MAPPINGS OF BMO-DISTORTION AND BELTRAMI-TYPE OPERATORS

By

TADEUSZ IWANIEC, PEKKA KOSKELA AND GAVEN MARTIN *

In memory of Tom Wolff

1 Introduction

This paper can be viewed as a sequel to the papers [29], [30], where the theory of quasiregular mappings in even dimensions is developed using singular integral operators and, in particular, the Beurling–Ahlfors transform. In those papers, we established sharp versions of the Liouville Theorem and more general regularity and higher integrability results, gave applications concerning removable singularities, and so forth. One of the key ideas was that the governing equations for the theory of quasiregular mapping linearise when lifted to the level of exterior algebra on middle dimensional forms, a fact first recognised by Donaldson and Sullivan [12].

In this paper, we continue this theme from a more general perspective. Here we are concerned with mappings with a distortion function bounded above by a BMO function of small BMO-norm. Since quasiregular mappings are assumed *a priori* to have bounded distortion and since the BMO-norm of a constant function is zero, the class of mappings we consider contains the quasiregular mappings and can be viewed as a natural extension of them. Since the results we prove often fail if the BMO-norm of no distortion function is small, this class of mappings seems optimal in many respects and illustrates the fact that something interesting is going on. There are also new phenomena. The inverse and compositions of such mappings need no longer be in the same class. We can deform an object (e.g., a ball) by a homeomorphism in this class but may not be able to deform it back. This irreversibility is reminiscent of some physical phenomena modelled

*Research of all authors supported in part by grants from the N.Z. Marsden Fund. Also the U.S. National Science Foundation (TI), DMS-0070807 and the Academy of Finland (PK), SA-34082.

using the theory of nonlinear elasticity. These mappings also allow various sorts of cusps to appear in the image of a hyperplane, a phenomenon not occurring for quasiconformal mappings.

Indeed, there has been a lot of interest recently in mappings of finite distortion [25, 26, 32, 33, 27, 40, 41] as there are natural and significant connections with the theory of nonlinear elasticity as developed by Antman [3] and Ball [5]. Our results extend and enhance these connections.

We shall show that to a large extent the regularity properties of mappings with a distortion function in BMO are controlled by the invertibility properties of a certain family of singular integral operators, closely related to the Beurling–Ahlfors transform. We call these operators Beltrami-type operators. We establish, in even dimensions, improved integrability and regularity beyond the *a priori* assumptions, and study the removable sets for bounded mappings in our class. Very recent progress in odd dimensions is presented in [27]. Such results are central to modern geometric function theory; see [33].

As with our earlier work, we view mappings of finite distortion as solutions to a Beltrami system. These equations form a highly nonlinear first order system which linearises in even dimensions when lifted to the exterior bundle. The reader should consult [29, 33, 36] for details, as we give only the briefest sketch below. The major difference between our earlier work and what we do here is that for mappings of finite distortion, uniform ellipticity bounds on the Beltrami system are no longer valid and the equations are degenerate.

It is natural in our setting to look at singular integral operators, since they have played an important role in the theory of partial differential equations from the very beginning, largely pertaining to the higher integrability theory of the gradient of weak solutions. It is only recently that we have become fully aware of their importance in determining the exact degrees of improved regularity and other geometric properties of solutions, their gradients and related nonlinear quantities for both linear and nonlinear PDEs in higher dimensions [30, 36].

There has been some recent and related work on mappings of finite distortion in the plane. This starts with the important work of G. David [11], where he established the existence of solutions to the Beltrami equation in the plane by assuming some degree of exponential integrability of the distortion function. David went on to give modulus of continuity estimates and an area distortion estimate similar to those we obtain in all even dimensions here. David's assumptions turn out to be the same as assuming that the distortion function is bounded above by some BMO function; see [36, 31, 33] for details concerning this equivalence. Later, P. Tukia [55] studied the compactness properties of this class of mappings.

Other properties of solutions in the plane, such as the Stoilow factorisation theorem, were discussed by T. Iwaniec and V. Šverák [37]. More recent work concerning the regularity properties of these mappings can be found in the work of Brakalova and Jenkins [6], Ryazanov, Srebro and Yacubov [53] and Migliaccio and Moscariello [45]. Our work extends and improves many of these results to all even dimensions. Even in the plane many of our results here were new at the time of writing. For the most recent and comprehensive results in the plane, the reader should consult our monograph [31].

Finally, we make a comment about the odd-dimensional case. There are various odd-dimensional nonlinear counterparts to the operators we utilise here. They are defined and discussed in [22], where partial counterparts to the results in [29] are found. As suspected, our results hold in all dimensions, although there are quite significant technical obstructions to overcome, see [27].

2 Definitions

Here we give the definitions of the function spaces, mention some of their properties, and record some useful terminology.

2.1 Hardy spaces and BMO A measurable function $a(x)$ supported in some ball B in $\mathbb{R}^n$ is called an *$\mathcal{H}^1$-atom* if it satisfies the conditions

$$(1)\qquad |a(x)| \leq \frac{1}{|B|} \qquad \text{a.e. } x \in \mathbb{R}^n,$$

$$(2)\qquad a_B = \frac{1}{|B|}\int_B a(x)dx = 0.$$

A function $f \in L^1(\mathbb{R}^n)$ belongs to $\mathcal{H}^1(\mathbb{R}^n)$ if and only if it can be written as a (possibly infinite) linear combination of $\mathcal{H}^1$-atoms, $f = \sum_{k=1}^{\infty} \lambda_k a_k$, with $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. The norm is then defined by

$$(3)\qquad \|f\|_{\mathcal{H}^1} = \inf\left\{\sum_{k=1}^{\infty} |\lambda_k| : f = \sum_{k=1}^{\infty} \lambda_k a_k\right\},$$

where the infimum is taken over all atomic decompositions of f. It is important to notice that such an f satisfies the *moment condition*

$$(4)\qquad \int_{\mathbb{R}^n} f(x)dx = 0.$$

Next, for a measurable function g on $\mathbb{R}^n$ and a ball $B \subset \mathbb{R}^n$, we define the average of g on B as

$$g_B = \frac{1}{|B|}\int_B g(x)dx. \tag{5}$$

If $g \in L^1_{loc}(\mathbb{R}^n)$ and

$$\|g\|_{BMO} = \sup_B \int_B |g(x) - g_B|dx < \infty, \tag{6}$$

then we say g is of *bounded mean oscillation*, $g \in BMO(\mathbb{R}^n)$.

We wish to make some observations here to show that standard approximation arguments based on our earlier work will fail. The first is that bounded functions lie in $BMO(\mathbb{R}^n)$, but they are not dense. In particular, $C_0^\infty(\mathbb{R}^n)$ functions are not dense in $BMO(\mathbb{R}^n)$. The closure of $C_0^\infty(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$ is the space $VMO(\mathbb{R}^n)$, the functions of *vanishing mean oscillation*.

Next recall the duality theorem of Fefferman, which states that $BMO(\mathbb{R}^n)$ is the dual space of $\mathcal{H}^1(\mathbb{R}^n)$. Also, a result of Sarason states that $\mathcal{H}^1(\mathbb{R}^n)$ is the dual space of $VMO(\mathbb{R}^n)$. In particular, we note the following [14].

Theorem 2.1. *There is a constant $C(n)$ such that if $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$, then*

$$\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| \le C(n)\|f\|_{\mathcal{H}^1}\|g\|_{BMO}. \tag{7}$$

In general, the integral (7) does not converge; however, there are a number of ways to give meaning to it [54].

Finally, the fundamental integrability properties of BMO-functions are found in the well-known John–Nirenberg lemma [39]

Lemma 2.1. *There exists a constant $\epsilon = \epsilon(n) > 0$ such that for every $h \in BMO(\mathbb{R}^n)$ and every ball or cube $B \subset \mathbb{R}^n$, we have*

$$\frac{1}{|B|}\int_B \exp\left(\frac{\epsilon|h(x) - h_B|}{\|h\|_{BMO}}\right)dx \le 2.$$

2.2 Orlicz and Zygmund spaces A continuous function $P : [0,\infty] \to [0,\infty]$ which is strictly increasing with $P(0) = 0$ and $P(\infty) = \infty$ is called an *Orlicz function*. It is customary to call P a *Young function* if P is convex. Given a measurable set $\Omega \subset \mathbb{R}^n$, the Orlicz space $L^P(\Omega)$ consists of all measurable functions h on Ω such that

$$\int_\Omega P(\lambda^{-1}|h(x)|)dx < \infty$$

for some $\lambda = \lambda(h) > 0$. In what follows, we usually assume that h assumes values in some finite-dimensional normed space, so $|h(x)|$ stands for the norm of $h(x)$. The space $L^P(\Omega)$ is a complete linear metric space [48]. In general, the Luxemburg functional

$$(8) \qquad \|h\|_P = \|h\|_{L^P(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega P(\lambda^{-1}|h|) \leq P(1)\right\}$$

need not be a norm, but it is if P is a Young function. In this case, $L^P(\Omega)$ is a Banach space.

Of particular importance in our applications are functions of the type

$$P(t) = t^p \log^\alpha(3+t) \log\log^\beta(8+t),$$

where $1 \leq p < \infty$ and α, β are real numbers. For such functions as this, the spaces $L^P(\Omega)$ are called *Zygmund spaces.*

The Orlicz–Sobolev space, denoted by $\mathcal{W}^{1,P}(\Omega)$, consists of functions in $W^{1,1}_{loc}(\Omega)$ whose distributional gradient belongs to $L^P(\Omega)$. Observe that we do not assume that the function itself lies in $L^P(\Omega)$, although this is the case if Ω is a sufficiently regular bounded domain in $\mathbb{R}^n$.

The Zygmund space $L^P(\Omega)$ with $P(t) = t\log(3+t)$, also denoted $L\log L(\Omega)$, will play a special role in what follows. When Ω is a bounded open region in $\mathbb{R}^n$, we can use the norm given by

$$(9) \qquad [h]_{L\log L(\Omega)} = \frac{1}{|\Omega|}\int_\Omega |h| \log\left(3 + \frac{|h|}{|h|_\Omega}\right).$$

This norm is equivalent to the Luxemburg norm. More generally, the nonlinear functional

$$[h]_{L^p\log^\alpha L(\Omega)} = \left[\frac{1}{|\Omega|}\int_\Omega |h|^p \log^\alpha\left(3 + \frac{|h|^p}{\frac{1}{\Omega}\int_\Omega |h|^p}\right)\right]^{1/p}$$

for $p \geq 1$ and $\alpha \geq 0$ is comparable with the Luxemburg norm via constants depending only on p and α. Actually, the functional $[\cdot]_{L^p\log^1 L(\Omega)}$ is a norm. The triangle inequality is somewhat surprising. Hölder's inequality in Zygmund spaces reads as

$$(10) \qquad \|fg\|_1 \leq C_\alpha(p,q)\|f\|_P\|g\|_Q,$$

where

$$P(t) = t^p\log^{\alpha p}(3+t) \quad \text{and} \quad Q(t) = t^q\log^{-\alpha q}(3+t)$$

for $1 < p, q < \infty$ with $1/p + 1/q = 1$ and α is any real number.

The most important of our preliminaries is the following $\mathcal{H}^1$-inequality for the wedge products of differential forms.

Lemma 2.2. *Let $\varphi \in L^P(\mathbb{R}^n, \Lambda^\ell)$ and $\psi \in L^Q(\mathbb{R}^n, \Lambda^k)$ be closed differential forms, and let P and Q be as at (10). Then $\varphi \wedge \psi$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n, \Lambda^{\ell+k})$, and we have*

$$\|\varphi \wedge \psi\|_{\mathcal{H}^1} \leq C_\alpha(p,q)\|\varphi\|_P\|\psi\|_Q. \tag{11}$$

For a proof of this fact and related estimates, we refer the reader to [38], Theorem 7.1.

We also note the following inequality which we use in Section 8 (see (164)),

$$\|fg\|_{L\log L} \leq C\ \|f\|_{L^2\log L}\|g\|_{L^2\log L}. \tag{12}$$

Now we recall a result of E. Stein [54].

Theorem 2.2. *If $h \in \mathcal{H}^1(\mathbb{R}^n)$ and if $h \geq 0$ in an open set Ω, then $h \in L\log L(G)$ for every G compactly contained in Ω.*

A duality between the spaces $L\log L(\Omega)$ and the exponential space $Exp(\Omega)$, corresponding to $P(t) = e^t - 1$, is emphasized by the following inequality for positive real numbers a and b:

$$ab \leq a\log(1+a) + (e^b - 1). \tag{13}$$

3 Beltrami-type operators

Let E and F be finite-dimensional inner product spaces. We assume that the inner product is Hermitian if E and F are complex. Let $S : L^2(\mathbb{R}^n, E) \to L^2(\mathbb{R}^n, F)$ be a bounded linear operator and μ a measurable function defined in $\mathbb{R}^n$ with values in $\mathcal{L}(F, E)$, the linear transformations from F to E. We define

$$|\mu(x)| = \sup\{|\mu(x)\zeta|_E : |\zeta|_F = 1\} \tag{14}$$

and assume $|\mu| \in L^\infty(\mathbb{R}^n)$.

In our applications, μ will be the Beltrami coefficient of a mapping of finite distortion and S will be a singular integral operator, the Beurling–Ahlfors transform. Here, with other applications in mind, we have chosen to keep the setting fairly general. With this background, it is natural to introduce the following concept. An operator $\mathcal{B}$ defined on $L^2(\mathbb{R}^n, E)$ is said to be of *Beltrami type* if $\mathcal{B}$ can be written as

$$\mathcal{B} = I - \mu S, \tag{15}$$

where I is the identity operator and μ and S are as above with the following additional hypotheses.

1. (Ellipticity) $|\mu(x)| < 1$ for almost all $x \in \mathbb{R}^n$.
2. (Isometry) The linear operator S is an isometry in L^2; in particular, $\|S\|_2 = 1$.
3. (Hardy condition) The bilinear operator $\mathcal{A}$ defined by

$$\mathcal{A}(\alpha,\beta) = \langle S\alpha, S\beta\rangle - \langle\alpha,\beta\rangle \tag{16}$$

maps $L^2(\mathbb{R}^n, E) \times L^2(\mathbb{R}^n, E)$ into the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ and satisfies the inequality

$$\|\mathcal{A}(\alpha,\beta)\|_{\mathcal{H}^1} \leq C_{\mathcal{A}}\|\alpha\|_2\|\beta\|_2. \tag{17}$$

Notice that because of the moment condition for $\mathcal{H}^1$ functions, we have that 3. implies 2. Of course 2. by itself implies the moment condition (4) for $f = \mathcal{A}(\alpha,\beta)$.

The principal question we wish to consider here is the invertibility of the operator $\mathcal{B}$. It is trivial that $\mathcal{B}$ is invertible in L^2 if $\|\mu\|_\infty < 1$. Our interest lies in the borderline case, that is, when

$$\|\mu\|_\infty = 1. \tag{18}$$

However, in order to achieve any sort of result, it is necessary to place restrictions on $|\mu|$ which guarantee that it is not too close to 1 locally in average. More precisely, we introduce the distortion function

$$K(x) = \frac{1+|\mu(x)|}{1-|\mu(x)|} \tag{19}$$

and require this function to be majorised by a $BMO(\mathbb{R}^n)$ function $M(x)$:

$$\frac{1+|\mu(x)|}{1-|\mu(x)|} \leq M(x). \tag{20}$$

In this case, we have the following theorem, which we regard as one of our central results.

Theorem 3.1 ***(Invertibilty).*** *Let $\mathcal{B} = I - \mu S$ be an operator of Beltrami type. Then there is a number $\epsilon = \epsilon(n, C_{\mathcal{A}}) > 0$ with the following property. If the distortion function is dominated by M satisfying $\|M\|_{BMO} \leq \epsilon$, then the operator $M\ \mathcal{B}$ has a right inverse in $L^2(\mathbb{R}^n, E)$. That is, there exists an operator $\mathcal{F} : L^2(\mathbb{R}^n, E) \to L^2(\mathbb{R}^n, E)$ such that*

$$M\ \mathcal{B} \circ \mathcal{F} = I : L^2(\mathbb{R}^n, E) \to L^2(\mathbb{R}^n, E). \tag{21}$$

Before proving this theorem, we mention that there are examples to show that the conclusion fails if the distortion function is simply a BMO function. We shall discuss these later. Thus the requirement that the norm is small is necessary.

To begin the proof, we first show that $M\mathcal{B}$ is coercive in $L^2(\mathbb{R}^n, E)$.

Lemma 3.1. *Under the hypotheses of Theorem 3.1, we have*

$$\|\omega\|_2 \leq 2\|M\mathcal{B}\omega\|_2 \tag{22}$$

whenever $\omega \in L^2(\mathbb{R}^n, E)$ *and* $M\mathcal{B}\omega \in L^2(\mathbb{R}^n, E)$.

Proof. We begin with some elementary pointwise estimates. Let $K(x)$ be the distortion function of $\mathcal{B}$. Then $K(x) \leq M(x)$ and

$$|\omega| - \frac{K-1}{K+1}|S\omega| = |\omega| - |\mu||S\omega| \leq |\omega - \mu S\omega| = |\mathcal{B}\omega|. \tag{23}$$

Hence

$$|\omega| + |S\omega| \leq K\,(|S\omega| - |\omega|) + (K+1)|\mathcal{B}\omega|, \tag{24}$$

and so

$$\begin{aligned}(|\omega| + |S\omega|)^2 &\leq K\,(|S\omega|^2 - |\omega|^2) + (K+1)|\mathcal{B}\omega|(|\omega| + |S\omega|)\\ &\leq K\,(|S\omega|^2 - |\omega|^2) + \tfrac{1}{2}(K+1)^2|\mathcal{B}\omega|^2 + \tfrac{1}{2}(|\omega| + |S\omega|)^2.\end{aligned}$$

This implies that

$$\tfrac{1}{2}(|\omega|^2 + |S\omega|^2) \leq \tfrac{1}{2}(|\omega| + |S\omega|)^2 \leq K\,(|S\omega|^2 - |\omega|^2) + \tfrac{1}{2}(K+1)^2|\mathcal{B}\omega|^2. \tag{25}$$

Now the right hand side of this equation is positive and only increases if we multiply by the function $M/K \geq 1$. After so doing, we obtain

$$\tfrac{1}{2}(|\omega|^2 + |S\omega|^2) \leq M\,(|S\omega|^2 - |\omega|^2) + 2M^2|\mathcal{B}\omega|^2. \tag{26}$$

Now we integrate this inequality and apply Theorem 2.1, to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |\omega|^2 &\leq \int_{\mathbb{R}^n} M\,(|S\omega|^2 - |\omega|^2) + 2\int_{\mathbb{R}^n} M^2|\mathcal{B}\omega|^2\\ &\leq C(n)\|M\|_{BMO}\||S\omega|^2 - |\omega|^2\|_{\mathcal{H}^1} + 2\int_{\mathbb{R}^n} M^2|\mathcal{B}\omega|^2\\ &\leq C(n)C_A\|M\|_{BMO}\|\omega\|_2^2 + 2\|M\mathcal{B}\omega\|_2^2.\end{aligned}$$

Now we choose

$$0 < \epsilon \leq \frac{1}{2C(n)C_A}. \tag{27}$$

This choice of ϵ implies that

$$\int_{\mathbb{R}^n} |\omega|^2 \leq 4 \int_{\mathbb{R}^n} M^2 |\mathcal{B}\omega|^2, \tag{28}$$

which is what we wanted to prove. □

Proof of Theorem 3.1. Given $\gamma \in L^2(\mathbb{R}^n, E)$, we want to solve (uniquely) the equation

$$M\mathcal{B}\omega = M(\omega - \mu S\omega) = \gamma \tag{29}$$

for $\omega \in L^2(\mathbb{R}^n, E)$. The first point to notice is that uniqueness is a direct consequence of Lemma 3.1. We establish the existence via an approximation argument. To this end, we introduce the sequence $\{\mu_j\}_{j=1}^{\infty}$ defined by

$$\mu_j(x) = \begin{cases} \mu(x) & \text{if } |\mu(x)| < 1 - 1/j, \\ \frac{\mu(x)}{|\mu(x)|}(1 - 1/j) & \text{otherwise.} \end{cases} \tag{30}$$

Obviously, we have $\mu_j \to \mu$ a.e. $\mathbb{R}^n$. Moreover, the distortion functions satisfy

$$K_j(x) = \frac{1 + |\mu_j(x)|}{1 - |\mu_j(x)|} \leq M(x). \tag{31}$$

Next consider the unique solution $\omega_j \in L^2(\mathbb{R}^n, E)$ of the equation

$$M(\omega_j - \mu_j S\omega_j) = \gamma. \tag{32}$$

By Lemma 3.1, we have the uniform bounds

$$||\omega_j||_2 \leq 2||\gamma||_2. \tag{33}$$

There is no loss of generality in assuming that ω_j converges weakly in $L^2(\mathbb{R}^n, E)$ to a function $\omega \in L^2(\mathbb{R}^n, E)$ by simply passing to a subsequence if necessary. What we need to show is that $M\mathcal{B}\omega = \gamma$. Equivalently, it suffices to show

$$\int_{\mathbb{R}^n} \langle \omega - \mu S\omega, \varphi \rangle = \int_{\mathbb{R}^n} \langle M^{-1}\gamma, \varphi \rangle \tag{34}$$

for every test function $\varphi \in C_0^{\infty}(\mathbb{R}^n, E)$. Let $S^* : L^2(\mathbb{R}^n, F) \to L^2(\mathbb{R}^n, E)$ denote the adjoint of the operator S. The identity (34) is then equivalent to

$$\int_{\mathbb{R}^n} \langle \varphi - S^* \mu\varphi, \omega \rangle = \int_{\mathbb{R}^n} \langle M^{-1}\varphi, \gamma \rangle. \tag{35}$$

We certainly have

$$\int_{\mathbb{R}^n} \langle \varphi - S^*\mu_j\varphi, \omega_j\rangle = \int_{\mathbb{R}^n} \langle M^{-1}\varphi, \gamma\rangle, \tag{36}$$

and our goal is now to pass to the limit. The integrals $\int_{\mathbb{R}^n}\langle\varphi,\omega_j\rangle$ pose no problem, as ω_j is assumed to converge weakly to ω in $L^2(\mathbb{R}^n, E)$. We decompose the integral

$$\int_{\mathbb{R}^n} \langle S^*(\mu_j\varphi), \omega_j\rangle = \int_{\mathbb{R}^n} \langle S^*((\mu_j - \mu)\varphi), \omega_j\rangle + \int_{\mathbb{R}^n} \langle S^*(\mu\varphi), \omega_j\rangle. \tag{37}$$

As before, the latter integral poses no problem in passing to the limit. Now we need only observe that

$$\begin{aligned}\int_{\mathbb{R}^n} \langle S^*((\mu_j - \mu)\varphi), \omega_j\rangle &\le \|S^*((\mu_j - \mu)\varphi)\|_2\|\omega_j\|_2 \\ &= \|(\mu_j - \mu)\varphi\|_2\|\omega_j\|_2 \\ &\le 2\|\gamma\|_2\|(\mu_j - \mu)\varphi\|_2\end{aligned}$$

and this last term tends to zero as $j \to \infty$, since $\mu_j \to \mu$ almost everywhere. Thus ω solves the equation (29). Finally, we define

$$\mathcal{F}\gamma = \omega. \tag{38}$$

It is quite clear that $\mathcal{F} : L^2(\mathbb{R}^n, E) \to L^2(\mathbb{R}^n, E)$ is a linear operator of norm less than or equal to 2. This concludes the proof of the Theorem. □

4 Examples

In this section, we give a number of examples of the sort of operators to which our Invertibility Theorem applies in the situations which are of greatest importance to us.

4.1 Hilbert transform We begin with the one-dimensional Hilbert transform defined on the real line by the formula

$$\mathcal{H}\omega(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\omega(y)}{x - y}\, dy. \tag{39}$$

Note that $\mathcal{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometry but that $\mathcal{H}$ is not bounded in $L^1(\mathbb{R})$. The Hardy space $\mathcal{H}^1(\mathbb{R}) \subset L^1(\mathbb{R})$ is the subspace of those integrable functions whose Hilbert transform is also integrable. The $\mathcal{H}^1$-norm of $\omega \in \mathcal{H}^1(\mathbb{R})$ is equivalent to the norm given by $\|\omega\|_1 + \|\mathcal{H}\omega\|_1$. The Hilbert transform is characterised by the property that the function $\omega + i\mathcal{H}\omega$ admits an analytic extension to the upper half-plane. Given $\alpha, \beta \in L^2(\mathbb{R})$, we can write

$$(\alpha + i\mathcal{H}\alpha)(\beta + i\mathcal{H}\beta) = (\alpha\beta - \mathcal{H}\alpha\mathcal{H}\beta) + i(\alpha\mathcal{H}\beta + \beta\mathcal{H}\alpha), \tag{40}$$

which simply means that

$$\alpha\mathcal{H}\beta + \beta\mathcal{H}\alpha = \mathcal{H}(\alpha\beta - \mathcal{H}\alpha\mathcal{H}\beta). \tag{41}$$

This identity shows that the bilinear expression

$$\mathcal{A}(\alpha, \beta) = \mathcal{H}\alpha\mathcal{H}\beta - \alpha\beta \tag{42}$$

lies in the Hardy space $\mathcal{H}^1(\mathbb{R})$. Indeed, we have the estimate

$$||\mathcal{A}(\alpha, \beta)||_1 \leq ||\mathcal{H}\alpha||_2||\mathcal{H}\beta||_2 + ||\alpha||_2||\beta||_2 = 2||\alpha||_2||\beta||_2 \tag{43}$$

and, similarly,

$$||\mathcal{H}\mathcal{A}(\alpha, \beta)||_1 = ||\alpha\mathcal{H}\beta + \beta\mathcal{H}\alpha||_2 \leq 2||\alpha||_2||\beta||_2. \tag{44}$$

Hence we conclude that

$$||\mathcal{A}(\alpha, \beta)||_{\mathcal{H}^1} \leq C||\alpha||_2||\beta||_2. \tag{45}$$

4.2 Complex Hilbert transform The complex Hilbert transform (sometimes called the Beurling–Ahlfors transform [1], [2]) is defined by the formula

$$\mathcal{S}\omega(z) = -\frac{1}{2\pi i}\int\int_{\mathbb{C}} \frac{\omega(\zeta)d\zeta}{(z-\zeta)^2}. \tag{46}$$

It has the characteristic property that it intertwines the Cauchy–Riemann derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$; that is,

$$S \circ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} : W^{1,2}(\mathbb{C}) \to L^2(\mathbb{C}). \tag{47}$$

Given a $\omega \in L^2(\mathbb{C})$ we can solve the nonhomogeneous Cauchy–Riemann equation $\frac{\partial}{\partial \bar{z}} f = \omega$, for an $f \in W^{1,2}(\mathbb{C})$. Then the bilinear expression

$$\mathcal{A}(\omega, \omega) = \langle \mathcal{S}\omega, \mathcal{S}\omega\rangle - \langle \omega, \omega\rangle = |\partial f/\partial z|^2 - |\partial f/\partial \bar{z}|^2 \tag{48}$$

is nothing other than the Jacobian determinant of f. We appeal to a result of Coifman, Lions, Meyer and Semmes [8] which asserts that the expression

$|\partial f/\partial z|^2 - |\partial f/\partial \bar{z}|^2$ belongs to $\mathcal{H}^1(\mathbb{C})$. For more discussion and the relevant estimates, we refer the reader to [23, 28]. We find there that

$$\|\mathcal{A}(\omega,\omega)\|_{\mathcal{H}^1} \leq C\int_{\mathbb{C}}(|f_z|^2+|f_{\bar{z}}|^2) = C\int_{\mathbb{C}}(|\omega|^2+|S\omega|^2) = 2C\|\omega\|_2^2. \tag{49}$$

In order to extract Condition 3 from this, we write

$$4\mathcal{A}(\alpha,\beta) = \mathcal{A}(\alpha+\beta,\alpha+\beta) - \mathcal{A}(\alpha-\beta,\alpha-\beta). \tag{50}$$

This leads easily to the estimate

$$\|\mathcal{A}(\alpha,\beta)\|_{\mathcal{H}^1} \leq C\left(\|\alpha\|_2^2 + \|\beta\|_2^2\right). \tag{51}$$

For real and positive t, we may apply this estimate to the function $t\beta$ in the place of β. We find

$$\|\mathcal{A}(\alpha,\beta)\|_{\mathcal{H}^1} \leq C\left(\frac{1}{t}\|\alpha\|_2^2 + t\|\beta\|_2^2\right) \tag{52}$$

and, putting $t = \|\alpha\|_2/\|\beta\|_2$, we arrive at the multiplicative estimate

$$\|\mathcal{A}(\alpha,\beta)\|_{\mathcal{H}^1} \leq C\|\alpha\|_2\|\beta\|_2. \tag{53}$$

4.3 The Beurling–Ahlfors transform The n-dimensional Beurling–Ahlfors transform acts on differential forms in $\mathbb{R}^n$ and exhibits many similarities with the complex Hilbert transform. As applications of this example will be central in our work on the regularity theory of mappings of finite distortion, we take a moment here to review the theory. We first need some notation associated with the exterior algebra. For an integer $1 \leq \ell \leq n$, we denote by Λ^ℓ the space of ℓ-covectors spanned by $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_\ell}$, where $I = (i_1, i_2, \ldots, i_\ell)$ is an ordered ℓ-tuple, $i_1 < i_2 < \cdots < i_\ell \leq n$. The inner product on Λ^ℓ is defined by requiring that the basis covectors be orthonormal. As a notational convenience, we set $\Lambda^0 = \mathbb{R}$ (or $\mathbb{C}$ in the complex case). The exterior algebra $\Lambda = \bigoplus_{\ell=0}^n \Lambda^\ell$ is an associative graded algebra with respect to the wedge product, in which $1 \in \Lambda^0$ acts as a unit element. The orientation of $\mathbb{R}^n$ and the inner product on Λ give rise to the Hodge star duality operator

$$* : \Lambda^\ell \to \Lambda^{n-\ell} \tag{54}$$

defined via the pairing

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dx, \tag{55}$$

where $dx = dx_1 \wedge dx_2 \wedge \cdots dx_n$ is the volume form. A differential ℓ-form defined on an open subset Ω of $\mathbb{R}^n$ is simply a function, or more generally a Schwarz distribution, with values in Λ^ℓ. Every ℓ-form can be uniquely expressed as

$$\omega(x) = \sum_{1 \le i_1 \le \cdots \le i_\ell \le n} \omega_{i_1, i_2, \ldots, i_\ell}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_\ell}. \tag{56}$$

The coefficients $\omega_{i_1, i_2, \ldots, i_\ell}(x)$ are either functions or distributions in $\mathcal{D}'(\Omega, \Lambda^\ell)$.

The exterior differential operator $d : \mathcal{D}'(\Omega, \Lambda^\ell) \to \mathcal{D}'(\Omega, \Lambda^{\ell+1})$ is defined by

$$d\omega(x) = \sum_{k=1}^{n} \sum_{1 \le i_1 \le \cdots \le i_\ell \le n} \frac{\partial \omega_{i_1, i_2, \ldots, i_\ell}}{\partial x_k}(x) dx_k \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_\ell}. \tag{57}$$

The formal adjoint of d is

$$d^* = (-1)^{n\ell+1} * d* : \mathcal{D}'(\Omega, \Lambda^{\ell+1}) \to \mathcal{D}'(\Omega, \Lambda^\ell). \tag{58}$$

Then $d \circ d = d^* \circ d^* = 0$. Of course, both d and d^* may act on Sobolev spaces of differential forms; we use the notation $W^{1,p}(\Omega, \Lambda^\ell)$ for such spaces. The duality between the operators d and d^* is characterised by the formula for integration by parts

$$\int_\Omega \langle d\alpha, \beta \rangle = \int_\Omega \langle \alpha, d^* \beta \rangle \tag{59}$$

for $\alpha \in W^{1,p}(\Omega, \Lambda^\ell)$ and $\beta \in W^{1,q}(\Omega, \Lambda^{\ell+1})$ both with compact support. Here, of course, p and q are assumed to be Hölder conjugate exponents. A fundamental fact in the L^p theory of differential forms which will be useful to us is the existence of the Hodge decomposition of a differential form $\omega \in L^p(\mathbb{R}^n, \Lambda^\ell)$, $1 < p < \infty$. This asserts the existence of differential forms $\alpha \in W^{1,p}(\mathbb{R}^n, \Lambda^{\ell-1})$ and $\beta \in W^{1,p}(\mathbb{R}^n, \Lambda^{\ell+1})$ such that

$$\omega = d\alpha + d^* \beta. \tag{60}$$

The forms $d\alpha$ and $d^*\beta$ are unique, but of course the forms α and β need not be. Moreover, $d\alpha$ and $d^*\beta$ can be explicitly expressed in terms of Riesz transforms of ω.

The Beurling–Ahlfors Transform, introduced in [29], is a singular integral operator $S : L^p(\mathbb{R}^n, \Lambda^\ell) \to L^p(\mathbb{R}^n, \Lambda^\ell)$ defined by the rule

$$S\omega = d\alpha - d^* \beta. \tag{61}$$

Formally, this operator can also be defined as

$$S = (dd^* - d^*d) \circ \Delta^{-1} \tag{62}$$

acting on forms with L^p coefficients in $\mathbb{R}^n$. Thus S acts as the identity on exact forms and as minus the identity on co-exact forms.

The Beurling–Ahlfors transform is actually a singular integral operator of a rather natural type, being a $2^n \times 2^n$ matrix of second order Riesz transforms. In [30], we identify the Fourier multiplier of this operator and give some estimates of the p-norms.

It follows from the formula for integration by parts that when $p = 2$, we have an orthogonal Hodge decomposition and, moreover,

$$\|S\omega\|_2^2 = \|d\alpha\|_2^2 + \|d^*\beta\|_2^2 = \|\omega\|_2^2, \tag{63}$$

so that S is an isometry in $L^2(\mathbb{R}^n, \Lambda^\ell)$. To verify the Hardy condition for this operator, we consider the expression

$$|S\omega|^2 - |\omega|^2 = |d\alpha - d^*\beta|^2 - |d\alpha + d^*\beta|^2 = -4\langle d\alpha, d^*\beta\rangle. \tag{64}$$

Thus

$$(|S\omega|^2 - |\omega|^2)dx = -4d\alpha \wedge d * \beta. \tag{65}$$

Now we are in a position to use some of the $\mathcal{H}^1$ theory of wedge products of closed differential forms. This theory is similar in many respects to the "div-curl" theory. We refer the reader to [8, 18, 7, 33, 38, 52] and the references therein. In particular, we note that for α and β as above, we have $d\alpha \wedge d*\beta \in \mathcal{H}^1(\mathbb{R}^n, \Lambda^n)$ with the uniform norm estimate

$$\begin{aligned}\|d\alpha \wedge d * \beta\|_{\mathcal{H}^1} &\le C(n)\|d\alpha\|_2\|d * \beta\|_2 \\ &\le C(n)\|d\alpha\|_2\|d^*\beta\|_2 \\ &\le \tfrac{1}{2}C(n)(\|d\alpha\|_2^2 + \|d^*\beta\|_2^2) \\ &= \tfrac{1}{2}C(n)\|\omega\|_2^2.\end{aligned}$$

This then yields the estimate

$$\|\, |S\omega|^2 - |\omega|^2 \,\|_{\mathcal{H}^1} \le 2C(n)\|\omega\|_2^2. \tag{66}$$

The slightly more general inequality

$$\|\, \langle S\alpha, S\beta\rangle - \langle \alpha, \beta\rangle \,\|_{\mathcal{H}^1} \le 2C(n)\|\alpha\|_2\|\beta\|_2 \tag{67}$$

follows from much the same sort of arguments as those given for (53).

4.4 Even dimensions Something special occurs in even dimensions which is the basis for our later results; we begin a discussion of this here. If n is even, say $n = 2\ell$, then the Hodge $*$-operator acts

$$*: \Lambda^\ell \to \Lambda^\ell. \tag{68}$$

Since $** = (-1)^\ell$, there are two eigenvalues of $*$: i^ℓ and $-i^\ell$. Their eigenspaces are denoted Λ^+ and Λ^-, respectively. Thus

$$\Lambda^+ = \{\zeta \in \Lambda^\ell : *\zeta = i^\ell\zeta\} \quad \text{and} \quad \Lambda^- = \{\zeta \in \Lambda^\ell : *\zeta = -i^\ell\zeta\}. \tag{69}$$

We obtain from these spaces the orthogonal decomposition $\Lambda^\ell = \Lambda^+ \oplus \Lambda^-$. The $\pm$ components of a differential form $\omega \in \Lambda^\ell$ are given by

$$\omega^+ = \tfrac{1}{2}(\omega + (-i)^\ell * \omega), \qquad \omega^- = \tfrac{1}{2}(\omega - (-i)^\ell * \omega). \tag{70}$$

This leads to a decomposition of the exterior derivative as $d = d^+ + d^-$, where

$$d^+ = \tfrac{1}{2}[1 + (-i)^\ell *]d : W^{1,p}(\mathbb{R}^n, \Lambda^{\ell-1}) \to L^p(\mathbb{R}^n, \Lambda^+) \tag{71}$$

and

$$d^- = \tfrac{1}{2}[1 - (-i)^\ell *]d : W^{1,p}(\mathbb{R}^n, \Lambda^{\ell-1}) \to L^p(\mathbb{R}^n, \Lambda^-). \tag{72}$$

After noting the identity

$$|d^+\alpha|^2 - |d^-\alpha|^2 = \langle d\alpha, (-i)^\ell * d\alpha\rangle = \Re\{i^\ell d\alpha \wedge \overline{d\alpha}\}, \tag{73}$$

we find as before that $|d^+\alpha|^2 - |d^-\alpha|^2$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ whenever $\alpha \in W^{1,2}(\mathbb{R}^n, \Lambda^{\ell-1})$. In this case, we have the uniform bound

$$\|\, |d^+\alpha|^2 - |d^-\alpha|^2 \,\|_{\mathcal{H}^1} \le C(n)\|d\alpha\|_2^2. \tag{74}$$

As the operators d^+ and d^- are known to be elliptic, we can solve the nonhomogeneous equation

$$d^+\alpha = \omega \tag{75}$$

for $\omega \in L^2(\mathbb{R}^n, \Lambda^+)$ and expect a solution $\alpha \in W^{1,2}(\mathbb{R}^n, \Lambda^{\ell-1})$. Again, the solution α is not unique but, as before, $d\alpha$ is. Now the Beurling–Ahlfors transform swaps the eigenspaces of $*$ on Λ^ℓ and intertwines the d^+ and d^- derivatives. Notice the complete analogy with the complex Beurling–Ahlfors transform in the plane. Restricting $\mathcal{S}$ to the eigenspace Λ^+, we find that

$$\mathcal{S} : L^2(\mathbb{R}^n, \Lambda^+) \to \Lambda^2(\mathbb{R}^n, \Lambda^-) \tag{76}$$

can be defined by

$$S\omega = d^{-}\alpha. \tag{77}$$

Then we have

$$\|\langle S\phi, S\psi\rangle - \langle \phi, \psi\rangle\|_{\mathcal{H}^1} \leq C(n)\|\phi\|_2\|\psi\|_2. \tag{78}$$

5 Mappings with finite distortion

Let Ω be an open subset of $\mathbb{R}^n$ and let $f = (f^1, f^2, \dots, .f^n) : \Omega \to \mathbb{R}^n$ be a mapping of Sobolev class $W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$. We denote by $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ the differential of f and by $J(x, f)$ the Jacobian determinant of f,

$$J(x, f) = \det Df(x). \tag{79}$$

The mapping f is said to have finite distortion if the Jacobian $J(x, f)$ is locally integrable and there is a function $K(x)$, $1 \leq K(x) < \infty$, defined almost everywhere in Ω such that

$$|Df(x)|^n \leq K(x)J(x, f) \quad \text{a.e. } \Omega. \tag{80}$$

Here $|Df(x)| = \sup_{|h|=1} |Df(x)h|$. The function $K(x)$ is called a *distortion* function for f (or a distortion function).

When the function $K(x)$ is bounded, $K \in L^\infty(\Omega)$, then $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ and we obtain the class of quasiregular mappings. There is a substantial literature on this subject which is deep and has significant applications in analysis and geometry. We refer the reader to the monographs [33, 50, 51] as a starting point. It is important to notice that here we are relaxing *both* of the usual requirements of $W^{1,n}$-regularity for f and the boundedness of the distortion function for f.

Examples. Let us now give a class of examples of mappings of finite distortion which in a sense are extremal. These examples, the radial stretchings, later motivate our conjectures regarding the sharp values of various constants which we estimate. Let $f : B^n(0, R) \to \mathbb{R}^n$ be defined by

$$f(x) = x\rho(|x|) \tag{81}$$

where $\rho : [0, R] \to \mathbb{R}$ is a smooth function with $\rho'(t) \leq 0$ and $(t\rho)'(t) \geq 0$. An easy calculation shows that

$$|Df(x)| = \rho'(|x|) \tag{82}$$

and

$$J(x,f) = \rho^n(|x|) + |x|\rho'(|x|)\rho^{n-1}(|x|). \tag{83}$$

As a first example, take

$$\rho(t) = \frac{1}{t|\log t|^\alpha} \tag{84}$$

for $0 < t < e^{-\alpha}$ and $\alpha > 0$. Then we have

$$|Df(x)|^n = \frac{1}{t^n|\log t|^{\alpha n}} \quad \text{and} \quad J(x,f) = \frac{\alpha}{t^n|\log t|^{\alpha n+1}}, \tag{85}$$

so that the function

$$K(x) = \frac{1}{\alpha}|\log|x|| \tag{86}$$

is a distortion function for f. Notice that $K \in BMO$ and that $\|K\|_{BMO} \to 0$ as $\alpha \to \infty$. Further notice that $|Df|^n \in L^1(B(0,e^{-\alpha}))$ if and only if $\alpha > 1/n$, while $J(x,f)$ is always integrable.

Next we calculate the inverse of this radial stretching mapping. If f is defined with ρ as at (84), then we find that f^{-1} is the smooth radial stretching

$$f^{-1}(x) = \frac{x}{|x|}\exp\left(-|x|^{-1/\alpha}\right).$$

Using polar coordinates seems to be the easiest way to verify that any distortion function is bounded below by the linear distortion function of this mapping, which we calculate to be

$$K_{f^{-1}}(x) = \max\left\{\frac{1}{\alpha r^{1/\alpha}}, \alpha r^{1/\alpha}\right\}.$$

Thus no distortion function for f^{-1} can lie in BMO if $\alpha > 1/n$. We record this observation in the following corollary.

Corollary 5.1. *For each $\epsilon > 0$, there is a homeomorphism $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ of finite distortion K with $\|K\|_{BMO} < \epsilon$ such that f^{-1} does not admit any BMO distortion function.*

We note that in the definition of a mapping of finite distortion, the assumption that the Jacobian $J(x,f)$ is at least locally integrable is always satisfied if $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ is a homeomorphism. Our principal aim is to discover the consequences if a distortion function has good integrability properties. From such an assumption, we deduce further properties of the map and derive improved regularity, for instance asserting the function is in the Sobolev space $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. Notice that this is the natural Sobolev space for the differential inequality (80). We now show that the Jacobian is locally integrable under fairly weak assumptions.

6 Integrability of the Jacobian

The following result and its corollary seem to be of independent interest. They improve and extend a result of Gehring and Lehto [16] to higher dimensions.

Theorem 6.1. *Let $B \subset \mathbb{R}^n$ be a ball and $f : B \to \mathbb{R}^n$ a function of Sobolev class $W^{1,1}(B, \mathbb{R}^n)$. Then there is a set $E \subset B$ of measure zero such that*

$$\int_U |J(x,f)|dx = \int_{\mathbb{R}^n} \#\{x \in U \setminus E : f(x) = y\}dy \tag{87}$$

for each measurable $U \subset B$, where $\#A$ denotes the cardinality of the set A.

Proof. There is no loss of generality in assuming $f \in W^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$. Then, by the approximation theorem for Sobolev functions by Lipschitz functions (see 6.6.3 of [13]), there is a sequence of Lipschitz maps $\{f_j\}_{j=1}^{\infty}$ with $f_j : \mathbb{R}^n \to \mathbb{R}^n$ and a nested sequence of subsets $\{G_j\}_{j=1}^{\infty}$ of B such that

$$G_j \subset \{x \in B : f_j(x) = f(x)\} \quad \text{and} \quad |B \setminus G_j| \to 0 \quad \text{as } j \to \infty. \tag{88}$$

Set $E = B \setminus \bigcup_{j=1}^{\infty} G_j$, so that $|E| = 0$. We assume $G_0 = \emptyset$. From the co-area formula (see 3.4.2 of [13]), we have

$$\begin{aligned}\int_{U\cap(G_{j+1}\setminus G_j)} |J(x,f_{j+1})|dx &= \int_{\mathbb{R}^n} \#\{x \in U \cap (G_{j+1} \setminus G_j) : f_{j+1}(x) = y\}dy \\ &= \int_{\mathbb{R}^n} \#\{x \in U \cap (G_{j+1} \setminus G_j) : f(x) = y\}dy.\end{aligned}$$

On the other hand, $J(x, f_{j+1}) = J(f, x)$ at the points of linear density one, along the coordinate axes in $G_{j+1} \setminus G_j$ and so almost everywhere in $G_{j+1} \setminus G_j$. The claim now follows by summing over j and using the monotone convergence theorem. □

It is the following corollary which is of importance to us.

Corollary 6.1. *Suppose that the function $f : \Omega \to \mathbb{R}^n$ belongs to the space $W^{1,1}_{loc}(\Omega, \mathbb{R}^n) \cap L^{\infty}_{loc}(\Omega)$ and that for some positive integer N and for some set A of measure zero $f : \Omega \setminus A \to \mathbb{R}^n$ is at most N-to-1. Then $J(x,f) \in L^1_{loc}(\Omega)$.*

In particular, $J(x,f) \in L^1_{loc}(\Omega)$ for each local homeomorphism $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$.

Proof. Let B be a ball compactly contained in Ω. By Theorem 6.1, there is a set E of measure zero such that

$$\int_B |J(x,f)|dx = \int_{B\setminus A} |J(x,f)|dx = \int_{\mathbb{R}^n} \#\{x \in (B \setminus A) \setminus E : f(x) = y\}dy.$$

The result now follows, as

$$\#\{x \in (B \setminus A) \setminus E : f(x) = y\} \leq N \quad \text{and} \quad \#\{x \in (B \setminus A) \setminus E : f(x) = y\} = 0$$

for almost every sufficiently large y since, by hypothesis, $f|B$ is bounded. □

Remark. Recall that a homeomorphism $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ of finite distortion $K \in L^\infty(\Omega)$ is called a quasiconformal mapping. It follows from Corollary 6.1 that the regularity assumption $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ can be relaxed to the assumption $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$. This result seems new in dimensions $n \geq 3$. In the plane, as mentioned above, it is a well-known result of Gehring and Lehto [16].

7 $W^{1,n}$-Regularity

We now concern ourselves with mappings $f : \Omega \to \mathbb{R}^n$, Ω a domain in $\mathbb{R}^n$, whose distortion function is bounded by a BMO-function; that is,

$$|Df(x)|^n \leq M(x)J(x,f) \quad \text{a.e. } \Omega \tag{89}$$

for some $M \in BMO(\mathbb{R}^n)$. As we have earlier remarked, the natural setting for this differential equation is in the Sobolev space $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. The reason for this is simply that in order to get interesting estimates and geometric information, one must use the formula for integration by parts

$$\begin{aligned} \int_\Omega \varphi(x)J(x,f)dx &= \int_\Omega \varphi \, df^1 \wedge df^2 \wedge \cdots \wedge df^n \\ &= \int_\Omega f^1 d\varphi \wedge df^2 \wedge \cdots \wedge df^n \end{aligned} \tag{90}$$

for test functions $\varphi \in C_0^\infty(\Omega)$. The right hand side of (90) converges under somewhat weaker integrability assumptions, defining the so-called *distributional Jacobian*. It is, of course, natural to try and impose minimal regularity conditions on f to ensure that this formula remains valid, viz. that the pointwise Jacobian $J(x,f) \in L^1_{loc}\Omega)$ coincides with the distributional Jacobian. One of the first things we wish to discuss is whether integrability of the Jacobian is sufficient. In light of Corollary 6.1, this situation arises for any $W^{1,1}(\Omega, \mathbb{R}^n)$ homeomorphism (or, more generally, if the mapping is N-to-1 for some fixed and finite N). On the other hand, if $J = J(x,f) \in L^1_{loc}(\Omega)$, by virtue of the differential inequality (89) and Hadamard's inequality we have

$$\frac{|Df|^n}{\log(3 + |Df|^n)} \leq \frac{MJ}{\log(3 + J)}. \tag{91}$$

Recalling the John–Nirenberg Lemma and applying (13) with $a = J$ and $b = \epsilon(n)M/\|M\|_{BMO}$, we find that

$$\frac{|Df|^n}{\log(3+|Df|^n)} \le \frac{\|M\|_{BMO}}{\epsilon(n)\log(3+J)}\left[J\log(1+J) - 1 + \exp\left(\frac{\epsilon(n)M}{\|M\|_{BMO}}\right)\right]$$
$$\le \frac{\|M\|_{BMO}}{\epsilon(n)}\left[J + \exp\left(\frac{\epsilon(n)M}{\|M\|_{BMO}}\right)\right].$$

This implies that $|Df|^n$ belongs to the Zygmund space $L^P_{loc}(\Omega)$, with $P(t) = t/\log(3+t)$. Conversely, the Jacobian of an orientation-preserving mapping $f : \Omega \to \mathbb{R}^n$ with $|Df|^n \in L^P_{loc}(\Omega)$ is locally integrable [35]. As a matter of fact, $J \in L\log\log L(G)$ for each G compactly contained in Ω [46]. More importantly, the distributional Jacobian coincides with the pointwise Jacobian [19], which means that the formula (90) holds, though $|Df|^n$ may not be integrable. The preceding discussions motivate the definitions we want to make.

Definition 7.1. A mapping $f : \Omega \to \mathbb{R}^n$ is said to have *BMO-bounded distortion* if

1. $f \in W^{1,P}_{loc}(\Omega, \mathbb{R}^n)$, with $P(t) = t^n/\log(3+t)$,
2. $|Df(x)|^n \le M(x)J(x,f)$ a.e. Ω,

where $1 \le M \in BMO(\mathbb{R}^n)$. Note that we have already seen that the first condition can be replaced by the requirement $J(\cdot, f) \in L^1_{loc}(\Omega)$. It is also worth recalling at this point that a function $K : \Omega \to [1,\infty)$ is controlled by $M \in BMO(\mathbb{R}^n)$ (that is, $K(x) \le M(x)$ for almost all $x \in \Omega$) if and only if $e^{K(x)} \in L^p(\Omega)$ for some $p > 0$. More precisely, the exponent p is inversely proportional to the norm $\|M\|_{BMO}$, [33, 36]. Therefore, "mappings of BMO-bounded distortion" are the same as "mappings of exponentially integrable distortion". One advantage of the latter terminology is that numerous qualitatively sharp results can be formulated in terms of the exponent p rather than in terms of the BMO-norm of K. For just such an example, see Conjecture 7.1 below.

It is also interesting to note that the class of mappings with BMO-bounded distortion is invariant under rescaling. That is, if $\lambda > 0$ and $f : \Omega \to \mathbb{R}^n$ has BMO-bounded distortion, then $f_\lambda(x) = f(x/\lambda) : \lambda\Omega \to \mathbb{R}^n$ has BMO-bounded distortion and admits a BMO distortion function of the same norm as that for f.

More generally, by Reimann's theorem [49] a quasiconformal change of variables preserves BMO-spaces. Accordingly, the class of mappings with BMO-bounded distortion is invariant under a quasiconformal change of variable. Indeed the norms involved are bounded by constants depending only on the distortion of the quasiconformal mapping.

As our earlier example shows, mappings of BMO-bounded distortion need not be in the space $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. However, closer examination of that example shows that the mapping in question is in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ if $\alpha > 1/n$. Another way of viewing this restriction is that the BMO-norm of M is small. Indeed, it appears that this is no accident.

Theorem 7.1 (***$W^{1,n}$-Regularity***). . *Let n be an even integer, $n = 2\ell$. Then there is a constant $\epsilon_{BMO} > 0$ depending only on the dimension n with the following properties. Suppose $f : \Omega \to \mathbb{R}^n$ is a mapping of BMO-bounded distortion with $\|M\|_{BMO} < \epsilon_{BMO}$. Then $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. Moreover, we have the uniform Caccioppoli-type estimate*

$$\int_\Omega |\psi Df|^n \le C(n) \int_\Omega (M\, |\nabla\psi|\, |f|)^n \tag{92}$$

for each $\psi \in C_0^\infty(\Omega)$.[1]

Proof. We refer the reader back to Section 4.4 for the even-dimensional theory of the Beurling–Ahlfors transform and the Hodge-* operator. One thing we want to recall is the even-dimensional Beltrami equation developed in [29] and, in particular, the Beltrami coefficient of a mapping of finite distortion. Given a mapping $f : \Omega \to \mathbb{R}^n$, there is an associated measurable function

$$\mu_f : \Omega \times \Lambda^- \to \Lambda^+ \tag{93}$$

called the *Beltrami coefficient* of f. This coefficient appears in the Beltrami equation, which we now want to describe. Given the $(\ell - 1)$-form

$$\eta(y) = y_1 dy_1 \wedge dy_2 \wedge \cdots \wedge dy_\ell - (-i)^\ell y_{\ell+1} dy_{\ell+1} \wedge dy_{\ell+2} \wedge \cdots \wedge dy_{2\ell}, \tag{94}$$

we note that $d^+\eta = 0$ and set

$$\alpha = f^*(\eta), \tag{95}$$

the pull-back of η via the mapping f. For the form η and those obtained from η by permuting the coordinates in the obvious fashion, we easily achieve the estimates

$$|\alpha| \le C\, |f|\, |Df|^{\ell-1}, \quad |d\alpha| \le C|Df|^\ell, \tag{96}$$

where C is a constant depending only on the dimension. Note that the coefficients of the ℓ-form $d\alpha = f^*(d\eta)$ are just linear combinations of the $\ell \times \ell$ minors of the differential matrix Df. Since $f \in W^{1,P}_{loc}(\Omega, \mathbb{R}^n)$, we see that

$$d\alpha \in \frac{L^2}{\log L}(G) \tag{97}$$

[1] Most recently [27] this result has been extended to all dimensions.

on every compact $G \subset \Omega$. The Beltrami equation then takes the form

$$d^+\alpha = \mu_f d^-\alpha. \tag{98}$$

We again refer to [29, 33] for details. We want to note the pointwise estimate

$$\frac{1+|\mu_f(x)|}{1-|\mu_f(x)|} \leq K(x) \quad \text{a.e. } \Omega \tag{99}$$

for any map of finite distortion and any distortion function $K(x)$ of f. (Actually, in [29, 33] we establish the above estimate for the sectional distortion function; however, it is an easy matter to see that any distortion function as defined at (80) provides an upper-bound on the sectional distortion.)

What we first want to achieve here is to show, with the above notation, that

$$d\alpha \in L^2_{loc}(\Omega). \tag{100}$$

Let φ be a test function $\varphi \in C_0^\infty(\Omega)$. We multiply equation (98) by φ and obtain

$$d^+(\varphi\alpha) = \mu_f d^-(\varphi\alpha) + \gamma, \tag{101}$$

where

$$\begin{aligned}\gamma &= [d^+(\varphi\alpha - \varphi d^+\alpha] - \mu_f[d^-(\varphi\alpha) - \varphi d^-\alpha]\\ &= (d\varphi \wedge \alpha)^+ - \mu_f(d\varphi \wedge \alpha)^-.\end{aligned}$$

We now understand equation (101) to hold in all of $\mathbb{R}^n$ and that $\mu = \mu_f$ is zero outside of Ω. In particular, the estimate

$$|\gamma| \leq C|d\varphi|\,|f|\,|Df|^{\ell-1} \tag{102}$$

holds in $\mathbb{R}^n$. Set

$$\omega = d^+(\varphi\alpha) \in \frac{L^2}{\log L}(\Omega). \tag{103}$$

From Section 4.4, we recall

$$S\omega = d^-(\varphi\alpha), \tag{104}$$

where S is the Beurling–Ahlfors transform. The Beltrami equation now takes the form

$$\omega - \mu S\omega = (I - \mu S)\omega = \gamma, \tag{105}$$

and $(I - \mu S)$ is an operator of Beltrami type. (It was, of course, this equation that motivated our earlier definition.)

Now observe that if we knew *a priori* that $\omega \in L^2(\mathbb{R}^n, \Lambda^+)$, we could deduce from Lemma 3.1 that

$$\|\omega\|_2 \leq 2\|M\gamma\|_2, \tag{106}$$

since it follows that $M\gamma \in L^2(\mathbb{R}^n, \Lambda^+)$ by virtue of inequality (102). In order to get around the possibility that $\omega \notin L^2(\mathbb{R}^n, \Lambda^+)$, we make use of the left inverse $\mathcal{F}$ of the Beltrami operator constructed in Section 3,

$$\mathcal{F} : L^2(\mathbb{R}^n, \Lambda^+) \to L^2(\mathbb{R}^n, \Lambda^+). \tag{107}$$

Hence we can uniquely solve the equation

$$\omega' - \mu S\omega' = \gamma \tag{108}$$

for some $\omega' \in L^2(\mathbb{R}^n, \Lambda^+)$. In fact, we take $\omega' = \mathcal{F}(M\gamma)$.

We now want to show that $\omega' = \omega$.

To this end, we solve the elliptic equation $d^+\beta = \omega'$ for $\beta \in \mathcal{W}^{1,2}(\mathbb{R}^n, \Lambda^{\ell-1})$. This simply means that the distributional gradient of β is square integrable on $\mathbb{R}^n$. As a matter of fact, $\nabla\beta$ can be expressed in terms of the Riesz transforms of ω'. Therefore,

$$\|\nabla\beta\|_2 \leq C(n)\|\omega'\|_2. \tag{109}$$

Of course, the form β is unique up to a closed form which we shall choose as the need arises. Then

$$\omega' - \omega = d^+\zeta, \qquad \zeta = \beta - \varphi\alpha. \tag{110}$$

Note that in general the $(\ell - 1)$-form ζ is complex-valued and, more importantly, that we have

$$d\zeta \in \frac{L^2}{\log L}(G, \Lambda^\ell), \quad d\zeta \in L^2(\mathbb{R}^n \setminus G, \Lambda^\ell), \tag{111}$$

where G is the support of φ. The two equations (105) and (108) yield a homogeneous Beltrami equation for ζ in $\mathbb{R}^n$, namely,

$$d^+\zeta = \mu\, d^-\zeta. \tag{112}$$

It will follow that $\omega' = \omega$ after we prove that $d\zeta = 0$.

After an elementary calculation via exterior algebra, equation (112) yields

$$\begin{aligned}\Re\{i^{\ell} d\zeta \wedge \overline{d\zeta}\} &= (|d^-\zeta|^2 - |d^+\zeta|^2)dx \\ &= \frac{1-|\mu|^2}{1+|\mu|^2}|d\zeta|^2\, dx.\end{aligned} \tag{113}$$

Formally, we could try to integrate both sides of this equation with the aid of Stokes' Theorem. We would find

$$\int_{\mathbb{R}^n} \frac{1-|\mu|^2}{1+|\mu|^2}|d\zeta|^2\, dx = 0, \tag{114}$$

which would be enough for us to conclude $d\zeta = 0$. The problem is that we do not yet know that $d\zeta \in L^2(\mathbb{R}^n)$, a fact needed for the application of Stokes' Theorem. We get around this technical obstruction by introducing additional factors in equation (113).

Consider the $(\ell-1)$-form $\zeta' = \chi\zeta$, where χ is an arbitrary real valued test function in $C_0^\infty(\mathbb{R}^n)$ with support B. We have, for every δ with $0 < \delta < 1$,

$$d\zeta' = \chi d\zeta + d\chi \wedge \zeta \in \frac{L^2}{\log L}(B) \subset L^{2-\delta}(B). \tag{115}$$

We now recall the following crucial inequality for the wedge product of exact differential forms:

$$\left|\int_B |d\zeta'|^{-\delta} d\zeta' \wedge \overline{d\zeta'}\right| \le C_n \delta \int_B |d\zeta'|^{2-\delta}. \tag{116}$$

See [23] for a proof of this estimate and other relevant material. Since $d\zeta' \in L^2 \log^{-1} L(B)$, we know that

$$\lim_{\delta \searrow 0} \delta \int_B |d\zeta'|^{2-\delta} = 0. \tag{117}$$

This fact was first observed and proved in [19, 35]. In particular, we have

$$\lim_{\delta \searrow 0} \int_B |d\zeta'|^{-\delta} d\zeta' \wedge \overline{d\zeta'} = 0. \tag{118}$$

Now we need to use the pointwise inequality

$$\begin{aligned}&\left||d\zeta'|^{-\delta} d\zeta' \wedge \overline{d\zeta'} - |\chi|^{2-\delta}|d\zeta|^{-\delta} d\zeta \wedge \overline{d\zeta}\right| \\ &= \left||\chi d\zeta + d\chi \wedge \zeta|^{-\delta}(\chi d\zeta + d\chi \wedge \zeta) \wedge (\chi\overline{d\zeta} + d\chi \wedge \overline{\zeta}) - |\chi d\zeta|^{-\delta}\chi d\zeta \wedge \chi\overline{d\zeta}\right| \\ &\le C\Big(|\chi d\zeta|^{1-\delta}|d\chi \wedge \zeta| + |d\chi \wedge \zeta|^{2-\delta}\Big),\end{aligned}$$

where C is some absolute constant. Accordingly, we have

$$\lim_{\delta\searrow 0}\left|\int |\chi|^{2-\delta}|d\zeta|^{-\delta}d\zeta\wedge\overline{d\zeta}\right| \leq C\int \left(|\chi d\zeta||\zeta\wedge d\chi| + |\zeta\wedge d\chi|^2\right). \tag{119}$$

It is important to realise that the integrand on the left hand side can be replaced by a non-negative expression. More precisely, in view of (113), we have

$$\lim_{\delta\searrow 0}\int_B \frac{1-|\mu|^2}{1+|\mu|^2}|\chi|^{2-\delta}|d\zeta|^{2-\delta} \leq C\int \left(|\chi d\zeta||\zeta\wedge d\chi| + |\zeta\wedge d\chi|^2\right). \tag{120}$$

By Fatou's Lemma, we can pass the limit through into the integral. Thus

$$\int \frac{1-|\mu|^2}{1+|\mu|^2}|\chi|^2|d\zeta|^2 \leq C\int \left(|\chi d\zeta||\zeta\wedge d\chi| + |\zeta\wedge d\chi|^2\right). \tag{121}$$

Our final step is to pass to the limit as we let our test function χ approach the constant function 1. Thus we choose R to be so large that $G \subset B(0,R)$. (recall G is the support of the initial test function φ) and $\chi \in C_0^\infty(B(0,2R))$ with

- $0 \leq \chi \leq 1$,
- $\chi(x) \equiv 1$ for all $x \in B(0,R)$,
- $|\nabla\chi(x)| \leq C(n)R^{-1}$.

Then $\zeta = \beta$ outside the ball $B(0,R)$, and we have

$$\begin{aligned}\int_{B(0,R)} \frac{1-|\mu|^2}{1+|\mu|^2}|d\zeta|^2 &\leq C\int_{B(0,2R)\setminus B(0,R)} \left(\frac{1}{R}|\zeta||d\zeta| + \frac{1}{R^2}|\zeta|^2\right)\\ &\leq C\int_{B(0,2R)\setminus B(0,R)} \left(\frac{|\beta|^2}{R^2} + |d\beta|^2\right).\end{aligned}$$

This inequality remains valid if we add any constant form to β.

Now the Poincaré inequality comes to our aid, and we find that

$$\int_{B(0,R)} \frac{1-|\mu|^2}{1+|\mu|^2}|d\zeta|^2 \leq C\int_{B(0,2R)\setminus B(0,R)} |\nabla\beta|^2; \tag{122}$$

this latter integral converges to zero as $R \to \infty$. Hence $d\zeta = 0$ in $\mathbb{R}^n$; and, as we have noted, we can conclude $\omega = \omega'$. Therefore,

1. $d^+(\varphi\alpha) = \omega \in L^2(\mathbb{R}^n, \Lambda^+)$,
2. $d^-(\varphi\alpha) = S\omega \in L^2(\mathbb{R}^n, \Lambda^-)$, and
3. $d(\varphi\alpha) \in L^2(\mathbb{R}^n, \Lambda^\ell)$.

We are now in a position to complete the proof of Theorem 7.1. Our first step is to show that

$$\int_{\mathbb{R}^n} \varphi^2(x)M(x)J(x,f)dx < \infty. \tag{123}$$

For this, we look more closely at the form

$$d\alpha = d[f^*(\eta)] = f^*(d\eta) = df^1 \wedge \cdots \wedge df^\ell - (-i)^\ell df^{\ell+1} \wedge \cdots \wedge df^{2\ell}. \tag{124}$$

A calculation using the exterior algebra shows us that

$$2(-1)^k J(x,f) = \begin{cases} i(d\alpha \wedge d\overline{\alpha}) & \text{if } \ell = 2k+1, \\ d\alpha \wedge d\alpha & \text{if } \ell = 2k. \end{cases} \tag{125}$$

After multiplying by φ^2, we write

$$2(-1)^k \varphi^2 J(x,f) = h - g, \tag{126}$$

where

$$h = \begin{cases} id(\varphi\alpha) \wedge d(\varphi\overline{\alpha}) & \text{if } \ell = 2k+1, \\ d(\varphi\alpha) \wedge d(\varphi\alpha) & \text{if } \ell = 2k; \end{cases} \tag{127}$$

$$g = \begin{cases} i\varphi d\varphi \wedge (\alpha \wedge d\overline{\alpha} - d\alpha \wedge \overline{\alpha}) & \text{if } \ell = 2k+1, \\ 2\varphi d\varphi \wedge \alpha \wedge d\alpha & \text{if } \ell = 2k. \end{cases} \tag{128}$$

It is essential to observe that h, being the wedge product of closed forms in $L^2(\mathbb{R}^n, \Lambda^\ell)$, belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. Consequently, by the BMO-$\mathcal{H}^1$ duality, we can give a meaning to the integral $\int Mh$ and moreover have the estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} Mh \right| &\leq C(n) \|M\|_{BMO} \|h\|_{\mathcal{H}^1} \\ &\leq C(n)\|M\|_{BMO} \int_{\mathbb{R}^n} |d(\varphi\alpha)|^2. \end{aligned} \tag{129}$$

On the other hand, we have the following pointwise estimate for g

$$\begin{aligned} |g| &\leq 2|\varphi||\nabla\varphi||\alpha||d\alpha| \\ &\leq C(n)|\varphi||\nabla\varphi||f||Df|^{n-1}. \end{aligned} \tag{130}$$

Hence

$$\int_{\mathbb{R}^n} M|g| \leq C(n) \int_{\mathbb{R}^n} M|\varphi||\nabla\varphi||f||Df|^{n-1} < \infty. \tag{131}$$

This shows that the function $\varphi^2 MJ(x,f)$ is at least integrable. So too is $\varphi^2|Df|^n$. Also, we have the estimate

$$\int \varphi^2|Df|^n \leq \int \varphi^2 MJ(\cdot,f)$$
$$\leq C(n)\left(\|M\|_{BMO}\int |d(\varphi\alpha)|^2 + \int M|\varphi||\nabla\varphi||f||Df|^{n-1}\right).$$

Next note that

$$|d(\varphi\alpha)|^2 = |\varphi d\alpha + d\varphi\wedge d\alpha|^2 \leq C(n)\left(\varphi^2|Df|^n + |d\varphi|^2|f|^2|Df|^{n-2}\right), \tag{132}$$

so that

$$\int \varphi^2|Df|^n \leq C(n)\epsilon_{BMO}\int \varphi^2|Df|^n + C(n)\epsilon_{BMO}\int |d\varphi|^2|f|^2|Df|^{n-2}$$
$$+ C(n)\int M|\varphi||\nabla\varphi||f||Df|^{n-1}. \tag{133}$$

Up to this point, the dimension-dependent constant $C(n)$ has been changing from line to line. Now we fix it and define

$$\epsilon_{BMO}(n) = \frac{1}{2C(n)}. \tag{134}$$

Thus the first term in the right hand side can be absorbed in the left hand side. We apply the resulting inequality to a test function of the form $\varphi = |\psi|^{n/2}$ to obtain

$$\int |\psi|^n|Df|^n \leq \frac{n^2}{4}\int |\psi|^{n-2}|Df|^{n-2}|f|^2|\nabla\psi|^2 + \frac{nC(n)}{2}\int M|\psi|^{n-1}|Df|^{n-1}|f||\nabla\psi|. \tag{135}$$

The remaining arguments necessary to complete the proof are standard. With the aid of the Young-type inequality

$$a^{n-2}b^2 + a^{n-1}b \leq \epsilon a^n + C_\epsilon(n)b^n, \tag{136}$$

which holds for every $\epsilon > 0$, we estimate the right hand side by

$$\epsilon\int |\psi|^n|Df|^n + C_\epsilon(n)\int M^n|f|^n|\nabla\psi|^n$$

and choose ϵ sufficiently small so as to be able to conclude

$$\int_{\mathbb{R}^n} |\psi|^n|Df|^n \leq C(n)\int_{\mathbb{R}^n} M^n|f|^n|\nabla\psi|^n, \tag{137}$$

which is what we were aiming for. □

In investigating the earlier examples and working through the proof of Theorem 7.1, we were led to make the following conjecture regarding the sharp constants. While our approach here may be refined, it seems a long way from this conjecture.

Conjecture 7.1. *A mapping of finite distortion which has a distortion function $K(x)$ satisfying*

$$\|K\|_{BMO} < \|\log|\cdot|^n\|_{BMO} \quad or \quad e^K \in L^p_{loc}(\Omega) \text{ with } p > 1 \tag{138}$$

lies in the Sobolev space $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$

8 Higher integrability

One of the central results of the theory of quasiregular mappings is the higher integrability of the differential first proved by Gehring [15] and developed by many others. Here we seek to achieve related results for mappings of finite distortion. One cannot expect quite the same sort of results as for the classical theory. Here we must content ourselves with only a very slight degree of improved integrability. The example of the radial stretching given in Section 5 defined by ρ at (84) shows that

$$|Df| \in L^n \log^\delta L(\Omega) \tag{139}$$

whenever $0 < \delta < n\alpha - 1$. This function is in no $L^p_{loc}(\Omega)$ space for any $p > n$, even though the BMO norm of the distortion function can be chosen to be arbitrarily small. In this paper, we show that $|Df| \in (L^n \log^\delta L)_{loc}(\Omega)$ for $\delta = 1$. We expected that in general δ may become arbitrarily large as the BMO norm of a distortion function tends to 0, and this was subsequently verified in every dimension in [27].

Theorem 8.1. *Let n be an even integer, $n = 2\ell$. Then there is a constant $\epsilon_{BMO} > 0$ depending only on the dimension n with the following properties. Suppose $f : \Omega \to \mathbb{R}^n$ is a mapping of BMO-bounded distortion with $\|M\|_{BMO} < \epsilon_{BMO}$. Then $f \in W^{1,P}_{loc}(\Omega, \mathbb{R}^n)$, where $P(t) = t^n \log(3 + t)$; that is,*

$$\int_G |Df|^n \log(3 + |Df|) < \infty \tag{140}$$

for every G compactly contained in Ω.[2]

Remarks. We first observe that since $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ and $J = J(x, f) \geq 0$, by a theorem of Müller [47] the Jacobian determinant J lies in the Zygmund class $L \log L(G)$. Also, we have the uniform bound

$$\|J\|_{L\log L(G)} \leq C(n, G) \int_\Omega |Df|^n. \tag{141}$$

[2] See [27] for all dimensions.

In particular, if f has bounded distortion we could immediately conclude (140). This fact was already observed by Gehring; see [24]. However, for mappings of BMO-bounded distortion, a stronger result is needed to be able to deduce (140). Namely, we need that $KJ \in L \log L(G)$; and this is what we really prove.

It is possible to give explicit estimates for the left hand side of (140). In fact, we can prove the Cacciopolli-type inequality

$$\|\psi Df\|_{L^n \log L(\Omega)} \leq C(n, M, \Omega)\|M|f||\nabla\psi|\|_{L^n \log L(\Omega)} \tag{142}$$

for every test function $\psi \in C_0^\infty(\Omega)$. However, we have settled on establishing (140) because of the rather messy technical details.

Proof for Theorem 8.1. Recall the Beltrami equation of Section 7

$$d^+\xi - \mu d^-\xi = \omega - \mu S\omega = \gamma, \tag{143}$$

where $\omega = d^+\xi$, $\xi = \varphi\alpha$ and α is the $(\ell - 1)$-form defined at (94) and (95), that is

$$\alpha = f^1 df^2 \wedge \cdots \wedge df^\ell - (-i)^\ell f^{\ell+1} df^{\ell+2} \wedge \cdots \wedge df^n,$$

$$d\alpha = df^1 \wedge df^2 \wedge \cdots \wedge df^\ell - (-i)^\ell df^{\ell+1} \wedge df^{\ell+2} \wedge \cdots \wedge df^n,$$

and

$$\gamma = (d\varphi \wedge \alpha)^+ - \mu(d\varphi \wedge \alpha)^-.$$

The test function $\varphi \in C_0^\infty(\Omega)$ is arbitrary. It suffices for our purposes to assume that Ω is the unit cube in $\mathbb{R}^n$. Our first goal is to establish the uniform bound

$$\|\omega\|_{L^2 \log L(\mathbb{R}^n)} \leq C_\Omega \|M\gamma\|_{L^2 \log L(\mathbb{R}^n)}, \tag{144}$$

where C_Ω depends on n and M and we use the subscript Ω to note the dependence of constants on Ω. Since $|d\xi|^2 = |d^+\xi|^2 + |d^-\xi|^2 \geq |\omega|^2$, it suffices to show that

$$\int_\Omega |d\xi|^2 \log\left(3 + \frac{|d\xi|^2}{\|d\xi\|_2^2}\right) \leq C_\Omega(n, M) \int_\Omega |M\gamma|^2 \log\left(3 + \frac{|M\gamma|^2}{\|M\gamma\|_2^2}\right), \tag{145}$$

where $C_\Omega(n, M) \leq C(n) M_\Omega$ and M_Ω is the L^1-mean of M over the cube Ω. The integrals above represent the $L \log L$ norms of $|d\xi|^2$ and $|M\gamma|^2$. Lemma 3.1 yields

$$\|d\xi\|_2^2 = 2\|d^+\xi\|_2^2 \leq 8\|M\gamma\|_2^2. \tag{146}$$

Note that $|M\gamma|^2 \leq C(n) M^2 |d\varphi|^2 |f|^2 |Df|^{n-2}$. Hence $M\gamma \in L^p(\mathbb{R}^n)$ for each p with $1 \leq p < 2n/(n-2)$ since we have already shown that $f \in W_{loc}^{1,n}(\Omega, \mathbb{R}^n)$.

Although it follows from above that $|M\gamma| \in L^2 \log L(\mathbb{R}^n)$, it is not immediately apparent that $|d\xi| \in L^2 \log L(\mathbb{R}^n)$. However, once the estimate (144) is established, it follows that $|d\xi| \in L^2 \log L(\mathbb{R}^n)$ by an approximation argument as in Theorem 3.1. To this effect, we introduce the sequence $\{\mu_j(x)\}$ defined at (30) for $x \in \Omega$ and set $\mu_j = 0$ outside of Ω. Next we solve (uniquely) the equation

$$\omega_j - \mu_j S\omega_j = \gamma$$

for $\omega_j \in L^{2\pm\epsilon_j}(\mathbb{R}^n)$, where ϵ_j is a sufficiently small positive number determined by the inequality

$$||S||_{2\pm\epsilon_j} < \frac{j}{j-1},$$

which ensures that the operator $I - \mu_j S$ is invertible in $L^{2\pm\epsilon_j}(\mathbb{R}^n)$. Now we show that ω_j converges weakly to ω in $L^2(\mathbb{R}^n)$. The uniform bounds

$$||\omega_j||_{L^2\log L(\mathbb{R}^n)} \leq C_\Omega(n, M)||M\gamma||_{L^2\log L(\mathbb{R}^n)}$$

are also available in this situation and follow in much the same way as those for (145), which we present carefully below. We leave the details of this approximation to the interested reader.

Next we assume, without loss of generality because of homogeneity with respect to ζ and γ at (143), that

$$||d\xi||_2 = 1. \tag{147}$$

Hence we have

$$1 \leq 8||M\gamma||_2^2 \leq 8\int_\Omega |M\gamma|^2 \log\left(3 + \frac{|M\gamma|^2}{||M\gamma||_2^2}\right). \tag{148}$$

Applying inequality (26) to $\omega = d^+\xi$, $S\omega = d^-\xi$ and $B\omega = \gamma$, we have

$$|d\xi|^2 = |d^+\xi|^2 + |d^-\xi|^2 \leq 2M(|d^-\xi|^2 - |d^+\xi|^2) + 4M^2|\gamma|^2. \tag{149}$$

Let $\widetilde{d\xi}$ denote the Hardy–Littlewood maximal function of $d\xi$. (Note that $\widetilde{d\xi}$ is not supported in Ω.) We multiply inequality (149) by $\log(3 + |\widetilde{d\xi}|^2)$ and integrate to obtain

$$\begin{aligned}\int_\Omega |d\xi|^2 \log(3 + |d\xi|^2) \leq& \int_\Omega |d\xi|^2 \log(3 + |\widetilde{d\xi}|^2) \\ \leq& 2\int_\Omega M(|d^-\xi|^2 - |d^+\xi|^2)\log(3 + |\widetilde{d\xi}|^2) \\ &+ 4\int_\Omega M^2|\gamma|^2 \log(3 + |\widetilde{d\xi}|^2).\end{aligned} \tag{150}$$

We begin with the easier estimate of the latter integral

$$\int_\Omega M^2|\gamma|^2 \log(3 + |\widetilde{d\xi}|^2) \leq 26\int_\Omega |M\gamma|^2 \log\left(3 + \frac{|M\gamma|^2}{||M\gamma||_2^2}\right). \tag{151}$$

To see this, note the elementary inequality

$$a\log(3+s) \le a\log(3+\frac{a}{\lambda}) + 25\lambda s. \tag{152}$$

Applying this to $a = |M\gamma|^2$, $s = |\widetilde{d\xi}|^2$ and $\lambda = \|M\gamma\|_2^2$ yields

$$\int_\Omega M^2|\gamma|^2\log(3+|\widetilde{d\xi}|^2) \le \int_\Omega |M\gamma|^2\log\left(3+\frac{|M\gamma|^2}{\|M\gamma\|_2^2}\right) + 25\|M\gamma\|_2^2\int_{\mathbf{R}^n}|\widetilde{d\xi}|^2. \tag{153}$$

Then, by the L^2-inequality for the maximal operator and in view of the normalisation (147) and inequality (148), we deduce (151).

Our task is now to estimate the first term in the right hand side of (150). We formulate this result as a lemma which is used again in the final step of the proof of Theorem 8.1.

Lemma 8.1. *With the notation above,*

$$\begin{aligned}\left|\int_\Omega M(|d^-\xi|^2 - |d^+\xi|^2)\log(3+|\widetilde{d\xi}|^2)\right| &\le C(n)\|M\|_{BMO}\int_\Omega |d\xi|^2\log(3+|d\xi|^2)\\ &+ C_\Omega(n,M)\int_\Omega |M\gamma|^2\log\left(3+\frac{|M\gamma|^2}{\|M\gamma\|_2^2}\right).\end{aligned} \tag{154}$$

Proof. For notational simplicity, set $b = \log(3+|\widetilde{d\xi}|^2)$. Observe that $b \in BMO(\mathbb{R}^n)$ and

$$|b - 2\log(1+|\widetilde{d\xi}|)| \le 2.$$

A result of Coifman and Rochberg [9] asserts that the logarithm of a maximal function belongs to BMO with norm bounded by a universal constant. Since $1+|\widetilde{d\xi}| = \widetilde{1+|d\xi|}$, we have

$$\|b\|_{BMO} \le C(n). \tag{155}$$

We now express the term $M(|d^-\xi|^2 - |d^+\xi|^2)b$ as a wedge product of $d\xi$ with $\overline{d\xi}$. Formula (113) yields

$$\begin{aligned}\int_\Omega M(|d^-\xi|^2 - |d^+\xi|^2)b &\le 4\Re\left\{ i^\ell\int_{\mathbf{R}^n} Mb\, d\xi\wedge\overline{d\xi}\right\}\\ &\le 4\left|\int_{\mathbf{R}^n} M(b d\xi\wedge\overline{d\xi})\right|.\end{aligned} \tag{156}$$

We now use the Hodge decomposition to split $b\, d\xi$ into exact and coexact components,

$$b\, d\xi = d\eta + d^*\beta. \tag{157}$$

The components $d\eta$ and $d^*\beta$ are found from the formulas

$$d\eta = \mathcal{R}(b\, d\xi), \quad d^*\beta = \mathcal{T}(b\, d\xi), \tag{158}$$

where $\mathcal{R} = \frac{1}{2}(I + \mathcal{S})$ and $\mathcal{T} = \frac{1}{2}(I - \mathcal{S})$ and $\mathcal{S}$ is the Beurling–Ahlfors transform acting on ℓ-forms in $\mathbb{R}^n$; see [29]. We have reduced the problem to considering the two integrals

$$\int_{\mathbb{R}^n} M d\eta \wedge \overline{d\xi} \tag{159}$$

and

$$\int_{\mathbb{R}^n} M d^*\beta \wedge \overline{d\xi}. \tag{160}$$

The first integral is estimated using the BMO-$\mathcal{H}^1$ duality and the $\mathcal{H}^1$-estimate as described in Lemma 2.2. Accordingly,

$$\begin{aligned} \int_{\mathbb{R}^n} M d\eta \wedge \overline{d\xi} &\le C(n) \|M\|_{BMO} \|d\eta \wedge \overline{d\xi}\|_{\mathcal{H}^1} \\ &\le C(n) \|M\|_{BMO} \|d\eta\|_{L^2 \log^{-1} L} \|d\xi\|_{L^2 \log L}. \end{aligned}$$

This is because the Zygmund spaces $L^2 \log^{-1} L$ and $L^2 \log L$ are dual to each other. Also, in view of the normalisation at (147), we have

$$\|d\xi\|^2_{L^2 \log L} \le C(n) \int_\Omega |d\xi|^2 \log(3 + |d\xi|^2). \tag{161}$$

As the operator $\mathcal{R}$ is bounded in $L^2 \log^{-1} L(\mathbb{R}^n)$ and $1 = \|d\xi\|_2 \le \|\widetilde{d\xi}\|_2 \le C(n)$, we can write

$$\begin{aligned} \|d\eta\|^2_{L^2 \log^{-1} L} &\le C(n) \|b\, d\xi\|^2_{L^2 \log^{-1} L} \\ &\le C(n) \|\, |\widetilde{d\xi}| \log(3 + |\widetilde{d\xi}|^2)\|^2_{L^2 \log^{-1} L} \\ &\le C(n) \|\widetilde{d\xi}\|^2_{L^2 \log L} \\ &\le C(n) \|d\xi\|^2_{L^2 \log L} \\ &\le C(n) \int_\Omega |d\xi|^2 \log(3 + |d\xi|^2). \end{aligned}$$

We have used here the maximal inequality in $L^2 \log L$. Therefore, we have the estimate

$$\left| \int_{\mathbb{R}^n} M d\eta \wedge \overline{d\xi} \right| \le C(n) \|M\|_{BMO} \int_{\mathbb{R}^n} |d\xi|^2 \log(3 + |d\xi|^2). \tag{162}$$

Now to prove the lemma, we need only find a suitable estimate for the integral at (160). Before making any estimates, we split the integrand of (160) as

$$M(d^*\beta \wedge \overline{d\xi}) = -M_\Omega(d\eta \wedge \overline{d\xi}) + M_\Omega(b\,d\xi \wedge \overline{d\xi}) + (M - M_\Omega)(d^*\beta \wedge \overline{d\xi}).$$

Next we integrate each term over $\mathbb{R}^n$. Stokes' theorem says that $\int d\eta \wedge \overline{d\xi} = 0$. Considerations similar to those used to establish (159) also apply to the second term, giving

$$\begin{aligned}\left|\int b\,d\xi \wedge \overline{d\xi}\right| &\le C(n)\|b\|_{BMO}\|d\xi \wedge \overline{d\xi}\|_{\mathcal{H}^1}\\ &\le C(n)\|d\xi\|_2^2\\ &\le C(n)\|M\gamma\|^2_{L^2\log L}.\end{aligned}$$

This is because of (155) and the estimate $\|d\xi \wedge \overline{d\xi}\| \le C(n)\|d\xi\|_2^2$ given by Lemma 2.2. Hence

$$M_\Omega\left|\int b\,d\xi \wedge \overline{d\xi}\right| \le C(n)M_\Omega\|M\gamma\|^2_{L^2\log L}. \tag{163}$$

The last term of the decomposition needs somewhat greater care in its handling. Using the inequality at (13), we can write

$$\begin{aligned}|M - M_\Omega||d^*\beta \wedge \overline{d\xi}| \le &\frac{\|M\|_{BMO}}{\epsilon(n)}|d^*\beta \wedge \overline{d\xi}|\log\left(3 + \frac{|d^*\beta \wedge \overline{d\xi}|}{\|d^*\beta \wedge \overline{d\xi}\|_{L\log L}}\right)\\ &+ \frac{\|M\|_{BMO}}{\epsilon(n)}\|d^*\beta \wedge \overline{d\xi}\|_{L\log L}\exp\left(\frac{\epsilon(M - M_\Omega)}{\|M\|_{BMO}}\right),\end{aligned}$$

where $\epsilon = \epsilon(n)$ is the constant in the John–Nirenberg Lemma. Upon integration, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}|M - M_\Omega||d^*\beta \wedge \overline{d\xi}| &\le C(n)\|M\|_{BMO}\|d^*\beta \wedge \overline{d\xi}\|_{L\log L}\\ &\le C(n)\|M\|_{BMO}\|d^*\beta\|_{L^2\log L}\|d\xi\|_{L^2\log L},\end{aligned} \tag{164}$$

where we have used the Hölder inequality $\|ab\|_{L\log L} \le \|a\|_{L^2\log L}\|b\|_{L^2\log L}$ in Zygmund spaces. We now establish the estimate

$$\|d^*\beta\|_{L^2\log L} \le C(n)\|d\xi\|_{L^2\log L}. \tag{165}$$

(Recall that $d^*\beta = \mathcal{T}(b\,d\xi)$.) Since $\mathcal{T}$ vanishes on exact forms, we can express $d^*\beta$ as

$$d^*\beta = \mathcal{T}(b\,d\xi) = (\mathcal{T}b - b\mathcal{T})(d\xi),$$

which is the commutator of $\mathcal{T}$ with the multiplication by $b \in BMO(\mathbb{R}^n)$. Coifman, Rochberg and Weiss [10] have shown that $(\mathcal{T}b - b\mathcal{T}) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded operator for all $1 < p < \infty$. As a matter of fact, its norm is proportional to $\|b\|_{BMO}$. Using a standard interpolation argument in Orlicz spaces [48, 36], we find that $\mathcal{T}b - b\mathcal{T}$ is bounded in $L^2 \log L$. Hence (155) shows that

$$\|(\mathcal{T}b - b\mathcal{T})(d\xi)\|_{L^2 \log L} \leq C(n)\|d\xi\|_{L^2 \log L}, \tag{166}$$

proving (165). Thus (164) follows. Finally, returning to (160), we take into account the above estimates and find

$$\begin{aligned} \left|\int_{\mathbf{R}^n} M d^*\beta \wedge \overline{d\xi}\right| &\leq C(n)\|M\|_{BMO}\|d\xi\|^2_{L^2 \log L} + C_\Omega(n, M)\|d\xi\|_2^2 \\ &\leq C(n)\|M\|_{BMO} \int_\Omega |d\xi|^2 \log(3 + |d\xi|^2) \\ &\quad + C_\Omega(n, M) \int_\Omega |M\gamma|^2 \log\left(3 + \frac{|M\gamma|^2}{\|M\gamma\|_2^2}\right). \end{aligned} \tag{167}$$

Now, when put together, the estimates (162) and (167) imply Lemma 8.1. □

We are now in a position to establish the inequality at (145).

Combining Lemma 8.1 with (150) and (151), we have

$$\begin{aligned} \int_\Omega |d\xi|^2 \log\left(3 + \frac{|d\xi|^2}{\|d\xi\|_2^2}\right) &\leq C(n)\epsilon_{BMO} \int_\Omega |d\xi|^2 \log\left(3 + \frac{|d\xi|^2}{\|d\xi\|_2^2}\right) \\ &\quad + C_\Omega(n, M) \int_\Omega |M\gamma|^2 \log\left(3 + \frac{|M\gamma|^2}{\|M\gamma\|_2^2}\right). \end{aligned}$$

We now chose $\epsilon_{BMO} = 1/2C(n)$ and, after rearranging the inequality, obtain (145).

Having disposed of inequality (145) and Lemma 8.1, we can now return to the proof of Theorem 8.1.

We recall the distortion inequality

$$|Df(x)|^n \leq K(x)J(x, f),$$

which implies

$$|Df|^n \log(3 + |Df|) \leq KJ\log(4 + KJ) \leq 2MJ\log(3 + J) + K^2.$$

Since K^2 is integrable on Ω, we need only establish that $MJ\log(3 + J) \in L^1_{loc}(\Omega)$. We have the $(\ell - 1)$-form $\xi = \varphi\alpha$ with exterior derivative $d\xi = \varphi d\alpha + d\varphi \wedge \alpha$, which we have already shown belongs to $L^2 \log L(\mathbb{R}^n)$. More importantly, by Lemma 8.1, we have

$$\left|\int_\Omega M(|d^-\xi|^2 - |d^+\xi|^2)\log(3+|\widetilde{d\xi}|^2)\right| < \infty. \tag{168}$$

We really want the finiteness of this integral with $d\xi$ in place of $\widetilde{d\xi}$. Notice that

$$\begin{aligned}\Big|(|d^-\xi|^2 - |d^+\xi|^2)\log(3+|\widetilde{d\xi}|^2) &- (|d^-\xi|^2 - |d^+\xi|^2)\log(3+|d\xi|^2)\Big| \\ &\le |d\xi|^2 \log\left(\frac{3+|\widetilde{d\xi}|^2}{3+|d\xi|^2}\right) \\ &\le |d\xi|^2 \log \frac{|\widetilde{d\xi}|^2}{|d\xi|^2} \\ &\le |\widetilde{d\xi}|^2, \end{aligned} \tag{169}$$

where we have used the simple fact that $\log(t) < t$ for $t > 0$. Then, by (13), we have

$$\int M|\widetilde{d\xi}|^2 \le \frac{\|M\|_{BMO}}{\epsilon(n)}\int |\widetilde{d\xi}|^2\log(3+|\widetilde{d\xi}|^2) + \frac{\|M\|_{BMO}}{\epsilon(n)}\int \exp\left(\frac{\epsilon(n)M}{\|M\|_{BMO}}\right). \tag{170}$$

The right hand side of inequality (170) is finite by the John–Nirenberg lemma and the maximal theorem applied to $d\xi \in L^2\log L$. So we have

$$\left|\int_\Omega M(|d^-\xi|^2 - |d^+\xi|^2)\log(3+|d\xi|^2)\right| < \infty. \tag{171}$$

This inequality remains valid if we replace $d\xi$ by $\varphi d\alpha$, which is regarded as the principal part of $d\xi = \varphi d\alpha + d\varphi\wedge\alpha$. Indeed, the estimate $|d\varphi\wedge\alpha| \le C(n)|\nabla\varphi||f||Df|^{\ell-1}$ shows that $d\varphi\wedge\alpha$ lies in $L^p(\mathbb{R}^n)$ for every $2 < p < 2\ell/(\ell-1)$, and so does not affect the convergence of the integral at (171). Thus

$$\int_\Omega \varphi^2 M(|d^-\alpha|^2 - |d^+\alpha|^2)\log(3+\varphi^2|d\alpha|^2) < \infty. \tag{172}$$

More precisely, the difference between the integrands at (171) and (172) can be estimated by the function

$$M|\alpha\wedge d\varphi|(|d\xi| + |\alpha\wedge d\varphi|)\log(3+|d\xi|+|\alpha\wedge d\varphi|),$$

which is clearly integrable.

From (124) and (73), we find the identity

$$(|d^-\alpha|^2 - |d^+\alpha|^2)dx = -i^\ell\langle d\alpha, *d\alpha\rangle dx = -i^\ell d\alpha\wedge d\bar{\alpha} = 2J(x,f)dx.$$

This gives

$$\int_{\Omega} \varphi^2 MJ \log(3 + \varphi^2 |d\alpha|^2) < \infty. \tag{173}$$

It remains only to observe that $|d\alpha|^2 \geq 2J(\cdot, f)$ and therefore $MJ \log(3 + J)$ is locally integrable on Ω. In conclusion,

$$|Df|^n \log(3 + |Df|) \in L^1_{loc}(\Omega).$$

This completes the proof of Theorem 8.1.

In order to achieve the estimate at (142), one has to examine the inequality (150) and the proof of Lemma 8.1 carefully. This involves elementary but rather lengthy and messy calculations. We leave it to the interested reader. □

It is worth mentioning that general mappings of BMO-bounded distortion, regardless of the size of $||M||_{BMO}$, also admit a slight improvement of integrability. This is due to the higher integrability properties of the Jacobian determinant of orientation preserving mappings as developed in [20]. For example, if $f \in W^{1,P}_{loc}(\Omega, \mathbb{R}^n)$ with

$$P(t) = P_\beta(t) = \frac{t^n \log\log^\beta(8 + t)}{\log(3 + t)}, \qquad \beta \in \mathbb{Z},$$

and if $J(x, f) \geq 0$, then $J(\cdot, f) \in L^Q_{loc}(\Omega)$ with

$$Q(t) = Q_{\beta+1}(t) = t \log\log^{1+\beta}(8 + t).$$

The case $\beta = -1$ was treated separately in [57]. On the other hand, it follows from the inequality (13) that if $J \in L^{Q_{\beta+1}}_{loc}(\Omega)$ and $|Df|^n \leq MJ$, then $f \in W^{1,P_{\beta+1}}_{loc}(\Omega, \mathbb{R}^n)$. Using mathematical induction, we deduce the following theorem.

Theorem 8.2. *In every dimension, each mapping of BMO-bounded distortion $f : \Omega \to \mathbb{R}^n$ belongs to the Orlicz–Sobolev space $W^{1,P}_{loc}(\Omega, \mathbb{R}^n)$ with*

$$P(t) = \frac{t^n \log\log^\beta(8 + t)}{\log(3 + t)} \qquad \text{all } \beta = 0, 1, 2, \ldots.$$

Remark. One interesting inference from the above concerns Definition 7.1. Instead of Condition 1, we could assume less to begin with. Namely, it suffices to assume that $f \in W^{1,R}_{loc}(\Omega, \mathbb{R}^n)$ for some R of the form

$$R(t) = \frac{t^n}{\log(3 + t) \log\log^m(8 + t)}, \quad m \geq 0.$$

When combined with the distortion inequality, we obtain Condition 1.

9 Topological and analytical properties

It will not have escaped the attention of any reader that there are topological and analytical consequences of the results we have established above for mappings of finite distortion in analogy with those for quasiregular mappings. If f is quasiregular, that is if $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ has bounded distortion, then f is Hölder continuous, differentiable almost everywhere and discrete and open in Ω (or constant). These results are of fundamental importance in the theory of quasiregular mappings, and it is our aim in this section to verify the analogous results for mappings of BMO-bounded distortion. We leave it as an exercise for the interested reader to write down the analogous compactness (or normal families) type results.

We begin with the following theorem.

Theorem 9.1. *Suppose that n is even and f is a mapping of BMO-bounded distortion in $\Omega \subset \mathbb{R}^n$ with distortion function $M(x)$ satisfying $\|M\|_{BMO} \le \epsilon_{BMO}(n)$. Then f is continuous, differentiable almost everywhere, and open and discrete in Ω (or constant).*

Remark. It is shown in [37] that one can obtain openness and discreteness under weaker assumptions on the distortion function K than boundedness: for $n = 2$ a nonconstant mapping of the class $W^{1,2}_{loc}(\Omega, \mathbb{R}^2)$, it suffices that $K \in L^1_{loc}(\Omega)$. In higher dimensions, it is known that $K \in L^p_{loc}(\Omega)$ for some $p > n - 1$ suffices to imply that nonconstant mappings in the Sobolev class $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ are discrete and open. This was shown by Heinonen and Koskela [21] under an additional topological hypothesis, which was later relaxed by Manfredi and Villamor [42]. The case $p = n - 1$ still remains open for $n \ge 3$. Our Theorem 9.1 seems to be the first of the type where the $W^{1,n}_{loc}$ assumption is also relaxed.

Proof. Each mapping f of finite distortion in the Sobolev class $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ is continuous and differentiable almost everywhere by results of Goldstein and Vodop'yanov [17] and Resetnyak; see [21] for a discussion. Thus, the first claim follows from Theorem 7.1. The second follows from Theorem 7.1 and [42]. □

From the argument of Goldstein and Vodop'yanov, the modulus of continuity achieved is no better than

$$|f(x) - f(y)| \le C_n \|Df\|_{L^n(B)} \left(\log \frac{\operatorname{dist}(x, \partial\Omega)}{2|x - y|}\right)^{-1/n}$$

for $|x - y| < \operatorname{dist}(x, \partial\Omega)/2$. However, in the setting of Theorem 8.1, we gain additional integrability of the differential:

$$\int_B |Df(x)|^n \log(3 + |Df(x)|)dx < \infty \tag{174}$$

whenever B is compactly contained in Ω. By itself, this inequality does not even guarantee continuity; take, for instance, $f(x) = \log\log 1/|x|$ near the origin. However, in our setting we obtain a better modulus of continuity estimate than described above. Note that our earlier examples of Section 5 show that Hölder continuity is not to be expected. We begin with a lemma.

Lemma 9.1. *Let $B = B(0,R)$ denote a ball in $\mathbb{R}^n$ and let $u \in W^{1,Q}(B) \cap C(B)$, where $Q(t) = t^n \log^p(3+t)$ for some $p \geq 0$. Suppose in addition that*

$$|u(x) - u(0)| \leq \max_{|z|=|w|=r} |u(z) - u(w)| \tag{175}$$

whenever $|x| \leq r$ and $0 < r < R$. Then

$$|u(x) - u(0)|^n \leq \frac{C(n,p)}{|\log^{p+1}(R/|x|)} \int_B |\nabla u|^n \log^p \left(3 + \frac{|\nabla u|^n}{|B|^{-1} \int |\nabla u|^n}\right). \tag{176}$$

Proof. Because of homogeneity, we may assume that $R = 1$ and

$$1 = \int_B |\nabla u|^n \leq \int_B |\nabla u|^n \log^p \left(3 + \frac{|\nabla u|^n}{|B|^{-1} \int |\nabla u|^n}\right). \tag{177}$$

Fix $x \in B$ and put $a = |u(x) - u(0)|$. The Sobolev Embedding Theorem on spheres implies that for almost every r

$$|u(z) - u(w)|^n \leq C_n\, r \int_{S(r)} |\nabla u|^n \tag{178}$$

whenever $|z| = |w| = r$. By our assumption on u, we conclude that for almost every r,

$$\frac{a^n}{r} \leq C_n \int_{S(r)} |\nabla u|^n \tag{179}$$

for all $r \geq |x|$. Let $E = \{y \in S(r) : |\nabla u(y)| \leq r^{(1-n)/n}\}$. Then

$$\begin{aligned} \int_{S(r)} |\nabla u|^n &\leq r^{1-n}\sigma(E) + \int_{S(r)\setminus E} |\nabla u|^n \\ &\leq C_n + \log^{-p}(3 + r^{1-n}) \int_{S(r)} |\nabla u|^n \log^p(3 + |\nabla u|^n). \end{aligned}$$

Combining these last two inequalities and integrating with respect to r, we obtain

$$\begin{aligned} a^n \int_{|x|}^1 \frac{\log^p(3+r^{1-n})}{r} dr &\leq C_n \int_{|x|}^1 \log^p(3 + r^{1-n}) dr + \int_B |\nabla u|^n \log^p(3 + |\nabla u|^n) \\ &\leq C(n,p) + \int_B |\nabla u|^n \log^p(3 + |\nabla u|^n) \\ &\leq (1 + C(n,p)) \int_B |\nabla u|^n \log^p(3 + |\nabla u|^n) \\ &\leq C'(n,p) \int_B |\nabla u|^n \log^p \left(3 + \frac{|\nabla u|^n}{|B|^{-1} \int |\nabla u|^n}\right). \end{aligned}$$

Since

$$\int_{|x|}^{1} \frac{\log^p(3 + r^{1-n})}{r} dr \geq c(n,p) \log^{p+1} \frac{1}{|x|},$$

the lemma follows. □

Theorem 9.2. *Suppose that n is even and f is a mapping of BMO-bounded distortion in a ball $\Omega = B(a, 3R) \subset \mathbb{R}^n$ with a distortion function $M(x)$ whose BMO norm satisfies $\|M\|_{BMO} \leq \epsilon_{BMO}(n)$. Then*

$$|f(x) - f(y)|^n \leq C(n) \int_\Omega |Df|^n \log\left(3 + \frac{|Df|^n}{\||Df|^n|_\Omega}\right) \log^{-2}\left(\frac{|x-y|}{2R}\right) \tag{180}$$

whenever $x, y \in B(a, R)$.

Since f is a continuous mapping of finite distortion in the Sobolev class $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$, we may apply a result of Goldstein and Vodopyanov [17]. The coordinate functions f_j of f are monotone: they satisfy both the minimum and maximum principles in Ω. In particular, they satisfy

$$\max_{y \in B} |u(x) - u(y)| \leq \max_{x,y \in S(r)} |u(x) - u(y)| \tag{181}$$

whenever B is compactly contained in Ω. Thus Lemma 9.1 implies the desired result. □

We finish this section by showing that L^p-integrability, for some $p > n - 1$, of a distortion function of a homeomorphism of finite distortion guarantees differentiability almost everywhere.

Theorem 9.3. *Let $f : \Omega \to \mathbb{R}^n$ be a homeomorphism of finite distortion with a distortion function $K \in L^p_{loc}(\Omega)$, $p > n - 1$. Then f is differentiable almost everywhere in Ω.*

Proof. By a result of Väisälä (see [56] or Section 4.4 of [51]), each homeomorphism of the Sobolev class $W^{1,q}_{loc}(\Omega, \mathbb{R}^n)$ is differentiable almost everywhere, provided $q > n-1$. Thus it suffices to show that $|Df| \in L^q_{loc}(\Omega)$ for some $q > n-1$. Using Hölder's inequality and Corollary 6.1, we have

$$\begin{aligned}\int_B |Df(x)|^q dx &\leq \int_B K(x)^{q/n} J(x,f)^{q/n} dx \\ &\leq \left(\int_B K(x)^{q/(n-q)} dx\right)^{1-q/n} \left(\int_B |J(x,f)| dx\right)^{q/n} < \infty\end{aligned}$$

whenever B is compactly contained in $\emptyset mega$ and $q/(n-q) = p$. The claim follows, as $q > n-1$ above if and only if $p > n-1$. □

10 Singularities

Another important family of results in the geometric function theory concerns the types of removable singularities. Combining the results of [29] and [22], one knows that sets of Hausdorff dimension 0 are always removable for bounded quasiregular mappings. Again, we do not achieve this result for mappings with small BMO distortion; in fact, we do not know if this holds or not. However, we get close to this result. We first define the sort of sets we can prove are removable.

Definition 10.1. A compact set $E \subset \mathbb{R}^n$ is said to have zero $L^n \log L$-capacity if there is a bounded open set Ω containing E such that

$$\inf \int_\Omega |\nabla \chi|^n \log(3 + |\nabla \chi|) = 0, \tag{182}$$

where the infimum is taken with respect to all functions $\chi \in C_0^\infty(\Omega)$ which are equal to 1 on some neighbourhood of the set E.

There are many elementary properties of this capacity which follow as in the classical setting for the more usual n-capacity. We do not dwell on these, but use them as necessary. We do note that points have zero $L^n \log L$-capacity, as do certain Cantor sets, and that zero $L^n \log L$-capacity implies Hausdorff dimension zero. A closed set has zero $L^n \log L$-capacity if and only if it can be written as the countable union of compact sets of zero $L^n \log L$-capacity. Our main result in this section is

Theorem 10.1 ***(Removability).*** *Let n be an even integer. There exists a number $\epsilon = \epsilon_{BMO}(n) > 0$ with the following properties. Let $E \subset \mathbb{R}^n$ be a closed set of zero $L^n \log L$-capacity and*

$$f : \Omega \setminus E \to \mathbb{R}^n \tag{183}$$

be a bounded BMO-bounded distortion mapping with distortion function M satisfying $\|M\|_{BMO(\mathbb{R}^n)} < \epsilon$. Then f extends to a BMO-bounded distortion mapping of Ω.

Proof. Since E has zero n-capacity, it suffices to show that $|Df| \in L^n_{loc}(\Omega)$. Let $\eta \in C_0^\infty(\Omega)$ be an arbitrary test function. We denote by E' the intersection of E with the support of η. There exists a sequence of functions $\{\chi_j\}_{j=1}^\infty$ such that for each j we have

1. $\chi_j \in C_0^\infty(\Omega)$,
2. $0 \leq \chi_j \leq 1$,
3. $\chi_j = 1$ on some neighbourhood U_j of E',
4. $\lim_{j\to\infty} \chi_j(x) = 0$ for almost all $x \in \mathbb{R}^n$,
5. $\lim_{j\to\infty} \int_\Omega |\nabla\chi_j|^n \log(3 + |\nabla\chi_j|) = 0$.

We set

$$\varphi_j = (1 - \chi_j)\eta \in C_0^\infty(\Omega \setminus E'). \tag{184}$$

The distortion inequality $|Df|^n \leq MJ$ and (13) imply

$$\begin{aligned} \frac{|\varphi_j Df|^n}{\log(3 + |\varphi_j Df|^n)} &\leq \frac{M|\varphi_j|^n J}{\log(3 + |\varphi_j|^n J)} \\ &\leq \frac{\|M\|_{BMO}}{\epsilon(n)} \left[|\varphi_j|^n J + \exp\left(\frac{\epsilon(n)M}{\|M\|_{BMO}} \right) \right], \end{aligned} \tag{185}$$

where $\epsilon(n)$ is the constant in the John–Nirenberg Lemma. We now integrate both sides of this inequality over Ω and use integration by parts on the term containing the Jacobian determinant to obtain

$$\begin{aligned} \int_\Omega \frac{|\varphi_j Df|^n}{\log(3 + |\varphi_j Df|^n)} \leq & C(n)\|M\|_{BMO} \int_\Omega |\varphi_j|^{n-1} |\nabla\varphi_j||f||Df|^{n-1} \\ & + C_\Omega(n, M). \end{aligned} \tag{186}$$

Next we use the inequality for non-negative numbers a and b

$$ab^{n-1} \leq C(n) \left[\frac{b^n}{\log(3 + b^n)} + a^n \log(3 + a) \right], \tag{187}$$

which yields

$$\begin{aligned} \int_\Omega \frac{|\varphi_j Df|^n}{\log(3 + |\varphi_j Df|^n)} \leq & C(n)\|M\|_{BMO} \int_\Omega \frac{|\varphi_j Df|^n}{\log(3 + |\varphi_j Df|^n)} \\ & + C(n)\|M\|_{BMO} \int_\Omega |\nabla\varphi_j|^n |f| \log(3 + |f||\nabla\varphi_j|) \\ & + C_\Omega(n, M). \end{aligned}$$

Then if the $\|M\|_{BMO}$ is sufficiently small, the first part of the right hand side is absorbed in the left and we have

$$\int_\Omega \frac{|\varphi_j Df|^n}{\log(3 + |\varphi_j Df|^n)} \leq C(n) \int_\Omega |\nabla\varphi_j|^n |f| \log(3 + |f||\nabla\varphi_j|) + C_\Omega(n, M). \tag{188}$$

Recall that $|f|$ is assumed bounded in Ω and also that $\nabla\varphi_j = (1-\chi_j)\nabla\eta - \eta\nabla\chi_j$. It follows from the conditions defining χ_j that we can pass to the limit (as $j \to \infty$) in equation (189) to obtain

$$\int_\Omega \frac{|\eta Df|^n}{\log(3+|\eta Df|^n)} \leq C(n)\int_\Omega |\nabla\eta|^n|f|\log(3+|f||\nabla\eta|) + C_\Omega(n,M), \tag{189}$$

so that $|Df|^n \in L\log L(G)$ for G any compact subset of Ω on which η does not vanish. Since $\eta \in C^{\infty})_0(\Omega)$ was arbitrary, by Theorem 7.1 we now have $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. It is clear that f satisfies the condition given in Definition 7.1 in the entire domain Ω. This proves the removability theorem. □

We actually believe that sets of conformal capacity zero are removable for bounded mappings with a distortion function whose BMO norm is small enough. However, this seems to require more subtle estimates in the Zygmund spaces $L^n\log^\alpha L$ for $\alpha < -1$ together with additional complications. Therefore, we do not pursue the matter here.

Remark. In recent joint work with Kari Astala [4], we have established that *planar* sets of conformal capacity zero are removable for bounded BMO-bounded distortion mappings with a distortion function whose BMO norm is small enough. This is a planar result, since we go through the Stoilow factorisation theorem. We also show that the removability result above is qualitatively sharp in all (even) dimensions. Other surprising phenomena are observed. There are planar and space domains which support bounded mappings of (arbitrarily) small BMO-distortion and yet admit no bounded quasiregular mappings. As noted there, this suggests that quasiregular mappings play a role within the class of mappings of BMO-bounded distortion similar to the role played by L^∞ inside BMO.

References

[1] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, Princeton, 1966; reprinted by Wadsworth Inc., Belmont, 1987.

[2] L. V. Ahlfors and A. Beurling, *Conformal invariants and function theoretic null sets*, Acta Math. **83** (1950), 101–129.

[3] S. S. Antman, *Fundamental mathematical problems in the theory of nonlinear elasticity*, in *Numerical Solution of Partial Differential Equations, III*, Academic Press, New York, 1976, pp. 35–54.

[4] K. Astala, P. Koskela, T. Iwaniec and G. Martin, *Mappings of BMO-bounded distortion II*, Math. Ann. **317** (2000), 703–726.

[5] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1977), 337–403.

[6] M. A. Brakalova and J. A. Jenkins, *On solutions of the Beltrami equation*, J. Analyse Math. **76** (1998), 67–92.

[7] R. R. Coifman and L. Grafkos, *Hardy space estimates for multilinear operators, I*, Rev. Mat. Iberoamericana **8** (1992), 45–67.

[8] R. R. Coifman, P. L Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993), 247–286.

[9] R. R. Coifman and R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc. **79** (1980), 249–254.

[10] R. R. Coifman, R. Rochberg and G. Weiss *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1978), 569–645.

[11] G. David, *Solutions de l'equation de Beltrami avec* $\|\mu\|_\infty = 1$, Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1988), 25–70.

[12] S. Donaldson and D. Sullivan, *Quasiconformal 4-manifolds*, Acta Math. **163** (1989), 181–252.

[13] L. C. Evans and R. F Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.

[14] C. Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. **77** (1971), 587–588.

[15] F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasiconformal mapping*. Acta Math. **130** (1973), 265–277.

[16] F. W. Gehring and O. Lehto, *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. A I **272** (1959), 3–9.

[17] V. M. Goldstein and S. K. Vodop'yanov, *Quasiconformal mappings and spaces of functions with generalised first derivatives*, Sb. Mat. Z. **17** (1976), 515–531.

[18] L. Grafkos, *Hardy space estimates for multilinear operators, II*, Rev. Mat. Iberoamericana **8** (1992), 69–92.

[19] L. Greco, *A remark on the equality* $\det Df = \mathrm{Det}\, Df$, Differential Integral Equations **6** (1993), 1089–1100.

[20] L. Greco, T. Iwaniec and G. Moscariello, *Limits of the improved integrability of the volume forms*, Indiana Univ. Math. J. **44** (1995), 305–339.

[21] J. Heinonen and P. Koskela *Sobolev mappings with integrable distortion*, Arch. Rational Mech. Anal. **125** (1993), 81–97.

[22] T. Iwaniec, *p-Harmonic tensors and quasiregular mappings*, Ann. of Math. **136** (1992), 651–685.

[23] T. Iwaniec, *Integrability theory of Jacobians*, Lipschitz Lectures, Preprint #36, Sonderforschungsbereich **256**, Bonn, 1995.

[24] T. Iwaniec, *The Gehring Lemma*, in *Quasiconformal Mappings and Analysis*, A collection of papers honoring F. W. Gehring (P. Duren, J. Heinonen, B. Osgood and B. Palka, eds.), Springer-Verlag, Berlin, 1998, pp. 181–204.

[25] T. Iwaniec, P. Koskela and J. Onninen, *Mappings of finite distortion: Monotonicity and continuity*, Invent. Math. **144** (2001), 507–531.

[26] T. Iwaniec, P. Koskela and J. Onninen, *Mappings of finite distortion: Compactness*, Ann. Acad. Sci. Fenn. Ser. A I Math., to appear.

[27] T. Iwaniec, P. Koskela, G. J. Martin and C. Sbordone, *Mappings of finite distortion: $L^n \log^\alpha L$-integrability*, Bull. London Math. Soc., to appear.

[28] T. Iwaniec and A. Lutoborski, *Integral estimates for null Lagrangians*, Arch. Rational Mech. Anal. **125** (1993), 25–79.

[29] T. Iwaniec and G. J. Martin, *Quasiregular mappings in even dimensions*, Acta Math. **170** (1993), 29–81.

[30] T. Iwaniec and G. J. Martin, *Riesz transforms and related singular integrals*, J. Reine Angew. Math. **473** (1993), 29–81.

[31] T. Iwaniec and G. J. Martin, *The Beltrami Equation*, Mem. Amer. Math. Soc., to appear.

[32] T. Iwaniec and G. J. Martin, *Squeezing the Sierpinski Sponge*, Studia Math. **149** (2002), 133–145.

[33] T. Iwaniec and G. J. Martin, *Geometric Function Theory and Non-linear Analysis*, Oxford University Press, 2001.

[34] T. Iwaniec, L. Migliaccio, L. Nania and C. Sbordone, *Integrability and removability results for quasiregular mappings in high dimensions*, Math. Scand. **75** (1994), 263–279.

[35] T. Iwaniec and C. Sbordonne, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rational Mech. Anal. **119** (1992), 129–143.

[36] T. Iwaniec and C. Sbordonne, *Quasiharmonic fields*, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), 519–572.

[37] T. Iwaniec and V. Šverák, *On mappings with integrable dilatation*, Proc. Amer. Math. Soc. **118** (1993), 181–188.

[38] T. Iwaniec and A. Verde, *A study of Jacobians in Hardy–Orlicz spaces* Proc. Edinburgh Math. Soc. **129** (1999), 539–570.

[39] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[40] J. Kauhanen, P. Koskela and J. Malý, *Mappings of finite distortion: Discreteness and openeness*, Arch. Rational Mech. Anal. **160** (2001), 135–151.

[41] J. Kauhanen, P. Koskela and J. Malý, *Mappings of finite distortion: Condition N*, Michigan Math. J. **49** (2001), 169–181.

[42] J. Manfredi and E. Villamor, *Mappings with integrable dilatation in higher dimensions*, Bull. Amer. Math. Soc. **32** (1995), 235–240.

[43] O. Martio, S. Rickman and J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I **448** (1969), 1–40.

[44] V. G. Maz'ja, *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.

[45] L. Migliaccio and G. Moscariello, *Mappings with unbounded dilatation*, Preprint, no. 3, DIIMA, Università di Salerno, 1997.

[46] G. Moscariello, *On the integrability of the Jacobian in Orlicz spaces*, Math. Japon. **40** (1992), 323–329.

[47] S. Müller, *A surprising higher integrability property of mappings with positive Jacobians*, Bull. Amer. Math. Soc. **21** (1989), 245–248.

[48] M. M Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Pure Appl. Math. **146**, Wiley, New York, 1991.

[49] M. Reimann, *Functions of bounded mean oscillation and quasiconformal mappings* Comment. Math. Helv. **49** (1974), 260–276.

[50] Yu. G. Reshetnyak, *Space Mappings with Bounded Distortion*, Transl. Math. Monographs **73**, Amer. Math. Soc., Providence, RI, 1989.

[51] S. Rickman, *Quasiregular Mappings*, Springer-Verlag, Berlin, 1993.

[52] J. W. Robin, R. C. Rogers and B. Temple, *On weak continuity and Hodge decomposition*, Trans. Amer. Math. Soc. **303** (1987), 609–618.

[53] V. Ryazanov, U. Srebro and E. Yacubov, *BMO-quasiregular mappings*, J. Analyse Math. **83** (2001), 1–20.

[54] E. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[55] P. Tukia, *Compactness properties of μ-homeomorphisms*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), 47–69.

[56] J. Väisälä, *Two new characterizations for quasiconformality*, Ann. Acad. Sci. Fenn. Ser. A I Math. **362** (1965), 1–12.

[57] S. Wu, *On the higher integrability of nonnegative Jacobians*, Comptes Rendus Acad. Sci. Paris, to appear.

Tadeusz Iwaniec
DEPARTMENT OF MATHEMATICS
SYRACUSE UNIVERSITY
SYRACUSE, NY 13244, USA
email: tiwaniec@mailbox.syr.edu

Pekka Koskela
DEPARTMENT OF MATHEMATICS
P.O.BOX 35 (MAD)
UNIVERSITY OF JYVÄSKYLÄ
FIN-40351, JYVÄSKYLÄ, FINLAND
email: pkoskela@math.jyu.fi

Gaven Martin
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF AUCKLAND
AUCKLAND, NEW ZEALAND
email: g.martin@mat.aukuni.ac.nz

(Received July 11, 2001)