ANGLES AND QUASICONFORMAL MAPPINGS IN SPACE(*)

By

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To Professor Heinrich Brinkmann on his 70th birthday.

1. Introduction. Quasiconformal mappings, by their very name, ought to be in some sense angle-preserving. One of the first problems which arises, however, in studying the behavior of angles under quasiconformal mappings, is the non-differentiable character of these mappings, allowable on a set of measure zero. An exceptional point P_0 may lie on a smooth curve which is mapped onto a curve not possessing a tangent line at the image P_0^*.

For example, given $k > 0$, let the transformation T be defined in the (x, y)-plane for $x > 0$, by

$$u = x,$$

$$v = k x \sin(\log x) + y.$$

T may be extended to be a homeomorphism of the plane by taking it to be the identity mapping for $x \leqq 0$, but it is then non-differentiable on the axis $x = 0$. The ray $y = c$, $x \geqq 0$ is mapped on the curve $v = k u \sin(\log u) + c$, which has no tangent line at $(0, c)$. T may further be extended to any number of dimensions by setting the other coordinates respectively equal to one another.

Such behavior is part of the price paid for compactness of various families of quasiconformal mappings, which is essential for their use in extremal problems. The compactness is obtained, for example, by adopting the definition that a homeomorphism T is quasiconformal if the ratios

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$$\frac{M(R^*)}{M(R)} \tag{1.1}$$

are bounded above and below. Here, R is any doubly connected region in the domain of T, R^* its image, and $M(R)$, $M(R^*)$ the respective moduli of R, R^*.

Various measures of quasiconformality have been investigated: the *inner* dilatation $K_I(T)$, which is the reciprocal of the infimum of the ratios in (1.1); the *outer* dilatation $K_0(T)$, which is the supremum; and the *maximal* dilatation $K(T)$, which is simply $\max\{K_I(T), K_0(T)\}$. For the general theory of quasiconformal mappings in space, the reader is referred in particular to Väisälä, [8], and Gehring, [4].

A somewhat different looking dilatation is the *linear* dilatation $K_L(T)$, defined by

$$K_L(T) = \operatorname{ess\,sup} D(P_0),$$

where

$$D(P_0) = \limsup_{r\to 0} \frac{\max\{\,|T(P) - T(P_0)| : |P - P_0| = r\}}{\min\{\,|T(Q) - T(P_0)| : |Q - P_0| = r\}}\,.$$

It will prove to be $K_L(T)$ which is most intimately connected with angles. Our results can easily be rephrased in terms of $K(T)$ through the sharp estimates for n-dimensional mappings,

$$K(T)^{1/(n-1)} \leqq K_L(T) \leqq K(T)^{2/n}\,.$$

Our example is quasiconformal, with $K_L(T) = 1 + k^2 + k\sqrt{2 + k^2}$, irrespective of the number of dimensions.

The problem of defining angle measure for curves, possibly without tangent lines at the vertex, has been studied extensively in the plane by F. W. Gehring and the present author, [1], and by O. Taari, [7]. The definition proposed in the former could have been formally adapted to any number of dimensions, although it was not immediately clear how it could be applied to obtain analogous theorems. It now appears that the key lies in the inner angles of homeomorphic images of cones. For clarity of exposition, we present the theory of angles and quasiconformal mappings in three dimensions, treating first necessary, then sufficient, and finally necessary and sufficient conditions for quasiconformality. The article closes with the few remarks necessary to recover the theory in higher dimensions.

2. Notation and elementary observations. We consider the 3-dimensional Euclidean space of points $P = (x, y, z)$. If, for $i = 1, 2$, $P_i = (x_i, y_i, z_i)$, we denote as usual by $P_1 \cdot P_2$ the real number $x_1x_2 + y_1y_2 + z_1z_2$, and by $|P|$ the nonnegative square root of $P \cdot P$. The point $(0, 0, 0)$ will be 0.

Central in our work will be homeomorphisms of domains in space. We will generally use the symbol T for such a mapping, and conversely, any transformation labelled T will always be a homeomorphism. Occasionally we write $(u, v, w) = T(x, y, z)$. Given T and any set E, point P, or topological angle α, the symbol E^*, P^*, α^* will denote the image of E, P, or α under T. T^{-1} will denote the inverse transformation, ∂E will denote the boundary of E, and $\rho(P, E)$ will denote the distance from P to the set E.

2.1. Topological angles and their measure. By a topological angle α at a point P_0, will be understood a pair of simple curves (γ_1, γ_2), with common initial point P_0, known as the vertex. It will not be necessary to consider parametrizations of these curves, as our subsequent definitions depend only on their traces. If we wish to indicate the curves explicitly, we will write $\alpha = (\gamma_1, \gamma_2)$. For the measure A of the topological angle $\alpha = (\gamma_1, \gamma_2)$ at P_0, we define

$$A(\alpha) = \liminf_{\substack{P_1 \in \gamma_1 \\ P_2 \in \gamma_2 \\ P_1, P_2 \to P_0}} 2 \sin^{-1} \left(\frac{|P_1 - P_2|}{|P_1 - P_0| + |P_2 - P_0|} \right). \tag{2.1}$$

It is clear from the triangle inequality that A is well-defined, with $0 \leqq A \leqq \pi$. We also note that if T is a similarity mapping, then A is invariant, which in our terminology we write briefly as $A(\alpha^*) = A(\alpha)$. Constant use will be made of this observation without further mention. We will say that α is non-zero if $A(\alpha) > 0$.

2.2. Ordinary angles. It is essential to know that the definition (2.1) coincides with the usual in the event that the curves γ_1 and γ_2 have unique tangent lines at the vertex. Such angles will be known as *ordinary* angles, and the following theorem shows that A provides the "ordinary" measure for an ordinary angle.

Theorem 2.2. *Suppose that* $\alpha = (\gamma_1, \gamma_2)$ *is a topological angle at* 0, *and suppose that*

$$\lim_{\substack{P_i \in \gamma_i \\ P_i \to 0}} \frac{P_i}{|P_i|} = Q_i\,; \qquad i = 1, 2.$$

Then $\cos A(\alpha) = Q_1 \cdot Q_2$.

Proof. Let ϕ be the true angle between P_1 and P_2, which is to say $0 \leqq \phi \leqq \pi$, $P_1 \cdot P_2 = |P_1||P_2|\cos\phi$, and let

$$\phi' = 2\sin^{-1}\left(\frac{|P_1 - P_2|}{|P_1| + |P_2|}\right).$$

The law of cosines gives

$$\cos^2\frac{\phi'}{2} = 1 - \sin^2\frac{\phi'}{2} = 1 - \left(\frac{|P_1 - P_2|}{|P_1| + |P_2|}\right)^2$$

$$= \frac{|P_1|^2 + |P_2|^2 + 2|P_1||P_2| - |P_1 - P_2|^2}{(|P_1| + |P_2|)^2}$$

$$= \frac{2|P_1||P_2|}{(|P_1| + |P_2|)^2}(1 + \cos\phi) = \frac{4|P_1||P_2|}{(|P_1| + |P_2|)^2}\cos^2\frac{\phi}{2},$$

and therefore,

$$\cos^2\frac{\phi'}{2} \leqq \cos^2\frac{\phi}{2},$$

with equality if $|P_1| = |P_2|$. Hence for $P_i \in \gamma_i$, $P_i \to 0$,

$$\cos^2\frac{A(\alpha)}{2} = \overline{\lim}\cos^2\frac{\phi'}{2} \leqq \lim\cos^2\frac{\phi}{2}. \tag{2.2}$$

However, if in addition $|P_1| = |P_2|$, (2.2) takes the form

$$\cos^2 \frac{A(\alpha)}{2} \geqq \lim \cos^2 \frac{\phi'}{2} = \lim \cos^2 \frac{\phi}{2}. \tag{2.3}$$

The inequalities (2.2) and (2.3) yield $\cos A(\alpha) = \lim \cos \phi = Q_1 \cdot Q_2$.

2.3. Cones. By a *regular* cone C with vertex P_0 and central angle β, $0 < \beta < \frac{\pi}{2}$, will be understood any similarity transform E^* of the graph E of the equation

$$z = \sqrt{x^2 + y^2} \cot \beta,$$

with $0^* = P_0$. The axis γ_0 of the cone C will be the image of the positive z-axis.

More generally, a *topological* cone will be the image C^* of a regular cone C under a homeomorphism of a neighborhood of the vertex. By analogy, we define the inner angle A' of the topological cone C^* by

$$A'(C^*) = \liminf_{\substack{P \in C \\ Q \in \gamma_0 \\ P, Q \to P_0}} 2 \sin^{-1} \left(\frac{|P^* - Q^*|}{|P^* - P_0^*| + |Q^* - P_0^*|} \right).$$

We note that $A'(C^*)$ depends on the image γ_0^* of the axis of C, as well as on the surface C^*. However, there will be no need to display this dependence.

A topological angle $\alpha^* = (\gamma^*, \gamma_0^*)$ will be said to *lie on* C^*, providing γ lies on C and forms a topological angle $\alpha = (\gamma, \gamma_0)$ with the axis γ_0 at the vertex P_0. We next prove a useful result relating $A'(C^*)$ to the measure of the angles lying on C^*.

Theorem 2.3. *$A'(C^*)$ is the minimum value of $A(\alpha^*)$, where α is any ordinary angle lying on C.*

Proof. It is clear that for any α lying on C, $A'(C^*) \leqq A(\alpha^*)$. On the other hand, select $P_n \in C$, $Q_n \in \gamma_0$, with

$$A'(C^*) = \lim_{n \to \infty} 2 \sin^{-1} \left(\frac{|P_n^* - Q_n^*|}{|P_n^* - P^*| + |Q_n^* - P_0^*|} \right).$$

By the compactness of the unit sphere, we may also assume $\{P_n/|P_n|\}$ converges. It is then a simple matter to run a smooth curve γ on C, through each P_n, and because of the convergence of $\{P_n/|P_n|\}$, forming an ordinary angle α_0 with the vertex γ_0. But then α_0 lies on C, and it is clear that $A(\alpha_0^*) \leqq A'(C^*)$. The theorem is proved.

Corollary 2.4. *Let C be a regular cone of central angle β. Then*
(a) $A(\alpha) = \beta$ *for every ordinary angle α lying on C,*
(a′) $A'(C) = \beta$.

Proof. The assertion (a) follows from Theorem 2.2, and (a′) follows from Theorem 2.3, together with part (a).

2.4. An example. From a certain standpoint, it would be desirable in Theorem 2.4 to use in place of ordinary angles on C, *segmental* angles on C, which would require that γ be a generator of the cone. We give a brief description of an example which shows that this is not possible.

For $0 \leqq \psi \leqq 1$, $0 \leqq z \leqq 1$, let $f = f(\ ;\psi)$ be the piecewise linear function of z, with successive vertices at (z,f) equal respectively to $(0,0)$; $\left(\frac{\psi}{4}, \frac{\psi}{4}\right)$; $\left(\frac{\psi}{2}, \frac{\psi^2}{4}\right)$; (ψ,ψ); $(1,1)$. Define $f(\ ;\psi)$ for $1 \leqq \psi \leqq 2$ by the relation $f(\ ;\psi) = f(\ ;2-\psi)$. Let $[r,\theta,z]$ denote cylindrical coordinates in (x,y,z)-space, and $[\rho,\phi,w]$ denote cylindrical coordinates in (u,v,w)-space.

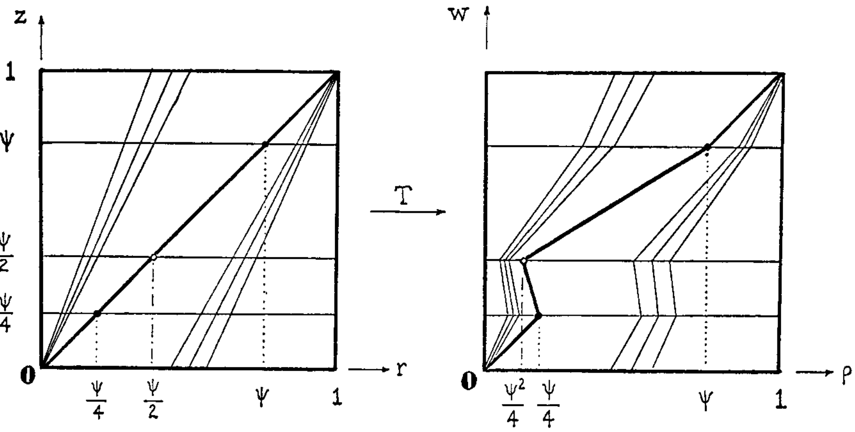
Define the mapping $[\rho,\phi,w] = T[r,\theta,z]$ as follows:

$$\phi = \theta$$

$$w = z$$

$$\rho = \begin{cases} \dfrac{r}{z} f\left(z;\dfrac{\theta}{\pi}\right) & : 0 \leqq r \leqq z \leqq 1 \\ 1 - \left(\dfrac{1-r}{1-z}\right)\left(1 - f\left(z;\dfrac{\theta}{\pi}\right)\right) & : 0 \leqq z \leqq r \leqq 1, \\ r & : -1 \leqq z \leqq 0. \end{cases}$$

The behavior of the mapping T in the half-plane determined by θ = constant $= \pi\psi$ may be observed on the following sketch.



We take C to be the vertical cone with vertex O and central angle $\frac{\pi}{4}$. The axis γ_0, is mapped on itself. For any fixed θ, the generator g_θ, parametrized by $g_\theta(t) = [t, \theta, t]$, is mapped onto g_θ^*, parametrized by $g_\theta^*(t) = \left[f\left(t;\frac{\theta}{\pi}\right), \theta, t\right]$, $t \geqq 0$. For $\theta = 0$, $f(t;0) \equiv t$, and for $0 < \theta < 2\pi$, $t < \min\left\{\frac{\theta}{4\pi}, \frac{2\pi-\theta}{4\pi}\right\}$ implies $f\left(t;\frac{\theta}{\pi}\right) = t$. Hence each generator is mapped onto itself for sufficiently small t, and letting $\alpha_\theta = (g_\theta, \gamma_0)$, it follows that $A(\alpha_\theta^*) = \frac{\pi}{4}$ for each θ. On the other hand, the curve γ on C, parametrized by $\gamma(t) = [t, 2\pi t, t]$ is tangent to the generator g_0, and is mapped on γ^*, parametrized by $\gamma^*(t) = [f(t;2t), 2\pi t, t] = [t^2, 2\pi t, t]$, tangent to γ_0^* at O. Setting $\alpha_0 = (\gamma, \gamma_0)$, it follows from Theorems 2.3 and 2.2 that $0 \leqq A'(C^*) \leqq A(\alpha_0^*) = 0$, while $\min_\theta A(\alpha_\theta^*) = \frac{\pi}{4}$.

3. The distortion of angle measure under transformations. In the plane, a theorem of Teichmüller made possible the calculation of the sharp lower bound for $A(\alpha^*)$ in terms of $A(\alpha)$ and $K(T)$. The methods available in space are not yet so refined, but we can nevertheless obtain a satisfactory estimate. At the points of differentiability a substantially better bound is obtained. We discuss these bounds in their natural setting, returning in Section 3.3 to a discussion of quasiconformal mappings.

3.1. Affine and locally affine transformations. We will discuss the distortion of angles with vertex O, for homeomorphisms T which can be written in the form

$$T(P) = T_0(P) + o(|P|),$$

where T_0 is one of the following affine transformations:

$$\begin{cases} T_1:\ T_1(x,y,z) = (x, Cy, Dz), & 1 \leqq C \leqq D; \\ T_2:\ T_2(x,y,z) = (0, Cy, z), & 0 \leqq C \leqq 1; \\ T_3:\ T_3(x,y,z) = (0,0,0). \end{cases} \tag{3.1}$$

Theorem 3.1. *With T as above, for each β, $0 < \beta < \pi$,*

(a) *In case T_0 is T_1, then for ordinary angles α at* O, *with $A(\alpha) = \beta$, we have the sharp bound*

$$A(\alpha^*) \geqq 2\tan^{-1}\left(\frac{1}{D}\tan\frac{\beta}{2}\right). \tag{3.2}$$

(b) *In case T_0 is T_2, there exists a segmental angle α_0 at* O, *with $A(\alpha_0) = \beta$, but with $A(\alpha_0^*) = 0$.*

Proof. Since Lemma 1 of [1] and its proof remain correct in higher dimensions, we may at once assume the error $o(|P|)$ is identically zero. Since the ordinary angle is determined by the tangent lines, we need only consider segmental angles. Part (b) is obvious, since the segments joining O respectively to $\left(\sin\frac{\beta}{2}, 0, \cos\frac{\beta}{2}\right)$, $\left(-\sin\frac{\beta}{2}, 0, \cos\frac{\beta}{2}\right)$ are both mapped onto the z-axis. For part (a), we may assume $A(\alpha^*) = \beta^*$, $0 < \beta^* < \pi$. Then with suitable choice of coordinates (ξ, η) in the plane determined by the segments forming α, and in the plane determined by the segments forming α^*, we find T takes the form $T(\xi, \eta) = (\xi, B\eta)$, $1 \leqq B \leqq D$. For some θ, α is formed by the segments joining $(0, 0)$ respectively to $\left(\cos\left(\theta + \frac{\beta}{2}\right), \sin\left(\theta + \frac{\beta}{2}\right)\right)$, $\left(\cos\left(\theta - \frac{\beta}{2}\right), \sin\left(\theta - \frac{\beta}{2}\right)\right)$, so that

$$\cos\beta^* = \frac{\cos\left(\theta + \frac{\beta}{2}\right)\cos\left(\theta - \frac{\beta}{2}\right) + B^2\sin\left(\theta + \frac{\beta}{2}\right)\sin\left(\theta - \frac{\beta}{2}\right)}{\sqrt{\cos^2\left(\theta + \frac{\beta}{2}\right) + B^2\sin^2\left(\theta + \frac{\beta}{2}\right)}\sqrt{\cos^2\left(\theta - \frac{\beta}{2}\right) + B^2\sin^2\left(\theta - \frac{\beta}{2}\right)}}.$$

The procedure is now elementary: one differentiates $\cos\beta^*$ with respect to θ, and finds the only extremal cases to be $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, of which the first and third yield equal minima, and the second and fourth equal maxima. Since $\cos\beta^*$ is decreasing in β^*, one may evaluate $\min\beta^*$ at $\theta = \frac{\pi}{2}$, obtaining (3.2) with B in place of D. One need only take the (x, z)-plane initially to obtain $B = D$.

3.2. Θ-mappings. For the class of Θ-mappings introduced by Gehring, [2, Section 8], a particularly simple estimate of angle distortion can be found. We recall that given a continuous, increasing, unbounded function Θ defined for $0 \leqq t < 1$ with $\Theta(0) = 0$, a transformation T is called a Θ-mapping if for every subdomain G_0 of the domain of T, the conditions

$$P_1 \in G_0, \ |P_1 - P_2| < \rho(P_1, \partial G_0),$$

imply

$$\frac{|P_2^* - P_1^*|}{\rho(P_1^*, \partial G_0^*)} \leqq \Theta\left(\frac{|P_2 - P_1|}{\rho(P_1, \partial G_0)}\right).$$

If T^{-1} is a Θ-mapping, then the starred and unstarred letters may be interchanged, and applying Θ^{-1} we find

$$\Theta^{-1}\left(\frac{|P_2 - P_1|}{\rho(P_1, \partial G_0)}\right) \leqq \frac{|P_2^* - P_1^*|}{\rho(P_1^*, \partial G_0^*)}, \tag{3.3}$$

in this form holding whether or not $|P_2^* - P_1^*| < \rho(P_1^*, \partial G_0^*)$.

Given P_0 in G, the domain of T, we take G_0 to be the domain $G - \{P_0\}$, and by continuity, for P_1, P_2 sufficiently close to P_0 we will have $\rho(P_i, \partial G_0) = |P_i - P_0|$, $\rho(P_i^*, \partial G_0^*) = |P_i^* - P_0^*|$, $i = 1, 2$. Then (3.3) may be applied with the roles of P_1 and P_2 interchanged, and we find for $i = 1, 2$,

$$(3.4) \quad \Theta^{-1}\left(\frac{|P_1 - P_2|}{|P_1 - P_0| + |P_2 - P_0|}\right) \leqq \Theta^{-1}\left(\frac{|P_1 - P_2|}{|P_i - P_0|}\right) \leqq \frac{|P_1^* - P_2^*|}{|P_i^* - P_0^*|}.$$

Taking reciprocals in (3.4), adding the cases $i = 1, 2$, and taking reciprocals again, we find

$$\frac{1}{2}\Theta^{-1}\left(\frac{|P_1 - P_2|}{|P_1 - P_0| + |P_2 - P_0|}\right) \leqq \frac{|P_1^* - P_2^*|}{|P_1^* - P_0^*| + |P_2^* - P_0^*|}.$$

Finally taking lower limits on both sides, we obtain the following theorem.

Theorem 3.2. *Let T be a transformation, such that T^{-1} is a Θ-mapping. Then for every point P_0 in the domain of T, and every topological angle α at P_0,*

$$A(\alpha^*) \geqq 2\sin^{-1}\left(\frac{1}{2}\Theta^{-1}\left\{\sin\frac{1}{2}A(\alpha)\right\}\right).$$

3.3. Quasiconformal mappings. For the present, we need two properties of quasiconformal mappings. First, by [4, Theorem 11], given $K \geqq 1$, there exists a distortion function Θ_K such that each quasiconformal mapping T with $K(T) \leqq K$ is a Θ_K-mapping. From (1.1) and the definition of the maximal dilatation it is evident that $K(T) \leqq K$ if and only if $K(T^{-1}) \leqq K$. Hence we may apply Theorem 3.2 with $\Theta = \Theta_K$ to quasiconformal mappings T with $K(T) \leqq K$.

Next, from [4, Theorem 6] it follows that a quasiconformal mapping T is almost everywhere differentiable with non-zero Jacobian. To say that T is differentiable at P_0 is to assert the existence of a linear transformation $DT(P_0;\ )$ with the property that

$$T(P) - T(P_0) = DT(P_0; P - P_0) + o(|P - P_0|).$$

With proper choice of coordinates, $DT(P_0;\ )$ becomes one of the trio T_1, T_2, or T_3 of (3.1), but in case of non-zero Jacobian, $DT(P_0;\ )$ becomes T_1 with $D = D(P_0)$. Combining these observations, we derive

Theorem 3.3. *Let T be a quasiconformal mapping of a domain G. Then*

(a) *For every point P_0 in G, and every topological angle α at P_0,*

$$A(\alpha^*) \geqq 2\sin^{-1}\left(\frac{1}{2}\Theta_K^{-1}\left\{\sin\frac{1}{2}A(\alpha)\right\}\right);\ K = K(T).$$

(b) *For every point P_0 of differentiability with non-zero Jacobian, we have the sharp lower bound for ordinary angles α at P_0,*

$$A(\alpha^*) \geqq 2\tan^{-1}\left(\frac{1}{D}\tan\frac{1}{2}A(\alpha)\right);\ D = D(P_0).$$

Theorem 2.3 enables us to convert these estimates into statements about cones.

Corollary 3.4. *Let T be a quasiconformal mapping of a domain G. Then*

(a′) *For every point P_0 in G, and every regular cone C at P_0,*

$$A'(C^*) \geqq 2\sin^{-1}\left(\frac{1}{2}\Theta_K^{-1}\left\{\sin\frac{1}{2}A'(C)\right\}\right);\ K = K(T).$$

(b′) *For every point P_0 of differentiability with non-zero Jacobian, we have the sharp lower bound for regular cones C at P_0,*

$$A'(C^*) \geqq 2\tan^{-1}\left(\frac{1}{D}\tan\frac{1}{2}A'(C)\right)\ ;\ D = D(P_0). \tag{3.5}$$

4. A criterion for absolute continuity on lines. This section will be substantially measure-theoretic. We will treat z as a real variable, and (x, y) as a point in the plane. We let $\mathscr{I}$, $\mathscr{S}$, $\mathscr{M}$ denote respectively the classes of Lebesgue measurable sets in 3-space, the (x, y)-plane, and the z-axis. The corresponding Lebesgue measures will be denoted by τ, σ, μ, and we will use the expressions τ-a.e., σ-a.e., or μ-a.e. to mean almost everywhere with respect to τ, σ, or μ. For $E \in \mathscr{S}$, $F \in \mathscr{M}$, we will denote by $E \times F$ the set of all (x, y, z); $(x, y) \in E$, $z \in F$.

A transformation T is said to be ACL in z if for σ-a.e. (x, y), $T(x, y, z)$ is an absolutely continuous function of z. Under such conditions, let us take a fixed, non-exceptional (x, y), and an interval I. Then [6, page 123] the curve traced out by $T(x, y, z)$: $z \in I$, is rectifiable with length Λ. The partial derivative $T_z = \left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}, \frac{\partial w}{\partial z}\right)$ exists for μ-a.e. z in I, and $\Lambda = \int_I |T_z(x, y, \zeta)| \, d\zeta$. From its expression as a limit of continuous difference quotients, and by Fubini's Theorem, T_z exists τ-a.e., and is $\mathscr{I}$-measurable.

T is further said to be ACL_p in z, $1 \leqq p$, if T is ACL in z, and if $|T_z|$ is locally p-integrable with respect to τ, and to be ACL_p if it is ACL_p in each of its variables. By a recent result due to Väisälä, [9], if T is ACL_3 then T is already totally differentiable τ-a.e., and we can as before form $DT(P_0;\ )$, with suitable choice of coordinates either T_1, T_2, or T_3 of (3.1). The discussion on page 4 of [9] may now be rephrased as follows: *A necessary and sufficient condition that T be quasiconformal is that T be ACL_3 with $DT(P_0;\ )$ either T_1 or T_3, and that at points of the former type, $D(P_0)$ be essentially bounded.* Because $DT(P_0;\ )$ is of type T_3 only on the τ-null set where the Jacobian $J(P_0)$ vanishes, $K_L(T)$ is in fact ess sup$\{D(P_0)\colon J(P_0) \neq 0\}$.

We first set out to give conditions under which T will be ACL_3. We select minimal hypotheses to emphasize only what is needed for our technique. Noting that for $0 < \beta < \pi$,

$$2\tan^{-1}\left(\frac{1}{K}\tan\frac{\beta}{2}\right) > \frac{\beta}{K},$$

it is clear that our hypotheses are necessary conditions for T to be quasiconformal with $K_L(T) \leqq K$. We let $\mathscr{C}_\beta$ denote the class of regular cones with axis parallel to one of the coordinate axes and central angle β.

Theorem 4.4. *Suppose* $0 < \beta < \frac{\pi}{2}$ *and* $K \geqq 1$ *are fixed numbers. Suppose that T is a homeomorphism of a domain G such that* (a′) *For each vertex* P_0 *in* G, *and each* C *in* $\mathscr{C}_\beta$ *at* P_0,

$$A'(C^*) > 0.$$

(b′) *For* τ-a.e. *vertex* P_0 *in* G, *and each* C *in* $\mathscr{C}_\beta$ *at* P_0,

$$A'(C^*) \geqq \frac{\beta}{K}. \tag{4.1}$$

Then T *is* ACL_3.

Since the proof is rather long, we break it into a number of lemmas.

4.1. Linear measure. Given any set E, and $\delta > 0$, we define

$$\Lambda[\delta, E] = \inf_{\mathscr{U}} \sum_{i=1}^{n} d(E_i),$$

where $\mathscr{U}$ is any covering of E by sets $E_1, E_2, \cdots, E_n$, each of which has diameter $d(E_i) < \delta$. $\Lambda[\delta, E]$ is evidently non-increasing in δ, and accordingly we may set

$$\Lambda[E] = \lim_{\delta \to 0} \Lambda[\delta, E)],$$

which is termed the linear measure of E. Λ is a regular Caratheodory outer measure, and determines a class $\mathscr{L}$ of measurable sets, on which Λ is a measure. A simple curve γ is $\mathscr{L}$-measurable, and γ is rectifiable if and only if $\Lambda[\gamma] < \infty$, in which case $\Lambda[\gamma]$ is the arc length. See [6, II§8].

Next, for a line segment L, and $r > 0$, we let L_r denote the closed right circular cylinder with axis L and radius r.

Lemma 4.1. *Let* T *be a transformation of a domain* G. *Let* L *be a segment in* G, *with*

$$\lim_{r \to 0} \frac{\tau[L_r^*]}{\pi r^2} = M < \infty. \tag{4.2}$$

Let E *be a compact subset of* L, *and suppose there exist integers* p, q, *such that for every regular cone* C *with central angle* β, *axis* $\gamma_0 \subset L$, *and vertex* $P_0 \in E$, *we have*

$$\frac{|P^* - Q^*|}{|P^* - P_0^*| + |Q^* - P_0^*|} \geqq \frac{1}{2p} \tag{4.3}$$

whenever

$$P \in C,\ |P - P_0| < \frac{1}{q},\ Q \in \gamma_0,\ |Q - P_0| < \frac{1}{q}. \tag{4.4}$$

Then

$$\{\Lambda[E^*]\}^2 \leqq 3p^2 M \tan^2 \beta \{\Lambda[E]\}^2. \tag{4.5}$$

Proof. First we remark that all cones in the subsequent discussion will be assumed to have central angle β and axis in the z-axis. We take L to be a segment in the z-axis. To any interval I, $I = [a, b] \subset L$, we associate the set $C(I)$, which is the region enclosed by the up-cone with vertex $P_a = (0, 0, a)$ and the down-cone with vertex $P_b = (0, 0, b)$. If $P_a \in E$, and $P_b \in E$, $b - a$ sufficiently small, the estimate (4.3) may be applied to any $P \in \partial C(I)$, $Q \in I$, and P_0 the appropriate endpoint of I.

For such an interval I, we wish to estimate $\tau[C(I)^*]$, and we may assume P_a^* is $(0, 0, 0)$ and P_b^* is $(0, 0, d)$, $d > 0$. We first estimate the cross-section area $g(w_0)$ of $C(I)^*$ in the plane $w = w_0$, $0 < w_0 < d$. Let $Q^* = T(Q)$ be a point of intersection of I^* with the plane $w = w_0$, and let η be the distance from Q^* to the nearest point $P^* = T(P)$ of $\partial C(I)^*$ in $w = w_0$. The alternative that P lies on the up-cone at P_a leads to the estimate

$$\eta = |P^* - Q^*| \geqq \frac{1}{2p}(|P^* - P_a^*| + |Q^* - P_a^*|) \geqq \frac{w_0}{p},$$

while the alternative that P lies on the down-cone at P_b leads to

$$\eta = |P^* - Q^*| \geqq \frac{1}{2p}(|P^* - P_b^*| + |Q^* - P_b^*|) \geqq \frac{d - w_0}{p}.$$

Thus, for $0 < w_0 < d$,

$$g(w_0) \geqq \pi\eta^2 \geqq \frac{\pi}{p^2}\min\{w_0^2, (d - w_0)^2\},$$

and

$$\tau[C(I)^*] \geqq \int_0^d g(w)dw \geqq \frac{2\pi}{p^2} \int_0^{d/2} w^2 dw = \frac{\pi d^3}{12p^2}. \tag{4.6}$$

We next note that we may assume our set E is perfect, since unless countable itself, it differs from a perfect subset of itself by a countable set. This will enable us to use the covering Lemma 7 of [1], which asserts that for all sufficiently small $t > 0$, there exist N (N depending on t) non-overlapping intervals I_n with endpoints in E, lengths not exceeding t, E contained in their union, and $Nt \leqq \Lambda[E(t)]$, where $E(t)$ is the set of points of L within distance t of E. Given these intervals I_n, we note that for t sufficiently small, the estimate (4.6) can be applied to any subinterval $I \subset I_n$, whose endpoints also lie in E, and since the associated set $C(I)$ will be a subset of $C(I_n)$, we may draw from (4.6) the conclusion that if d_n is the diameter of $(I_n \cap E)^*$

$$\pi d_n^3 \leqq 12\, p^2 \tau[C(I_n)^*], \quad n = 1, 2, \cdots, N. \tag{4.7}$$

Setting $\delta = \max\{d_1, d_2, \cdots, d_N\}$, it follows from the uniform continuity of T in a compact neighborhood of L that δ tends to zero with t. Since E is compact, we also know $\Lambda[E(t)] \to \Lambda[E]$. Setting $r = \frac{1}{2} t \tan\beta$ assures that $C(I_n) \subset L_r$, $n = 1, 2, \cdots, N$. Using Hölder's inequality, we find from (4.7),

$$\begin{aligned} \{\Lambda[\delta, E^*]\}^3 &\leqq \left\{\sum_{n=1}^{N} d_n\right\}^3 \leqq N^2 \left\{\sum_{n=1}^{N} d_n^3\right\} \\ &\leqq \frac{12\, N^2 p^2}{\pi} \left\{\sum_{n=1}^{N} \tau[C(I_n)^*]\right\} \\ &\leqq \frac{12\, N^2 p^2 \tau[L_r^*]}{\pi} \\ &= 3(Nt)^2 p^2 \tan^2\beta \left(\frac{\tau[L_r^*]}{\pi r^2}\right) \\ &\leqq 3p^2 \left(\frac{\tau[L_r^*]}{\pi r^2}\right) \tan^2\beta \{\Lambda[E(t)]\}^2, \end{aligned}$$

from which (4.5) follows upon letting $t \to 0$.

4.2. Differentiation of measures. Regrettably, we must introduce still more notation. R will be a fixed rectangle in the (x, y)-plane, and L a fixed segment in the z-axis, such that $R \times L$ lies in the domain of T. In the (x, y)-plane, $B_r(x, y)$ will denote the disk of radius r, centered at (x, y). $L(x, y)$ will denote the vertical segment $\{(x, y)\} \times L$. We introduce the set function ϕ, defined for $E \in \mathscr{S}$, $F \in \mathscr{M}$, by

$$\phi[E, F] = \tau[(E \times F)^*].$$

It is clear that for fixed E, $\phi[E, \ ]$ is a measure on $\mathscr{M}$, and that for fixed F, $\phi[\ , F]$ is a measure on $\mathscr{S}$. Lebesgue's theorem on differentiation, [6], asserts that for our fixed segment L, the $\mathscr{S}$-measurable function $\phi'(\ ; L)$, defined for (x, y) in R by

$$\phi'(x, y; L) = \lim_{r \to 0} \frac{\phi[B_r(x, y), L]}{\pi r^2}$$

exists and is finite σ-a.e., and for $E \in \mathscr{S}$, $E \subset R$,

$$\iint_E \phi'(\ ; L) d\sigma \leqq \phi[E, L].$$

Lemma 4.2. *Under the hypotheses of Theorem* 4.4, *there exists a positive constant ξ, depending only on K and β, such that for σ-a.e. (x, y), the inequality*

$$\{\Lambda[E^*]\}^3 \leqq \xi \phi'(x, y; L) \{\mu[E]\}^2, \tag{4.8}$$

holds for every compact linear subset $E \subset L(x, y)$.

Proof. We set H_{pq} equal to the set of all vertices P_0 in $R \times L$ for which (4.3) holds whenever (4.4) holds. Hypothesis (a') assures us that $R \times L = \bigcup_{p,q} H_{pq}$. Moreover, the continuity of T assures us that H_{pq} is compact for large enough q.

Fix an integer p_0 so that $\sin \dfrac{\beta}{2K} > \dfrac{1}{2p_0}$. By hypothesis (b′) and Fubini's theorem, we know that for σ-a.e. (x, y),

$$\mu[L(x,y) - \bigcup_q H_{p_0 q}] = 0.$$

Let us fix a non-exceptional (x,y) and one for which also $\phi'(x,y;L) < \infty$. We may as well assume (x,y) is $(0,0)$, so that $L(x,y)$ becomes L, and write M for $\phi'(x,y;L)$. Since $\mu = \Lambda$ for linear sets, we will be done if we can re-establish (4.5) with $p = p_0$.

If F_{pq} is a compact μ-null set in $L \cap H_{pq}$, then Lemma 4.1 implies $\Lambda[F_{pq}^*] = 0$. If F_1 is a compact μ-null subset of L, the preceding case applies to $F_{pq} = F_1 \cap H_{pq}$, hence $\Lambda[F_1^*] \leqq \sum_{p\,q} \Lambda[F_{pq}^*] = 0$. If F_2 is a μ-null G_δ-Borel set in L, then since $L \cap H_{pq}$ is compact, $\Lambda[(F_2 \cap H_{pq})^*] \leqq \Lambda[(L \cap H_{pq})^*] < \infty$, which shows F_2^* is a countable union of sets of finite linear measure, and by [3, Lemma 2] and the preceding case, $\Lambda[F_2^*] = \sup\{\Lambda[F_1^*]\colon F_1 \text{ compact}, F_1 \subset F_2\} = 0$. Generally, if F_3 is any μ-null subset of L, there exists a G_δ-Borel set F_2, $F_3 \subset F_2$, $\mu[F_2] = \mu[F_3] = 0$, so by the preceding case $\Lambda[F_3^*] \leqq \Lambda[F_2^*] = 0$.

Finally, given our compact set E in L, we write

$$E = \{E \bigcap \bigcup_q H_{p_0 q}\} \bigcup \{E - \bigcup_q H_{p_0 q}\}.$$

The latter, as a subset of $L - \bigcup_q H_{p_0 q}$, is μ-null, and by the preceding argument, its image under T is Λ-null. Since the $H_{p_0 q}$ are expanding as $q \to \infty$, Lemma 4.1 gives

$$\begin{aligned} \{\Lambda[E^*]\}^3 &= \lim_{q\to\infty} \{\Lambda[(E \cap H_{p_0 q})^*]\}^3 \\ &\leqq \lim_{q\to\infty} 3p_0^2 M \tan^2\beta \{\Lambda[E \cap H_{p_0 q}]\}^2 \\ &= 3p_0^2 M \tan^2\beta \{\Lambda[E]\}^2, \end{aligned}$$

and Lemma 4.2 is proved.

4.3. The cube-integrability of $|T_z|$, and the proof of Theorem 4.4. We are now in a position to prove the final lemma.

Lemma 4.3. *Let T satisfy the conclusion of Lemma 4.2 for every segment L in a fixed segment I. Then*

$$\iiint_{R\times I} |T_z|^3 \, d\tau \leqq \xi\phi[R,I]. \tag{4.9}$$

Proof. Indeed, (4.8) already shows that T is ACL in z. With the exception of (x,y) in a σ-null set F, we have

$$\Lambda[L(x,y)^*] = \int_L |T_z(x,y,\zeta)|\, d\zeta$$

holding for all subintervals $L \subset I$.

Define the $\mathscr{I}$-measurable functions g_n, $n = 1, 2, \cdots$ by

$$g_n(x,y,z) = \frac{n}{2} \int_{I_n(z)} |T_z(x,y,\zeta)|\, d\zeta,$$

where $I_n(z)$ denotes the interval $\left\{\zeta : z - \frac{1}{n} \leqq \zeta \leqq z + \frac{1}{n}\right\}$. For (x,y) not in F, it follows that for μ-a.e. z,

$$\lim_{n\to\infty} g_n(x,y,z) = |T_z(x,y,z)|,$$

and hence by Fubini's theorem, τ-a.e. we have $|T_z| = g$, where

$$g = \liminf_{n\to\infty} g_n.$$

Now let us fix z and n, and denote by $E_n(x,y,z)$ the segment $\{(x,y)\} \times I_n(z)$. By Lemma 4.2 with $E = L = I_n(z)$, we have for σ-a.e. (x,y),

$$g_n(x,y,z)^3 = \left(\frac{\Lambda[E_n(x,y,z)^*]}{\mu[E_n(x,y,z)]}\right)^3 \leqq \frac{\xi\phi'(x,y;I_n(z))}{\mu[E_n(x,y,z)]}.$$

Integrating over R, we find

$$\iint_R g_n(x,y,z)^3 d\sigma(x,y) \leqq \frac{n\xi}{2}\phi[R,I_n(z)],$$

and letting $n \to \infty$, by Fatou's lemma, we conclude

$$\iint_R g(x,y,z)^3 d\sigma(x,y) \leqq \liminf_{n\to\infty} \frac{n\xi}{2}\phi[R,I_n(z)].$$

But with another application of Lebesgue's theorem, we know that for μ-a.e. z,

$$\lim_{n\to\infty} \frac{n}{2}\phi[R,I_n(z)] = \phi''(z)$$

where ϕ'' is $\mathscr{M}$-measurable, and $\int_I \phi'' d\mu \leqq \phi[R,I]$. Hence for μ.-ae. z,

$$\iint_R g(x,y,z)^3 d\sigma(x,y) \leqq \xi\phi''(z),$$

and (4.9) follows by Fubini's theorem.

Regarding Theorem 4.4, it is now clear that T is ACL_3 in z. But by the symmetry of the hypotheses of Theorem 4.4, T is ACL_3. The proof is completed.

5. Characterizations of quasiconformality. Suppose that T satisfies the hypotheses of Theorem 4.4, not only for cones in $\mathscr{C}_\beta$, but for all regular cones of central angle β. By Theorem 4.4 and the above mentioned differentiation theorem of Väisälä, T is almost everywhere differentiable, and we may compute $DT(P_0;\ )$ — of type T_1, T_2, or T_3 of (3.1). Hypothesis (a′) together with Theorems 3.1 (b) and 2.3 rules out the possibility T_2. Further, almost everywhere that $DT(P_0;\ )$ is of type T_1, we may compare the bound (4.1) with the sharp bound (3.5). It follows that

$$2\tan^{-1}\left(\frac{1}{D(P_0)}\tan\frac{\beta}{2}\right) \geqq \frac{\beta}{K},$$

and hence

$$K_L(T) = \operatorname{ess\,sup}\{D(P_0): J(P_0) \neq 0\} \leqq \tan\frac{\beta}{2}\cot\frac{\beta}{2K}.$$

Next, we may wish to require that T satisfy (4.1) for all cones and *all* β, in which case it follows that

$$K_L(T) \leqq \inf_{0<\beta<\pi/2} \tan\frac{\beta}{2}\cot\frac{\beta}{2K} = K.$$

We thus have the following characterizations:

Theorem 5.1. *A homeomorphism T is quasiconformal in a domain G, with $K_L(T) \leqq K$, if and only if*

(a′) *For every vertex P_0 in G, and every non-zero regular cone at P_0,*

$$A'(C^*) > 0.$$

(b′) *For almost every vertex P_0 in G, and every regular cone C at P_0,*

$$A'(C^*) \geqq \frac{1}{K} A'(C).$$

Theorem 5.2. *A homeomorphism T of a domain G is quasiconformal, with $K_L(T) \leqq K$, if and only if*

(a) *For every vertex P_0 in G, and every non-zero ordinary angle α at P_0,*

$$A(\alpha^*) > 0,$$

(b) *For almost every vertex P_0 in G, and every ordinary angle α at P_0,*

$$A(\alpha^*) \geqq \frac{1}{K} A(\alpha).$$

From a *methodological* standpoint, Theorem 5.1 is the extension of Theorem 4 of [1]. We note that only regular cones are needed, and they are used as were the segments in the latter theorem. From a *formal* standpoint, the analogue to Theorem 4 of [1] would be the alternative Theorem 5.2, but requiring only that (a) and (b) apply to *segmental* angles. Unfortunately, the example in Section 2.4 will not allow us to use these methods to obtain a theorem using only segmental angles. On the other hand, although the example is not quasiconformal, neither does it satisfy the hypothesis (b) of Theorem 5.2 for segmental angles, and hence it casts no light on the possible truth or falsity of such a strengthened theorem.

One result in this direction is due to Kopylov, [5]: *A mapping which preserves angles between rays is conformal.* The hypothesis means that every segmental angle is mapped on an ordinary angle of equal measure. The methods are quite different, apparently using Baire Category and Rademacher-Stepanov Theorems. For contrast, we close this section with the best result obtainable by our methods in the conformal case. An *oriented* ordinary angle is an ordinary angle, one of whose sides is a segment parallel to a coordinate axis.

Theorem 5.3. *If T is a homeomorphism, and if there exists β, $0 < \beta < \frac{\pi}{2}$, such that for oriented ordinary angles α of size β,*

(a) *$A(\alpha^*) > 0$ at every vertex,*

(b) *$A(\alpha^*) \geqq \beta$ at almost every vertex,*

then T is conformal.

Proof. Our hypotheses, together with Theorem 2.3 and Corollary 2.4, assure that the hypotheses of Theorem 4.4 are satisfied with $K = 1$. Hence T is ACL_3, and totally differentiable almost everywhere. It is then an exercise in elementary trigonometry, not unlike the proof of Theorem 3.1, to show that the partial derivatives T_x, T_y, T_z are mutually orthogonal and of equal length almost everywhere, from which the result follows.

6. Extension of the theory to $n+1$ dimensions, $n \geqq 3$. The preceding sections have been designed as much as possible to permit immediate extension to higher dimensions. Many of the remarks remain formally correct, given the new interpretations for the symbols. For example, we will denote the points by $P = (X, z)$, where $X = (x_1, x_2, \cdots, x_n)$. The scalar product is defined as usual, and $|P|^2 = |X|^2 + z^2$, where $|X|^2 = \sum_{j=1}^{n} x_j^2$. Section 2 needs only the remark that a regular cone is a similarity transform of the graph of the equation $z = |X| \cot \beta$ to remain in force. The idea of the example in Section 2.4 is easy to visualize, though it becomes tedious to write down and we will not do so.

In Section 3, the trio of affine mappings becomes

$$
\begin{aligned}
T_1(X,z) &= (x_1, C_2x_2, \cdots, C_nx_n, Dz)\colon 1 \leqq C_2 \leqq \cdots \leqq C_n \leqq D, \\
T_2(X,z) &= (0, C_2x_2, \cdots, C_nx_n, z)\colon\ 0 \leqq C_2 \leqq \cdots \leqq C_n \leqq 1, \qquad (3.1') \\
T_3(X,z) &= (0, 0, \cdots, 0, 0),
\end{aligned}
$$

and Theorem 3.1, because of its essentially 2-dimensional character, still applies. The distortion theory of Section 3.2 is known to be true in $n+1$ dimensions, although no references are as yet available. Theorem 3.3 and its Corollary 3.4 are thus verified conditionally.

For Section 4, we need to interpret τ as $(n+1)$-measure, σ as n-measure, and μ as 1–measure. We let Ω_n be the n-measure of the unit ball in n-space, and by a right circular cylinder with axis L and radius r, we understand the image under a rigid motion of the set $\{(X,z)\colon |X| \leqq r,\ 0 \leqq z \leqq \text{length } L\}$ in which the z-axis is carried onto L. Lemma 4.1 requires the hypothesis

$$(4.2') \qquad \lim_{r \to 0} \frac{\tau[L_r^*]}{r^n \Omega_n} = M < \infty,$$

in order to draw the conclusion

$$\{\Lambda[E^*]\}^{n+1} \leqq (n+1)p^n M \tan^n \beta \{\Lambda[E]\}^n.$$

Indeed, the "plane" $w = w_0$ is now an n-dimensional hyperplane, and the inequality above (4.6) reads

$$g(w_0) \geqq \eta^n \Omega_n \geqq \Omega_n p^{-n} \min\{w_0^n, (d - w_0)^n\}.$$

Inequality (4.6) and the long chain following (4.7) condense respectively to

$$(4.6') \qquad \tau[C(I)^*] \geqq \frac{\Omega_n d^{n+1}}{(n+1)(2p)^n}$$

and

$$\{\Lambda[\delta, E^*]\}^{n+1} \leqq (n+1)p^n M \tan^n \beta \{\Lambda[E(t)]\}^n,$$

from which Lemma 4.1 is recovered.

For Lemma 4.2, we let $B_r(X)$ denote the n-ball in n-space, centered at X with radius r. Then setting

$$\phi'(X;L) = \lim_{r \to 0} \frac{\phi[B_r(X), L]}{r^n \Omega_n},$$

Lemma 4.2 goes through, with the conclusion

$$\{\Lambda[E^*]\}^{n+1} \leqq \xi \phi'(X;L)\{\mu[E]\}^n. \tag{4.8'}$$

In Section 4.3 we now obtain the $(n+1)$-integrability of $|T_z|$. This is essential since ACL_{n+1} is now needed in place of ACL_3, and indeed the differentiability follows as before from [9], which is already phrased for n dimensions. The characterizations of Section 5 hold verbatim.

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Note added in proof: During the preceding year the following additional references have come to my attention:

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The theory of Θ — mappings in n-dimensions is discussed extensively in [A, Section 9]. In [B], the authors announce the following theorem: If the conditions (a) and (b) of the present Theorem 5.2 are fulfilled for segmental angles, and if (c) T is totally differentiable a.e., then T is quasiconformal. The first author has noted in correspondence that there is an oversight in the proof of this theorem, and so its status remains in doubt.

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