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ON THE RIESZ BASISNESS OF THE ROOT FUNCTIONS OF THE NONSELF-ADJOINT STURM-LIOUVILLE OPERATOR

BY

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ABSTRACT

In this article we obtain the asymptotic formulas for eigenfunctions and eigenvalues of the nonself-adjoint Sturm–Liouville operators with periodic and antiperiodic boundary conditions, when the potential is an arbitrary summable complex-valued function. Then using these asymptotic formulas, we find the conditions on Fourier coefficients of the potential for which the eigenfunctions and associated functions of these operators form a Riesz basis in $L_2(0,1)$.

Let $L_t(q)$ be the operator generated in $L_2[0,1]$ by the expression

$$-y'' + q(x)y, \tag{1}$$

and the boundary conditions

$$y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0), \tag{2}$$

where $q(x)$ is a complex-valued summable function.

In this article we obtain asymptotic formulas of order $O(n^{-l})$ $(l = 1,2,\ldots,)$ for the n-th eigenvalue and corresponding eigenfunction of the operator $L_t(q)$

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with $q(x) \in L_1(0,1)$, for $t = 0, \pi$, that is, for periodic and antiperiodic boundary conditions. Note that in classical investigations, in order to obtain the asymptotic formulas of order $O(n^{-l})$ it is required that $q(x)$ be $(l-1)$ times differentiable (see [2, 3]). Then using these asymptotic formulae, we find the conditions on Fourier coefficients $q_n = (q(x), e^{i2\pi nx})$ where $(.,.)$ denotes inner product in $L_2(0,1)$ of $q(x)$ for which the root functions (the eigenfunctions and associated functions) of $L_t(q)$, for $t = 0, \pi$, form a Riesz basis in $L_2(0,1)$. Note that the periodic and antiperiodic boundary conditions are regular boundary conditions, but are not strongly regular boundary conditions. Therefore, in general, the eigenfunctions and associated functions of $L_0(q)$ and $L_\pi(q)$ do not form a Riesz basis; they form a basis with brackets in $L_2(0,1)$ (see [4], [5]). In this paper we prove that if

$$\lim_{n\to\infty} \frac{\ln|n|}{nq_{2n}} = 0, \quad q_{2n} \sim q_{-2n}, \tag{3}$$

then the eigenfunctions and associated functions of $L_0(q)$ form a Riesz basis in $L_2(0,1)$, where $a_n \sim b_n$ means that $c_1 \mid b_n \mid < \mid a_n \mid < c_2 \mid b_n \mid$, for all $n = 1, 2, \ldots$. Here and in subsequent relations, we denote by c_m ($m = 1, 2, \ldots$) the positive constants whose exact values are inessential. Similarly, if

$$\lim_{n\to\infty} \frac{\ln|n|}{nq_{2n+1}} = 0, \quad q_{2n+1} \sim q_{-2n-1}, \tag{4}$$

then the eigenfunctions and associated functions of $L_\pi(q)$ form a Riesz basis in $L_2(0,1)$.

It is well-known that (see formulas (47a), (47b) on page 65 of [3]) the eigenvalues of the operators $L_0(q)$ and $L_\pi(q)$ consist of the sequences $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}$ and $\{\mu_{n,1}\}, \{\mu_{n,2}\}$ satisfying

$$\lambda_{n,j} = (2n\pi)^2 + O(n^{1/2}) \tag{5}$$

and

$$\mu_{n,j} = (2n\pi + \pi)^2 + O(n^{1/2}), \tag{6}$$

for $j = 1, 2$. From these formulas one can easily obtain the following inequalities:

$$|\lambda_{n,j} - (2\pi k)^2| > |2(n-k)\pi||2(n+k)\pi| - c_3 n^{1/2} > c_4 n, \tag{7}$$

$$|\mu_{n,j} - (2\pi k + \pi)^2| > |2(n-k)\pi||2(n+k+1)\pi| - c_5 n^{1/2} > c_6 n \tag{8}$$

for $j = 1, 2$; $k \neq n$; $k = 0, 1, \ldots$; and $n \geq N$, where we denote by N a sufficiently large positive integer, that is, $N \gg 1$. To obtain the asymptotic formula for eigenvalues $\lambda_{n,j}$ and corresponding normalized eigenfunctions $\Psi_{n,j}(x)$ of $L_0(q)$, we use (7) and the well-known relation

$$(\lambda_{N,j} - (2\pi n)^2)(\Psi_{N,j}(x), e^{i2\pi nx}) = (q(x)\Psi_{N,j}(x), e^{i2\pi nx}). \tag{9}$$

Moreover, we use the following relations:

$$(q(x)\Psi_{N,j}(x), e^{i2\pi nx}) = \sum_{m=-\infty}^{\infty} q_{-m}(\Psi_{N,j}(x), e^{i2\pi(n+m)x}), \tag{10}$$

$$|(q(x)\Psi_{N,j}(x), e^{i2\pi nx})| < 4M, \tag{11}$$

for $n \in Z$ and $N \gg 1$, where $M = \sup_{n \in Z} |q_n|$. These relations are obvious for $q(x) \in L_2(0,1)$. For $q(x) \in L_1(0,1)$, see Lemma 1 of [6]. Note that $q_n \to 0$ as $|n| \to \infty$ and, without loss of generality, we assume that $q_0 = 0$.

Using (10) in (9) we get

$$(\lambda_{n,j} - (2\pi n)^2)(\Psi_{n,j}, e^{i2\pi nx}) = \sum_{\substack{n_1=-\infty, \\ n_1 \neq 0}}^{\infty} q_{n_1}(\Psi_{n,j}, e^{i2\pi(n-n_1)x}). \tag{12}$$

Now isolating the term in the right-hand side of (12) containing the multiplicant $(\Psi_{n,j}(x), e^{-i2\pi nx})$ (i.e., case $n_1 = 2n$), replacing

$$(\Psi_{n,j}(x), e^{i2\pi(n-n_1)x}) \quad \text{by} \quad \frac{(q(x)\Psi_{n,j}(x), e^{i2\pi(n-n_1)x})}{\lambda_{n,j} - (2\pi(n-n_1))^2} \quad \text{for } n_1 \neq 2n$$

(this replacement can be obtained from (9) by taking n and $n - n_1$ instead of N and n) and using (10) for the numerator of the last fraction, we obtain

$$\begin{aligned}&(\lambda_{n,j} - (2\pi n)^2)(\Psi_{n,j}(x), e^{i2\pi nx}) - q_{2n}(\Psi_{n,j}(x), e^{-i2\pi nx}) \\ &\quad = \sum_{\substack{n_1,n_2=-\infty \\ n_1 \neq 2n}}^{\infty} \frac{q_{n_1}q_{n_2}(\Psi_{n,j}(x), e^{i2\pi(n-n_1-n_2)x})}{\lambda_{n,j} - (2\pi(n-n_1))^2}.\end{aligned} \tag{13}$$

Note that, since $q_0 = 0$, here and in the subsequent relations the sums are taken under conditions $n_1, n_2, \ldots \neq 0$. Now isolating the terms on the right-hand side of (13) containing one of the multiplicants $(\Psi_{n,j}(x), e^{i2\pi nx})$, $(\Psi_{n,j}(x), e^{-i2\pi nx})$ (i.e., cases $n_1 + n_2 = 0, 2n$) and replacing

$$(\Psi_{n,j}(x), e^{i2\pi(n-n_1-n_2)x}) \quad \text{by} \frac{(q(x)\Psi_{n,j}(x), e^{i2\pi(n-n_1-n_2)x})}{\lambda_{n,j} - (2\pi(n-n_1-n_2))^2} \quad \text{for } n_1+n_2 \neq 0, 2n$$

we obtain

$$(14)\quad \begin{aligned}&(\lambda_{n,j}-(2\pi n)^2)(\Psi_{n,j}(x),e^{i2\pi nx})-q_{2n}(\Psi_{n,j}(x),e^{-i2\pi nx})\\ &-\sum_{\substack{n_1=-\infty\\ n_1\neq 2n}}^{\infty}\frac{q_{n_1}q_{-n_1}(\Psi_{n,j}(x),e^{i2\pi nx})}{\lambda_{n,j}-(2\pi(n-n_1))^2}-\sum_{\substack{n_1=-\infty\\ n_1\neq 2n}}^{\infty}\frac{q_{n_1}q_{2n-n_1}(\Psi_{n,j}(x),e^{-i2\pi nx})}{\lambda_{n,j}-(2\pi(n-n_1))^2}\\ &=\sum_{\substack{n_1,n_2=-\infty\\ n_1\neq 2n,n_1+n_2\neq 0,2n}}^{\infty}\frac{q_{n_1}q_{n_2}(q(x)\Psi_{n,j}(x),e^{i2\pi(n-n_1-n_2)x})}{(\lambda_{n,j}-(2\pi(n-n_1))^2)(\lambda_{n,j}-(2\pi(n-n_1-n_2))^2)}.\end{aligned}$$

Repeating this process m-times (i.e., using (10) in the numerator of the fractions on the right-hand side of (14) and isolating the terms containing one of the multiplicants $(\Psi_{n,j}(x),e^{i2\pi nx})$, $(\Psi_{n,j}(x),e^{-i2\pi nx})$ etc.), we get

$$(15)\quad \begin{aligned}&(\lambda_{n,j}-(2\pi n)^2-A_m(\lambda_{n,j}))(\Psi_{n,j}(x),e^{i2\pi nx})\\ &-(q_{2n}+B_m(\lambda_{n,j}))(\Psi_{n,j}(x),e^{-i2\pi nx})=R_m,\end{aligned}$$

where $A_m(\lambda_{n,j})=\sum_{k=1}^m a_k(\lambda_{n,j})$, $B_m(\lambda_{n,j})=\sum_{k=1}^m b_k(\lambda_{n,j})$,

$$a_k(\lambda_{n,j})=\sum_{n_1,n_2,\ldots,n_k}\frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n-n_1-\cdots-n_k))^2]},$$

$$b_k(\lambda_{n,j})=\sum_{n_1,n_2,\ldots,n_k}\frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{2n-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n-n_1-\cdots-n_k))^2]},$$

$$R_m=\sum_{n_1,n_2,\cdots,n_{m+1}}\frac{q_{n_1}q_{n_2}\cdots q_{n_m}q_{n_{m+1}}(q(x)\Psi_{n,j}(x),e^{i2\pi(n-n_1-\cdots-n_{m+1})x})}{[\lambda_{n,j}-(2\pi(n-n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n-n_1-\cdots-n_{m+1}))^2]}.$$

Here, the sums are taken under the conditions

$$n_s\neq 0,\quad \sum_{j=1}^{s}n_j\neq 0,2n$$

for $s=1,2,\ldots,m+1$. Using (7) we get the estimations

$$(16)\quad a_k=O\Big(\Big(\frac{\ln|n|}{n}\Big)^k\Big),\quad b_k=O\Big(\Big(\frac{\ln|n|}{n}\Big)^k\Big),\quad R_m=O\Big(\Big(\frac{\ln|n|}{n}\Big)^{m+1}\Big).$$

In the same way, the relation

$$(17)\quad \begin{aligned}&(\lambda_{n,j}-(2\pi n)^2-A'_m(\lambda_{n,j}))(\Psi_{n,j}(x),e^{-i2\pi nx})\\ &-(q_{-2n}+B'_m(\lambda_{n,j}))(\Psi_{n,j}(x),e^{i2\pi nx})=R'_m\end{aligned}$$

can be obtained, where $A'_m(\lambda_{n,j})=\sum_{k=1}^m a'_k(\lambda_{n,j})$, $B'_m(\lambda_{n,j})=\sum_{k=1}^m b'_k(\lambda_{n,j})$,

$$a'_k(\lambda_{n,j})=\sum_{n_1,n_2,\ldots,n_k}\frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n+n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n+n_1+\cdots+n_k))^2]},$$

$$b_k^{'}(\lambda_{n,j}) = \sum_{n_1,n_2,\ldots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-2n-n_1-n_2-\cdots-n_k}}{[\lambda_{n,j}-(2\pi(n+n_1))^2]\cdots[\lambda_{n,j}-(2\pi(n+n_1+\cdots+n_k))^2]},$$

$$a_k^{'} = O((\frac{\ln|n|}{n})^k), \quad b_k^{'} = O((\frac{\ln|n|}{n})^k), \quad R_m^{'} = O((\frac{\ln|n|}{n})^{m+1}). \tag{18}$$

Here, the sums are taken under the conditions

$$n_s \neq 0, \quad \sum_{j=1}^{s} n_j \neq 0, -2n$$

for $s = 1, 2, \ldots, k$.

It follows from (7), (9) and (11) that

$$\sum_{k\in Z, k\neq n,-n} |(\Psi_{n,j}(x), e^{i2\pi kx})|^2 = \sum_{k\in Z, k\neq n,-n} \frac{(4M)^2}{[\lambda_{n,j}-(2\pi k)^2]^2} = O\Big(\frac{1}{n^2}\Big).$$

Therefore, $\Psi_{n,t}(x)$ has an expansion of the form

$$\Psi_{n,j}(x) = u_{n,j}e^{i2\pi nx} + v_{n,j}e^{-i2\pi nx} + h(x), \tag{19}$$

where $u_{n,j} = (\Psi_{n,j}(x), e^{i2\pi nx})$, $v_{n,j} = (\Psi_{n,j}(x), e^{-i2\pi nx})$, $\|h(x)\| = O(1/n)$,

$$|u_{n,j}|^2 + |v_{n,j}|^2 = 1 + O\Big(\frac{1}{n^2}\Big). \tag{20}$$

Now using (16), (18), (20) in the system of equations (15), (17) we find asymptotic formulas for eigenfunctions $\Psi_{n,j}(x)$ and eigenvalues $\lambda_{n,j}$ of the operator $L_0(q)$ with $q(x) \in L_1(0,1)$, and prove that if conditions (3) hold then the eigenfunctions and associated functions of this operator form a Riesz basis in $L_2(0,1)$.

THEOREM 1: *Let the conditions (3) hold. Then:*

(a) *All sufficiently large eigenvalues of the operator $L_0(q)$ are simple. They consists of two sequences $\{\lambda_{n,1} : n > N\}$ and $\{\lambda_{n,2} : n > N\}$ satisfying*

$$\lambda_{n,j} = (2\pi n)^2 + (-1)^j p_n + O\Big(\frac{\ln|n|}{n}\Big), \tag{21}$$

for $j = 1, 2$, where $p_n = (q_{2n}q_{-2n})^{1/2}$.

The corresponding eigenfunction $\varphi_{n,j}(x)$ satisfies

$$\varphi_{n,j}(x) = e^{i2\pi nx} + \alpha_{n,j}e^{-i2\pi nx} + O(1/n), \tag{22}$$

where

$$\alpha_{n,j} \sim 1, \tag{23}$$

$$\alpha_{n,j} = \frac{(-1)^j p_n}{q_{2n}} + O\Big(\frac{\ln|n|}{nq_{2n}}\Big), \tag{24}$$

for $j = 1, 2$.

(b) *The root functions of* $L_0(q)$ *form a Riesz basis in* $L_2(0,1)$.

Proof: First we prove that if conditions (3) hold, then

$$u_{n,j} \sim v_{n,j} \sim 1, \tag{25}$$

$$\lambda_{n,j} - (2\pi n)^2 = a_j p_n + O\Big(\frac{\ln|n|}{n}\Big), \tag{26}$$

for $j = 1, 2$, where $a_j = \pm 1$ and $u_{n,j}$, $v_{n,j}$ are defined in (19). From (15)–(18) we obtain the system of equations

$$(\lambda_{n,j} - (2\pi n)^2) u_{n,j} = q_{2n} v_{n,j} + O\Big(\frac{\ln|n|}{n}\Big), \tag{27}$$

$$(\lambda_{n,j} - (2\pi n)^2) v_{n,j} = q_{-2n} u_{n,j} + O\Big(\frac{\ln|n|}{n}\Big). \tag{28}$$

It follows from (20) that at least one of the numbers $|v_{n,j}|$, $|v_{n,j}|$ is greater than $\frac{1}{2}$. Assume that $|v_{n,j}| > \frac{1}{2}$. Then from (28) and (3) we obtain $|\lambda_{n,j} - (2\pi n)^2| < c_7 |q_{-2n}|$. Moreover, the inequalities $|u_{n,j}| < 2$ (see (20)), $|v_{n,j}| > \frac{1}{2}$, and (27) imply that $|\lambda_{n,j} - (2\pi n)^2| > c_8 |q_{2n}|$. Hence using these inequalities and (3) we obtain

$$\lambda_{n,j} - (2\pi n)^2 \sim q_{2n} \sim q_{-2n} \sim p_n, \quad \lim_{n\to\infty} \frac{\ln|n|}{n p_n} = 0. \tag{29}$$

This with (27) implies that $u_{n,j} \sim v_{n,j}$. Therefore (25) follows from (20). Now dividing both sides of (27) and (28) by $u_{n,j}\, q_{2n}$ and $(\lambda_{n,j} - (2\pi n)^2) u_{n,j}$, respectively, we get

$$\frac{v_{n,j}}{u_{n,j}} = \frac{\lambda_{n,j} - (2\pi n)^2}{q_{2n}} + O\Big(\frac{\ln|n|}{n q_{2n}}\Big) = \frac{q_{-2n}}{\lambda_{n,j} - (2\pi n)^2} + O\Big(\frac{\ln|n|}{n q_{2n}}\Big). \tag{30}$$

From here, using (29), we get (26).

The proof of (a). Suppose $\lambda_{n,j}$ (where $n > N \gg 1$) is multiple. There are two possible cases:

1. There is an associated function $\Psi^1_{n,j}(x)$ corresponding to the eigenfunction $\Psi_{n,j}(x)$, that is,

$$(L_0 - \lambda_{n,j}) \Psi^1_{n,j}(x) = \Psi_{n,j}(x). \tag{31}$$

Since the boundary condition (2) for $t = 0$ is selfadjoint, $\overline{\lambda_{n,j}}$ and $\overline{\Psi_{n,j}}$ are an eigenvalue and an eigenfunction of the adjoint operator L_0^*. Therefore, multiplying both sides of (31) by $\overline{\Psi_{n,j}}$ we get $(\Psi_{n,j}(x), \overline{\Psi_{n,j}}) = 0$. This with (19) implies the equality $u_{n,j} v_{n,j} = O(1/n)$, which contradicts (25).

2. There are two eigenfunctions corresponding to $\lambda_{n,j}$. Then all solutions of the equation

$$-y'' + q(x)y = \lambda_{n,j}y$$

are eigenfunctions. In particular, the solution

$$y(x, \lambda_{n,j}) = e^{\rho_{n,j}x} + O(1/n) = e^{2\pi n x} + O(1/n),$$

where $\rho_{n,j} = (\lambda_{n,j})^{1/2} = 2\pi n + (1/n)$ (see page 52 of [3]), and (26)), is an eigenfunction. In (27), instead of using $\Psi_{n,j}$, taking this eigenfunction we get

$$\lambda_{n,j} = (2\pi n)^2 + O\Big(\frac{\ln|n|}{n}\Big),$$

which contradicts (29). Thus the eigenvalues $\lambda_{n,j}$ for $n > N$ are simple and satisfy (26). Now, in order to obtain (21) from (26) it remains to prove that $a_1 \neq a_2$, say $a_1 = -1, a_2 = 1$. Suppose $a_1 = a_2 = -1$. Then (26), (27) yield

$$(32) \qquad -p_n u_{n,1} = q_{2n}v_{n,1} + O\Big(\frac{\ln|n|}{n}\Big), -p_n u_{n,2} = q_{2n}v_{n,2} + O\Big(\frac{\ln|n|}{n}\Big).$$

Multiplying both sides of the first and second equalities by $-v_{n,2}$ and $v_{n,1}$, respectively, and then summing the equalities, we get

$$(33) \qquad u_{n,1}v_{n,2} - u_{n,2}v_{n,1} = O\Big(\frac{\ln|n|}{np_n}\Big).$$

On the other hand, since $\overline{\lambda_{n,j}}$ and $\overline{\Psi_{n,j}}$ are an eigenvalue and an eigenfunction of adjoint operator L_0^* and $\lambda_{n,1} \neq \lambda_{n,2}$, we have

$$(34) \qquad 0 = (\Psi_{n,1}, \overline{\Psi_{n,2}}) = u_{n,1}v_{n,2} + u_{n,2}v_{n,1} + O(1/n)$$

(see (19)). It follows from (33), (34), (29) that

$$u_{n,1}v_{n,2} = O\Big(\frac{\ln|n|}{np_n}\Big) = o(1),$$

which contradicts (25). If we suppose that $a_1 = a_2 = 1$, then, in the same way, we obtain the same contradiction. Thus one of these numbers, say a_1, is -1 and the other is 1. The formulas (21) are proved. Since $\lambda_{n,1}$ satisfies (21), for $j = 1$, the first equality in (32) holds. Therefore

$$(35) \qquad \frac{v_{n,1}}{u_{n,1}} = \frac{-p_n}{q_{2n}} + O\Big(\frac{\ln|n|}{nq_{2n}}\Big).$$

Denoting $\alpha_{n,j} = v_{n,j}/u_{n,j}$ and $\varphi_{n,j}(x) = \Psi_{n,j}(x)/u_{n,j}$, from (19), (25) and (35) we obtain (22), (23) and (24) for $j = 1$. Taking into account that $\lambda_{n,2}$ satisfies (21) for $j = 2$, in the same way we get the proof of (22), (23) and (24) for $j = 2$.

The proof of (b). Now we prove that the eigenfunctions $\varphi_{n,j} \equiv \varphi^0_{n,j}$ and associated functions $\varphi^k_{n,j}$ (where $n = 1, 2, \ldots$, and $k = 1, 2, \ldots, s(n, j)$) of $L_0(q)$ form a Riesz basis in $L_2(0,1)$. For every $f(x) \in L_2(0,1)$, the asymptotic formulas (22), (23) yield

$$\sum_{n=1}^{N}\Big(\sum_{k=0}^{s(n,j)}\sum_{j=1}^{2}|(f,\varphi^k_{n,j})|^2\Big) + \sum_{j=1}^{2}\sum_{n=N+1}^{\infty}|(f,\varphi_{n,j})|^2 < \infty. \tag{36}$$

Let $\chi_{n,j} \equiv \chi^0_{n,j}$ and $\chi^k_{n,j}$, where $k = 1, 2, \ldots, t(n, j)$ be the biorthonormal system of eigenfunctions and associated functions of L_0^*. Clearly $\chi_{n,j}(x) = \overline{\varphi_{n,j}(x)}/(\varphi_{n,j}, \overline{\varphi_{n,j}})$ for $n > N$. Therefore, using (22), (23) we get

$$\sum_{n=1}^{N}\Big(\sum_{k=0}^{t(n,j)}\sum_{j=1}^{2}|(f,\chi^k_{n,j})|^2\Big) + \sum_{j=1}^{2}\sum_{n=N+1}^{\infty}|(f,\chi_{n,j})|^2 < \infty. \tag{37}$$

Since

$$\{\varphi^k_{n,j} : k = 0, 1, 2, \ldots, s(n, j); j = 1, 2; n = 1, 2, \ldots\},$$

and the biorthonormal system

$$\{\chi^k_{n,j} : k = 0, 1, 2, \ldots, t(n, j); j = 1, 2; n = 1, 2, \ldots\}$$

are total (see chap. 1, sec. 3 of [2]), by the well-known theorem of Bari (see [1], chap. 6), the inequalities (36) and (37) imply that the system of the eigenfunctions and associated functions of $L_0(q)$ forms a Riesz basis in $L_2(0,1)$. The theorem is proved. ∎

If instead of (3), (7), and (9) we use (4), (8) and

$$(\mu_{N,j} - (2\pi n + \pi)^2)(\Phi_{N,j}(x), e^{i\pi(2n+1)x}) = (q(x)\Phi_{N,j}(x), e^{i\pi(2n+1)x}), \tag{38}$$

where $\Phi_{N,j}(x)$ is a normalized eigenfunction of $L_\pi(q)$, and arguing as in the proof of Theorem 1, we obtain

THEOREM 2: *Let conditions (4) hold. Then:*

(a) *All sufficiently large eigenvalues of the operator L_π are simple. They consist of two sequences $\{\mu_{n,1} : n > N\}$ and $\{\mu_{n,2} : n > N\}$ satisfying*

$$\mu_{n,j} = (2\pi n + \pi)^2 + (-1)^j p'_n + O\Big(\frac{\ln|n|}{n}\Big) \tag{39}$$

for $j = 1, 2$, *where* $p_n' = (q_{2n+1}q_{-2n-1})^{1/2}$. *The corresponding eigenfunction* $\phi_{n,j}(x)$ *satisfies*

$$\phi_{n,j}(x) = e^{i(2\pi n+\pi)x} + \alpha_{n,j}' e^{-i(2\pi n+\pi)x} + O(1/n) \tag{40}$$

for $j = 1, 2$, *where*

$$\alpha_{n,j}' \sim 1, \quad \alpha_{n,j}' = \frac{(-1)^j p_n'}{q_{2n+1}} + O\Big(\frac{\ln|n|}{nq_{2n+1}}\Big).$$

(b) *The root functions of* $L_\pi(q)$ *form a Riesz basis in* $L_2(0,1)$.

Now to derive the asymptotic formulas of arbitrary order for eigenfunctions and eigenvalues of the operators $L_0(q)$, $L_\pi(q)$ with $q(x) \in L_1(0,1)$, we define successively the following functions:

$$F_{0,j} = 0, \quad G_{0,j} = p_n, \quad E_{n,m,j} = (2\pi n)^2 + F_{m,j} + (-1)^j G_{m,j},$$
$$F_{k,j} = \frac{1}{2}(A_k(E_{n,k-1,j}) + A_k'(E_{n,k-1,j})),$$
$$G_{k,j} =$$
$$[\frac{1}{4}(A_k(E_{n,k-1,j}) - A_k'(E_{n,k-1,j}))^2 + (B_k(E_{n,k-1,j}) + q_{2n})(B_k'(E_{n,k-1,j}) + q_{-2n})]^{\frac{1}{2}}$$

for $m = 0, 1, 2, \ldots$, and $k = 1, 2, \ldots$, where $j = 1, 2$ and the functions A_k', A_k are defined in (15), (17). Moreover, we use the functions A_m^*, B_m^* which are obtained from A_m, B_m respectively by replacing q_{n_1} with $e^{i2\pi(n-n_1)x}$.

THEOREM 3: *If conditions (3) hold, then the eigenvalue* $\lambda_{n,j}$ *and the eigenfunction* $\varphi_{n,j}(x)$ *of* $L_0(q)$ *satisfy the following formulas:*

$$\lambda_{n,j} = E_{n,m,j} + O\Big(\Big(\frac{\ln|n|}{n}\Big)^{m+1}\Big), \tag{41}$$

$$\varphi_{n,j}(x) = e^{i2\pi nx} + \alpha_{n,j}e^{-i2\pi nx} + A_m^*(E_{n,m,j}) + \alpha_{n,j}B_m^*(E_{n,m,j}) + O\Big(\Big(\frac{\ln n}{n}\Big)^{m+1}\Big) \tag{42}$$

for $j = 1, 2$ *and* $m = 1, 2, \ldots$, *where*

$$\alpha_{n,j} = \frac{v_{n,j}}{u_{n,j}} = \frac{F_{m,j} + (-1)^j G_{m,j} - A_m(E_{n,m,j})}{q_{2n} + B_m(E_{n,m,j})} + O\Big(\frac{1}{q_{2n}}\Big(\frac{\ln|n|}{n}\Big)^{m+1}\Big).$$

If conditions (4) hold, then the eigenvalue $\mu_{n,j}$ *and the eigenfunction* $\phi_{n,j}(x)$ *of* $L_\pi(q)$ *satisfy formulas which are similar to (41) and (42).*

Proof: Arguing as in the proof of (30), using (15), (17) instead of (27), (28), and denoting $\lambda = \lambda_{n,j} - (2\pi n)^2$, we get

$$\begin{aligned}\frac{v_{n,j}}{u_{n,j}} &= \frac{\lambda - A_m(\lambda_{n,j})}{(q_{2n} + B_m(\lambda_{n,j}))} + O\Big(\frac{1}{q_{2n}}\Big(\frac{\ln|n|}{n}\Big)^{m+1}\Big)\\ &= \frac{(q_{-2n} + B'_m(\lambda_{n,j}))}{\lambda - A'_m(\lambda_{n,j})} + O\Big(\frac{1}{q_{2n}}\Big(\frac{\ln|n|}{n}\Big)^{m+1}\Big).\end{aligned} \tag{43}$$

Solving this square equation with respect to λ we obtain

$$\begin{aligned}\lambda_{n,j} - (2\pi n)^2 = \frac{1}{2}(A_m(\lambda_{n,j}) + A'_m(\lambda_{n,j})) \mp \Big[\frac{1}{4}(A_m(\lambda_{n,j}) - A'_m(\lambda_{n,j}))^2\\ + (B_m(\lambda_{n,j}) + q_{2n})(B'_m(\lambda_{n,j}) + q_{-2n})) + O\Big(q_{2n}\Big(\frac{\ln|n|}{n}\Big)^{m+1}\Big)\Big]^{1/2}.\end{aligned} \tag{44}$$

Here, the upper and lower sign is taken for $j = 1$ and $j = 2$, respectively, since the expression in the square brackets is

$$q_{2n}q_{-2n} + O\Big(q_{2n}\frac{\ln|n|}{n}\Big) = q_{2n}q_{-2n}(1 + o(1))$$

(see (3), (16), (18)), and (21) holds. Thus in (44) the $\mp$ can be replaced by $(-1)^j$. Now we prove (41) by induction. It is proved for $m = 1$ (see (21)). Assume that it is true for $m = k - 1$. Substituting the value of $\lambda_{n,j}$, given by this formula for $m = k - 1$, in the right-hand side of (44) for $m = k$ and using the relations

$$\begin{aligned}A_k(E_{n,k-1,j} + O\Big(\Big(\frac{\ln|n|}{n}\Big)^k\Big)) &= A_k(E_{n,k-1,j}) + O\Big(\Big(\frac{\ln|n|}{n}\Big)^{k+1}\Big),\\ A'_k(E_{n,k-1,j} + O\Big(\Big(\frac{\ln|n|}{n}\Big)^k\Big)) &= A'_k(E_{n,k-1,j}) + O\Big(\Big(\frac{\ln|n|}{n}\Big)^{k+1}\Big),\\ B_k(E_{n,k-1,j} + O\Big(\Big(\frac{\ln|n|}{n}\Big)^k\Big)) &= B_k(E_{n,k-1,j}) + O\Big(\Big(\frac{\ln|n|}{n}\Big)^{k+1}\Big),\\ B'_k(E_{n,k-1,j}) + O\Big(\Big(\frac{\ln|n|}{n}\Big)^k\Big)) &= B'_k(E_{n,k-1,j}) + O\Big(\Big(\frac{\ln|n|}{n}\Big)^{k+1}\Big),\end{aligned} \tag{45}$$

we get the proof of (41) for $m = k$. Note that the validity of these relations can be easily verified by using the obvious equality

$$\sum_{\substack{n_1=-\infty,\\ n_1\neq 0,2n}}^{\infty} \Big|\frac{1}{E_{n,k-1,j} + O((\frac{\ln|n|}{n})^k) - (2\pi(n-n_1))^2} - \frac{1}{E_{n,k-1,j} - (2\pi(n-n_1))^2}\Big| = O\Big(\Big(\frac{\ln|n|}{n}\Big)^{k+1}\Big).$$

Now we prove (42). Writing the decomposition of $\Psi_{n,j}(x)$ as

$$\{e^{i2\pi(n-n_1)x} : n_1 \in Z\},$$

we obtain

$$\Psi_{n,j}(x) - u_{n,j}e^{i2\pi nx} = \sum_{\substack{n_1=-\infty, \\ n_1\neq 0}}^{\infty} (\Psi_{n,j}(x), e^{i2\pi(n-n_1)x})e^{i2\pi(n-n_1)x}. \tag{46}$$

The right-hand side of (46) can be obtained from the right-hand side of (12) by replacing q_{n_1} with $e^{i2\pi(n-n_1)x}$. Since we obtained (15) from (12) by iteration, by doing the same iteration we obtain

$$\begin{aligned}\Psi_{n,j}(x) =& u_{n,j}e^{i2\pi nx} + v_{n,j}e^{-i2\pi nx} \\ &+ u_{n,j}A_m^*(\lambda_{n,j}) + v_{n,j}B_m^*(\lambda_{n,j}) + O\Big(\Big(\frac{\ln|n|}{n}\Big)^{m+1}\Big)\end{aligned} \tag{47}$$

from (46). Dividing both sides of (47) by $u_{n,j}$ and using (43), (41), (45) we get (42).

To obtain the asymptotic formulas for the eigenvalue $\mu_{n,j}$ and the eigenfunction $\phi_{n,j}(x)$ of $L_\pi(q)$, we use (4), (8), (38) and Theorem 2 instead of (3), (7), (9) and Theorem 1 and repeat the proof of (41) and (42). ∎

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