbits in $\mathfrak{g}$ to the set of conjugacy classes in the Weyl group.

This section will be concerned with the study of the map (a). Note that a map like (a) has been defined earlier (in quite different ways) on certain subsets of the set of nilpotent orbits by Carter–Elkington [CE] and Springer [Sp 1].

9.2. Proposition. *If N is a nilpotent element of $\mathfrak{g}$ not contained in any Levi subalgebra of a proper parabolic subalgebra of $\mathfrak{g}$, then σ_N is not contained in any proper parabolic subgroup of the Weyl group.*

Proof. Let $\mathscr{U} \subset N + \mathfrak{n}$ be as in 9.1. By 8.2 there exists a non-empty open subset $\mathscr{U}'$ of $N + \mathfrak{n}$ such that $\dim \hat{\mathscr{B}}_{\tilde{N}} = \dim {}^1\mathscr{B}_N$ for all $\tilde{N} \in \mathscr{U}'$ (notation of 7.9 with ${}^1\mathfrak{g} = \mathfrak{g}$). The intersection $\mathscr{U} \cap \mathscr{U}'$ must be a non-empty open subset of $N + \mathfrak{n}$; by 8.3 any $\tilde{N} \in \mathscr{U} \cap \mathscr{U}'$ is elliptic. Since $\tilde{N}$ is contained in a Cartan subalgebra of type (σ_N), the proposition follows.

9.3. Proposition. *Assume that $\mathfrak{g} = \mathfrak{sl}(V_{\mathbf{C}})$, where $V_{\mathbf{C}}$ is a vector space of dimension $n \geqq 2$ over $\mathbf{C}$. Let $N \in \mathfrak{g}$ be a nilpotent element with Jordan blocks of sizes $n_1 \geqq n_2 \geqq \cdots \geqq n_r > 0$ $(\Sigma\, n_i = n)$. Then the conjugacy class (σ_N) in the symmetric group $\mathfrak{S}_n$ is represented by a product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$.*

We first prove the following.

9.4. Lemma. *With the notations in the previous proposition, let $X \in \mathfrak{n}$, and let $\lambda_1, \ldots, \lambda_n \in \bar{F}$ be the eigenvalues of $\tilde{N} = N + X : V_F \to V_F$ $(V_F = V_{\mathbf{C}} \otimes F)$. Let $s = n_1 + n_2 + \cdots + n_{a-1} + j$ $(1 \leqq a \leqq r,\ 0 < j \leqq n_a)$. Then, for any $1 \leqq i_1 < i_2 < \cdots < i_s \leqq n$, we have*

$$v(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s}) \geqq a - 1 + \frac{j}{n_a}.$$

Proof. If is enough to show that $\Lambda^s(\tilde{N}^{n_a}) : \Lambda^s(V_F) \to \Lambda^s(V_F)$ maps $\Lambda^s(V_A)$ into $\varepsilon^{n_a(a-1)+j} V_A$ $(V_A = V_{\mathbf{C}} \otimes A)$.

We have $\tilde{N}^{n_a} = N^{n_a} + \varepsilon Y$, $y \in \mathfrak{g}_A$. Note also that the image of $N^{n_a}: V_{\mathbf{C}} \to V_{\mathbf{C}}$ has dimension

$$d = (n_1 - n_a) + (n_2 - n_a) + \cdots + (n_{a-1} - n_a).$$

Let $e_1, \ldots, e_s \in V_A$. We have

$$\begin{aligned}\Lambda^s(\tilde{N}^{n_a})(e_1 \Lambda \cdots \Lambda e_s) &= (N^{n_a}e_1 + \varepsilon Ye_1)\Lambda \cdots \Lambda(N^{n_a}e_s + \varepsilon Ye_s)\\ &= \sum (\pm \varepsilon^{s-k}) N^{n_a}e_{j_1}\Lambda \cdots \Lambda N^{n_a}e_{j_k}\Lambda Ye_{h_1}\Lambda \cdots \Lambda Ye_{h_{s-k}}\end{aligned}$$

sum over all subsets $j_1 < \cdots < j_k$ of $[1, s]$; $h_1 < \cdots < h_{s-k}$ is the complementary subset. We may restrict this sum to those terms for which $k \leqq d$ (the other terms are clearly zero). For such terms we have

$$\begin{aligned}s - k \geqq s - d &= n_1 + \cdots + n_{a-1} + j - (n_1 - n_a) - \cdots - (n_{a-1} - n_a)\\ &= n_a(a-1) + j;\end{aligned}$$

hence these terms are in $\varepsilon^{n_a(a-1)+j}V_A$. The lemma is proved.

9.5. Lemma. *Consider the $n \times n$ matrix (a_{ij}) $(n \geqq 2)$ with $a_{i,i+1} \in \mathbf{C}^\times$ for $1 \leqq i \leqq n-1$, $a_{n,1} \in \varepsilon \cdot \mathbf{C}^*$, $a_{ij} = 0$ for all other (i, j). Then the eigenvalues of this matrix are $\alpha\zeta\varepsilon^{1/n}$ where $\alpha \in \mathbf{C}^*$ is fixed and ζ runs through the n-th roots of 1.*

Proof. It is clear that the n-th power of our matrix is α' times identity where $\alpha' = a_{1,2}a_{2,3} \cdots a_{n-1,n}a_{n,1}$. The lemma follows.

9.6. We recall that $\Delta: \mathfrak{g}_F \to F$ is the map $\Delta(x) = \mathrm{Tr}(\Lambda^{2\nu}((\mathrm{ad})x))$ where 2ν is the number of roots and $\Lambda^{2\nu}$ denotes exterior power. If $x \in \mathfrak{g}_F$ is regular semisimple then $\Delta(x)$ is the product of non-zero eigenvalues of $\mathrm{ad}(x): \mathfrak{g}_F \to \mathfrak{g}_F$; if $x \in \mathfrak{g}_F$ is not regular semisimple then $\Delta(x) = 0$. Recall (Lemma 6.1) that for any constructible set X in $\mathfrak{g}_A$ and any integer i, the subset

$$\{x \in X \mid v(\Delta(x)) \leqq i\}$$

is open in X.

9.7. Proof of Proposition 9.3. Let $\tilde{N} = N + X \in N + \mathfrak{n}$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\tilde{N}$ arranged so that $v(\lambda_1) \leqq v(\lambda_2) \leqq \cdots \leqq v(\lambda_n)$. We have

$$v(\Delta(\tilde{N})) = 2\sum_{i<j} v(\lambda_i - \lambda_j) \geqq 2\sum_{i<j} v(\lambda_i) = 2v(\lambda_1^{n-1}\lambda_2^{n-2}\cdots\lambda_{n-1}).$$

Moreover, if $v(\lambda_i - \lambda_j) > v(\lambda_i)$ for some $i < j$, then the same argument shows that $v(\Delta(\tilde{N})) > 2v(\lambda_1^{n-1}\lambda_2^{n-2}\cdots\lambda_{n-1})$.

We take a direct sum of blocks of sizes $n_1, n_2, \ldots, n_r$, as in Lemma 9.5, when $n_i > 1$, and of form $(a_i\varepsilon)$, $a_i \in \mathbf{C}$, when $n_i = 1$ $(\Sigma\, a_i = 0)$. We obtain a nil-element $N + X_0 \in N + \mathfrak{n}$; we can assume that it is regular semisimple. We arrange its eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ so that $v(\mu_1) \leqq v(\mu_2) \leqq \cdots \leqq v(\mu_n)$; we then have $v(\mu_1\mu_2\cdots\mu_s) = a - 1 + j/n_a$ where s, a, j are as in Lemma 9.4 and $s \leqq n - 1$. From Lemma 9.4, it follows that $v(\lambda_1\lambda_2\cdots\lambda_s) \geqq v(\mu_1\mu_2\cdots\lambda_s)$ for $s = 1, 2, \ldots, n-1$. Adding these inequalities term by term, we get

$$v(\lambda_1^{n-1}\lambda_2^{n-2}\cdots\lambda_{n-1}) \geqq v(\mu_1^{n-1}\mu_2^{n-2}\cdots\mu_{n-1}),$$

hence $v(\Delta(\tilde{N})) \geqq v(\Delta(N + X_0))$ with strict inequality unless $v(\lambda_i - \lambda_j) = v(\lambda_i)$ for all $i < j$, and

$$v(\lambda_1\lambda_2\cdots\lambda_s) = v(\mu_1\mu_2\cdots\mu_s) \quad \text{for } s = 1, 2, \ldots, n-1$$

(which implies $v(\lambda_i) = v(\mu_i)$ for $1 \leqq i \leqq n - 1$).

It follows that the set

$$\mathcal{S} = \{\tilde{N} \in N + \mathfrak{n} \mid v(\tilde{N}) = v(\Delta(N + X_0))\}$$

is open in $N + \mathfrak{n}$. It is non-empty since it contains $N + X_0$. Let $\tilde{N} \in \mathcal{S}$. As we have seen above, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\tilde{N}$ satisfy $v(\lambda_i) = v(\mu_i)$ for $1 \leqq i \leqq n - 1$. Hence

$$(*) \qquad \begin{aligned} &v(\lambda_1) = \cdots = v(\lambda_{n_1}) = 1/n_1, \\ &v(\lambda_{n_1+1}) = \cdots = v(\lambda_{n_1+n_2}) = 1/n_2, \quad \text{etc.,} \end{aligned}$$

except that, when $n_r = 1$, $v(\lambda_n)$ is an integer (or ∞) not necessarily equal to $v(\mu_n)$. We have also seen that $v(\lambda_i - \lambda_j) = v(\lambda_i)$ for all $i < j$. Let λ_i be such that $v(\lambda_i) = 1/m$ and assume that $\lambda_i \notin \mathbf{C}[[\varepsilon^{1/m}]]$. Let $\alpha\varepsilon^{r/t}$ $(\alpha \in \mathbf{C}^*)$ be the first term in the Puiseux series of λ_i such that r/t is an irreducible fraction with t not divisible by m. Since $v(\lambda_i) = 1/m$ we have $1/m < r/t$. We can find $\gamma \in \mathrm{Gal}(\bar{F}/F)$ such that $\gamma(\varepsilon^{1/m}) = \varepsilon^{1/m}$, $\gamma(\varepsilon^{r/t}) = \zeta\varepsilon^{r/t}$ where $\zeta \neq 1$ is a root of 1. Then clearly $\lambda_i - \gamma(\lambda_i) = \alpha(1 - \zeta)\varepsilon^{r/t} +$ higher powers, hence $v(\lambda_i - \gamma(\lambda_i)) = r/t > 1/m = v(\lambda_i)$; this is a contradiction since $\gamma(\lambda_i) = \lambda_j$ for some j. Thus we must have

$$v(\lambda_i) = 1/m \Rightarrow \lambda_i \in \mathbf{C}[[\varepsilon^{1/m}]].$$

This and $(*)$ imply that $\tilde{N}$ is contained in a Cartan subalgebra of $\mathfrak{g}_F$ whose type is a permutation of $1, 2, \ldots, n$ which is a product of disjoint cycles of lengths

$n_1, n_2, \ldots, n_r$. Since this holds for all $\tilde{N}$ in the non-empty open set $\mathcal{S}$, the proposition follows.

9.8. PROPOSITION. *Assume that* $\mathfrak{g} = \mathfrak{sp}(V_{\mathbf{C}})$ *where* $V_{\mathbf{C}}$ *is a vector space of dimension* $2n$ *over* $\mathbf{C}$ *with a non-singular symplectic form. Let* $N \in \mathfrak{g}$ *be a nilpotent element with Jordan blocks of even distinct sizes* $2n_1 > 2n_2 > \cdots > 2n_r$ $(\Sigma n_i = n)$. *Then the conjugacy class* (σ_N) *in* W *(imbedded naturally in the symmetric group* $\mathfrak{S}_{2n}$*) is represented by a product of disjoint cycles of lengths* $2n_1, 2n_2, \ldots, 2n_r$.

PROOF. Let $\tilde{N} = N + X \in N + \mathfrak{n}$ and let $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ be the eigenvalues of $\tilde{N}$ arranged so that $v(\lambda_1) \leqq v(\lambda_2) \leqq \cdots \leqq v(\lambda_{2n})$. We can assume that $\lambda_2 = -\lambda_1, \lambda_4 = -\lambda_3, \ldots, \lambda_{2n} = -\lambda_{2n-1}$. We have

$$v(\Delta(\tilde{N})) = 2 \sum_{1 \leqq i < j \leqq n} v(\lambda_{2i} - \lambda_{2j}) + 2 \sum_{1 \leqq i \leqq j \leqq n} v(\lambda_{2i} + \lambda_{2j})$$

$$\geqq 2v(\lambda_2^{2n-1}\lambda_4^{2n-3} \cdots \lambda_{2n}).$$

Moreover, if the last inequality is equality then $v(\lambda_{2i} - \lambda_{2j}) = v(\lambda_{2i})$ for all $1 \leqq i < j \leqq n$ and $v(\lambda_{2i} + \lambda_{2j}) = v(\lambda_{2i})$ for all $1 \leqq i \leqq j \leqq n$; hence $v(\lambda_i - \lambda_j) = v(\lambda_i)$ for all $1 \leqq i < j \leqq 2n$.

Taking a direct sum of blocks as in Lemma 9.5, of sizes $2n_1, \ldots, 2n_r$ (representing nil-elements in symplectic Lie algebras), we obtain a nil-element $N + X_0 \in N + \mathfrak{n}$ which can be assumed to be regular semisimple and whose eigenvalues $\mu_1, \ldots, \mu_{2n}$ satisfy

$$\mu_2 = -\mu_1, \quad \mu_4 = -\mu_3, \quad \cdots, \quad \mu_{2n} = -\mu_{2n-1},$$

$$v(\mu_1) = \cdots = v(\mu_{2n_1}) = \frac{1}{2n_1}, \quad v(\mu_{2n_1+1}) = \cdots = v(\mu_{2n_1+2n_2}) = \frac{1}{2n_2}, \quad \text{etc.}$$

From Lemma 9.4, it follows that $v(\lambda_1\lambda_2 \cdots \lambda_{2s}) \geqq v(\mu_1\mu_2 \cdots \mu_{2s})$ for $s = 1, \ldots, n$. It follows that $v(\Delta(\tilde{N})) \geqq v(\Delta(N + X_0))$. It follows that the set $\mathcal{S} = \{\tilde{N} \in N + \mathfrak{n} \mid v(\Delta(\tilde{N})) = v(N + X_0)\}$ is non-empty, open in $N + \mathfrak{n}$. As in 9.7, we see that for $\tilde{N} \in \mathcal{S}$, the inequalities $v(\lambda_1\lambda_2 \cdots \lambda_{2s}) \geqq v(\mu_1\mu_2 \cdots \mu_{2s})$ must be equalities for $s = 1, \ldots, n$. It follows that $v(\lambda_i) = v(\mu_i)$ for $i = 1, \ldots, n$. As in 9.7, we see that for $\tilde{N} \in \mathcal{S}$, we have:

$$v(\lambda_i) = \frac{1}{2m} \Rightarrow \lambda_i \in \mathbf{C}[[\varepsilon^{1/2m}]].$$

The proposition follows.

9.9. PROPOSITION. *Assume that* $\mathfrak{g} = \mathfrak{so}(V_{\mathbf{C}})$ *where* $V_{\mathbf{C}}$ *is a vector space of dimension* $2n$ *over* $\mathbf{C}$ *with a non-singular symmetric bilinear form* $\langle\ ,\ \rangle$. *Let* $N \in \mathfrak{g}$ *be a nilpotent element with two Jordan blocks of odd sizes* $2p + 1 > 2q - 1$ $(p + q = n)$. *Then the conjugacy class* (σ_N) *in* W *(imbedded naturally in the symmetric group* $\mathfrak{S}_{2n}$) *is represented by a product of two disjoint cycles of lengths* $2p$, $2q$.

PROOF. Let $f_i : \mathfrak{g}_F \to F$ $(i = 1, 2)$ be the polynomials which attach to $x \in \mathfrak{g}$ the trace of $\Lambda^{2p}x$ on $\Lambda^{2p}V_F$ (for $i = 1$) and the determinant of x on V_F (for $i = 2$).

For any $\tilde{N} \in N + \mathfrak{n}$, we have $v(f_1(\tilde{N})) \geqq 1$, $v(f_2(\tilde{N})) \geqq 2$. (The second inequality follows directly from Lemma 9.4 applied to $\mathfrak{sl}(V_{\mathbf{C}})$. Using Lemma 9.4 we see also that $v(f_1(\tilde{N})) \geqq 2p/(2p + 1)$, hence $v(f_1(\tilde{N})) \geqq 1$ since $v(f_1(\tilde{N}))$ is an integer or ∞.)

It follows that the set

$$\mathcal{U} = \{\tilde{N} \in N + \mathfrak{n} \mid v(f_1(\tilde{N})) = 1, v(f_2(\tilde{N})) = 2\}$$

is an open subset of $N + \mathfrak{n}$.

We now show that $\mathcal{U}$ is non-empty. Consider a basis

$$\{e_i \mid 1 \leqq i \leqq 2p + 1\} \amalg \{f_j \mid 1 \leqq j \leqq 2q - 1\}$$

of $V_{\mathbf{C}}$ such that

$$\langle e_i, e_{i'} \rangle = \begin{cases} 1 & \text{if } i + i' = 2p + 2, \\ 0 & \text{otherwise,} \end{cases} \qquad \langle f_j, f_{j'} \rangle = \begin{cases} 1 & \text{if } j + j' = 2q, \\ 0 & \text{otherwise,} \end{cases}$$

and $\langle e_i, f_j \rangle = 0$ for all i, j. We may assume that

$$Ne_i = e_{i+1}\ (1 \leqq i \leqq p), \quad Ne_i = -e_{i+1}\ (p < i \leqq 2p), \quad Ne_{2p+1} = 0,$$

$$Nf_j = f_{j+1}\ (1 \leqq j \leqq q - 1), \quad Nf_j = -f_{j+1}\ (q - 1 < j \leqq 2q - 2), \quad Ne_{2q-1} = 0.$$

Let $M : V_{\mathbf{C}} \to V_{\mathbf{C}}$ be defined by $Me_{2p} = ae_1$, $Me_{2p+1} = -ae_2 - bf_1$, $Mf_{2q-1} = be_1$ $(a, b \in \mathbf{C}^*)$ and M is zero on all other basis elements. Then $N + \varepsilon M \in N + \mathfrak{n}$ and a simple computation shows that $f_1(N + \varepsilon M) \in \varepsilon \cdot \mathbf{C}^*$ $f_2(N + \varepsilon M) \in \varepsilon^2\mathbf{C}^*$. Thus, $N + \varepsilon M \in \mathcal{U}$ so that $\mathcal{U}$ is non-empty.

Now let $\tilde{N} \in \mathcal{U}$ and let $\lambda_1, \ldots, \lambda_{2n}$ be its eigenvalues on V_F. We may assume that $\lambda_2 = -\lambda_1$, $\lambda_4 = -\lambda_3, \ldots, \lambda_{2n} = -\lambda_{2n-1}$ and that $v(\lambda_1) \leqq v(\lambda_2) \leqq \cdots \leqq v(\lambda_{2n})$. We will show that:

$$(*)\qquad \begin{aligned} v(\lambda_i) &= \frac{1}{2p} \quad \text{for } 1 \leqq i \leqq 2p, \\ v(\lambda_i) &= \frac{1}{2q} \quad \text{for } 2p+1 \leqq i \leqq 2p+2q. \end{aligned}$$

Since $\tilde{N} \in \mathcal{U}$, we have $v(\lambda_1\lambda_2\cdots\lambda_{2p}) \leqq 1$. It follows that $v(\lambda_1) \leqq 1/2p$. Using 9.4, we have $v(\lambda_1) \geqq 1/(2p+1)$. If $v(\lambda_1) = 1/(2p+1)$ then multiplying λ_1 with the various $(2p+1)$-th roots of 1 and with ± 1 will yield $2(2p+1)$ distinct eigenvalues for $\tilde{N}$; this is impossible since $2(2p+1) > 2n$. Thus, we have $1/(2p+1) < v(\lambda_1) \leqq 1/2p$. If $v(\lambda_1) < 1/2p$, then $v(\lambda_1)$ would be a rational number with denominator $\geqq 4p+1$; this would imply again that there are at least $4p+1$ distinct eigenvalues for $\tilde{N}$, which is impossible since $4p+1 > 2n$. Thus, we must have $v(\lambda_1) = 1/2p$. It follows that there are at least $2p$ distinct eigenvalues λ for $\tilde{N}$ with $v(\lambda) = 1/2p$, hence

$$v(\lambda_1) = v(\lambda_2) = \cdots = v(\lambda_{2p}) = \frac{1}{2p},$$

and $\lambda_1, \ldots, \lambda_{2p}$ are distinct and contained in a single orbit of $\mathrm{Gal}(\bar{F}/F)$.

Since $\tilde{N} \in \mathcal{U}$, we have $v(\lambda_1\lambda_2\cdots\lambda_{2n}) = 2$ hence $v(\lambda_{2p+1}\lambda_{2p+2}\cdots\lambda_{2p+2q}) = 1$. It follows tht $v(\lambda_{2p+1}) \leqq 1/2q$. If $v(\lambda_{2p+1}) < 1/2q$, then the orbit of λ_{2p+1} under $\mathrm{Gal}(\bar{F}/F)$ would contain at least $2q+1$ elements, hence this orbit must meet the set $\{\lambda_1, \ldots, \lambda_{2p}\}$. Hence the Galois orbit of λ_1 contains strictly more than $2p$ elements; its cardinal is a multiple of $2p$ since $v(\lambda_1) = 1/2p$, hence it is at least $4p$. This implies that $p = q$ and $v(\lambda_{2q+1}) = v(\lambda_1) = 1/2p$, in contradiction with $v(\lambda_{2p+1}) < 1/2q$. Thus, we have $v(\lambda_{2q+1}) = 1/2q$. Since

$$v(\lambda_{2q+1}) \leqq \cdots \leqq v(\lambda_{2p+2q}) \quad \text{and} \quad v(\lambda_{2q+1}) + \cdots + v(\lambda_{2p+2q}) = 1,$$

it follows that $v(\lambda_{2q+1}) = \cdots = v(\lambda_{2p+2q}) = 1/2q$ and $(*)$ follows.

If $p > q$, then $\{\lambda_1, \ldots, \lambda_{2p}\}$, $\{\lambda_{2q+1}, \ldots, \lambda_{2p}\}$ must be two orbits of the Galois group and the desired description of (σ_N) follows. If $p = q$, then there are three possible cases:

(a) $\{\lambda_1, \ldots, \lambda_{2p}\}$, $\{\lambda_{2p+1}, \ldots, \lambda_{4p}\}$ form two distinct orbits of the Galois group,

(b) $\{\lambda_1, \ldots, \lambda_{2p}\} = \{\lambda_{2p+1}, \ldots, \lambda_{4p}\}$,

(c) $\{\lambda_1, \ldots, \lambda_{4p}\}$ is a single orbit of the Galois group;

we must exclude the cases (b), (c).

The case (c) cannot arise since the Weyl group of type D_{2p} does not contain a

$4p$-cycle of the symmetric group $\mathfrak{S}_{4p}$. The case (b) can be excluded by shrinking $\mathcal{U}$ to a non-empty open subset of $\mathcal{U}$ in which all elements are regular semisimple. The proposition follows.

More generally we have the following result.

9.10. PROPOSITION. *Let N be a nilpotent element in $\mathfrak{g} = \mathfrak{so}(V_{\mathbf{C}})$, with Jordan blocks of odd, distinct sizes*

$$2p_1 + 1 > 2q_1 - 1 > 2p_2 + 1 > 2q_2 - 1 > \cdots > 2p_{2r} + 1 > 2q_{2r} - 1$$

($\Sigma p_i + \Sigma q_i = n$). Then the conjugacy class (σ_N) in $W \subset \mathfrak{S}_{2n}$ is represented by a product of disjoint cycles of lengths $2p_1, 2q_1, 2p_2, 2q_2, \ldots, 2p_{2r}, 2q_{2r}$.

We shall omit the proof; we only note that the proof involves considering a direct sum of blocks M as in the previous proposition.

9.11. PROPOSITION. *Let N be a nilpotent element of $\mathfrak{g}$ which is not contained in any Levi subalgebra of a proper parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{g}_{2i}$ be the grading provided by the Jacobson–Morozov theorem for N. (See* [BC].*) Let a be the largest integer such that $\mathfrak{g}_{2a} \neq 0$. Let $M_{2i} \in A\mathfrak{g}_{2i}$, $M = \Sigma_i M_{2i} \in \mathfrak{g}_A$; then $N + \varepsilon M \in N + \mathfrak{n}$.*

(a) *Any eigenvalue λ of* $\mathrm{ad}(N + \varepsilon M) : \mathfrak{g}_F \to \mathfrak{g}_F$ *satisfies $v(\lambda) \geq 1/(a+1)$.*

(b) *If $M \in \mathfrak{g}_{-2a} - \{0\}$, then $N + \varepsilon M$ is not nilpotent* [Sp 1, 9.3(i)] *and any non-zero eigenvalue λ of* $\mathrm{ad}(N + \varepsilon M) : \mathfrak{g}_F \to \mathfrak{g}_F$ *is of form $\alpha\varepsilon^{1/(a+1)}$ ($\alpha \in \mathbf{C}^*$).*

(c) *There exists a non-empty open subset $\mathcal{U} \subset N + \mathfrak{n}$ such that for any $\tilde{N} \in \mathcal{U}$,* $\mathrm{ad}\,\tilde{N} : \mathfrak{g}_F \to \mathfrak{g}_F$ *has some eigenvalue λ with $v(\lambda) = 1/(a+1)$.*

(d) *Assume that N satisfies Springer's condition* $\dim \mathfrak{g}_4 = \dim(\mathfrak{g}_2) - 1$, [Sp 1, 9.6(d)]. *If $M_{-2a} \notin \varepsilon A\mathfrak{g}_{-2a}$ then $N + \varepsilon M$ is regular semisimple and any non-zero eigenvalue λ of* $\mathrm{ad}(N + \varepsilon M) : \mathfrak{g}_F \to \mathfrak{g}_F$ *satisfies $v(\lambda) = 1/(a+1)$; moreover, we have $v(\Delta(N + \varepsilon M)) = 2\upsilon/(a+1)$, where 2υ is the number of roots of $\mathfrak{g}$. On the other hand, if $M_{-2a} \in \varepsilon A\mathfrak{g}_{-2a}$, then $v(\Delta(N + \varepsilon M)) > 2\upsilon/(a+1)$.*

PROOF. Let $V_0 = \mathfrak{g}_0$, $V_i = \mathfrak{g}_{2i} \oplus \varepsilon\mathfrak{g}_{2i-2a-2}$ $(1 \leq i \leq a)$, $V = \bigoplus_{i=0}^{a} V_i \subset \mathfrak{g}_A$. Let $\mathcal{V}_i = A \cdot V_i$, $\mathcal{V} = A \cdot V = \bigoplus_{i=0}^{a} \mathcal{V}_i \subset \mathfrak{g}_A$. If $\tilde{N} = N + \varepsilon M$ then

(e) $\mathrm{ad}(\tilde{N})v_j = \Sigma_{0 \leq i \leq j}\, \varepsilon\phi_{ij}^{(M)}(v_j) + \Sigma_{j < i \leq a}\, \phi_{ij}^{(M)}(v_j)$ for all $v_j \in \mathcal{V}_j$,

where $\phi_{i,j}^{(M)} : \mathcal{V}_j \to \mathcal{V}_i$ are A-linear maps $(0 \leq i, j \leq a)$; moreover, if $M^0 \in \mathfrak{g}_{-2a}$ is defined by $M_{-2a} - M^0 \in \varepsilon A\mathfrak{g}_{-2a}$, we have:

(f) $\begin{cases} (\phi^{(M)}_{j+1,j} - \phi^{(M^0)}_{j+1,j})(\mathscr{V}_j) \subset \varepsilon \mathscr{V}_{j+1} \\ (\phi^{(M)}_{0,a} - \phi^{(M^0)}_{0,a})(\mathscr{V}_a) \subset \varepsilon^2 \mathscr{V}_0 \end{cases} \quad (0 \leqq j \leqq a-1).$

From (e) we deduce that

(g) $\mathrm{ad}(\tilde{N})^{a+1}(\mathscr{V}_j) \subset \varepsilon \mathscr{V}$ for all j, hence $\mathrm{ad}(\tilde{N})^{a+1}(\mathscr{V}) \subset \varepsilon \mathscr{V}$.

Let $\tilde{N}^0 = N + \varepsilon M^0$. Using also (f), we see that

(h) $\mathrm{ad}(\tilde{N})^{a+1} v_j = \mathrm{ad}(\tilde{N}^0)^{a+1} v_j + \varepsilon \sum_{i>j} \psi_{i,j}(v_j) + \varepsilon^2 \sum_i \rho_{i,j}(v_j)$, $(v_j \in \mathscr{V}_j)$,

where $\psi_{i,j}$, $\rho_{i,j}$ are A-linear maps $\mathscr{V}_j \to \mathscr{V}_i$. From (g) it follows that any non-zero eigenvalue λ of $\mathrm{ad}(\tilde{N})^{a+1}$ satisfies $v(\lambda) \geqq 1$ and (a) follows.

Next, we note that $\mathrm{ad}(\tilde{N}^0)(V_j) \subset V_{j+1}$ for $0 \leqq j < a$ and $\mathrm{ad}(\tilde{N}^0)V_a \subset \varepsilon V_0$. It follows that $\mathrm{ad}(\tilde{N}^0)^{a+1} V_j \subset \varepsilon V_j$ for all j. Hence there exists a $\mathbf{C}$-linear map $\phi : V \to V$ such that $\mathrm{ad}(\tilde{N}^0)^{a+1} = \varepsilon\phi$ on V. It follows that the eigenvalues of $\mathrm{ad}(\tilde{N}^0)^{a+1} : \mathfrak{g}_F \to \mathfrak{g}_F$ are $\varepsilon\lambda$ where $\lambda \in \mathbf{C}$ are the various eigenvalues of $\phi : V \to V$; from this, (b) follows.

We have

$$\varepsilon^{-1}\mathrm{ad}(\tilde{N})^{a+1} = \varepsilon^{-1}(\mathrm{ad}(\tilde{N}^0)^{a+1} + T) + \varepsilon T'$$

as endomorphisms of $\mathscr{V}$; here $T' \in \mathrm{End}_A(\mathscr{V})$ and $\varepsilon^{-1}(\mathrm{ad}(\tilde{N}^0)^{a+1} + T)$ is an upper triangular endomorphism of $\mathscr{V}$ (with respect to the block decomposition $\bigoplus \mathscr{V}_i$) with $\varepsilon^{-1}\mathrm{ad}(\tilde{N}^0)^{a+1}$ representing the diagonal blocks. Hence if P (resp. $P^{(0)}$) is the characteristic polynomial of the endomorphism $\varepsilon^{-1}\mathrm{Ad}(\tilde{N})^{a+1}$ resp. $\varepsilon^{-1}\mathrm{Ad}(\tilde{N}^0)^{a+1}$ of $\mathfrak{g}_F$ then:

(i) $P \in A[X]$, $P^{(0)} \in \mathbf{C}[X]$ and the difference $P - P^{(0)}$ is a polynomial of lower degree with coefficients in εA.

Assume now that N is as in (d) and that $M_{-2a} \notin \varepsilon A \mathfrak{g}_{-2a}$. Then, according to [Sp 1, 9.5, 9.6], $\tilde{N}^0$ is regular semisimple. Hence $P^{(0)}$ is divisible by X^l but not by X^{l+1} where l is the rank of $\mathfrak{g}$. From (i) it follows that P is not divisible by X^{l+1}. Hence $\mathrm{ad}(\tilde{N})^{a+1}$ has at most l eigenvalues equal to zero. The same must hold for $\mathrm{ad}(\tilde{N})$. Hence $\mathrm{ad}(\tilde{N})$ has exactly l zero eigenvalues hence $\tilde{N}$ is regular semisimple. From (i) we see also that the product $\lambda_1\lambda_2 \cdots \lambda_{2v}$ of the non-zero eigenvalues of $\varepsilon^{-1}\mathrm{ad}(\tilde{N})^{a+1}$ is an element of A whose constant term is the product of the non-zero eigenvalues of $\varepsilon^{-1}\mathrm{ad}(\tilde{N}^0)^{a+1}$, an element of $\mathbf{C}^*$. In particular, this constant term is non-zero so that $v(\lambda_1\lambda_2 \cdots \lambda_{2v}) = 0$. Hence if $\mu_1, \ldots, \mu_{2v}$ are the non-zero eigenvalues of $\mathrm{ad}(\tilde{N})$ then $v(\mu_1\mu_2 \cdots \mu_{2v}) = 2v/(a+1)$. Using now (a) it follows that $v(\mu_1) = \cdots = v(\mu_{2v}) = 1/(a+1)$.

Assume now that $M_{-2a} \in \varepsilon A \mathfrak{g}_{-2a}$ and that $\tilde{N}$ is regular semisimple. Then $\tilde{N}^0$ is nilpotent, hence $P^{(0)} = X^{\dim \mathfrak{g}}$. From (i), it follows that the coefficient of X^l in P

is in εA. This coefficient must be (up to sign) the product of the $2v$ non-zero eigenvalues of $\varepsilon^{-1}\mathrm{ad}(\tilde{N})^{a+1}$. Thus, if $\lambda_1, \dots, \lambda_{2v}$ are the non-zero eigenvalues of $\mathrm{ad}(\tilde{N})$ then $\Pi_{i=1}^{2v}(\varepsilon^{-1}\lambda_i^{a+1}) \in \varepsilon A$. It follows that $v(\lambda_1 \cdots \lambda_{2v}) > 2v/(a+1)$; in other words, we have $v(\Delta(\tilde{N})) > 2v/(a+1)$.

The last inequality holds trivially when $\tilde{N}$ is not regular semisimple; we then have $v(\Delta(\tilde{N})) = \infty$. Thus, (d) is proved.

We now prove (c). We fix $M^{(0)} \in \mathfrak{g}_{-2a} - \{0\}$ and let s be the number of non-zero eigenvalues of $\mathrm{ad}(N + \varepsilon M_0)$ (including repetitions). Let $R : \mathfrak{g}_F \to F$ be the polynomial which attaches to an element $x \in \mathfrak{g}$, the s-th elementary symmetric function in the eigenvalues of $\mathrm{ad}(x)$.

By (a) we have $v(R(\tilde{N})) \geqq s/(a+1)$ for any $\tilde{N} \in N + \mathfrak{n}$. Hence the set

$$\mathscr{U} = \left\{ \tilde{N} \in N + \mathfrak{n} \mid v(R(\tilde{N})) = \frac{s}{a+1} \right\}$$

is open in $N + \mathfrak{n}$. It is non-empty since $v(R(N + \varepsilon M^{(0)})) = s/(a+1)$, by (b).

Let $\tilde{N} \in \mathscr{U}$. Assume that all eigenvalues λ of $\mathrm{ad}(\tilde{N})$ satisfy $v(\lambda) > 1/(a+1)$. From the definition of R, it follows that $v(R(\tilde{N})) > s/(a+1)$, a contradiction. Thus, at least one eigenvalue λ of $\mathrm{ad}(\tilde{N})$ satisfies $v(\lambda) \leqq 1/(a+1)$. It then satisfies $v(\lambda) = 1/(a+1)$, by (a). This completes the proof.

9.12. We now consider a nilpotent element N in $\mathfrak{g}$ such that the centralizer of N in the adjoint group of $\mathfrak{g}$ is a unipotent group. The classification of such N is due to Dynkin. We shall describe in each case the conjugacy class (σ_N) in W.

(a) Assume that N is regular in $\mathfrak{g}$. From 9.11(d) it follows that σ_N is a Coxeter element of W.

(b) Assume that N is subregular in $\mathfrak{g}$ of type E_6, E_7 or E_8. From 9.11(d) it follows that (σ_N) is the class of a regular element in W (in the sense of [Sp 1]) of order 9, 14, 24 respectively.

(c) Assume that N is sub-subregular in $\mathfrak{g}$ of type E_8. From 9.11(d) it follows that σ_N is the class of a regular element in W of order 20.

(d) Assume that N is sub-subregular in $\mathfrak{g}$ of type E_7. From 9.11(c) it follows that σ_N is the class of an element of W with characteristic polynomial $\Phi_{12}(X)\Phi_6(X)\Phi_2(X)$ ($\Phi_d = d$-th cyclotomic polynomial).

(e) Assume that N is as in 9.9 in $\mathfrak{g}$ of type D_n. Then (σ_N) is as described in 9.9.

This exhausts all cases. We find that the restriction of our map $N \to (\sigma_N)$ to N as above coincides with the map defined by Carter–Elkington [CE]. (See also [Sp 1, 9.13].) The connection between regular nilpotent elements and Coxeter elements has been first pointed out by Kostant [Ko].

9.13. It is likely that our map $N \to (\sigma_N)$ is an injection of the set of nilpotent orbits in $\mathfrak{g}$ to conjugacy classes in W. It is a bijection for $\mathfrak{sl}_n$, but not in the general case. Note however that our map extends naturally to a map $\amalg_{\mathfrak{p}}$ {nilpotent orbits in ${}^1\mathfrak{p}$} $\to$ conjugacy classes in W where $\mathfrak{p}$ runs over the classes of maximal parahoric subalgebras of $\mathfrak{g}_F$ and ${}^1\mathfrak{p}$ is the finite-dimensional reductive quotient of $\mathfrak{p}$. It is likely that this extended map is surjective.

§10. Remarks on characters

In this section we shall change our notations: $\boldsymbol{G}$ will denote a split semisimple simply connected algebraic group over a finite field $k = F_q$, $G = \boldsymbol{G}(k)$, $\boldsymbol{T} \subset \boldsymbol{G}$ a maximal torus defined over F_q, $T = \boldsymbol{T}(k)$. To any character $\theta : T \to \mathbf{C}^*$ in general position one can associate an irreducible representation $\pi_{T,\theta}$ of G (see [DL]). Fix a unipotent element u in G. The character value tr $\pi_{T,\theta}(u)$ does not depend on the choice of θ. We denote it $a_w(u)$, where w is the conjugacy class in the Weyl group W of $\boldsymbol{G}$ corresponding to $\boldsymbol{T}$. In large characteristic one has the following algebro-geometric interpretation for $a_w(u)$ (see [Sp 3], [K]). Let $\mathscr{B}$ be the variety of Borel subgroups in $\boldsymbol{G}$, $\mathscr{B}_u \subset \mathscr{B}$ the variety of Borel subgroups containing u. The group W acts naturally on $H^*(\mathscr{B}_u) = H^*_{et}(\mathscr{B}_u, Q_l)$ (see [Sp 3]) where l is a prime, $(l, q) = 1$. We have

$$a_w(u) = (-1)^{r - r(T)} \sum_{i=0}^{2 \dim \mathscr{B}_u} (-1)^i \mathrm{Tr}(w \circ \mathrm{Fr}, H^i(\mathscr{B}_u))$$

where $r = k$-rank of $\boldsymbol{G}$, $r(\boldsymbol{T}) = k$-rank of $\boldsymbol{T}$ and Fr denotes the Frobenius map. In other words the computation of $a_w(u)$ is reduced to the study of the action of $W \times \mathrm{Fr}^{\mathbf{Z}}$ on $H^*(\mathscr{B}_u)$. It is known that $H^{\mathrm{odd}}(\mathscr{B}_u) = 0$ (see for example [CLP]) and that the eigenvalues of Fr on $H^*(\mathscr{B}_u)$ are integral powers of q times roots of 1. This gives us some fairly explicit formulas for $a_w(u)$.

We now consider the local field $F = k((\varepsilon))$ with residue field k and with ring of integers $A = k[[\varepsilon]]$. Let $\hat{G} = \boldsymbol{G}(F)$, $K = \boldsymbol{G}(A)$; let $p : K \to G$ be the natural projection. To any representation π of G we associate the representation $\Pi(\pi)$ of $\hat{G}$ defined by $\Pi(\pi) = \mathrm{Ind}_K^{\hat{G}}(\pi \circ p)$. Let $\Pi_{T,\theta} = \Pi(\pi_{T,\theta})$. If $\boldsymbol{T}$ is a k-anisotropic torus in $\boldsymbol{G}$, then $\Pi_{T,\theta}$ is a finite sum of irreducible cuspidal representations of $\hat{G}$ (see [H-Ch]). Therefore its character $\chi_{T,\theta}$ is well defined and is a locally constant function on the set of regular semisimple elements in $\hat{G}$. Let $\mathscr{U} \subset K$ be the set of regular semisimple elements u in K such that $p(u)$ is unipotent. It is clear that the restriction of $\chi_{T,\theta}$ to $\mathscr{U}$ does not depend on θ. We denote it by χ_w where as before w is the conjugacy class in W corresponding to $\boldsymbol{T}$. An element $u \in \hat{G}$ is said to be elliptic if its centralizer is a torus anisotropic over the maximal unramified extension of F. (This is stronger than the usual notion of ellipticity.) Let u be an elliptic element in $\mathcal{U}$. We denote by $\hat{\mathcal{B}}_u(k)$ the set of Iwahori subgroups in $\hat{G}$ containing u. One can show as in §3, Corollary 2, that $\hat{\mathcal{B}}_u(k)$ is the set of k-points of a natural projective k-variety $\hat{\mathcal{B}}_u$. The group W acts naturally on $H^*(\hat{\mathcal{B}}_u)$ and

$$\chi_w(u) = \pm \sum_{i=0}^{2\dim\hat{\mathcal{B}}_u} (-1)^i \operatorname{Tr}(w \circ \operatorname{Fr}, H^i(\hat{\mathcal{B}}_u)).$$

For $G = \mathrm{GL}_n$, $\chi_w(u)$ is given by an explicit elementary formula [BDKV], [CH] (in large characteristic); this suggests that $H^{\mathrm{odd}}(\hat{\mathcal{B}}_u) = 0$ and that the eigenvalues of Fr on $H^{2i}(\hat{\mathcal{B}}_u)$ are powers of q^i.

In the Appendix it will be seen that for $G = \mathrm{Sp}_6$ there exists a family $u_\lambda \in \mathcal{U}$, $\lambda \in k$, of elliptic elements such that the motive of $\hat{\mathcal{B}}_u$ contains the motive of an elliptic curve E_λ with j-invariant λ. Therefore $\chi_w(u_\lambda)$ has the form $\chi_w(u_\lambda) = A(q) + B(q)|E_\lambda(k)|$ where $A(q)$, $B(q)$ are non-zero polynomials in q. In particular, we see that there is no "elementary" formula for $\chi_w(u)$.

APPENDIX

An example of a non-rational variety $\hat{\mathcal{B}}_N$ for $G = \mathrm{Sp}(6)$

BY

J. BERNSTEIN AND D. KAZHDAN

Let L be a 6-dimensional C-vector space with symplectic scalar product $\langle\ ,\ \rangle$, $G(=\mathrm{Sp}(6))$ the corresponding symplectic group $\mathrm{Aut}(L, \langle\ ,\ \rangle)$, $\mathfrak{g}$ its Lie algebra. We put $L_A = A \otimes_{\mathbf{C}} L$, $\hat{L} = F \otimes L$, $\hat{G} = G(F) = \mathrm{Aut}(\hat{L}, \langle\ ,\ \rangle)$, $K = G(A)$.

Consider the nilpotent conjugacy class in $\mathfrak{g}$, which consists of elements S such that $\operatorname{Ker} S = \operatorname{Im} S$ is a Lagrangian subspace in L. We fix one such element S and denote by N an induced element in $\mathfrak{g}_A$. By definition $N = S + \varepsilon Y$, where $Y \in \mathfrak{g}_A$ is an element in the general position (see the end of the introduction and §8).

We claim that the variety $\hat{\mathcal{B}}_N$ is not rational. More precisely, we will prove that one of its components admits a dominant morphism onto an elliptic curve.

First of all, let us describe the curve. Put $V = \operatorname{Ker} S = \operatorname{Im} S$, $V' = L/V$.

Since V is Lagrangian, $\dim V = \dim V' = 3$ and V' is canonically isomorphic to the dual of the space V.

Operators S and Y define morphisms $\bar{S}: V' \to V$, $v' \mapsto Sv' \in V \subset L$ and $\bar{Y}: V \to V'$, $v \to Yv \pmod{V + \varepsilon L_A} \in L_A/(V + \varepsilon L_A) \simeq V'$. Since $S, Y \in \mathfrak{g}_A$ these morphisms are symmetric, i.e., they define quadratic forms B_S and B_Y on V' and V respectively, namely, $B_S(v') = \langle Sv', v' \rangle$, $B_Y(v) = \langle Yv, v \rangle \pmod{\varepsilon A}$. We denote by $Q \subset \mathbf{P}(V)$ and $Q' \subset \mathbf{P}(V')$ quadrics, defined by these forms and consider a curve $E = \{(l, l') \subset Q \times Q' \mid \langle l, l' \rangle = 0\}$.

It is easy to see that E is connected and the natural projection $E \to Q \approx \mathbf{P}^1$ is a 2-fold covering ramified at 4 points, which are points of intersection of the quadric Q with the quadric Q_S^* dual to the quadric Q'. This shows that E is an elliptic curve and allows us to compute the Jacobi invariant $j(E)$. Namely, let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of the operator $\bar{S}\bar{Y}: V \to V$; since Y is generic, we assume that they are nonzero and distinct. Then in some coordinate systems on V and V' we can write $B_Y = \Sigma x_i^2$, $B_S = \Sigma \lambda_i y_i^2$, $\langle \quad \rangle = \Sigma x_i y_i$. The dual quadric is given by $B_S^* = \Sigma \mu_i x_i^2$ with $\mu_i = \lambda_i^{-1}$. Using this it is easy to check that

$$j(E) = 32 \left(\sum_{i<j} (\mu_i - \mu_j)^2 \right)^3 \Big/ \left(\prod_{i<j} (\mu_i - \mu_j) \right)^2 .$$

PROPOSITION 1. *There exists an irreducible component Z of $\hat{\mathscr{B}}_N$ and a fibration $\varphi: Z \to E$, with fibers isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.*

COROLLARY 1. *Variety $\hat{\mathscr{B}}_N$ is not rational. Moreover, its birational type essentially depends on N, i.e., on $(\lambda_1, \lambda_2, \lambda_3)$.*

REMARKS. (1) One can show that $\hat{\mathscr{B}}_N \setminus Z$ is a rational variety, so the birational type of $\hat{\mathscr{B}}_N$ completely determines the curve E.

(2) Our proof depends on the fact that $\dim \hat{\mathscr{B}}_N = 3$, which we easily deduce from §8 for generic Y. In fact, one can prove by direct calculations (similar to ones in §9) that $\dim \hat{\mathscr{B}}_N = 3$ for any Y such that λ_i are nonzero and distinct.

PROOF OF PROPOSITION. First of all, let us give a more geometric description of varieties $X = \hat{G}/K$ and $\hat{\mathscr{B}} = \hat{G}/\hat{\mathscr{B}}_0$.

DEFINITION. A lattice is a finitely generated A-submodule $\Lambda \subset \hat{L}$, such that $F \cdot \Lambda = \hat{L}$.

A *chain* is a sequence of lattices $C = \{C(i)\}$, $i \in \mathbf{Z}$, such that for all i, $C(i+1) \subsetneq C(i)$ and $C(i+6) = \varepsilon C(i)$.

It is easy to see that in a chain $\dim_{\mathbf{C}} C(i)/C(j) = j - i$ for $j \geqq i$. We define the dual lattice $\Lambda^{\#}$ by $\Lambda^{\#} = \{x \in \hat{L} \mid \langle x, \Lambda \rangle \in A\}$ and the dual chain $C^{\#}$ by $C^{\#}(i) = C(-i)^{\#}$.

LEMMA 1. *X and $\hat{\mathscr{B}}$ as $\hat{G}$-sets are naturally isomorphic to the sets of selfdual lattices and selfdual chains, respectively.*

PROOF. Denote by Λ_0 the selfdual lattice $\Lambda_0 = L_A = A \otimes L$. Its stabilizer in $\hat{G}$ coincides with K. It is known that $\hat{G}$ acts transitively on the set of selfdual lattices, so this set is isomorphic to $\hat{G}/K$. Now let $C = \{C(i)\}$ be a selfdual chain. Replacing C by gC for some $g \in \hat{G}$ we can assume that $C(0) = \Lambda_0$. Consider the flag $\mathscr{F} = \{L = F_0 \supset F_1 \supset \cdots \supset F_6 = 0\}$ in $L = \Lambda_0/\varepsilon\Lambda_0$ given by $F_i = C(i)/\varepsilon\Lambda_0$. Since $\varepsilon\Lambda_0 = C(6)$, this flag is a complete flag in L, $\dim F_i = 6 - i$. Moreover, the condition $C = C^{\#}$ implies that the flag $\mathscr{F}$ is selfdual, i.e., $F_i^{\perp} = F_{6-i}$. Conversely, any selfdual flag $\mathscr{F}$ in L determines a selfdual chain C by $C(i) = F_i + \varepsilon\Lambda_0$ for $i = 0, 1, \ldots, 6$. Since K acts transitively on selfdual flags and the stationary subgroup of one of them coincides with Iwahori subgroup $\hat{B}_0$, the set of selfdual chains is isomorphic to $\hat{G}/\hat{B}_0 = \hat{\mathscr{B}}$.

Henceforth we will identify $\hat{\mathscr{B}}_N$ with the variety of selfdual N-invariant chains. Note that since N is nil and $\dim(C(i)/C(i+1)) = 1$ the condition $NC(i) \subset C(i)$ in fact is equivalent to $NC(i) \subset C(i+1)$.

Fix lattices $\Lambda_0 = L_A = A \otimes L$ and $\Lambda' = V + \varepsilon\Lambda_0$. Clearly $\Lambda_0^{\#} = \Lambda_0$ and $(\Lambda')^{\#} = \varepsilon^{-1}\Lambda' = \varepsilon^{-1}V + \Lambda_0$. We define a subvariety $Z \subset \hat{\mathscr{B}}_N$ by

$$Z = \{C = \{C(i)\} \mid \Lambda_0 \supset C(1) \supset C(2) \supset \Lambda', NC(-1) \subset C(1), NC(2) \subset C(4)\}.$$

LEMMA 2. *There is a natural fibration $\varphi: Z \to E$ with fibers isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. In particular, Z is irreducible nonsingular of dimension S.*

LEMMA 3. *All irreducible components of $\hat{\mathscr{B}}_N$ have dimension* 3.

Clearly, these two lemmas imply the proposition.

PROOF OF LEMMA 2. Let $C = \{C(i)\} \in Z$. We have $\Lambda_0 \supset C(1) \supset \Lambda' = V + \varepsilon\Lambda_0$ so $\varepsilon^{-1}V + \Lambda_0 \supset C(-1) \supset \Lambda_0$. Since $\dim C(-1)/C(1) = 2$ we have $\dim C(-1)/\Lambda_0 = 1$, i.e., $C(-1) = \varepsilon^{-1}l + \Lambda_0$, where $l \in \mathbf{P}(V)$. Note that l is completely determined by $C(-1)$ as $l = \varepsilon C(-1)/\varepsilon\Lambda_0 \subset \Lambda'/\varepsilon\Lambda_0 = V$.

Choose a nonzero element $a \in l$ and consider $b = \varepsilon^{-1}a \in C(-1)$. Then $(Nb, b) = \varepsilon^{-2}(Sa, a) + \varepsilon^{-1}(Ya, a) = \varepsilon^{-1}B_Y(a) \pmod{A}$. Since $b \in C(-1)$ and $Nb \in C(1)$ by definition of Z this implies that $B_Y(a) = 0$, i.e., $l \in Q$.

Similarly $C(2) = l' \oplus \Lambda'$ where $l' \in Q'$. The line l' is determined by $C(2)$ as

$l' = C(2) \bmod \Lambda' \subset \Lambda_0/\Lambda' \approx V'$. Since $\langle C(-1), C(2)\rangle \subset A$, we see that $\langle l, l'\rangle = 0$, i.e., $(l, l') \in E$.

Thus we have described a morphism $\varphi : Z \to E$. Let us describe the fiber $\varphi^{-1}(l, l')$. By definition, it consists of selfdual chains $C = \{C(i)\} \in Z$ such that $C(-1) = \varepsilon^{-1} l + \Lambda_0$, $C(1) = C(-1)^{\#}$, $C(2) = l' + \Lambda'$, $C(4) = \varepsilon C(-2) = \varepsilon C(2)^{\#}$. Hence such a chain is completely determined by lattices $C(0)$ and $C(3)$. Since $C(-1) \subsetneq C(0) \subsetneq C(1)$ the lattice $C(0)$ is determined by a line in the 2-plane $C(-1)/C(1)$. Similarly $C(3)$ is determined by a line in the 2-plane $C(2)/C(4)$. We claim that any choice of these two lines gives a chain $C = \{C(i)\} \in Z$, which shows that $\varphi^{-1}(l, l') \approx \mathbf{P}^1 \times \mathbf{P}^1$.

Let us check that any lattice $C(0)$ between $C(-1)$ and $C(1)$ satisfies all conditions for a chain in Z. Since $NC(-1) \subset C(1)$ we automatically have $NC(0) \subset C(1)$. So we only have to check that $C(0)$ is selfdual.

Clearly $C(0)^{\#}$ also lies strictly in-between $C(-1)$ and $C(1)$. So it is enough to show that $C(0)^{\#} \supset C(0)$, i.e., that for each $a, b \in C(0)$ we have $\langle a, b\rangle \in A$. Since $\langle\ \rangle$ is skew symmetric and $\dim C(0)/C(1) = 1$, we can always assume that $b \in C(1)$. But then $\langle a, b\rangle \in A$ since $a \in C(-1) = C(1)^{\#}$.

Similarly one checks that any choice of $C(3)$ between $C(2)$ and $C(4)$ satisfies all conditions.

Proof of Lemma 3. As was shown in §4 all components of $\hat{\mathscr{B}}_N$ have the same dimension. Since N is induced by a nilpotent element $S \in \mathfrak{g}$, $\dim \hat{\mathscr{B}}_N$ equals $\dim \mathscr{B}_S$, where $\mathscr{B}_S$ is the variety of S-invariant flags in the flag variety of $\mathfrak{g}$. It is easy to check that $\dim \mathscr{B}_S = 3$, which proves the lemma.

References

[B] N. Bourbaki, *Groupes et algèbres de Lie*, Ch. IV, V, VI, Hermann, Paris, 1968.

[BC] P. Bala and R. W. Carter, *Classes of unipotent elements in simple algebraic groups*, Math. Proc. Camb. Phil. Soc. **79** (1976), 401–425 and **80** (1976), 1–18.

[BDKV] J. Bernstein, P. Deligne, D. Kazhdan and M-F. Vigneras, *Représentations des groups réductifs sur un corps local*, Hermann, Paris, 1984.

[BT] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local, I*, Publ. Math. IHES, **41** (1972), 5–251.

[CE] R. W. Carter and G. B. Elkington, *A note on the parametrization of conjugacy classes*, J. Alg. **20** (1972), 350–354.

[CH] L. Corwin and R. Howe, *Computing characters of tamely ramified p-adic division algebras*, Pac. J. Math. **73** (1977), 461–477.

[CLP] C. De Concini, G. Lusztig and C. Procesi, *Homology of the zero-set of a nilpotent vector field on a flag manifold*, J. Am. Math. Soc. **1** (1988).

[DG] M. Demazure and P. Gabriel, *Groupes algébriques*, North-Holland, Amsterdam, 1970.

[DL] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*. Ann. of Math. **103** (1976), 103–161.

[H-Ch] Harish-Chandra, *Harmonic Analysis on Reductive p-adic Groups*, Lecture Notes in Mathematics **162**, Springer-Verlag, Berlin, 1970.

[K] D. Kazhdan, *Proof of Springer's hypothesis*, Isr. J. Math. **28** (1977), 272–286.

[Ka] V. Kac, *Constructing groups associated to infinite-dimensional algebras*, in *Infinite Dimensional Groups with Applications*, MSRI Publications, Springer-Verlag, Berlin, 1985.

[K–L] D. Kazhdan and G. Lusztig, *A topological approach to Springer's representations*, Adv. Math. **38** (1980), 222–228.

[Ko] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Am. J. Math. **81** (1959), 973–1032.

[L] G. Lusztig, *Singularities, characters, formulas and a q-analog of weight multiplicities*, Astérisque **101–102** (1983), 208–229.

[LS] G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, J. London Math. Soc. **19** (1979), 41–52.

[S1] J.-P. Serre, *Corps locaux*, Hermann, Paris, 1962.

[S2] J.-P. Serre, *Cohomologie galoisienne*, Lecture Notes in Math. **5**, Springer-Verlag, Berlin, 1965.

[Sp 1] T. A. Springer, *Regular elements of finite reflection groups*, Invent. Math. **25** (1974), 159–198.

[Sp 2] T. A. Springer, *A construction of representations of Weyl groups*, Invent. Math. **44** (1978), 278–293.

[Sp 3] T. A. Springer, *Generalization of Green's polynomials*, in Proc. Symp. in Pure Math., Vol. 21, Am. Math. Soc., Providence, 1971, 149–154.

[Spa] N. Spaltenstein, *On the fixed point set of a unipotent element on the variety of Borel subgroups*, Topology **16** (1977), 203–204.

[St] R. Steinberg, *Conjugacy classes in algebraic groups*, Lecture Notes in Math. **366**, Springer-Verlag, Berlin, 1974.