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ON THE LINDENSTRAUSS–ROSENTHAL THEOREM*

BY

JESÚS M. F. CASTILLO AND YOLANDA MORENO

Departamento de Matemáticas, Universidad de Extremadura
Avenida de Elvas, 06071 Badajoz, Spain
e-mail: castillo@unex.es, ymoreno@unex.es

ABSTRACT

We present a homological principle that governs the behaviour of couples of exact sequences of quasi-Banach spaces. Three applications are given: (i) A unifying method of proof for the results of Lindenstrauss, Rosenthal, Kalton, Peck and Kislyakov about the extension and lifting of isomorphisms in c_0, l_∞, l_p and L_p for $0 < p \leq 1$; (ii) A study of the Dunford–Pettis property in duals of quotients of $\mathcal{L}_\infty$-spaces; and (iii) New results on the extension of $C(K)$-valued operators.

1. Introduction

The paper [25] of Lindenstrauss and Rosenthal contains a proof of the following fundamental results in Banach space theory (see also [27, Thms. 2.f.8, 2.f.10, 2.f.12]):

PROPOSITION 1.1 (case l_1): *Let q and Q be two operators from l_1 onto a Banach space X not isomorphic to l_1. There exists an automorphism τ of l_1 such that $q\tau = Q$.*

The paper also contains the "dual" results for c_0 and l_∞:

PROPOSITION 1.2 (case c_0): *Let i and j be two injective isomorphisms from a Banach space Y into c_0 in such a way that both c_0/iY and c_0/jY are infinite dimensional. There is an automorphism τ of c_0 such that $\tau i = j$.*

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PROPOSITION 1.3 (case l_∞): *Let i and j be two injective isomorphisms from a Banach space Y into l_∞ in such a way that both l_∞/iY and l_∞/jY are not reflexive. Then there is an automorphism τ of l_∞ such that $\tau i = j$. If the quotients l_∞/iY and l_∞/jY are both reflexive, then the automorphism τ exists if and only if the Fredholm index of any extension of j to l_∞ through i is 0. If one of the quotients is reflexive but the other is not, no such automorphism exists.*

Kalton extended in [16] the first of those results to l_p-spaces for $0 < p < 1$:

PROPOSITION 1.4 (case l_p): *Let $0 < p < 1$ and let q and Q be two quotient maps onto a quasi-Banach space X not isomorphic to l_p. If $\ker q$ and $\ker Q$ contain copies of l_p complemented in l_p, then there exists an automorphism τ of l_p such that $q\tau = Q$.*

In [18] Kalton and Peck obtained some variations of this result for $L_p(0,1)$, $0 \le p < 1$.

PROPOSITION 1.5 (case L_p): *Let $0 < p < 1$ and let q and Q be two quotient maps onto a quasi-Banach space X in such a way that $\ker q$ and $\ker Q$ are either q-Banach spaces for some $q > p$ or ultrasummand spaces. Then there exists an automorphism τ of L_p such that $q\tau = Q$.*

In [21] Kislyakov considered the case L_1, obtaining:

PROPOSITION 1.6 (case L_1): *Let A and B be two reflexive subspaces of $L_1(\mu)$ such that $L_1(\mu)/A = L_1(\mu)/B$. Then one of the subspaces A, B is isomorphic to the product of the other one with a finite dimensional space*

Lindenstrauss showed in [23] a partial converse of the l_1-result and applied it to solve a problem raised in [26] about the existence of infinitely many isomorphy types of $\mathcal{L}_1$-spaces.

PROPOSITION 1.7 ($\mathcal{L}_1$ subspaces of l_1): *Let A, B be two $\mathcal{L}_1$-spaces, and let $q_A\colon l_1 \to A$ and $q_B\colon l_1 \to B$ be two quotient operators with infinite-dimensional kernels. Then $\ker q_A$ is isomorphic to $\ker q_B$ if and only if A and B are isomorphic.*

Our purpose in sections 2 and 3 is to obtain a basic principle that governs the behaviour of exact sequences of quasi-Banach spaces. Using what we have called the diagonal principle we obtain in section 4 a unifying method of proof for all the previous results 1.1–1.7, unveiling in this way their common homological nature. Section 5 contains a study of the Dunford–Pettis property in duals of quotients of $\mathcal{L}_\infty$ spaces, completing results of Kislyakov [21] and Kalton and Pelczynski [20]. In the final section 6 we obtain some new results on the extension of $C(K)$-valued operators from subspaces of $\mathcal{L}_1$-spaces that complement the Johnson–Zippin theorem [14].

2. Preliminaries

2.1. NOTATION AND BACKGROUND. For a sound background on homological algebra we suggest [13, 28]. We assume familiarity with the basic theory of exact sequences of (quasi-) Banach spaces and quasi- and zero-linear mappings as described in [7, 18, 1]. Let us however recall briefly the basic facts the reader should have in mind for the rest of the paper.

In what follows we shall mostly work in the category **B** of Banach spaces and operators. An exact sequence $0 \to Y \to X \to Z \to 0$ is a diagram in which the kernel of each arrow coincides with the image of the preceding; it is also called an **extension** of Z by Y; the middle space X is also called a **twisted sum** of Y and Z. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be equivalent if there exists an operator $T\colon X \to X_1$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

An exact sequence is said to split if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. The vector space (when endowed with suitable defined operations) of all extensions of Z by Y, modulo the equivalence relation, is denoted $\mathrm{Ext}_{\mathbf{B}}(Z, Y)$. A Banach space X is said to be projective if $\mathrm{Ext}_{\mathbf{B}}(X, A) = 0$ for all Banach spaces A. The projective spaces in **B** are the $l_1(\Gamma)$-spaces. A Banach space X is said to be injective if $\mathrm{Ext}_{\mathbf{B}}(A, X) = 0$ for all Banach spaces A. The spaces $l_\infty(\Gamma)$ are injective in **B**. An exact sequence $0 \to Y \to X \to Z \to 0$ with X projective is called a projective presentation of Z; if X is injective, then it is called an injective presentation of Y. Of course, the same notions can be defined for the bigger category **Q** of quasi-Banach spaces and operators and for its subcategory $\mathbf{Q}_p$ of p-Banach spaces. There is a correspondence (see [15, 18, 7]) between exact sequences $0 \to Y \to X \to Z \to 0$ of Banach spaces and the so-called zero-linear maps which are homogeneous maps $F\colon Z \to Y$ with the property that there exists some constant $Z(F) > 0$ such that for all finite sets $x_1, \ldots, x_N \in Z$ one has $\|F(\sum_{n=1}^N x_n) - \sum_{n=1}^N F(x_n)\| \leq Z(F) \sum_{n=1}^N \|x_n\|$. Thus, in what follows we shall use the notation

$$0 \to Y \to X \to Z \to 0 \equiv F$$

to indicate that the exact sequence $0 \to Y \to X \to Z \to 0$ is represented by the zero-linear map $F\colon Z \to Y$. Two zero-linear maps $F, G\colon Z \to Y$ are said to be equivalent, and we write $F \equiv G$, if the induced exact sequences are equivalent. Extensions of quasi-Banach spaces come represented by the so-called quasi-linear maps, which are homogeneous maps satisfying the given inequality for $N = 2$. In [3, 9] we introduced the notion of isomorphically equivalent sequences, which is the key for our approach in this paper.

Definition: We shall say that two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y_1 \to X_1 \to Z_1 \to 0$ of Banach or quasi-Banach spaces are **isomorphically equivalent** if there exist isomorphisms $\alpha\colon Y \to Y_1$, $\beta\colon X \to X_1$ and $\gamma\colon Z \to Z_1$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y_1 & \longrightarrow & X_1 & \longrightarrow & Z_1 & \longrightarrow & 0. \end{array}$$

In terms of quasi-linear maps it turns out (see [9]) that $0 \to Y \to X \to Z \to 0 \equiv F$ and $0 \to Y_1 \to X_1 \to Z_1 \to 0 \equiv G$ are isomorphically equivalent if and only if there exist isomorphisms $\alpha\colon Y \to Y_1$ and $\gamma\colon Z \to Z_1$ such that $\alpha F = G\gamma$.

2.2. THE PULL-BACK AND PUSH-OUT DIAGONAL SEQUENCES. The basic theory of the pull-back and push-out constructions in the categories **B** and **Q**, and their associated exact sequences, can be seen in [7, 4, 5, 3]. For the sake of clarity, and to ease the introduction of several new results which are essential for the rest of the paper, let us briefly describe the pull-back and push-out constructions. Given operators $q\colon X \to Z$ and $T\colon W \to Z$ the pull-back space of $\{q, T\}$ is $PB = \{(x, w) : qx = Tw\} \subset X \oplus W$ endowed with the relative product topology. If $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$ is an exact sequence, then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F \\ & & \| & & \uparrow & & \uparrow{\scriptstyle T} & & \\ 0 & \longrightarrow & Y & \xrightarrow[i]{} & PB & \longrightarrow & W & \longrightarrow & 0 \equiv FT \end{array}$$

in which the operators $PB \to X$ and $PB \to W$ are the restrictions of the canonical projections of $X \oplus W$ into, respectively, X and W, and $i(y) = (jy, 0)$. It can be checked that the composition $FT\colon W \to Y$ is a quasi-linear map corresponding to the lower pull-back sequence. We are especially interested in another exact sequence

$$0 \to PB \to X \oplus W \overset{p}{\to} Z \to 0,$$

where $p(x,w) = qx - Tw$, which we shall call the **diagonal pull-back sequence**. It is not hard to verify that its associated quasi-linear map is iF. The dual notion of pull-back is that of *push-out*. The push-out space PO of two operators $S\colon Y \to M$ and $j\colon Y \to X$ is the quotient space $M \oplus X/\overline{\Delta}$ where

$$\Delta = \{(Sy, -jy) \in M \oplus X\}.$$

If $0 \to Y \overset{j}{\to} X \overset{q}{\to} Z \to 0$ is an exact sequence, then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \overset{j}{\longrightarrow} & X & \overset{q}{\longrightarrow} & Z & \longrightarrow & 0 \equiv F \\ & & \downarrow S & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & PO & \underset{Q}{\longrightarrow} & Z & \longrightarrow & 0 \equiv SF \end{array}$$

where the arrows $M \to PO$ and $X \to PO$ are the restrictions to M and X of the quotient map $M \oplus X \to PO$, and $Q[(m,x) + \Delta] = qx$. It is not hard to verify that the composition SF is a quasi-linear map corresponding to the lower push-out sequence. The **diagonal push-out sequence** is now

$$0 \to Y \overset{d}{\to} M \oplus X \to PO \to 0,$$

where $d(y) = (Sy, -jy)$; it is not hard to verify that its associated quasi-linear map is FQ.

Definition: We shall say that two exact sequences $0 \to Y \to X \to Z \to 0 \equiv F$ and $0 \to Y \to X_1 \to Z_1 \to 0 \equiv F_1$ (or two quasi-linear maps F and F_1 as before) are **semi-equivalent** if one is the pull-back of the other, and vice-versa; i.e., if there exist two operators $\alpha\colon Z_1 \to Z$ and $\alpha_1\colon Z \to Z_1$ such that $F\alpha \equiv F_1$ and $F_1\alpha_1 \equiv F$. For instance, such is the case of any two injective presentations of a given space. Two exact sequences (or two quasi-linear maps F and F_1) $0 \to Y \to X \to Z \to 0 \equiv F$ and $0 \to Y_1 \to X_1 \to Z \to 0 \equiv F_1$ are said to be semi-equivalent if one is the push-out of the other, and vice-versa; i.e., if there exist two operators $\alpha\colon Y \to Y_1$ and $\alpha_1\colon Y_1 \to Y$ such that $\alpha F \equiv F_1$ and

$\alpha_1 F_1 \equiv F$. For instance, such is the case of any two injective presentations of a given space.

With a slight abuse of notation, if $0 \to Y \to X \to Z \to 0 \equiv F$ we shall denote $(F, 0)$ the quasi-linear map $z \to (Fz, 0)$ associated with the sequence $0 \to Y \oplus A \to X \oplus A \to Z \to 0$ obtained multiplying by a fixed space A. It is clear that F and $(F, 0)$ are semi-equivalent. The constructions on the right are analogous. To detect when a given sequence F has been obtained from another G via pull-back or push-out is not as hard as it seems. We do that in the next result, whose proof is a direct application of the homology sequence (see [5]) plus the previous observations about the diagonal pull-back and push-out sequences.

LEMMA 1: *Let $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$ and $0 \to Y_1 \to X_1 \to Z \to 0 \equiv G$. Then G is the push-out of F if and only if $Gq \equiv 0$. If $0 \to Y \to X_1 \to Z_1 \to 0 \equiv G$, then G is the pull-back of F if and only if $jG \equiv 0$.*

3. The diagonal principles

We establish now a basic principle asserting that two semi-equivalent sequences are "essentially" isomorphically equivalent.

THEOREM 1 (Diagonal principle: projective case): *Let $0 \to Y \to X \xrightarrow{q} Z \to 0 \equiv F$ and $0 \to Y_1 \to X_1 \xrightarrow{Q} Z \to 0 \equiv G$ be two exact sequences such that $FQ \equiv 0 \equiv Gq$. Then the exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y \oplus X_1 & \longrightarrow & X \oplus X_1 & \longrightarrow & Z & \longrightarrow & 0 \equiv (F, 0) \\ & & & & & & \| & & \\ 0 & \longrightarrow & Y_1 \oplus X & \longrightarrow & X_1 \oplus X & \longrightarrow & Z & \longrightarrow & 0 \equiv (G, 0) \end{array}$$

are isomorphically equivalent.

Proof: The hypothesis $Gq \equiv 0$ means, by Lemma 1, that there exists an operator α such that $G \equiv \alpha F$. Thus, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y_1 & \longrightarrow & X_1 & \xrightarrow[Q]{} & Z & \longrightarrow & 0 \equiv G. \end{array}$$

The hypothesis $FQ = 0$ implies that the diagonal push-out diagonal sequence $0 \to Y \xrightarrow{d} Y_1 \oplus X \to X_1 \to 0$ splits, which yields an isomorphism $\phi\colon Y \oplus X_1 \to$

$Y_1 \oplus X$ making the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{d} & Y_1 \oplus X & \longrightarrow & X_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & Y \oplus X_1 & \longrightarrow & X_1 & \longrightarrow & 0
\end{array}$$

commutative. We have $\phi d(y) = (y, 0)$ and therefore $\phi dF \equiv (F, 0)$. Recalling that $d(y) = (\alpha y, -jy)$ we get $dF \equiv (\alpha F, -jF) \equiv (G, 0)$, and thus $(F, 0) \equiv \phi(G, 0)$, which means the existence of a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y_1 \oplus X & \longrightarrow & X_1 \oplus X & \longrightarrow & Z & \longrightarrow & 0 \equiv (G, 0) \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \eta} & & \| & & \\
0 & \longrightarrow & Y \oplus X_1 & \longrightarrow & X \oplus X_1 & \longrightarrow & Z & \longrightarrow & 0 \equiv (F, 0)
\end{array}$$

in which the operator η is an isomorphism by the 3-lemma (see [7]). ∎

The name **diagonal principle** comes from the fact that there is a still stronger principle behind it: that the diagonal pull-back sequence is a well-defined object, in a very precise sense we briefly describe now. If $0 \to Y \to X \xrightarrow{q} Z \to 0 \equiv F$ and $0 \to Y_1 \to X_1 \xrightarrow{q_1} Z \to 0 \equiv G$ are exact sequences and we construct the associated diagonal pull-back sequence $0 \to PB \to X \oplus X_1 \to Z \to 0 \equiv \Delta$ corresponding to the operators (q, q_1), then the process $(F, G) \to \Delta$ does as follows: if one replaces F, G by equivalent extensions F', G' then the resulting Δ' is isomorphically equivalent to Δ. That correspondence $\Delta(\cdot)$ actually defines a functor under the appropriate categories. The categorical foundations of the theory along with many of their applications can be found in [29]. The same remark as before can be made about the second diagonal principle:

THEOREM 2 (Diagonal principle: injective case): *Let* $0 \to Y \xrightarrow{j} X \to Z \to 0 \equiv F$ *and* $0 \to Y \xrightarrow{i} X_1 \to Z_1 \to 0 \equiv G$. *If* $iF = 0 = jG$, *then the sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X \oplus X_1 & \longrightarrow & Z \oplus X_1 & \longrightarrow & 0 \\
 & & \| & & & & & & \\
0 & \longrightarrow & Y & \longrightarrow & X_1 \oplus X & \longrightarrow & Z_1 \oplus X & \longrightarrow & 0
\end{array}$$

are isomorphically equivalent.

The proof is simple dualization of the previous one. Observe that the diagonal principles work in a general category in which the statements have meaning.

We postpone for a different place the discussion of either their categorical background or their applications to topological groups or Banach algebras.

4. Applications

4.1. Extension and lifting of isomorphisms. Of course, two projective or injective presentations are not, in general, isomorphically equivalent. The l_1 case of the Lindenstrauss–Rosenthal theorem easily follows from the projective case of the diagonal principle; indeed, two exact sequences $0 \to A \to l_1 \to Z \to 0$ and $0 \to B \to l_1 \to Z \to 0$ are isomorphically equivalent since the sequences $0 \to A \to l_1 \to Z \to 0$ and $0 \to A \oplus l_1 \to l_1 \oplus l_1 \to Z \to 0$ are isomorphically equivalent: this is due to the fact that every closed infinite dimensional subspace of l_1 contains a copy of l_1 complemented in l_1 (see [27]). Recalling that the $l_p(\Gamma)$-spaces are precisely the projective spaces in $\mathbf{Q}_p$, the preceding proof yields also a proof for Kalton's result about quotients of l_p for $0 < p < 1$.

The results of Kalton and Peck for L_p-spaces, $0 < p < 1$, follow from the diagonal principle using that $\mathrm{Ext}(L_p, A) = 0 = \mathcal{L}(L_p, A)$ when A is either an ultrasummand or a q-Banach space for $p < q$ (see [19]). The interested reader may find in [9] a proof for this result based on the study of the natural transformations of the functor $\mathrm{Ext}(Z, \cdot)$.

The c_0 case of the Lindenstrauss–Rosenthal theorem is a consequence of the injective case of the diagonal principle using that the sequences $0 \to Y \to c_0 \to Z \to 0$ and $0 \to Y \to c_0 \oplus c_0 \to Z \oplus c_0 \to 0$ are isomorphically equivalent: this follows from Pelczynski's result (see [30]) that c_0 admits no strictly singular quotient maps. The l_∞ (in general, the injective) case has to be treated, as in [25], with some care since injective spaces admit strictly singular quotient maps (actually, l_2 is a quotient of l_∞). So, if $0 \to Y \to I_1 \to Z_1 \to 0$ and $0 \to Y \to I_2 \to Z_2 \to 0$ are two injective presentations of Y, we have to distinguish three possibilities:

(1) Both quotient operators are not strictly singular. In that case they are isomorphisms on some copy of l_∞ (see [32]) and we conclude that the sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & I_i & \longrightarrow & Z_i & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & Y & \longrightarrow & I_i \oplus l_\infty & \longrightarrow & Z_i \oplus l_\infty & \longrightarrow & 0 \end{array}$$

are isomorphically equivalent for $i = 1, 2$. So, when $I_i = l_\infty$ the diagonal principle yields that the two injective presentations are isomorphically equivalent.

(2) One quotient map is strictly singular but not the other. In such a case the two sequences cannot be isomorphically equivalent.

(3) Both quotient maps are strictly singular. Then Z and W are reflexive and all operators $I_i \to Z_i$ are strictly singular. Since $I_1 \oplus Z_2$ is isomorphic to $I_2 \oplus Z_1$, Z_2 must be isomorphic to a complemented subspace of $I_2 \oplus Z_1$. Applying the Edelstein–Wojtasczyk decomposition principle [12] it must occur that $Z_2 = F_2 \oplus Z_{21}$ with F_2 a complemented subspace of I_2 and Z_{21} a complemented subspace of Z_1. Necessarily then F_2 must be finite dimensional. The same reasoning with Z_1 yields that $Z_1 = F_1 \oplus Z_{12}$, and F_1 must be finite dimensional. So, in this case the conclusion of the diagonal principle can be improved to obtain that the sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & I_1 \oplus F_2 & \longrightarrow & Z_1 \oplus F_2 & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & Y & \longrightarrow & I_2 \oplus F_1 & \longrightarrow & Z_2 \oplus F_1 & \longrightarrow & 0 \end{array}$$

are isomorphically equivalent. It is now easily achieved that the two starting sequences are isomorphically equivalent if and only if F_1 and F_2 have the same dimension.

A similar (dual) reasoning yields an improvement for Kislyakov's results [21, Thm. 4 and Cor]: given two exact sequences $0 \to A \to \mathcal{L}_1 \to Z \to 0$ and $0 \to B \to \mathcal{L}_1' \to Z \to 0$ with A and B reflexive there exist finite dimensional spaces F and G so that the sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus F & \longrightarrow & \mathcal{L}_1 \oplus F & \longrightarrow & Z & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & B \oplus G & \longrightarrow & \mathcal{L}_1' \oplus G & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

are isomorphically equivalent.

The result of Lindenstrauss [23] about $\mathcal{L}_1$-subspaces of l_1 can also be easily achieved. Let $\mathcal{L}_1$ and $\mathcal{L}_1'$ be two unspecified $\mathcal{L}_1$-spaces. Consider the couple of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & l_1 & \longrightarrow & \mathcal{L}_1 & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & K_2 & \longrightarrow & l_1 & \longrightarrow & \mathcal{L}_1' & \longrightarrow & 0. \end{array}$$

It is clear that K_1 and H_2 are $\mathcal{L}_1$-spaces. By Lindenstrauss's lifting principle, $\mathrm{Ext}(\mathcal{L}_1, l_1) = 0$, and therefore applying the diagonal principle we get, in particular, that $\mathcal{L}_1 \oplus l_1$ and $\mathcal{L}_1' \oplus l_1$ are isomorphic. Since every $\mathcal{L}_1$-space contains l_1 complemented we conclude: $\mathcal{L}_1 \simeq \mathcal{L}_1 \oplus l_1 \simeq \mathcal{L}_1' \oplus l_1 \simeq \mathcal{L}_1'$.

4.2. The Dunford–Pettis property on quotients of $\mathcal{L}_\infty$-spaces. The background we will use about the Dunford–Pettis property, which can be found in [10] and in [7, Chapter 6], can be resumed as: A Banach space X is said to have the Dunford-Pettis property (in short, DPP) if weakly compact operators $X \to Y$ are completely continuous. The DPP passes to complemented subspaces but not to dual spaces as Stegall's example $l_1(l_2^n)$ has a dual $l_\infty(l_2^n)$ which contains complemented copies of l_2 (see [34, 10, 8]). A result of Diestel [10] establishes that if in an exact sequence $0 \to Y \to X \to Z \to 0$ the space X has DPP and Y does not contain l_1, then X/Y has DPP. A consequence of this is Kislyakov's result [21] that quotients of an $\mathcal{L}_\infty$-space by a reflexive subspace have DPP, as well as all their higher duals. And the same occurs with the kernels of quotient maps from an $\mathcal{L}_1$-space onto a reflexive space. A warning is in order for this and the next section: we shall adopt the slightly improper custom of writing $\mathcal{L}_\infty$ and $\mathcal{L}_1$ to denote an unspecified $\mathcal{L}_\infty$- or $\mathcal{L}_1$-space.

The following proposition extends Kislyakov's result and completes those of Kalton and Pelczynski in [20]:

Proposition 4.1: *Let $0 \to K \to \mathcal{L}_1 \to X \to 0$ be an exact sequence in which X is an ultrasummand (i.e., it is complemented in its bidual) with the Radon–Nikodym property (in short, RNP). Then K has the DPP.*

Proof: Let us observe the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(X) & \longrightarrow & l_1(\Gamma) & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & \mathcal{L}_1 & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

The kernel $K(X)$ is an ultrasummand as proved by Kalton and Pelczynski [20, Prop. 2.3] (see below for another proof). Thus, $\operatorname{Ext}(\mathcal{L}_1, K(X)) = 0$ by Lindenstrauss's lifting principle [22] (see also [4] for several extensions of the principle); hence the diagonal principle yields that $K(X) \oplus \mathcal{L}_1$ is isomorphic to $K \oplus l_1(\Gamma)$. The former space has the DPP since $K(X)$ is Schur; hence, the latter space also has the DPP, as well as K. ∎

If X is just a separable dual (instead of reflexive), then K^* need not have the DPP as the example $0 \to l_1(l_2^n) \to l_1 \to X \to 0$ shows. A slightly more general proof than those of [20, Lemma 2.1 and Prop. 2.3] of the fact that $K(X)$ is an ultrasummand follows.

LEMMA 2: *If Z is an ultrasummand with RNP then, for every $L_1(\mu)$-space, the kernel of a quotient map $L_1(\mu) \to Z$ is an ultrasummand.*

Proof: A careful observation of the following diagram (in which the names of the zero-linear maps have been chosen following the duality theory developed in [3] plus the pull-back and push-out agreements in section 2; this does not affect the understanding of the proof):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & L_1(\mu) & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F \\
 & & \downarrow{\scriptstyle \delta_K} & & & & \| & & \\
0 & \longrightarrow & K^{**} & \longrightarrow & PO & \longrightarrow & Z & \longrightarrow & 0 \equiv \delta_K F \\
 & & \| & & & & \| & & \\
0 & \longrightarrow & K^{**} & \longrightarrow & PB & \xrightarrow[v]{} & Z & \longrightarrow & 0 \equiv F^{**}\delta_Z \\
 & & \| & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \delta_Z} & & \\
0 & \longrightarrow & K^{**} & \longrightarrow & L_1(\mu)^{**} & \xrightarrow[q^{**}]{} & Z^{**} & \longrightarrow & 0 \equiv F^{**}
\end{array}$$

reveals that the second and third rows are equivalent. The existence of a projection $p\colon Z^{**} \to Z$ and the fact that $q^{**}w = \delta_Z v$ imply $pq^{**}w = p\delta_Z v = v$; and thus, since Z has the RNP and $L_1(\mu)^{**}$ is an abstract L-space, v factorizes through some $l_1(\Gamma)$. Therefore $Fv = 0$. Moreover, $F^{**}\delta_Z q = 0$ quite obviously. The diagonal principle implies that $K \oplus PB$ is isomorphic to $K^{**} \oplus L_1(\mu)$, hence K is complemented in some dual, and it is therefore an ultrasummand. ∎

It is perhaps worth remarking that the $L_1(\mu)$-space cannot be replaced by an arbitrary $\mathcal{L}_1$-space. For instance, if $0 \to D_1 \to l_1 \to L_1 \to 0$ then D_1 is an $\mathcal{L}_1$-space that is not an ultrasummand. Hence, the kernel K in a sequence $0 \to K \to D_1 \to l_2 \to 0$ is not an ultrasummand since, otherwise, D_1 would be an ultrasummand using [3].

In [20], Kalton and Pelczynski asked if the kernel K_1 of a sequence $0 \to K_1 \to L_1 \to c_0(\Gamma) \to 0$ has the DPP. In the same paper they give one interesting quotient map $L_1(G) \to c_0(S)$, namely the Fourier transform when G is a locally compact abelian group and S a Sidon set of the dual group of G, whose kernel has the DPP. Of course, it is not difficult to give quotient maps $L_1 \to c_0$ with

DPP kernels. It is an open question if there exists one with non-DPP kernel. A reformulation of the question is then:

QUESTION A: *Are two sequences* $0 \to \ker Q \to L_1(\mu) \xrightarrow{Q} X$ *and* $0 \to \ker q \to L_1(\mu) \xrightarrow{q} X$ *semi-equivalent?*

Observe that if Q: $L_1(\mu) \to X$ and q: $L_1(\nu) \to X$ are two quotient maps so that Q factorizes through q, if $\ker q$ has DPP then also $\ker Q$ has DPP. This is so, because when Q factorizes through q we obtain a commutative push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker Q & \longrightarrow & L_1(\mu) & \xrightarrow{Q} & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \ker q & \longrightarrow & L_1(\nu) & \xrightarrow{q} & Z & \longrightarrow & 0. \end{array}$$

Its diagonal push-out sequence $0 \to \ker Q \to \ker q \oplus L_1(\mu) \to L_1(\nu) \to 0$ locally splits (which means that its dual sequence splits; see [16]). It is an elementary observation that if a sequence $0 \to Y \to X \to Z \to 0$ locally splits and X has DPP, then so does Y. Thus, if $\ker q$ has DPP then so does $\ker Q$. Moreover, since $(\ker Q)^* \oplus L_1(\mu)^* = (\ker q)^* \oplus L_1(\mu)^*$, it is clear that the duals of the kernels have DPP simultaneously.

This observation about the duals touches the question of when quotients of $\mathcal{L}_\infty$-spaces have DPP, which we consider now. Recall that an Asplund space is one all of whose separable subspaces have separable duals or, equivalently, the predual of a space with RNP. Hence, an Asplund space cannot contain l_1 and thus every quotient of an $\mathcal{L}_\infty$-space by an Asplund subspace has the DPP. We can extend Diestel and Kislyakov's results as follows:

PROPOSITION 4.2: *Let A be an Asplund space. The dual of every quotient $\mathcal{L}_\infty/A$ has the DPP.*

Proof: Since A^* has RNP, Proposition 4.1 applies to the dual sequence $0 \to A^\perp \to \mathcal{L}_1 \to A^* \to 0$. ∎

The bidual of $\mathcal{L}_\infty/A$ does not need to have the DPP, as the sequence $0 \to c_0(K_n) \to c_0 \to c_0(l_2^n) \to 0$ shows. The general situation about which quotients of an $\mathcal{L}_\infty$-space have DPP is much more elusive. In particular

QUESTION B: *Does l_∞/l_1 have the DPP?*

By the Lindenstrauss–Rosenthal theorem, the space l_∞/l_1 is uniquely defined. In [7, 8] the question was posed if the bidual $K(c_0)^{**}$ of the kernel of a projective presentation of c_0 has the DPP; which is the same as: does $(l_\infty/l_1)^*$ have the DPP? We show now that, as regards this last question, the space l_∞ can be replaced by any other $\mathcal{L}_\infty$-space.

Proposition 4.3: *Let E be a subspace of $\mathcal{L}_\infty$ and $\mathcal{L}'_\infty$.*

(1) *If $\mathcal{L}_\infty$ and $\mathcal{L}'_\infty$ are injective, the quotients $\mathcal{L}_\infty/E$ and $\mathcal{L}'_\infty/E$ have DPP simultaneously.*
(2) *If $\mathcal{L}'_\infty$ is a subspace of $\mathcal{L}_\infty$ and $\mathcal{L}_\infty/E$ has DPP, then $\mathcal{L}'_\infty/E$ has DPP.*
(3) *The dual spaces $(\mathcal{L}_\infty/E)^*$ and $(\mathcal{L}'_\infty/E)^*$ have DPP simultaneously.*

Proof: The first assertion is a direct consequence of the diagonal principle (injective case). As for the second assertion, observe the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & E & = & E & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{L}'_\infty & \to & \mathcal{L}_\infty & \to & \mathcal{L}_\infty/\mathcal{L}'_\infty & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{L}'_\infty/E & \to & \mathcal{L}_\infty/E & \to & \mathcal{L}_\infty/\mathcal{L}'_\infty & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Since the middle row locally splits, the same is true for the lower sequence. This also proves (3) since $(\mathcal{L}_\infty/E)^* = (\mathcal{L}'_\infty/E)^* \oplus (\mathcal{L}_\infty/\mathcal{L}'_\infty)^*$ and $\mathcal{L}_\infty/\mathcal{L}'_\infty$ is an $\mathcal{L}_\infty$-space. ∎

In general, a quotient $\mathcal{L}_\infty/E$ is not uniquely defined. Nonetheless, we conjecture the following:

Conjecture C: *Given X and Y two isomorphic subspaces of $C[0,1]$, then $C[0,1]/X$ has DPP if and only if $C[0,1]/Y$ has DPP.*

Recalling that the Bourgain–Pisier construction [2] yields for every separable Banach space E an $\mathcal{L}_\infty$-space, say $\mathcal{L}_\infty(E)$, such that the quotient $\mathcal{L}_\infty(E)/E$ has the Schur property, we see that the implication (2) in 4.3 cannot, in general, be reversed since $C[0,1]$ has non-DPP quotients, such as l_2. Nevertheless, there is a connection between A, B and C which clarifies the situation:

PROPOSITION 4.4: *Let X^* be a separable Banach space. The following are equivalent:*

(1) *Given a projective presentation $0 \to K(X) \to l_1 \to X \to 0$ of X, the space $K(X)^*$ has DPP.*
(2) *Given a sequence $0 \to W \to \mathcal{L}_1 \to X \to 0$, the space W^* has DPP.*
(3) *For all exact sequences $0 \to X^* \to C[0,1] \to C[0,1]/X^* \to 0$, the space $C[0,1]/X^*$ has the DPP.*

Proof: We already know that (1) and (2) are equivalent, and that (1) implies (3) has been proved in 4.3. To show that (3) implies (1) it is clearly enough to show that under the hypothesis (3) every separable subspace of l_∞/X^* is contained in some subspace with DPP. To this end, let $S \to l_\infty/X^*$ be a separable subspace. The pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^* & \longrightarrow & l_\infty & \longrightarrow & l_\infty/X^* & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & X^* & \longrightarrow & PB & \longrightarrow & S & \longrightarrow & 0 \end{array}$$

shows the existence of a separable subspace $PB \subset l_\infty$ such that $S = PB/X^*$. Let A be the commutative C^*-algebra that PB spans in l_∞. Necessarily $A = C(K)$ with K metric, which implies, using Milutin's theorem, that A is isomorphic to a complemented subspace of $C[0,1]$. Hence A/X^* is isomorphic to a complemented subspace of $C[0,1]/X^*$, and thus it has the DPP. ∎

4.3. THE EXTENSION OF $C(K)$-VALUED OPERATORS. Lindenstrauss and Pelczynski [24] proved that $C(K)$-valued operators defined on subspaces H of c_0 can be extended to the whole c_0; hence, see [24, cor.4], to any separable $\mathcal{L}_\infty$-space; of course the separability assumption cannot be dropped. Later, Johnson and Zippin [14] proved that $C(K)$-valued operators defined on a weak*-closed subspace of l_1 (i.e., the orthogonal of a subspace of c_0) can be extended to the whole l_1. From this they show [14, Cor.1.1] that if Q: $S \to H^*$ is a quotient operator from a separable space onto H^*, then every operator $\ker Q \to \mathcal{L}_\infty$ can be extended to S. The separability assumption can be dropped.

LEMMA 3: *Let H be a subspace of c_0 and let $0 \to D \to \mathcal{L}_1 \to H^* \to 0$ be an exact sequence. Every operator $D \to \mathcal{L}_\infty$ can be extended to $\mathcal{L}_1$.*

Proof: Since H is an Asplund space, by 4.1 the sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(H^*) & \longrightarrow & l_1 & \longrightarrow & H^* & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & \mathcal{L}_1 & \longrightarrow & H^* & \longrightarrow & 0 \end{array}$$

are semi-equivalent and thus the sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(H^*) \oplus \mathcal{L}_1 & \longrightarrow & l_1 \oplus \mathcal{L}_1 & \longrightarrow & H^* & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & D \oplus l_1 & \longrightarrow & \mathcal{L}_1 \oplus l_1 & \longrightarrow & H^* & \longrightarrow & 0 \end{array}$$

are isomorphically equivalent. Everything we need is to show that the lower sequence is isomorphically equivalent to the starting sequence $0 \to D \xrightarrow{j} \mathcal{L}_1 \to H^* \to 0$. In order to prove this, observe that the middle space in its dual sequence $0 \to H^{**} \to (\mathcal{L}_1)^* \xrightarrow{j^*} D^* \to 0$ is a $C(K)$-space, and thus the quotient map j^* is an isomorphism on some copy of c_0 which, by standard arguments (see [11]), produces a complemented copy of l_1 inside D on which j acts as an isomorphism. ∎

The following lemma is essentially [14, Prop. 1.1].

LEMMA 4: *Let $0 \to Y \to X \to Z \to 0 \equiv F$ be an exact sequence with the $\mathcal{L}_\infty$-extension property; i.e., every operator $Y \to \mathcal{L}_\infty$ can be extended to X. Let α: $Y \to Y_1$ a quotient operator. The push-out sequence αF also has the $\mathcal{L}_\infty$-extension property.*

Putting the two lemmata in combination with the observation that every exact sequence $0 \to Y \to X \to Z \to 0$ can be obtained taken push-out from a certain projective presentation of Z via a surjective operator, we get:

PROPOSITION 4.5: *Let H be a subspace of c_0. Each sequence $0 \to Y \to X \to H^* \to 0$ has the $\mathcal{L}_\infty$-extension property.*

It is therefore obvious that there are non-weak*-closed subspaces of l_1 admitting extension of $\mathcal{L}_\infty$-valued operators. Kalton has shown in [17] that if a subspace K of l_1 has the extension property for $C(K)$-valued operators, then l_1/K must have the strong Schur property. The weak*-character of the subspace has to be taken with respect to c_0, or at least with respect to a predual with the hereditary Dunford–Pettis property: otherwise, if X is a predual of l_1 and

$0 \to S \to X \to Z \to 0$ is an exact sequence in which S does not have the DPP, then S^* is not Schur and thus the kernel $K(S^*)$ of a quotient operator $l_1 \to S^*$ cannot have the extension property for $C(K)$-valued operators.

Definition: Let Y,X be Banach spaces. We shall say that X is Y-**automorphic** if any two exact sequences $0 \to Y_0 \to X \to Z_0 \to 0$ and $0 \to Y_1 \to X \to Z_1 \to 0$ in which Y_0 and Y_1 are isomorphic to Y, and Z, Z_1 are infinite dimensional, are isomorphically equivalent. We say that X is automorphic if it is Y-automorphic for every inifinite dimensional subspace Y of X.

Lindenstrauss and Pelczynski prove in [24] that $C[0,1]$ is H-automorphic for all subspaces H of c_0 and pose the question of whether this property characterizes c_0. The automorphic character of c_0 is at the basis of the "partially automorphic" character of $C[0,1]$. Since Hilbert spaces are also automorphic, it makes sense to pose the question: Is l_2-automorphic the space $C[0,1]$? Let us show that this question actually is not different from problem 4.2 in [14] about the extension of $C(K)$-valued operators:

PROPOSITION 4.6: *Let X be a separable Banach space not containing l_1. The space $C[0,1]$ is X-automorphic if and only if every $C(K)$-valued operator on X can be extended to $C[0,1]$.*

Proof: Since $C(K)$-valued operators on a Banach space X can be extended to $C(B_{X^*})$ through the canonical embedding, if $C[0,1]$ was X-automorphic, Milutin's theorem yields the result. To prove the converse, observe that, since X is separable, it is enough to work with $C[0,1]$-valued operators. Let $0 \to X \to C[0,1] \stackrel{q_1}{\to} Q_1 \to 0$ and $0 \to X \to C[0,1] \stackrel{q_2}{\to} Q_2 \to 0$ be two exact sequences. The hypothesis yields that they are semi-equivalent and thus, applying the diagonal principle, the sequences $0 \to X \to C[0,1] \oplus C[0,1] \to Q_1 \oplus C[0,1] \to 0$ and $0 \to X \to C[0,1] \oplus C[0,1] \to Q_2 \oplus C[0,1] \to 0$ are isomorphically equivalent. Now, since Q_1 (and Q_2) contain l_1, the dual spaces are not separable and thus (see [33]) q_1 is an isomorphism on some copy C_1 of $C[0,1]$ inside $C[0,1]$. By a result of Pelczynski [31], C_1 contains another copy of $C[0,1]$ complemented in $C[0,1]$; and the same occurs in Q_2. In other words, the sequences $0 \to X \to C[0,1] \to Q_i \to 0$ and $0 \to X \to C[0,1] \oplus C[0,1] \to Q_i \oplus C[0,1] \to 0$ are isomorphically equivalent ($i = 1,2$), which gives the result. ∎

Thus, either the Johnson–Zippin question has a negative answer (i.e., not every $C(K)$-valued operator on l_2 extends to $C[0,1]$) or the Lindenstrauss–Pelczynski conjecture has a negative answer.

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