LIOUVILLE-TYPE THEOREMS AND HARNACK-TYPE INEQUALITIES FOR SEMILINEAR ELLIPTIC EQUATIONS

By

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Dedicated with admiration to L. Nirenberg on his seventy-fifth birthday

1 Introduction

In this paper, we study properties of positive solutions of semilinear elliptic equations with critical exponent. We give different proofs, improvements, and extensions to some previously established Liouville-type theorems and Harnack-type inequalities.

For $\mu > 0, \bar{x} \in \mathbb{R}^n, \quad n \geq 3,$

$$u(x) = \Big(\frac{\mu}{1+\mu^2|x-\bar{x}|^2}\Big)^{\frac{n-2}{2}} \tag{1}$$

satisfies

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}}, \qquad u > 0, \quad in\ \mathbb{R}^n. \tag{2}$$

The following celebrated Liouville-type theorem was established by Caffarelli, Gidas and Spruck.

Theorem 1.1 *([12]). A C^2 solution of (2) is of the form (1).*

Under the additional hypothesis $u(x) = O(|x|^{2-n})$ for large $|x|$, the result was established earlier by Obata [49] and Gidas, Ni and Nirenberg ([30]). The proof of Obata is more geometric, while the proof of Gidas, Ni and Nirenberg is by the method of moving planes. The proof of Caffarelli, Gidas and Spruck is by a "measure theoretic" variation of the method of moving planes. Such Liouville-type

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theorems have played a fundamental role in the study of semilinear elliptic equations with critical exponent, which include the Yamabe problem and the Nirenberg problem. The method of moving planes (and its variants including the method of moving spheres, etc.) goes back to A. D. Alexandroff in his study of embedded constant mean curvature surfaces. It was then used and developed through the work of Serrin ([54]) and Gidas, Ni and Nirenberg ([30] and [31]). In recent years, and stimulated by a series of beautiful papers of Berestycki, Caffarelli and Nirenberg ([1]–[8]), the method has been widely used and has become a powerful and user-friendly tool in the study of nonlinear partial differential equations. In this paper, we develop a rather systematic, and simpler, approach to Liouville-type theorems and Harnack-type inequalities along the lines of [42] and [26] using the method of moving spheres.

For $n \geq 3$, let $\mathbb{R}^n_+ = \{x = (x', t) \; ; \; x' \in \mathbb{R}^{n-1}, t > 0\}$ denote Euclidean half space. For $\mu > 0, \bar{x} = (\bar{x}', \bar{t}) \in \mathbb{R}^n$,

$$u(x', t) = \left(\frac{\mu}{1 + \mu^2 |(x', t) - (\bar{x}', \bar{t})|^2} \right)^{\frac{n-2}{2}} \tag{3}$$

satisfies

$$\begin{cases} -\Delta u = n(n-2) u^{\frac{n+2}{n-2}}, & u > 0, \qquad \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial t} = c u^{\frac{n}{n-2}}, & \qquad\qquad \text{on } \partial \mathbb{R}^n_+, \end{cases} \tag{4}$$

where $c = (n-2)\mu \bar{t}$.

The following theorem was established by Li and Zhu.

Theorem 1.2 *([42]). A C^2 solution of (4) is of the form (3) for some $\mu > 0$, $\bar{x}' \in \mathbb{R}^{n-1}$, and $\bar{t} = \frac{c}{(n-2)\mu}$.*

Under an additional hypothesis $u(x) = O(|x|^{2-n})$ for large $|x|$, the result was established earlier by Escobar ([28]). The proof of Escobar is along the lines of the proof of Obata, while the proof of Li and Zhu is by the method of moving spheres, a variant of the method of moving planes.

Liouville-type theorems in dimension $n = 2$ were established in [22], [27], [42], and the references therein. Analogues for systems were established in [14]. Improvements to the results in [42] can be found in recent papers of Ou ([51]) and the second author ([55]).

For $n \geq 3$, Liouville-type theorems for more general semilinear equations

$$-\Delta u = g(u), \quad u > 0, \qquad \text{in } \mathbb{R}^n, \tag{5}$$

and

$$(6) \qquad \begin{cases} -\Delta u = g(u), & u > 0, \quad \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial t} = h(u), & t = 0, \end{cases}$$

have been studied in [32], [22], [9], [15], [20], [26], [42], and the references therein.

The following two Liouville-type theorems concerning (5) and (6) are improvements of previous results.

Assume that

$$(g1) \qquad g \text{ is locally bounded in } (0,\infty),$$

$$(g2) \qquad g(s)s^{-\frac{n+2}{n-2}} \text{ is non-increasing in } (0,\infty).$$

Theorem 1.3. *Let g satisfy (g1) and (g2), and let u be a (continuous) solution of (5). Then either*

for some $b > 0$, bu is of the form (1) and

$$s^{-\frac{n+2}{n-2}} g(s) \equiv n(n-2)b^{\frac{4}{n-2}} \quad on\ (0, \max_{\mathbb{R}^n} u];$$

or

$u \equiv a$ for some constant $a > 0$ satisfying $g(a) = 0$.

Remark 1.1. Radial symmetry of solutions was established, under additional hypotheses, by Caffarelli, Gidas and Spruck ([12]). Under the additional hypotheses that $g \geq 0$ and g is locally Lipschitz in $(0, \infty)$, Theorem 1.3 was established by Chen and Lin ([15]) and by Bianchi ([9]). The locally Lipschitz assumption of g was weakened to locally boundedness of g by Chen and Lin in [20]. Theorem 1.3 gives a further improvement by dropping the extra hypothesis that $g \geq 0$. For $g(s) = s^p, 1 \leq p < \frac{n+2}{n-2}$, the non-existence of positive entire solutions was established by Gidas and Spruck ([32]). See also a closely related work [37] by Congming Li.

Remark 1.2. Taking $g(s) = -s^p$, we recover the following well-known result (a very special case of the results in [10]): for $n \geq 1$ and $p > 1$, there is no positive solution of $\Delta u = u^p$ in $\mathbb{R}^n$. Indeed, u can be viewed as a solution of the same equation in $\mathbb{R}^m$ with $p > (m+2)/(m-2)$ and $m > n$, and the result follows from Theorem 1.3.

For Euclidean half space, we assume that h satisfies

$$(h1) \qquad h \text{ is locally Hölder continuous in } (0,\infty),$$

$$(h2) \qquad h(s)s^{-\frac{n}{n-2}} \text{ is non-decreasing.}$$

Theorem 1.4. *Let g satisfy (g1) and (g2), and let h satisfy (h1) and (h2). Assume that u is a (continuous) solution of (6). Then one of the following two alternatives holds.*

Alternative One: u depends only on t and satisfies the ordinary differential equation

$$\begin{cases} u'' = -g(u), \quad u > 0, & \text{in } [0,\infty), \\ u'(0) = h(u(0)). \end{cases}$$

Alternative Two: there exist some constants a and b, with $b < -\sqrt{-\frac{(n-2)a}{n}}$ when $a \leq 0$, such that

$$\begin{cases} g(s) = as^{\frac{n+2}{n-2}}, & \text{for} \quad 0 < s \leq max_{\mathbb{R}^n_+} u, \\ h(s) = bs^{\frac{n}{n-2}}, & \text{for} \quad 0 < s \leq \max_{\partial\mathbb{R}^n_+} u. \end{cases}$$

Moreover,

$$u(x) = \frac{\alpha}{(|x-\bar{x}|^2 + \beta)^{\frac{n-2}{2}}} \qquad \alpha > 0, \qquad \bar{x} \in \mathbb{R}^n,$$

where $\bar{x}_n = \frac{b}{n-2}\alpha^{\frac{2}{n-2}}$, and $\beta = \frac{a}{(n-2)n}\alpha^{\frac{4}{n-2}}$.

Remark 1.3. Under the additional hypotheses that g is locally Lipschitz, non-negative and non-decreasing, Theorem 1.4 was established by Bianchi ([9]). For $g(s) = as^{\frac{n+2}{n-2}}$ and $h(s) = bs^{\frac{n}{n-2}}$, see [42] and [26].

Remark 1.4. If we further assume $g(s) \geq 0$ for $s > 0$, we have the following observation:

1°. If $g \equiv 0$, there exist $a \geq 0$ and $b > 0$ such that

$$u(x) = u(t) = at + b \quad \text{and} \quad h(b) = a.$$

2°. If $\liminf_{s\to\infty} g(s) > 0$, then Alternative One does not occur. This follows from an elementary phase plane argument for ODE (see Appendix C).

3°. If $\liminf_{s\to\infty} g(s) = 0$, Alternative One may occur. Indeed, we can take $u(x) = u(t) = (1+t)^{\frac{1}{2}}$, $g(s) = \frac{1}{4}s^{-3}$, and $h(s) = \frac{1}{2}s^{\frac{n}{n-2}}$.

We point out that Theorem 1.4 and Remark 1.4 include a number of previously established results of various authors as consequences.

Corollary 1.1. *For $n \geq 3$, $-\infty < q < \frac{n}{n-2}$, there is no positive classical solution of*

$$\begin{cases} -\Delta u = 0, & \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial t} = -u^q, & \partial\mathbb{R}^n_+. \end{cases}$$

Proof. Let $g(s) = 0$ and $h(s) = -s^q$. Clearly, Alternative Two in Theorem 1.4 does not occur. By Remark 1.4, Alternative One cannot occur either. □

Remark 1.5. Corollary 1.1 in the case $1 \leq q < \frac{n}{n-2}$ was established by B. Hu in [35].

Corollary 1.2. *Suppose $0 \leq p \leq \frac{n+2}{n-2}$, $-\infty < q \leq \frac{n}{n-2}$, and $p + q < (\frac{n+2}{n-2}) + (\frac{n}{n-2})$. Then for any positive constant a, there is no positive classical solution of*

$$\begin{cases} \Delta u + au^p = 0, & \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial t} = -u^q, & \partial\mathbb{R}^n_+. \end{cases}$$

Proof. Let $g(s) = as^p$ and $h(s) = -s^q$. By the assumptions on p and q, we know that (g1), (g2), (h1), (h2) are satisfied. The conclusion follows easily from Theorem 1.4 and Remark 1.4. □

Remark 1.6. Corollary 1.2 under the additional hypothesis $p, q > 1$ was established by Chipot, Chlebik, Fila and Shafrir in [25].

Corollary 1.3 ***(Lou and Zhu [48]).*** *For $n \geq 1$ and $p, q > 1$, there is no positive classical solution of*

$$\begin{cases} \Delta u = u^p, & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial t} = u^q, & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Proof. Let $g(s) = -s^p$ and $h(s) = s^q$. Then u can be viewed as a solution of the same equation in $\mathbb{R}^m$ with $m > n$ so large that $q > m/(m-2)$. Then (g1), (g2), (h1), (h2) are satisfied (with n replaced by m). Clearly, Alternative Two of Theorem 1.4 does not occur. By Remark 1.4, Alternative One does not occur either. □

Corollary 1.4 ***(Lou and Zhu, [48]).*** *For $q > 1$, the only positive classical solutions of*

$$\begin{cases} \Delta u = 0, & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial u}{\partial t} = u^q, & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

are $u = at + b$ with some positive constants a, b satisfying $a = b^q$.

Proof. Choose large m such that $q > m/(m-2)$, and view u as a solution in $\mathbb{R}^m_+$. □

Corollary 1.5 ***(Hu and Yin [36], Ou [50]).*** *Let $n \geq 3, q \leq \frac{n}{n-2}$, and let u be a positive classical solution of*

$$\begin{cases} -\Delta u = 0, & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial t} = -u^q & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Then $q = \frac{n}{n-2}$, and for some $\bar{x}' \in \mathbb{R}^{n-1}$ and $\bar{t} < 0$,

$$u(x) = \left(\frac{-(n-2)\bar{t}}{|x - (\bar{x}', \bar{t})|^2} \right)^{\frac{n-2}{2}}.$$

Proof. Apply Theorem 1.4. □

Based on the Liouville-type theorem of Caffarelli, Gidas and Spruck (Theorem 1.1), Schoen established the following groundbreaking Harnack-type inequality.

Theorem 1.5 ***([52]).*** *For $n \geq 3$, let B_{3R} be a ball of radius $3R$ in R^n, and let $u \in C^2(B_{3R})$ be a positive solution of*

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}}, \quad \text{in } B_{3R}. \tag{7}$$

Then

$$(\max_{\overline{B}_R} u)(\min_{\overline{B}_{2R}} u) \leq C(n)R^{2-n}. \tag{8}$$

A consequence is the following energy estimate.

Corollary 1.6 *([52]). Let u be as in Theorem 1.5. Then*

$$\int_{B_R} \left(|\nabla u|^2 + u^{\frac{2n}{n-2}}\right) \leq C(n). \tag{9}$$

Harnack-type inequalities of this nature in dimension $n = 2$ were established by Brezis, Li and Shafrir ([11]), Chen and Lin ([17]), and Li ([41]). For $n \geq 3$, Chen and Lin ([15], [16]) established such Harnack-type inequalities for more general right hand sides $g(x, u)$. In particular, they established a slightly weaker version of the following theorem.

Assume that g satisfies

$$g \text{ is continuous and positive in } (0,\infty), \text{ and } \sup_{0<s\leq t} g(s) < \infty, \ \forall\, t < \infty; \tag{10}$$

then

$$s^{-\frac{n+2}{n-2}} g(s) \text{ is non-increasing in } (0,\infty) \tag{11}$$

and

$$\lim_{s\to\infty} s^{-\frac{n+2}{n-2}} g(s) \text{ exists and belongs to } (0,\infty). \tag{12}$$

Theorem 1.6. *Let g satisfy the above, and let u be a (continuous) solution of*

$$-\Delta u = g(u), \qquad u > 0, \qquad on\ B_{3R}, \tag{13}$$

with

$$\max_{\overline{B}_R} u \geq 1.$$

Then

$$(\max_{\overline{B}_R} u)(\min_{\overline{B}_{2R}} u) \leq C R^{2-n},$$

where C depends only on n and g.

Remark 1.7. Under the slightly stronger hypothesis that g is locally Lipschitz in $(0,\infty)$, the result was established by Chen and Lin (theorem 1.2 in [15]).

Remark 1.8. If we allow $\lim_{s\to\infty} s^{-\frac{n+2}{n-2}} g(s) = 0$ in (12), the result no longer holds. For instance, let $g(s) = \frac{1}{4}(s+1)^{-3}$; then g satisfies (10), (11) and $\lim_{s\to\infty} s^{-\frac{n+2}{n-2}} g(s) = 0$. However $u_j(x) = \sqrt{x_1 + j} - 1$ satisfies $-\Delta u_j = g(u_j)$ in B_3, and $\min_{\overline{B}_2} u_j \to \infty$. On the other hand, as shown in Appendix D, if $\lim_{s\to\infty} s^{-p} g(s) \in (-\infty, 0)$ for some $p > 1$, and $\sup_{0<s<t} |g(s)| < \infty$ for every t, then any positive solution of $-\Delta u = g(u)$ in B_3 satisfies $\max_{\overline{B}_1} u \leq C(n, g)$.

Harnack-type inequalities are closely related to works on pointwise estimates of blow-up solutions to Yamabe-type and scalar-curvature-type equations (e.g., [52], [53], [38], [39], [40], [16], [18], [19], [46], [20], [21], [23], [34], [45], and the references therein). They are also related to the work in [13].

The following theorem is an extension of the Harnack-type inequality and the energy estimate of Schoen to half Euclidean balls under geometrically natural boundary conditions. We will use notation $B_R^+ = \{x = (x', t) \in B_R : t > 0\}$ to denote the half ball, and $\partial' B_R^+ = \partial B_R^+ \cap \{t = 0\}$. For $n \geq 3$ and $c \in \mathbb{R}$, consider

$$(14)\qquad \begin{cases} \Delta u + n(n-2)u^{\frac{n+2}{n-2}} = 0, \quad u > 0, & \text{in } B_{3R}^+, \\ \dfrac{\partial u}{\partial t} = cu^{\frac{n}{n-2}}, & \text{on } \partial' B_{3R}^+. \end{cases}$$

Theorem 1.7. *For $n \geq 3$, $c \in \mathbb{R}$, let $u \in C^1(\overline{B_{3R}^+}) \cap C^2(B_{3R}^+)$ be a solution of (14). Then, for some constant $C = C(n, c)$,*

$$(15)\qquad (\max_{B_R^+} u)(\min_{\partial B_{2R}^+} u) \leq CR^{2-n}$$

and

$$(16)\qquad \int_{B_R^+} (|\nabla u|^2 + u^{\frac{2n}{n-2}})dx \leq C.$$

Remark 1.9. It is easy to see from the proof that for all $c \leq A$, the constant C in Theorem 1.7 depends only on n and A.

Remark 1.10. For $c \leq 0$, the energy estimate (16) can easily be deduced from (15) as in the derivation of (9) from (8) (see, e.g., pages 974–975 of [16]). However, our proof of (16) for $c > 0$ is surprisingly elaborate. See Section 9 for details.

Remark 1.11. The difference between Theorem 1.7 and the results in [34] is that Theorem 1.7 is purely local (no assumption is made on the other part of the boundary of ∂B_{3R}^+). The difference is the same as that between [38] and [16]. The Harnack-type inequality (15) plays an important role in deducing the energy estimate (16). It implies that all the large local maxima of u must have comparable magnitudes if they are not too close to $\partial B_{3R}^+ \cap \mathbb{R}_+^n$. Once the energy estimate (16) is established, the results in [34] can be applied, i.e., any blow-up solutions $\{u_j\}$ must have isolated simple blow-ups in $\overline{B_{(3-\beta)R}^+}$ for any $\beta > 0$, and the distance between any two blow-up points is bounded below by dR, $d = d(n, c, \beta) > 0$. Moreover,

$$\inf_{R\Lambda_1} u \leq C(n, c, \beta, \Lambda_1, \Lambda_2) \inf_{R\Lambda_2} u, \tag{17}$$

for any solution u of (14) and any infinite subsets Λ_1 and Λ_2 of $\overline{B^+_{(3-\beta)}}$. In particular, $\min_{\overline{\partial B^+_{2R}}} u$ in (15) can be replaced by $\inf_{R\Lambda} u$ for any infinite subset Λ of $\overline{B^+_2}$ (the C in (15) then depends also on Λ). Estimate (17) will be established towards the end of Section 9.

We have also established the Harnack-type inequality (15) for more general right hand sides g and h.

We assume that h is locally Hölder continuous in $(0, \infty)$ and g is continuous in $(0, \infty)$, and that they satisfy

$(G1)$ $\quad g(s) > 0$ and $\sup_{0<s\leq t} g(s) < \infty \ \forall\, t < \infty$,

$(G2)$ $\quad s^{-\frac{n+2}{n-2}} g(s)$ is non-increasing and $\lim_{s\to\infty} s^{-\frac{n+2}{n-2}} g(s) > 0$,

$(H1)$ $\quad \inf_{0<s\leq 1} h(s) > -\infty$,

$(H2)$ $\quad s^{-\frac{n}{n-2}} h(s)$ is non-decreasing and $\lim_{t\to\infty} s^{-\frac{n}{n-2}} h(s) < \infty$.

Theorem 1.8. *Let g and h satisfy the above, and let u be a (continuous) positive solution of*

$$\begin{cases} -\Delta u = g(u), & B^+_{3R}, \\ \dfrac{\partial u}{\partial t} = h(u), & t = 0, \end{cases} \tag{18}$$

with $\max_{\overline{B^+_R}} u \geq 1$. Then

$$(\max_{B^+_R} u)(\min_{\partial B^+_{2R}} u) \leq CR^{2-n},$$

where C depends only on g, h and dimension n.

Harnack-type inequalities for

$$\begin{cases} \Delta u + K(x) u^{\frac{n+2}{n-2}} = 0, \quad u > 0, & \text{in } B^+_{3R}, \\ \dfrac{\partial u}{\partial t} = c(x') u^{\frac{n}{n-2}}, & \text{on } \partial' B^+_{3R}, \end{cases}$$

with appropriate $K(x)$ and $c(x')$, will be given in a subsequent paper of the second author ([56]).

Recent works on pointwise estimates of blow-up solutions of critical exponent equations with boundary conditions can be found in works of Li ([39]), Li and Zhu ([43] and [44]), Han and Li ([34]), Zhu ([57] and [58]), Chen and Li ([24]), Ghoussoub, Gui and Zhu ([29]), Lin ([47]), Gui and Lin ([33]), Zhang ([56]), and the references therein.

Our paper is organized as follows. In Section 2, we give a different proof of the Liouville-type theorem of Caffarelli, Gidas and Spruck (Theorem 1.1). For instance, we do not reduce it to the radial symmetry of u and conclude by using ODE; rather, we catch the form of solutions using the method of moving spheres. This approach was suggested in [42], while we have made significant simplifications in this paper. Using the same approach, we prove Theorem 1.3 in Section 3, and Theorem 1.4 in Section 4. In Section 5, we give a different proof of the Harnack-type inequality of Schoen (Theorem 1.5). In particular, our proof does not rely on the Liouville-type theorem of Caffarelli, Gidas and Spruck. In Section 6, we establish Theorem 1.6 by essentially the same arguments in Section 5. Our proof is different from the one given by Chen and Lin in [15]. In Section 7, we establish the Harnack-type inequality (15) in Theorem 1.7. In Section 8, we prove Theorem 1.8. In Section 9, we establish the energy estimate (16), thus completing the proof of Theorem 1.7. In Appendix A, we prove a boundary lemma for linear second-order elliptic equations. In Appendix B, we include some calculus lemmas taken from [42] and [26]. In Appendix C, we present an elementary proof of some statement concerning ODE. In Appendix D, we present a result concerning Remark 1.8.

2 A different proof of the Liouville-type theorem of Caffarelli, Gidas and Spruck

In this section, we give a different proof of the Liouville-type theorem of Caffarelli, Gidas and Spruck (Theorem 1.1). Our proof makes use of ideas in [42] and [26]. The theorem will be deduced from a number of lemmas. For $x \in \mathbb{R}^n$ and $\lambda > 0$, consider the Kelvin transformation of u:

$$u_{x,\lambda}(y) = \frac{\lambda^{n-2}}{|y-x|^{n-2}} u\Big(x + \frac{\lambda^2(y-x)}{|y-x|^2}\Big), \qquad y \in \mathbb{R}^n \setminus \{x\}.$$

Our first lemma says that the method of moving spheres can get started.

Lemma 2.1. *For every $x \in \mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that $u_{x,\lambda}(y) \le u(y)$, for all $0 < \lambda < \lambda_0(x)$ and $|y - x| \ge \lambda$.*

Set, for $x \in \mathbb{R}^n$,

$$\bar{\lambda}(x) = \sup\{\mu > 0 : u_{x,\lambda}(y) \le u(y), \text{for all } |y-x| \ge \lambda, 0 < \lambda \le \mu\}.$$

By Lemma 2.1, $\bar{\lambda}(x)$ is well-defined and $0 < \bar{\lambda}(x) \le \infty$ for $x \in \mathbb{R}^n$.

Then we show

Lemma 2.2. *If* $\bar{\lambda}(x) < \infty$ *for some* $x \in \mathbb{R}^n$, *then* $u_{x,\bar{\lambda}(x)} \equiv u$ *on* $\mathbb{R}^n \setminus \{x\}$.

Lemma 2.3. *If* $\bar{\lambda}(\bar{x}) = \infty$ *for some* $\bar{x} \in \mathbb{R}^n$, *then* $\bar{\lambda}(x) = \infty$ *for all* $x \in \mathbb{R}^n$.

Lemma 2.4. $\bar{\lambda}(x) < \infty$ *for all* $x \in \mathbb{R}^n$.

Proof of Theorem 1.1. It follows from Lemma 2.2 and Lemma 2.4 that for every $x \in \mathbb{R}^n$, there exists $\bar{\lambda}(x) > 0$ such that $u_{x,\bar{\lambda}(x)} \equiv u$. Then by a calculus lemma in Appendix A (Lemma 11.1), for some $a, d > 0$ and some $\bar{x} \in \mathbb{R}^n$,

$$u(x) \equiv \left(\frac{a}{d + |x - \bar{x}|^2}\right)^{\frac{n-2}{2}}.$$

Theorem 1.1 follows from the above and the fact that u is a solution of (2). □

In the rest of this section, we establish the above lemmas.

Proof of Lemma 2.1. Without loss of generality we may take $x = 0$. We use u_λ to denote $u_{0,\lambda}$. Clearly, there exists $r_0 > 0$ such that

$$\frac{d}{dr}(r^{\frac{n-2}{2}} u(r,\theta)) > 0, \qquad 0 < r < r_0, \quad \theta \in \mathbb{S}^{n-1}.$$

Consequently,

$$u_\lambda(y) < u(y), \qquad 0 < \lambda < |y| < r_0. \tag{19}$$

By the super-harmonicity of u and the maximum principle,

$$u(y) \ge (\min_{\partial B_{r_0}} u) r_0^{n-2} |y|^{2-n}, \qquad |y| \ge r_0. \tag{20}$$

Let

$$\lambda_0 = r_0 \left(\frac{\min_{\partial B_{r_0}} u}{\max_{\overline{B}_{r_0}} u}\right)^{\frac{1}{n-2}} \le r_0.$$

Then for every $0 < \lambda < \lambda_0$, and $|y| \ge r_0$, we have

$$u_\lambda(y) \le \frac{{\lambda_0}^{n-2}}{|y|^{n-2}} (\max_{\overline{B}_{r_0}} u) \le \frac{r_0^{n-2} \min_{\partial B_{r_0}} u}{|y|^{n-2}}. \tag{21}$$

It follows from (20), (21) and (19) that for every $0 < \lambda < \lambda_0$,

$$u_\lambda(y) \le u(y), \qquad |y| \ge \lambda.$$

□

Proof of Lemma 2.2. Without loss of generality, we take $x = 0$ and let $\bar\lambda = \bar\lambda(0)$, $u_\lambda = u_{0,\lambda}$, and $\Sigma_\lambda = \{y : \ |y| > \lambda\}$. We wish to show $u_{\bar\lambda} \equiv u$ in $R^n \backslash \{0\}$. Clearly, it suffices to show

$$u_{\bar\lambda} \equiv u \quad \text{on } \Sigma_{\bar\lambda}.$$

From the definition of $\bar\lambda$,

$$u_{\bar\lambda} \le u \quad \text{on } \Sigma_{\bar\lambda}.$$

A simple calculation yields

$$\Delta u_\lambda(y) = \Big(\frac{\lambda}{|y|}\Big)^{n+2} \Delta u\Big(\frac{\lambda^2 y}{|y|^2}\Big)$$

and, in view of (2),

$$-\Delta u_\lambda = n(n-2) u_\lambda^{\frac{n+2}{n-2}}, \quad \lambda > 0.$$

Therefore,

$$-\Delta(u - u_{\bar\lambda}) = n(n-2)(u^{\frac{n+2}{n-2}} - u_{\bar\lambda}^{\frac{n+2}{n-2}}) \ge 0 \quad \text{in } \Sigma_{\bar\lambda}. \tag{22}$$

If $u - u_{\bar\lambda} \equiv 0$ on $\Sigma_{\bar\lambda}$, we stop. Otherwise, by the Hopf lemma and the compactness of $\partial B_{\bar\lambda}$, we have

$$\frac{d}{dr}(u - u_{\bar\lambda})|_{\partial B_{\bar\lambda}} \ge b > 0. \tag{23}$$

By the continuity of ∇u, there exists $R > \bar\lambda$ such that

$$\frac{d}{dr}(u - u_\lambda) \ge \frac{b}{2} > 0 \qquad \text{for } \bar\lambda \le \lambda \le R, \quad \lambda \le r \le R.$$

Consequently, since $u - u_\lambda = 0$ on ∂B_λ, we have

$$u(y) - u_\lambda(y) > 0 \qquad \text{for } \bar\lambda \le \lambda < R, \quad \lambda < |y| \le R. \tag{24}$$

Set $c = \min_{\partial B_R}(u - u_{\bar\lambda}) > 0$. It follows from the super-harmonicity of $u - u_{\bar\lambda}$ that

$$u(y) - u_{\bar\lambda}(y) \ge \frac{cR^{n-2}}{|y|^{n-2}}, \quad |y| \ge R. \tag{25}$$

Therefore,

$$u(y) - u_\lambda(y) \geq \frac{cR^{n-2}}{|y|^{n-2}} - (u_\lambda(y) - u_{\bar\lambda}(y)), \quad |y| \geq R. \tag{26}$$

By the uniform continuity of u on $\bar B_R$, there exists $0 < \epsilon < R - \bar\lambda$ such that for all $\bar\lambda \leq \lambda \leq \bar\lambda + \epsilon$,

$$\left|\lambda^{n-2} u\Big(\frac{\lambda^2 y}{|y|^2}\Big) - \bar\lambda^{n-2} u\Big(\frac{\bar\lambda^2 y}{|y|^2}\Big)\right| < \frac{cR}{2} \qquad \text{for } |y| \geq R.$$

It follows from (26) and the above that

$$u(y) - u_\lambda(y) > 0 \qquad \text{for } \bar\lambda \leq \lambda \leq \bar\lambda + \epsilon, \quad |y| \geq R. \tag{27}$$

Estimates (24) and (27) violate the definition of $\bar\lambda$. □

Proof of Lemma 2.3. Since $\bar\lambda(\bar x) = \infty$, we have

$$u(y) \geq u_{\bar x,\lambda}(y) \quad \text{for all } \lambda > 0 \text{ and } |y - \bar x| \geq \lambda.$$

It follows that

$$\lim_{|y|\to\infty} |y|^{n-2} u(y) = \infty.$$

On the other hand, if $\bar\lambda(x) < \infty$ for some $x \in \mathbb{R}^n$, then by Lemma 2.2,

$$\lim_{|y|\to\infty} |y|^{n-2} u(y) = \lim_{|y|\to\infty} |y|^{n-2} u_{x,\bar\lambda(x)}(y) = \bar\lambda(x)^{n-2} u(x) < \infty.$$

Contradiction. □

Proof of Lemma 2.4. We prove this by contradiction. If $\bar\lambda(\bar x) = \infty$ for some $\bar x$, then by Lemma 2.3, $\bar\lambda(x) = \infty$ for all x, i.e.,

$$u_{x,\lambda}(y) \leq u(y), \qquad \text{for all } \lambda > 0 \text{ and } x \in \mathbb{R}^n, |y - x| \geq \lambda.$$

By a calculus lemma in Appendix A (Lemma 11.2), this implies that $u \equiv \mathit{constant}$, a contradiction to (2). □

3 Proof of Theorem 1.3, a Liouville-type theorem for more general equations in $\mathbb{R}^n$

In this section, we establish Theorem 1.3. The proof is along the lines of the proof of Theorem 1.1, first establishing Lemmas 2.1–2.3. Our proof makes use of ideas in [42], [26] and [20].

Proof of Lemma 2.1 under the hypothesis of Theorem 1.3. We follow the proof of Lemma 2.1. Since we cannot use the super-harmonicity of u (g is allowed to change signs), we need to prove that

$$\liminf_{|y|\to\infty} \left(|y|^{n-2}u(y)\right) > 0. \tag{28}$$

Once (28) is proved, we have, instead of (20),

$$u(y) \geq c_0|y|^{2-n} \quad \text{for some } c_0 > 0 \text{ and } \forall\, |y| \geq r_0.$$

Then we pick some $\lambda_0 \in (0, r_0)$ such that $\lambda_0^{n-2} \max_{\overline{B}_{r_0}} u) \leq c_0$ to complete the proof as in the proof of Lemma 2.1.

In the following, we establish (28). Let

$$O = \{y : u(y) < |y|^{2-n}\}.$$

By (g2),

$$u(y)^{-\frac{n+2}{n-2}} g(u(y)) \geq (|y|^{2-n})^{-\frac{n+2}{n-2}} g(|y|^{2-n}) \geq g(1), \qquad y \in O \setminus B_1.$$

It follows that

$$\frac{g(u(y))}{u(y)} \geq g(1)u(y)^{\frac{4}{n-2}} \geq \frac{\min\{0, g(1)\}}{|y|^4}, \qquad y \in O \setminus B_1,$$

and therefore

$$-\Delta u(y) + \frac{C}{|y|^4}u(y) \geq 0, \quad y \in O \setminus B_1,$$

where $C = \max\{0, -g(1)\} \geq 0$. Let

$$\xi(y) = |y|^{2-n} + |y|^{1-n}. \tag{29}$$

A simple calculation yields

$$-\Delta\xi(y) + \frac{C}{|y|^4}\xi(y) = -(n-1)|y|^{-n-1} + C(|y|^{-n-2} + |y|^{-n-3}).$$

Thus, for large $\bar{R}$,

$$-\Delta\xi(y) + \frac{C}{|y|^4}\xi(y) \leq 0 \qquad \text{for } |y| \geq \bar{R}.$$

Pick some small $\bar{\epsilon} > 0$ such that

$$u(y) > \bar{\epsilon}\xi(y) \qquad \text{for } |y| = \bar{R}$$

and

$$u(y) = |y|^{2-n} > \bar{\epsilon}\xi(y) \qquad \text{on } \partial O.$$

As a result, $u - \bar{\epsilon}\xi$ satisfies

$$\begin{cases} -\Delta(u - \bar{\epsilon}\xi) + \dfrac{C}{|y|^4}(u - \bar{\epsilon}\xi) \geq 0 & \text{in } O \setminus B_{\bar{R}}, \\ u - \bar{\epsilon}\xi \geq 0 & \text{on } \partial(O \setminus B_{\bar{R}}), \\ \liminf_{|y|\to\infty}(u(y) - \bar{\epsilon}\xi(y)) \geq 0. \end{cases}$$

By the maximum principle, $u - \bar{\epsilon}\xi \geq 0$ on $\overline{O \setminus B_R}$, and therefore

$$\liminf_{y\in\bar{O},|y|\to\infty} \left(|y|^{n-2}u(y)\right) \geq \liminf_{y\in\bar{O},|y|\to\infty} \left(\bar{\epsilon}|y|^{n-2}\xi(y)\right) > 0.$$

Estimate (28) follows immediately. □

Proof of Lemma 2.2 under the hypothesis of Theorem 1.3. We follow the proof of Lemma 2.2 and only provide necessary changes.

The equation of u_λ now is

$$-\Delta u_\lambda = (\frac{\lambda}{|y|})^{n+2} g((\frac{|y|}{\lambda})^{n-2} u_\lambda(y)), \quad y \in \Sigma_\lambda.$$

Let

$$O := \left\{ y \in \Sigma_{\bar{\lambda}} : u(y) < \min\left\{ \left(\frac{|y|}{\bar{\lambda}}\right)^{n-2}, 2 \right\} u_{\bar{\lambda}}(y) \right\}.$$

By (g2),

$$u^{-\frac{n+2}{n-2}} g(u) \geq u_{\bar{\lambda}}^{-\frac{n+2}{n-2}} \left(\frac{\bar{\lambda}}{|y|}\right)^{n+2} g\left(\left(\frac{|y|}{\bar{\lambda}}\right)^{n-2} u_{\bar{\lambda}}\right) \qquad \text{in } O.$$

So, instead of (22), we have,

$$u^{-\frac{n+2}{n-2}} \Delta u \leq u_{\bar{\lambda}}^{-\frac{n+2}{n-2}} \Delta u_{\bar{\lambda}} \qquad \text{in } O. \tag{30}$$

Writing $u_s = su + (1-s)u_{\bar{\lambda}}$, we have by (30) that

$$\begin{aligned} 0 &\geq \int_0^1 \frac{d}{ds}\left(u_s^{-\frac{n+2}{n-2}} \Delta u_s\right) ds \\ &= \left(\int_0^1 u_s^{-\frac{n+2}{n-2}} ds\right) \Delta(u - u_{\bar{\lambda}}) - \frac{n+2}{n-2}\left(\int_0^1 u_s^{-\frac{2n}{n-2}} \Delta u_s ds\right)(u - u_{\bar{\lambda}}) \text{ in } O. \end{aligned} \tag{31}$$

We establish (23) as follows. For $y_0 \in \partial B_{\bar\lambda}$, if $\frac{d}{dr}(u - u_{\bar\lambda})(y_0) < (n-2)u(y_0)$, then

$$\frac{d}{dr}\left(\left(\frac{|y|}{\bar\lambda}\right)^{n-2} u_{\bar\lambda}(y) - u(y)\right)\Bigg|_{y=y_0} = (n-2)u(y_0) - \frac{d}{dr}(u - u_{\bar\lambda})(y_0) > 0.$$

So for some $\bar\delta > 0$, $B_{\bar\delta}(y_0) \cap \Sigma_{\bar\lambda} \subset O$. By the Hopf lemma (see (31)), $\frac{d}{dr}(u - u_{\bar\lambda})(y_0) > 0$. Estimate (23) is established. Clearly, (24) still follows from (23). Next we establish, instead of (25), the estimate

$$\liminf_{|y|\to\infty} |y|^{n-2}(u - u_{\bar\lambda})(y) > 0. \tag{32}$$

Once (32) is established, the rest of the proof of Lemma 2.2 is the same (note that on $\Sigma_{\bar\lambda} \setminus (O \cup B_R)$, $u \geq a u_{\bar\lambda}$ with $a =: \min\{(R/\bar\lambda)^{n-2}, 2\} > 1$; moreover, by (31) and the strong maximum principle, $u - u_{\bar\lambda} > 0$ in O).

To prove (32), we observe that for large $\bar R$,

$$u_{\bar\lambda}(y) \leq u(y) \leq 2u_{\bar\lambda}(y) \leq C|y|^{2-n} < 1 \qquad \text{in } O \setminus B_{\bar R}.$$

It follows by (g2) and the the equation of u that

$$\Delta u = -g(u) \leq -g(1) u^{\frac{n+2}{n-2}} \leq \frac{C}{|y|^{n+2}} \qquad \text{in } O \setminus B_{\bar R}.$$

Since both $(|y|/\bar\lambda)^{n-2} u_{\bar\lambda}(y)$ and $(|y|/\bar\lambda)^{n-2} u(y)$ stay in a compact subset of $(0, \infty)$ for $y \in O \setminus B_{\bar R}$,

$$\frac{1}{C|y|^{n-2}} \leq u_s(y) \leq \frac{C}{|y|^{n-2}}, \qquad y \in O \setminus B_{\bar R},\ 0 \leq s \leq 1;$$

and, by the equation of $u_{\bar\lambda}$,

$$|\Delta u_{\bar\lambda}| \leq \frac{C}{|y|^{n+2}} \qquad \text{in } O \setminus B_{\bar R}.$$

By (31) and the above estimates, we have, for some positive constant C,

$$-\Delta(u - u_{\bar\lambda}) + \frac{C}{|y|^4}(u - u_{\bar\lambda}) \geq 0 \qquad \text{in } O \setminus B_{\bar R}.$$

Let ξ be given in (29). Then for a possibly larger $\bar R$,

$$-\Delta\xi(y) + \frac{C}{|y|^4}\xi(y) \leq 0 \qquad \text{for } |y| \geq \bar R.$$

Since, $u - u_{\bar\lambda} > 0$ in O and

$$(u - u_{\bar\lambda})(y) \geq u_{\bar\lambda}(y) \geq \frac{\bar\lambda^{n-2} \min_{\overline{B_{\bar\lambda}}} u}{|y|^{n-2}} \qquad \text{in } \partial O \setminus B_{\bar R},$$

there exists some $\bar{\epsilon} > 0$ such that

$$(u - u_{\bar{\lambda}} - \bar{\epsilon}\xi)(y) \geq 0 \qquad \text{on } \partial(O \setminus B_{2\bar{R}}).$$

By the maximum principle,

$$(u - u_{\bar{\lambda}} - \bar{\epsilon}\xi)(y) \geq 0 \qquad \text{in } O \setminus B_{2\bar{R}}.$$

It follows that

$$\liminf_{y \in \bar{O}, |y| \to \infty} |y|^{n-2}(u - u_{\bar{\lambda}})(y) \geq \bar{\epsilon} > 0.$$

On the other hand, by the note below (32), for some $a > 1$,

$$\liminf_{y \in \mathbb{R}^n \setminus O, |y| \to \infty} |y|^{n-2}(u - u_{\bar{\lambda}})(y) \geq (a-1) \lim_{|y| \to \infty} |y|^{n-2} u_{\bar{\lambda}}(y) > 0.$$

Estimate (32) is established. □

Proof of Theorem 1.3. It follows from Lemma 2.1 and Lemma 2.3 that either $\bar{\lambda}(x) = \infty$ for all x in $\mathbb{R}^n$, or $0 < \bar{\lambda}(x) < \infty$ for all x in $\mathbb{R}^n$. In the first case, $u \equiv b$ for some constant a by Lemma 11.2. In the second case, it follows from Lemma 2.2 that $u_{x, \bar{\lambda}(x)} \equiv u$ for all x in $\mathbb{R}^n$. Consequently, in view of Lemma 11.1,

$$u(x) \equiv \left(\frac{a}{d + |x - \bar{x}|^2} \right)^{\frac{n-2}{2}}$$

where $a, d > 0$. So, for some constant $c > 0$,

$$-\Delta u = c u^{\frac{n+2}{n-2}} = g(u).$$

Theorem 1.3 follows easily. □

4 Proof of Theorem 1.4, a Liouville-type theorem on $\mathbb{R}^n_+$

In this section, we establish Theorem 1.4. Our proof makes use of ideas in [42], [26], [9] and [20]. We still use $u_{x,\lambda}$ to denote the Kelvin transformation of u, as in Section 2, but mainly work with $x \in \partial\mathbb{R}^n_+$. We use notations $B_\lambda(x) = \{y \in \mathbb{R}^n : |y - x| < \lambda\}$ and $B_\lambda = B_\lambda(0)$.

Lemma 4.1. *For every $x \in \partial\mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that $u_{x,\lambda}(y) \leq u(y)$, for all $0 < \lambda < \lambda_0(x)$ and $y \in \mathbb{R}^n_+ \setminus B_\lambda(x)$.*

Set, for $x \in \partial\mathbb{R}^n_+$,

$$\bar{\lambda}(x) = \sup\{\mu > 0 : u_{x,\lambda}(y) \leq u(y), \text{for all } y \in \mathbb{R}^n_+ \setminus B_\lambda(x), 0 < \lambda \leq \mu\}.$$

Lemma 4.2. *If* $\bar{\lambda}(\bar{x}) < \infty$ *for some* $\bar{x} \in \partial\mathbb{R}^n_+$, *then* $u_{\bar{x},\bar{\lambda}(\bar{x})} \equiv u$ *on* $\mathbb{R}^n_+ \setminus \{\bar{x}\}$.

Lemma 4.3. *If* $\bar{\lambda}(\bar{x}) = \infty$ *for some* $\bar{x} \in \partial\mathbb{R}^n_+$, *then* $\bar{\lambda}(x) = \infty$ *for all* $x \in \partial\mathbb{R}^n_+$.

By Lemma 4.3, either $\bar{\lambda}(x) = \infty$ for all $x \in \partial\mathbb{R}^n_+$, or $\bar{\lambda}(x) < \infty$ for all $x \in \partial\mathbb{R}^n_+$. Theorem 1.4 then follows from the following two lemmas.

Lemma 4.4. *If* $\bar{\lambda}(x) = \infty$ *for all* $x \in \partial\mathbb{R}^n$, *we have Alternative One in Theorem 1.4.*

Lemma 4.5. *If* $\bar{\lambda}(x) < \infty$ *for all* $x \in \partial\mathbb{R}^n_+$, *we have Alternative Two in Theorem 1.4.*

Proof of Lemma 4.1. Without loss of generality, we let $x = 0$ and write $u_\lambda = u_{0,\lambda}$, $\bar{\lambda} = \bar{\lambda}(0)$.

A direct calculation gives

$$\Delta u_\lambda(y) + \left(\frac{\lambda}{|y|}\right)^{n+2} g\left(\left(\frac{|y|}{\lambda}\right)^{n-2} u_\lambda(y)\right) = 0 \tag{33}$$

and

$$\frac{\partial u_\lambda(y)}{\partial t} = \left(\frac{\lambda}{|y|}\right)^{n} h\left(\left(\frac{|y|}{\lambda}\right)^{n-2} u_\lambda(y)\right) \qquad \text{on } t = 0. \tag{34}$$

By the argument in the proof of Theorem 1.3, we only need to show that

$$\liminf_{|y|\to\infty} |y|^{n-2} u(y) > 0. \tag{35}$$

Let

$$O = \{y \in \mathbb{R}^n_+ \ ; \ u(y) < |y|^{2-n}\}.$$

By $(g2)$ and $(h2)$,

$$\frac{g(u(y))}{u(y)} \geq \frac{\min\{0, g(1)\}}{|y|^4}, \quad y \in O \setminus B_1^+$$

and

$$\frac{h(u(y))}{u(y)} \leq h(1) u(y)^{\frac{2}{n-2}} \leq \frac{\max\{0, h(1)\}}{|y|^2}, \quad y \in O \setminus B_1^+.$$

It follows that

$$\begin{cases} -\Delta u + \dfrac{C_1}{|y|^4} u \ \geq \ 0, & y \in O, \\[2ex] \dfrac{\partial u}{\partial t} - \dfrac{C_2}{|y|^2} u \ \leq \ 0, & y \in \partial' O, \end{cases}$$

where $C_1 = \max\{0, -g(1)\}$ and $C_2 = \max\{0, h(1)\}$, and $\partial' O = \partial O \cap \{t = 0\}$.

For $A > 1$, let

$$\xi(y) = |y - Ae_n|^{2-n} + |y|^{1-n}, \tag{36}$$

where $e_n = (0, \dots, 0, 1)$. It is easy to see that for large A and $R = A^2$, we have

$$\begin{cases} -\Delta\xi + \dfrac{C_1}{|y|^4}\xi \leq 0, & y \in \mathbb{R}^n_+ \setminus B_R, \\ \dfrac{\partial\xi}{\partial t} - \dfrac{C_2}{|y|^2}\xi \geq 0, & \{t = 0\}. \end{cases}$$

Picking some small $\bar{\epsilon} > 0$ such that $u \geq \bar{\epsilon}\xi$ on $\partial(O \setminus B_R)$ we have, by the maximum principle,

$$u \geq \bar{\epsilon}\xi \qquad \text{on } O \setminus B_R.$$

Estimate (35) follows from the above. □

Proof of Lemma 4.2. Without loss of generality, $\bar{x} = 0$. The equations of $u_{\bar{\lambda}}$ are given in (33) and (34). Let

$$O := \left\{ y \in \mathbb{R}^n_+ \setminus \overline{B_{\bar{\lambda}}}\ ;\ u(y) < \min\left\{ \left(\frac{|y|}{\bar{\lambda}}\right)^{n-2}, 2\right\} u_{\bar{\lambda}}(y) \right\}.$$

By $(g2)$ and $(h2)$,

$$u^{-\frac{n+2}{n-2}} g(u) \geq u_{\bar{\lambda}}^{-\frac{n+2}{n-2}} \left(\frac{\bar{\lambda}}{|y|}\right)^{n+2} g\left(\left(\frac{|y|}{\bar{\lambda}}\right)^{n-2} u_{\bar{\lambda}}\right) \qquad \text{in } O$$

and

$$u^{-\frac{n}{n-2}} h(u) \leq u_{\bar{\lambda}}^{-\frac{n}{n-2}} \left(\frac{\bar{\lambda}}{|y|}\right)^{n} h\left(\left(\frac{|y|}{\bar{\lambda}}\right)^{n-2} u_{\bar{\lambda}}\right) \qquad \text{in } \partial' O,$$

where $\partial' O = \partial O \cap \{t = 0\}$. Thus, by the equations of u and $u_{\bar{\lambda}}$, we have

$$\begin{cases} u^{-\frac{n+2}{n-2}} \Delta u \leq u_{\bar{\lambda}}^{-\frac{n+2}{n-2}} \Delta u_{\bar{\lambda}}, & \text{in } O, \\ u^{-\frac{n}{n-2}} \dfrac{\partial u}{\partial t} \leq u_{\bar{\lambda}}^{-\frac{n}{n-2}} \dfrac{\partial u_{\bar{\lambda}}}{\partial t}, & \text{on } \partial' O. \end{cases} \tag{37}$$

Letting $w_\lambda = u - u_\lambda$ and $u_s = su + (1 - s)u_{\bar{\lambda}}$ we have, by the arguments following (30), that

$$\begin{cases} \Delta w_{\bar{\lambda}} \leq \left(\frac{n+2}{n-2}\right) \left(\int_0^1 u_s^{-\frac{n+2}{n-2}} ds\right)^{-1} \left(\int_0^1 u_s^{-\frac{2n}{n-2}} \Delta u_s ds\right) w_{\bar{\lambda}}, \text{ in } O, \\ \dfrac{\partial w_{\bar{\lambda}}}{\partial t} \leq \left(\frac{n}{n-2}\right) \left(\int_0^1 u_s^{-\frac{n}{n-2}} ds\right)^{-1} \left(\int_0^1 u_s^{-\frac{2(n-1)}{n-2}} \frac{\partial u_s}{\partial t} ds\right) w_{\bar{\lambda}}, \text{ on } \partial' O. \end{cases} \tag{38}$$

Our goal is to show $w_{\bar\lambda} \equiv 0$ in $\mathbb{R}^n_+ \setminus B_{\bar\lambda}$. We prove it by contradiction. Suppose $w_{\bar\lambda} \not\equiv 0$. Let ν denote the unit outer normal of $\partial B_{\bar\lambda}$. For $y_0 \in \partial B_{\bar\lambda} \cap \overline{\mathbb{R}^n_+}$, if $\partial w_{\bar\lambda}(y_0)/\partial\nu < (n-2)u(y_0)$, by arguments similar to those following (30), we have for some $\bar\delta > 0$, $B_{\bar\delta}(y_0) \cap (\overline{\mathbb{R}^n_+} \setminus \overline{B_{\bar\lambda}}) \subset \overline{O}$. By the Hopf lemma and Lemma 10.1, $\partial w_{\bar\lambda}(y_0)/\partial\nu > 0$. So we have shown that

$$\partial w_{\bar\lambda}(y)/\partial\nu > 0, \qquad \text{for } y \in \partial B_{\bar\lambda} \cap \overline{\mathbb{R}^n_+}. \tag{39}$$

By the maximum principle,

$$w_{\bar\lambda}(y) > 0, \qquad \text{for } y \in O \cup (\partial' O \setminus \partial B_{\bar\lambda}).$$

Following the arguments in the proof of Theorem 1.3, we reach a contradiction once we show

$$\liminf_{|y|\to\infty} |y|^{n-2} w_{\bar\lambda}(y) > 0. \tag{40}$$

As in the proof of Theorem 1.3, for some large $\bar R$ and some positive constants C_1 and C_2,

$$\begin{cases} -\Delta w_{\bar\lambda} + \dfrac{C_1}{|y|^4} w_{\bar\lambda} \geq 0, & y \in O \setminus B_{\bar R}, \\ \dfrac{\partial w_{\bar\lambda}}{\partial t} - \dfrac{C_2}{|y|^2} w_{\bar\lambda} \leq 0, & y \in \partial'(O \setminus B_{\bar R}). \end{cases}$$

Let ξ be given in (36) for sufficiently large A, and let $\bar\epsilon > 0$ be such that

$$w_{\bar\lambda} \geq \bar\epsilon\xi \qquad \text{on } \partial(O \setminus B_{2\bar R}).$$

Applying the maximum principle in $O \setminus B_{2\bar R}$ as in the proof of Lemma 4.1, we have

$$w_{\bar\lambda} \geq \bar\epsilon\xi \qquad \text{on } (O \setminus B_{2\bar R}).$$

Estimate (40) follows from the above. □

The proof of Lemma 4.3 is the same as that of Lemma 2.3.

Proof of Lemma 4.4. Suppose that $\bar\lambda(x) = \infty\ \forall\, x \in \partial\mathbb{R}^n_+$. Then by a calculus lemma (Lemma 11.3 with $\nu = n-2$), u depends only on t, and we have Alternative One. □

Proof of Lemma 4.5. By Lemma 4.2, $u \equiv u_{x,\bar\lambda(x)}\ \forall\, x \in \partial\mathbb{R}^n_+$. In particular,

$$a := \lim_{|y|\to\infty} |y|^{n-2} u(y) = \bar\lambda(x)^{n-2} u(x) < \infty \qquad \forall\, x \in \partial\mathbb{R}^n_+. \tag{41}$$

Applying a calculus lemma (Lemma 11.1), on $\partial\mathbb{R}^n_+$, we have for all x'

$$u(x',0) = \frac{a}{(|x'-\bar{x}'|^2+d^2)^{\frac{n-2}{2}}}, \tag{42}$$

where $\bar{x}' \in \partial\mathbb{R}^n_+$ and $a, d > 0$.

The following arguments are taken from [26] and [9]. Consider the spheres $B(x, \bar{\lambda}(x))$ for $x \in \partial\mathbb{R}^n_+$. From (41) and (42), we see that all these spheres pass through $(\bar{x}', \pm d)$. Let $P = (\bar{x}', -d)$ and define

$$v(z) = \Big(\frac{2d}{|z-P|}\Big)^{n-2} u\Big(P + \frac{4d^2(z-P)}{|z-P|^2}\Big).$$

Then by direct computation and the properties of conformal transformations, $Q := (\bar{x}', d)$ is mapped into itself and P is mapped to ∞, and $\mathbb{R}^n_+$ is mapped to $|z-Q| < 2d$. Since $u \equiv u_{x,\bar{\lambda}(x)}$ for all $x \in \partial\mathbb{R}^n_+$, v is symmetric with respect to all hyperplanes through Q, so v is radially symmetric about Q in $|z-Q| \le 2d$. The equations that v satisfies are

$$\begin{cases} \Delta v(z) + \Big(\frac{2d}{|z-P|}\Big)^{n+2} g\Big(\Big(\frac{|z-P|}{2d}\Big)^{n-2} v(z)\Big) = 0, & B(Q,2d), \\ \frac{(2-n)v(z)}{4d} - \frac{\partial v(z)}{\partial \nu} = \Big(\frac{|z-P|}{2d}\Big)^{-n} h\Big(\Big(\frac{|z-P|}{2d}\Big)^{n-2} v(z)\Big), & \partial B(Q,2d), \end{cases} \tag{43}$$

where ν denotes the unit outer normal to the boundary of $|z-Q| \le 2d$. Since v is radially symmetric about Q, the right hand side of the second equation of (43) is a constant C on $|z-Q| = 2d$. Thus we have

$$h(s) = bs^{\frac{n}{n-2}} \quad \text{for } 0 < s \le \max_{\partial\mathbb{R}^n_+} u.$$

Here we have used the fact that $\{|z-P|/2d)^{n-2}v(z) : |z-Q| = 2d\} = (0, \max_{\partial\mathbb{R}^n_+} u]$.

From the first equation of (43), we can deduce that for some $c \in \mathbb{R}$,

$$g(s) = cs^{\frac{n+2}{n-2}}, \qquad 0 < s \le \max_{\mathbb{R}^n_+} u. \tag{44}$$

Indeed, since v is radially symmetric about Q, and $Q \ne P$,

$$u\Big(P + \frac{4d^2(z-P)}{|z-P|^2}\Big) = \Big(\frac{|z-P|}{2d}\Big)^{n-2} v(z)$$

is not constant on $\{|z-Q| = r\}$, i.e., $\min_{S_r} u < \max_{S_r} u$, where

$$S_r = \Big\{P + \frac{4d^2(z-P)}{|z-P|^2} \;;\; |z-Q| = r\Big\}.$$

Thus, by the radial symmetry of v and the first equation of (43), we have for every $r \in (0, 2d)$, $g(s) = C(r)s^{\frac{n+2}{n-2}}$ for $\min_{S_r} u \le s \le \max_{S_r} u$. It is clear that $C(r)$ is locally constant and therefore independent of r. Thus (44) follows from the fact that $\bigcup_{0<r<2d} S_r = \mathbb{R}^n$. Therefore, the first equation of (43) becomes

$$\Delta v(z) + cv(z)^{\frac{n+2}{n-2}} = 0 \qquad \text{in } |z - Q| < 2d.$$

Since v is radially symmetric about Q, by an ODE argument, we have Alternative Two. □

5 A different proof of the Harnack-type inequality of R. Schoen

In this section, we give a different proof of the Harnack inequality of R. Schoen (Theorem 1.5). Our proof, making use of ideas in [42], [26] and [52], is more direct and does not rely on the Liouville-type theorem of Caffarelli, Gidas and Spruck. The transformation $u(y) \to R^{\frac{n-2}{2}} u(Ry)$ shows that it suffices to prove Theorem 1.5 for $R = 1$; hence, for the rest of this section, we assume $R = 1$.

First we have the following elementary lemma.

Lemma 5.1. *Let $u \in C^0(\overline{B}_1)$ be a positive function. Then for every $a > 0$, there exists $|x| < 1$ such that*

$$u(x) \ge \frac{1}{2^a} \max_{B_\sigma(x)} u \quad \textit{and} \quad \sigma^a u(x) \ge \frac{1}{2^a} u(0),$$

where $\sigma = (1 - |x|)/2$.

Proof. Consider

$$v(y) = (1 - |y|)^a u(y).$$

Let $x \in B_1$ be a maximum point of v and let $\sigma = (1 - |x|)/2$. It is easy to see that x and σ have the desired properties.

Proof of Theorem 1.5. The proof is by contradiction. Suppose the contrary; then there exist solutions of (7) u_j, $j = 1, 2, \ldots$, such that

$$u_j(\bar{x}_j) \min_{\overline{B}_2} u_j > j, \tag{45}$$

where $u_j(\bar{x}_j) = \max_{\overline{B}_1} u_j(y)$.

Applying Lemma 5.1 to $u = u_j(\cdot + \bar{x}_j)$ and $a = (n-2)/2$, we find $x_j \in B_1(\bar{x}_j)$ such that

$$u_j(x_j) \ge 2^{\frac{2-n}{2}} \max_{B_{\sigma_j}(x_j)} u_j \quad \text{and} \quad (\sigma_j)^{\frac{n-2}{2}} u_j(x_j) \ge 2^{\frac{2-n}{2}} u_j(\bar{x}_j),$$

where

$$\sigma_j = \tfrac{1}{2}(1 - |x_j - \bar{x}_j|) \le \tfrac{1}{2}.$$

It follows that

$$u_j(x_j) \ge u_j(\bar{x}_j); \tag{46}$$

also, using (45), we have

$$\gamma_j := u_j(x_j)^{\frac{2}{n-2}}\sigma_j \ge \tfrac{1}{2}u_j(\bar{x}_j)^{\frac{2}{n-2}} \ge \tfrac{1}{2}[u_j(\bar{x}_j)\min_{\overline{B}_2} u_j]^{\frac{1}{n-2}} \ge \tfrac{1}{2}j^{\frac{1}{n-2}} \to \infty. \tag{47}$$

Set

$$w_j(y) = \frac{1}{u_j(x_j)}u_j\Big(x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}}\Big), \qquad |y| < \Gamma_j,$$

where

$$\Gamma_j := u_j(x_j)^{\frac{2}{n-2}}.$$

Then

$$-\Delta w_j = n(n-2)w_j^{\frac{n+2}{n-2}}, \qquad w_j > 0, \ \text{ on } B_{\Gamma_j} \tag{48}$$

and

$$1 = w_j(0) \ge 2^{\frac{2-n}{2}} \max_{B_{\gamma_j}} w_j. \tag{49}$$

On $|y| = \Gamma_j$ we have, by (45) and (46), that

$$\min_{\partial B_{\Gamma_j}} w_j \ge \frac{\min_{\overline{B}_2} u_j}{u_j(x_j)} > \frac{j}{u_j(x_j)u_j(\bar{x}_j)} \ge \frac{j}{u_j(x_j)^2} = j\Gamma_j^{2-n}. \tag{50}$$

For every fixed $x \in \mathbb{R}^n$, as in the derivation of (19), we can find $0 < r_{x,j} < 1$ such that

(51)

$$w_{j,x,\lambda}(y) := \Big(\frac{\lambda}{|y-x|}\Big)^{n-2} w_j\Big(x + \frac{\lambda^2(y-x)}{|y-x|^2}\Big) \le w_j(y), \quad 0 < \lambda \le |y-x| \le r_{x,j}.$$

It is then easy to find some $\lambda_{x,j} \in (0, r_{x,j}]$ such that

$$w_{j,x,\lambda}(y) \le w_j(y) \qquad \text{for all } 0 < \lambda \le \lambda_{x,j}, y \in B_{\Gamma_j} \setminus B_{r_{x,j}}(x). \tag{52}$$

Indeed, the above can be verified as in (21) with

$$\lambda_{x,j} = r_{x,j}\Big(\frac{\min_{B_{\Gamma_j}\setminus B_{r_{x,j}}(x)} w_j}{\max_{|y-x|\le r_{x,j}} w_j}\Big)^{\frac{1}{n-2}} \le r_{x,j}.$$

Because of (51) and (52), we can define

(53)

$$\bar{\lambda}_j(x) = \sup\{0 < \mu < \Gamma_j - |x| : w_{j,x,\lambda}(y) \le w_j(y), \ y \in B_{\Gamma_j} \setminus B_\lambda(x), \ 0 < \lambda < \mu\}.$$

Lemma 5.2. *For every $x \in \mathbb{R}^n$,*

$$\lim_{j\to\infty} \bar{\lambda}_j(x) = \infty.$$

Proof. For simplicity, we take $x = 0$. Suppose the contrary; then (along a subsequence)

$$\bar{\lambda}_j \le C < \gamma_j \tag{54}$$

for some constant C independent of j. Here we have used the fact $\gamma_j \to \infty$ (see (47)).

By the definition of $\bar{\lambda}_j$,

$$w_{j,\bar{\lambda}_j} \le w_j \quad \text{in } \Sigma_j := \{y : \bar{\lambda}_j < |y| < \Gamma_j\},$$

and therefore

$$-\Delta(w_j - w_{j,\bar{\lambda}_j}) = n(n-2)\left(w_j^{\frac{n+2}{n-2}} - w_{j,\bar{\lambda}_j}^{\frac{n+2}{n-2}}\right) \ge 0, \qquad \text{in } \Sigma_j. \tag{55}$$

Also, by (49) and (54),

$$\max_{\partial B_{\Gamma_j}} w_{j,\bar{\lambda}_j} \le C\Gamma_j^{2-n}$$

for some constant C independent of j. Therefore, by (55) and (50), for large j,

$$\min_{\partial B_{\Gamma_j}} (w_j - w_{j,\bar{\lambda}_j}) > 0.$$

Recall that

$$w_j - w_{j,\bar{\lambda}_j} = 0 \quad \text{on } \partial B_{\bar{\lambda}_j}.$$

An application of the Hopf Lemma and the strong maximum principle yields

$$(w_j - w_{j,\bar{\lambda}_j})(y) > 0, \quad \bar{\lambda}_j < |y| \le \Gamma_j \tag{56}$$

and

$$\frac{\partial(w_j - w_{j,\bar{\lambda}_j})}{\partial r}\Big|_{\partial B_{\bar{\lambda}_j}} > 0. \tag{57}$$

Consequently (see the derivation of (19)), for some $\epsilon_j > 0$,

$$w_{j,\lambda}(y) \le w_j(y), \quad \text{for } \bar{\lambda}_j \le \lambda \le \bar{\lambda}_j + \epsilon_j, \quad \lambda \le |y| \le \Gamma_j.$$

This violates the definition of $\bar{\lambda}_j$. □

Since $\gamma_j \to \infty$, one easily deduces from (48) and (49) that (along a subsequence)

$$w_j \to w \qquad \text{in } C^2_{loc}(\mathbb{R}^n)$$

for some solution w of

(58) $$-\Delta w = n(n-2)w^{\frac{n+2}{n-2}}, \quad w > 0, \quad \mathbb{R}^n.$$

By Lemma 5.2 and the convergence of w_j to w, we have

(59) $$w_{x,\lambda}(y) \le w(y) \qquad \text{for } |y-x| \ge \lambda > 0.$$

It follows, by Lemma 11.2, $w \equiv$ *constant*. This violates (58). Theorem 1.5 is established. □

6 Proof of Theorem 1.6, a Harnack-type inequality for more general equations in $\mathbb{R}^n$

Essentially the same proof of Theorem 1.5 yields a

Proof of Theorem 1.6. The proof is by contradiction. Suppose the contrary; then there exist solutions of (13) $u_j, j = 1, 2, \ldots$, such that

(60) $$u_j(\bar{x}_j) \min_{\overline{B}_{2R_j}} u_j > j/R_j^{n-2},$$

where

(61) $$u_j(\bar{x}_j) = \max_{\overline{B}_{R_j}} u_j \ge 1.$$

Applying Lemma 5.1 to $u = u_i(R_j \cdot + \bar{x}_j)$ and $a = (n-2)/2$, we can find $x_j \in B_{R_j}(\bar{x}_j)$ such that

$$u_j(x_j) \ge 2^{\frac{2-n}{2}} \max_{B_{\sigma_j}(x_j)} u_j(x)$$

and

$$(\sigma_j)^{\frac{n-2}{2}} u_j(x_j) \ge (R_j/2)^{\frac{n-2}{2}} u_j(\bar{x}_j),$$

where

$$\sigma_j = \tfrac{1}{2}(R_j - |x_j - \bar{x}_j|) \le R_j/2.$$

It follows that

(62) $$u_j(x_j) \ge u_j(\bar{x}_j)$$

and, by (60),

$$(63)\quad \gamma_j := u_j(x_j)^{\frac{2}{n-2}}\sigma_j \geq \frac{R_j}{2}u_j(\bar{x}_j)^{\frac{2}{n-2}} \geq \frac{R_j}{2}[u_j(\bar{x}_j)\min_{\overline{B}_{2R_j}} u_j]^{\frac{1}{n-2}} > \frac{1}{2}j^{\frac{1}{n-2}} \to \infty.$$

Set

$$w_j(y) = \frac{1}{u_j(x_j)}u_j\Big(x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}}\Big), \qquad |y| < \Gamma_j,$$

where

$$\Gamma_j = u_j(x_j)^{\frac{2}{n-2}}R_j.$$

Then

$$(64)\qquad -\Delta w_j = u_j(x_j)^{-\frac{n+2}{n-2}}g(u_j(x_j)w_j) \quad \text{on } B_{\Gamma_j},$$

$$(65)\qquad 1 = w_j(0) \geq 2^{\frac{2-n}{2}}\max_{\overline{B}_{\gamma_j}} w_j.$$

On $|y| = \Gamma_j$, we have by (60) and (62)

$$\min_{\partial B_{\Gamma_j}} w_j \geq \frac{\min_{\overline{B}_{2R_j}} u_j}{u_j(x_j)} > \frac{j}{u_j(x_j)u_j(\bar{x}_j)R_j^{n-2}} \geq \frac{j}{u_j(x_j)^2R_j^{n-2}} = j\Gamma_j^{2-n}.$$

As in the proof of Theorem 1.5, for every $x \in \mathbb{R}^n$, we can find $0 < \lambda_{x,j} < 1$ such that

$$w_{j,x,\lambda}(y) := \Big(\frac{\lambda}{|y-x|}\Big)^{n-2}w_j\Big(x + \frac{\lambda^2(y-x)}{|y-x|^2}\Big) \leq w_j(y), \qquad \text{forall } y \in B_{\Gamma_j}\setminus B_\lambda(x).$$

Define $\bar{\lambda}_j(x)$ as in (5.3); then Lemma 5.2 still holds. Indeed, only one change is needed in the proof: the derivation of (56) and (57). Consider

$$O = \{y \in B_{\Gamma_j}\setminus\overline{B_{\bar{\lambda}_j}} : w_j(y) < (|y|/\bar{\lambda}_j)^{n-2}w_{j,\bar{\lambda}_j}(y)\}.$$

As in the proof of (30), we have

$$w_j^{-\frac{n+2}{n-2}}\Delta w_j \leq w_{j,\bar{\lambda}_j}^{-\frac{n+2}{n-2}}\Delta w_{j,\bar{\lambda}_j}, \qquad \text{in } O.$$

Since Δw_j and $\Delta w_{j,\bar{\lambda}_j}$ are negative in O and $w_j \geq w_{j,\bar{\lambda}_j}$ in O, we have, instead of (55),

$$\Delta w_j \leq \Delta w_{j,\bar{\lambda}_j}, \qquad \text{in } O.$$

(56) and (57) follow from the arguments following (30).

Next we show

Lemma 6.1.

$$\|w_j\|_{C^1(B_{\gamma_j/2})} \le C.$$

Proof. It follows from (11) and (10) that

$$g(s) \le C(1 + s^{\frac{n+2}{n-2}}), \qquad s > 0.$$

Therefore, by (64), (65) and (61),

$$|\Delta w_j| \le C \qquad \text{on } B_{\gamma_j}.$$

Lemma 6.1 follows from standard $W^{2,p}$ estimates and Sobolev embedding theorems. □

By Lemma 6.1, we know that along a subsequence,

$$w_j \to w \qquad \text{in } C^0_{loc}(\mathbb{R}^n),$$

where w satisfies $w \ge 0$, $w(0) = 1$.

By the convergence of w_j to w and the fact that $\bar{\lambda}_j(x) \to \infty$ for every $x \in \mathbb{R}^n$, we have (59). Again, by Lemma 11.2, $w \equiv$ *constant*.

Let

$$c = \limsup_{j \to \infty} u_j(x_j) \ge 1.$$

If $c = \infty$, we see easily by (64), (12), and the convergence of w_j to w, that for some $a > 0$,

$$-\Delta w = a w^{\frac{n+2}{n-2}}, \quad w > 0, \qquad \text{on } \mathbb{R}^n.$$

If $c < \infty$, then

$$-\Delta w = c^{-\frac{n+2}{n-2}} g(cw) \quad w > 0, \qquad \text{on } \mathbb{R}^n.$$

Neither of the above is possible since w is identically a constant. Theorem 1.6 is established. □

7 A Harnack-type inequality on half Euclidean balls, the first part of Theorem 1.7

In this section, we establish the Harnack-type inequality (15) in Theorem 1.7. For $x \in \mathbb{R}^n$, $n \ge 3$, we write $x = (x', t)$, where $x' = (x_1, \ldots, x_{n-1})$. We also use the following notation:

$$B_R(x) = B(x, R) = \{y \in \mathbb{R}^n : |y - x| < R\}, \qquad B_R = B_R(0),$$

$$B_R^T(x) = B(x,R) \cap \{t > T\}, \quad B_R^+(x) = B(x,R) \cap \{t > 0\}, \quad B_R^+ = B_R^+(0),$$
$$\partial'' B_R^T(x) = \partial B_R^T(x) \cap \{t > T\}, \qquad \partial' B_R^T(x) = \partial B_R^T(x) \cap \{t = T\},$$
$$\partial' B_\sigma^+(x) = \partial B_\sigma^+(x) \cap \partial \mathbb{R}_+^n, \qquad \partial'' B_\sigma^+(x) = \partial B_\sigma^+(x) \cap \mathbb{R}_+^n, \qquad B_\sigma^+ = B_\sigma^+(0).$$

In this section, we give a proof of the Harnack-type inequality (15) in Theorem 1.7.

Proof of (15) in Theorem 1.7. We argue it by contradiction. If (15) were not true, we would have solutions $\{u_j\}$ of (14) on $B_{3R_j}^+$ such that

$$u_j(x_j) \inf_{\partial B_{2R_j}^+} u_j > jR_j^{2-n},$$

where $u_j(x_j) = \max_{\overline{B_{R_j}^+}} u_j$. It follows that

$$u_j(x_j) R_j^{\frac{n-2}{2}} \to \infty. \tag{66}$$

Before proceeding further, we record the following elementary lemma, which is similar to Lemma 5.1, and whose simple proof we leave to the reader.

Lemma 7.1. *Let $u \in C^0(\overline{B_1^{-T}})$ be a positive function, $T \geq 0$. Then for every $a > 0$, there exists $x \in B_1 \cap \{t \geq -T\}$ such that for $\sigma = (1 - |x|)/2$,*

$$u(x) \geq \frac{1}{2^a} \max_{\overline{B_\sigma^{-T}(x)}} u \quad \text{and} \quad \sigma^a u(x) \geq \frac{1}{2^a} u(0).$$

Applying Lemma 7.1 to $u_j(x_j + R_j/4\cdot)$ with $a = (n-2)/2$ and $T = 4x_{jn}/R_j$ (x_{jn} denotes the n-th component of x_j), we find $z_j \in \overline{B(x_j, R_j/4) \cap \mathbb{R}_+^n}$ such that

$$u_j(z_j) \geq 2^{\frac{2-n}{2}} u_j(x) \qquad \text{for } x \in B(z_j, \sigma_j) \cap \overline{\mathbb{R}_+^n} \tag{67}$$

and

$$(2\sigma_j)^{\frac{n-2}{2}} u_j(z_j) \geq u_j(x_j)(R_j/4)^{\frac{n-2}{2}} \to \infty, \tag{68}$$

where $\sigma_j = \frac{1}{2}(\frac{1}{4}R_j - |z_j - x_j|) \leq \frac{1}{8}R_j$. Set $\gamma_j := u_j(z_j)^{\frac{2}{n-2}}\sigma_j$ and $\Gamma_j := u_j(z_j)^{\frac{2}{n-2}}R_j$. It follows from (66), (67) and (68) that

$$u_j(z_j) \geq u_j(x_j), \qquad \Gamma_j \geq 8\gamma_j \to \infty. \tag{69}$$

Consequently,

$$u_j(z_j) \inf_{\partial'' B_{2R_j}^+} u_j > jR_j^{2-n}. \tag{70}$$

Let

$$T_j := u_j(z_j)^{\frac{2}{n-2}} z_{jn}$$

and set

$$v_j(y) = \frac{1}{u_j(z_j)} u_j\Big(z_j + \frac{y}{u_j(z_j)^{\frac{2}{n-2}}}\Big), \qquad y \in \Omega_j,$$

where

$$\Omega_j = \Big\{y \ ; \ z_j + \frac{y}{u_j(z_j)^{\frac{2}{n-2}}} \in B^+_{2R_j}\Big\}.$$

Clearly v_j satisfies

$$\begin{cases} \Delta v_j + n(n-2)v_j^{\frac{n+2}{n-2}} = 0, & \text{in } \Omega_j, \\ \dfrac{\partial v_j}{\partial t} = c v_j^{\frac{n}{n-2}}, & \text{on } t = -T_j, \\ v_j(0) = 1, \ \text{ and } v_j(y) \le 2^{\frac{n-2}{2}} & \text{for } \ y \in \Omega_j \text{ and } |y| \le \gamma_j. \end{cases}$$

Let $\partial''\Omega_j = \partial\Omega_j \cap \{y \ ; \ y_n > -T_j\}$. It is clear that

$$\frac{1}{10}\Gamma_j \le dist(0, \partial''\Omega_j) \le 10\Gamma_j,$$

and by (70) and the above,

$$\text{(71)} \qquad \inf_{y\in\partial''\Omega_j} (|y|^{n-2} v_j(y)) \ge \frac{u_j(z_j)\inf_{\partial'' B^+_{2R_j}} u_j}{u_j(z_j)^2} \inf_{y\in\partial''\Omega_j} (|y|^{n-2}) \to \infty.$$

We divide the remaining proof of Theorem 1.7 into two cases (after passing to a subsequence).

Case 1: $\lim_{j\to\infty} T_j = \infty$.

Case 2: $\lim_{j\to\infty} T_j = T \in [0, \infty)$.

Reaching a contradiction in Case 1. Since $\min\{\gamma_j, T_j\} \to \infty$, $\{v_j\}_{j=1,2,\ldots}$ is uniformly bounded on compact subsets of $\mathbb{R}^n$. It follows from standard elliptic estimates that v_j (or a subsequence) converges in C^2 norm to some U on compact subsets of $\mathbb{R}^n$, where U is a positive solution of (2).

For $x \in \mathbb{R}^n$ and $\lambda < T_j/2$, let $v^\lambda_{j,x}$ denote the Kelvin transformation of v_j with respect to the $B_\lambda(x)$, i.e.,

$$v^\lambda_{j,x}(y) = \Big(\frac{\lambda}{|y-x|}\Big)^{n-2} v_j\Big(x + \frac{\lambda^2(y-x)}{|y-x|^2}\Big), \qquad y \in \Sigma^\lambda_{j,x} := \Omega_j \backslash \overline{B_\lambda(x)}.$$

Clearly $v^\lambda_{j,x}$ satisfies the same equation of v_j in $\Sigma^\lambda_{j,x}$.

As in the proof of Theorem 1.5, we can find $\lambda_{j,x} > 0$ such that

$$v_{j,x}^{\lambda}(y) < v_j(y) \quad \text{for } y \in \Sigma_{j,x}^{\lambda} \text{ and } 0 < \lambda \leq \lambda_{j,x}.$$

Define

$$\bar{\lambda}_j(x) := \sup\{\mu > 0 : v_{j,x}^{\lambda}(y) \leq v_j(y) \text{ for } y \in \overline{\Sigma_{j,x}^{\lambda}} \text{ and } 0 < \lambda \leq \mu\}.$$

Lemma 7.2. *$\bar{\lambda}_j(x) \to \infty$ as $j \to \infty$.*

Proof. Without loss of generality, we take $x = 0$. Suppose the contrary; then along a subsequence, $\bar{\lambda}_j \leq C$. Let $w_\lambda = v_j - v_j^\lambda$. To reach a contradiction, we only need to show that

$$\frac{\partial w_{\bar{\lambda}_j}}{\partial \nu}(y) > 0 \qquad \text{for } y \in \partial B_{\bar{\lambda}_j} \tag{72}$$

and

$$w_{\bar{\lambda}_j}(y) > 0 \qquad \text{for } y \in \overline{\Sigma_{\bar{\lambda}_j}} \setminus \partial B_{\bar{\lambda}_j}, \tag{73}$$

where ν denotes the unit outer normal of $\partial B_{\bar{\lambda}_j}$.

Indeed, we easily deduce from (72) and (73) that $w_\lambda \geq 0$ on $\overline{\Sigma_\lambda}$ for λ close to $\bar{\lambda}_j$, violating the definition of $\bar{\lambda}_j$.

It is clear that

$$w_{\bar{\lambda}_j} \geq 0 \quad \text{in } \Sigma_{\bar{\lambda}_j}$$

and

$$\Delta w_{\bar{\lambda}_j}(y) + b_j(y) w_{\bar{\lambda}_j}(y) = 0 \quad \text{in } \Sigma_{\bar{\lambda}_j},$$

where

$$b_j(y) = n(n-2) \frac{v_j(y)^{\frac{n+2}{n-2}} - v_j^{\bar{\lambda}_j}(y)^{\frac{n+2}{n-2}}}{v_j(y) - v_j^{\bar{\lambda}_j}(y)}.$$

By (71) and the boundedness of $\bar{\lambda}_j$, $w_{\bar{\lambda}_j} > 0$ on $\overline{\partial'' \Omega_j}$; thus, by the strong maximum principle and the Hopf lemma, we have (72) and

$$w_{\bar{\lambda}_j}(y) > 0 \qquad \text{for } y \in \Sigma_{\bar{\lambda}_j}.$$

To show (73), we only need to establish

$$w_{\bar{\lambda}_j}(y) > 0 \quad \text{on } \{t = -T_j\} \cap \partial \Omega_j.$$

This follows from the following

Lemma 7.3. *Suppose $T_j \to \infty$ and $\{\bar\lambda_j\}$ are bounded. Then for any $N > 0$, there exists $j_0 > 1$ such that for $j > j_0$,*

$$\frac{\partial v_j^{\bar\lambda_j}(z)}{\partial t} > N v_j^{\bar\lambda_j}(z)^{\frac{n}{n-2}}, \qquad \textit{for } z \in \partial\Omega_j \cap \{t = -T_j\}.$$

Indeed, if for some z with $z_n = -T_j$,

$$w_{\bar\lambda_j}(z) = 0,$$

then z is a minimum point; and by Lemma 7.3 and for large j,

$$0 \le \frac{\partial w_{\bar\lambda_j}}{\partial t}(z) = c v_j(z)^{\frac{n}{n-2}} - \frac{\partial v_j^{\bar\lambda_j}}{\partial t}(z) = c(v_j^{\bar\lambda_j}(z))^{\frac{n}{n-2}} - \frac{\partial v_j^{\bar\lambda_j}}{\partial t}(z) < 0,$$

a contradiction.

Proof of Lemma 7.3. Since $T_j \to \infty$ and $\{\bar\lambda_j\}$ is bounded from above by positive constants, we have for large j,

$$\frac{1}{2}U(0) < v_j\Big(\frac{\bar\lambda_j^2 z}{|z|^2}\Big) < 2U(0) \ \text{ and } \ \Big|\nabla v_j\Big(\frac{\bar\lambda_j^2 z}{|z|^2}\Big)\Big| < |\nabla U(0)|+1, \ z \in \partial\Omega_j \cap \{t = -T_j\}.$$

By a direct computation,

$$\begin{aligned}\frac{\partial v_j^{\bar\lambda_j}}{\partial t}(z) &\ge (n-2)\bar\lambda_j^{n-2} T_j |z|^{-n} v_j\Big(\frac{\bar\lambda_j^2 z}{|z|^2}\Big) - \bar\lambda_j^n |z|^{-n}\Big|\nabla v_j\Big(\frac{\bar\lambda_j^2 z}{|z|^2}\Big)\Big| \\ &\ge m\bar\lambda_j^{n-2} T_j |z|^{-n} > N v_j^{\bar\lambda_j}(z)^{\frac{n}{n-2}},\end{aligned}$$

where m is a positive constant independent of j. Lemma 7.3 is established. So is Lemma 7.2. □

It follows from Lemma 7.2 and the convergence of v_j to U that for every $x \in \mathbb{R}^n$,

$$U_{x,\lambda}(y) := \Big(\frac{\lambda}{|y-x|}\Big)^{n-2} U\Big(x + \frac{\lambda^2(y-x)}{|y-x|^2}\Big) \le U(y), \qquad \text{for } |y-x| \ge \lambda > 0.$$

By Lemma 11.2, $U \equiv \textit{constant}$, a contradiction.

We have reached a contradiction in Case 1. Now we

Reach a contradiction in Case 2. For convenience, let $\hat v_j$ be a translation of v_j given by

$$\hat v_j(y) = v_j(y - T_j e_n), \qquad y \in \hat\Omega_j,$$

where $e_n = (0', 1)$ and $\hat\Omega_j = \Omega_j + T_j e_n$.

Clearly $\hat{v}_j$ satisfies

$$\begin{cases} \Delta \hat{v}_j + n(n-2)\hat{v}_j^{\frac{n+2}{n-2}} = 0, & \text{in } \hat{\Omega}_j, \\ \dfrac{\partial \hat{v}_j}{\partial t} = c\hat{v}_j^{\frac{n}{n-2}}, & \text{on } t = 0, \\ \hat{v}_j(T_j e_n) = 1, \text{ and } \hat{v}_j(y) \leq 2^{\frac{n-2}{2}} & \text{for } y \in \hat{\Omega}_j \text{ and } |y| \leq \gamma_j - T_j. \end{cases}$$

Let $\partial'' \hat{\Omega}_j = \partial \hat{\Omega}_j \cap \{y \,;\, y_n > 0\}$. Then for some positive constant C,

$$C^{-1}\Gamma_j \leq dist(0, \partial'' \hat{\Omega}_j) \leq C\Gamma_j$$

and

$$\inf_{\partial'' \hat{\Omega}_j} \left(\hat{v}_j(y)|y|^{n-2}\right) \to \infty. \tag{74}$$

It follows from standard elliptic estimates that after passing to a subsequence, $\hat{v}_j$ converges in C^2 norm to some $\hat{U}$ on compact subsets of $\overline{\mathbb{R}^n_+}$, where $\hat{U}$ is a solution of (4).

For every fixed $x \in \partial R^n_+$, consider the Kelvin transformation of $\hat{v}_j$

$$\hat{v}^\lambda_{j,x}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-2} \hat{v}_j\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right), \qquad y \in \hat{\Sigma}_{\lambda,x},$$

where $\hat{\Sigma}_{\lambda,x} := \hat{\Omega}_j \backslash \overline{B_\lambda(x)}$. As usual, there exists $\bar{\lambda}_{x,j} > 0$ such that

$$\hat{v}^\lambda_{j,x}(y) \leq \hat{v}_j(y) \qquad \text{for } y \in \overline{\hat{\Sigma}_{\lambda,x}} \quad \text{and} \quad 0 < \lambda \leq \lambda_{x,j}.$$

Define

$$\bar{\lambda}_j(x) := \sup\{\mu > 0 : \ \hat{v}_j(y) \geq \hat{v}^\lambda_{j,x}(y) \ \text{ for } \ y \in \overline{\hat{\Sigma}_{\lambda,x}}, \ \text{ and } \ 0 < \lambda \leq \mu\}.$$

Lemma 7.4. *$\bar{\lambda}_j(x) \to \infty$ as $j \to \infty$.*

Proof. For simplicity we take $x = 0$. Suppose, to the contrary, that along a subsequence, $\bar{\lambda}_j \leq C$. Let $w_\lambda = \hat{v}_j - \hat{v}^\lambda_j$. To reach a contradiction, we only need to show that

$$\begin{cases} w_{\bar{\lambda}_j} > 0 \text{ in } \Sigma_{\bar{\lambda}_j}, \\ \dfrac{\partial w_{\bar{\lambda}_j}}{\partial \nu} > 0 \text{ on } \partial'' B^+_{\bar{\lambda}_j} \\ w_{\bar{\lambda}_j} > 0 \text{ on } \partial' \hat{\Sigma}_{\bar{\lambda}_j} \backslash \overline{B_{\bar{\lambda}_j}}, \end{cases} \tag{75}$$

and

$$\frac{\partial w_{\bar{\lambda}_j}}{\partial \nu}(y) > 0, \qquad \text{for } y \in \partial \mathbb{R}^n_+ \cap \partial B_{\bar{\lambda}_j}, \tag{76}$$

where ν denotes the unit outer normal of the sphere $\partial B_{\bar{\lambda}_j}$, and

$$\partial' \hat{\Sigma}_{\bar{\lambda}_j} = \partial \hat{\Sigma}_{\bar{\lambda}_j} \cap \{t = 0\}.$$

Indeed, we easily deduce from (75) and (76) that $w_\lambda \geq 0$ on $\overline{\Sigma_\lambda}$ for λ bigger and close to $\bar{\lambda}_j$, violating the definition of $\bar{\lambda}_j$.

It is clear that w_λ satisfies

$$\begin{cases} \Delta w_\lambda + b_\lambda w_\lambda = 0 & \text{in } \hat{\Sigma}_\lambda, \\ \dfrac{\partial w_\lambda}{\partial t} = \dfrac{cn}{n-2} \xi^{\frac{2}{n-2}} w_\lambda & \text{on } t = 0, \end{cases}$$

where $\xi(y)$, given by the mean value theorem, is between $\hat{v}_j(y)$ and $\hat{v}_j^\lambda(y)$ and

$$b_\lambda(y) = n(n-2) \frac{\hat{v}_j(y)^{\frac{n+2}{n-2}} - \hat{v}_j^\lambda(y)^{\frac{n+2}{n-2}}}{\hat{v}_j(y) - \hat{v}_j^\lambda(y)}.$$

Since $\{\lambda_j\}$ is bounded and $\hat{v}_j$ converges to $\hat{U}$ uniformly on compact subsets, we have $\hat{v}_j^{\lambda_j}(y)|y|^{n-2} \leq C$ on $\partial'' \hat{\Omega}_j$. It follows from (74) that for large j,

$$\inf_{\partial'' \hat{\Omega}_j} w_{\bar{\lambda}_j}(y) > 0. \tag{77}$$

Estimate (75) follows from the strong maximum principle and the Hopf lemma, and estimate (76) follows from Lemma 10.1 in Appendix A. Lemma 7.4 is established. □

By Lemma 7.4 and the convergence of $\hat{v}_j$ to $\hat{U}$, we have, for every $x \in \partial \mathbb{R}^n_+$,

$$\begin{aligned} \hat{U}_{x,\lambda}(y) &:= \Big(\frac{\lambda}{|y-x|}\Big)^{n-2} \hat{U}\Big(x + \frac{\lambda^2 (y-x)}{|y-x|^2}\Big) \\ &\leq \hat{U}(y), \qquad \text{for all } y \in \mathbb{R}^n_+ \text{ and } |y-x| \geq \lambda > 0. \end{aligned}$$

By Lemma 11.3 in Appendix B, $\hat{U}$ depends only on t, a contradiction (see 2° in Remark 1.4). □

8 Harnack-type inequality for more general equations on $\mathbb{R}^n_+$, proof of Theorem 1.8

In this section, we establish Theorem 1.8. The proof is similar to the proof of (15) in Theorem 1.7.

Proof of Theorem 1.8. We follow the same line of proof of Theorem 1.7 and often use the same notation without explicit mention. Suppose the contrary; then there exist solutions $\{u_j\}$ of (18) on $B^+_{3R_j}$ such that

$$u_j(x_j) \inf_{\partial B^+_{2R_j}} u_j > jR_j^{2-n},$$

where $u_j(x_j) = \max_{\overline{B^+_{R_j}}} u_j \geq 1$. In the proof, we need to pass to subsequences several times; and we just do so without any explicit mention. Following the same selection process in the proof of Theorem 1.7, we can find $\{z_j\} \in \overline{B(x_j, R_j/4) \cap \mathbb{R}^n_+}$ such that (67), (68), (69) and (70) hold. Define v_j as in the proof of Theorem 1.7; then $v_j(y)$ satisfies (71) and

$$\begin{cases} \Delta v_j(y) + u_j(z_j)^{-\frac{n+2}{n-2}} g(u_j(z_j)v_j(y)) = 0, & \text{in } \Omega_j, \\ \dfrac{\partial v_j(y)}{\partial t} = u_j(z_j)^{-\frac{n}{n-2}} h(u_j(z_j)v_j(y)), & \text{on } t = -T_j, \\ v_j(0) = 1, \text{ and } v_j(y) \leq 2^{\frac{n-2}{2}}, & \text{for } y \in \Omega_j \text{ and } |y| \leq \gamma_j. \end{cases}$$

We divide the situation into two cases.

Case 1: $\lim_{j\to\infty} T_j = \infty$.
Case 2: $\lim_{j\to\infty} T_j = T \in [0, \infty)$.

Reaching a contradiction in Case 1. Most of the reasoning is like that in the proof of Theorem 1.7. We point out the necessary changes.

We know that $\min\{\gamma_j, T_j\} \to \infty$, so on any given compact subset of $\mathbb{R}^n$, $\{v_j\}$ is bounded by $2^{\frac{n-2}{2}}$ for j large. It follows from (G1) and (G2) that on any given compact subset K of $\mathbb{R}^n$, we have, for large j,

$$u_j(z_j)^{-\frac{n+2}{n-2}} g(u_j(z_j)v_j(y)) \leq g(v_j(y)) \leq C(K).$$

Here we have used the fact that $u(z_j) \geq u(x_j) \geq 1$. By standard elliptic estimates, v_j (after passing to a subsequence) converges in C^1 norm to some U on compact subsets of $\mathbb{R}^n$. Clearly $U(0) = 1$. Since v_j is super-harmonic, so is U; and therefore $U > 0$ on $\mathbb{R}^n$.

For a fixed $x \in \mathbb{R}^n$, let $v^\lambda_{j,x}$ be the Kelvin transformation of v_j, as in the proof of Theorem 1.7. As usual, for every x, we can find $\lambda_{j,x} > 0$ such that

$$v^\lambda_{j,x}(y) < v_j(y) \quad \text{for } y \in \Sigma^\lambda_{j,x} \text{ and } 0 < \lambda \leq \lambda_{j,x}.$$

Define $\bar{\lambda}_j(x)$ as in Section 7.

Lemma 8.1. *For every* $x \in \mathbb{R}^n$, $\bar\lambda_j(x) \to \infty$ *as* $j \to \infty$.

Proof. Without loss of generality, we take $x = 0$. Suppose, to the contrary, that along a subsequence $\bar\lambda_j \le C$. Let $w_\lambda = v_j - v_j^\lambda$. To reach a contradiction, we only need to show (72) and (73).

Let

$$O = \{y \in \Sigma_{\bar\lambda_j} \setminus \overline{B_{\bar\lambda_j}}\ ;\ v_j(y) < (|y|/\bar\lambda_j)^{n-2} v_j^{\bar\lambda_j}(y)\}.$$

The derivation of (37) yields

$$\begin{cases} v_j^{-\frac{n+2}{n-2}} \Delta v_j \le (v_j^{\bar\lambda_j})^{-\frac{n+2}{n-2}} \Delta v_j^{\bar\lambda_j}, & \text{in } O, \\ v_j^{-\frac{n}{n-2}} \dfrac{\partial v_j}{\partial t} \le (v_j^{\bar\lambda_j})^{-\frac{n}{n-2}} \dfrac{\partial v_j^{\bar\lambda_j}}{\partial t}, & \text{on } \partial' O, \end{cases}$$

where $\partial' O = \partial O \cap \{t = 0\}$. Since Δv_j and $\Delta v_j^{\bar\lambda_j}$ are negative in O, we have

$$\Delta(v_j - v_j^{\bar\lambda_j}) \le 0, \qquad \text{in } O.$$

The derivation of the second line in (38) yields, for some function $c_j(x')$,

$$\frac{\partial}{\partial t}(v_j - v_j^{\bar\lambda_j}) \le c(x')(v_j - v_j^{\bar\lambda_j}), \quad \text{on } \partial' O.$$

By (71) and the boundedness of $\bar\lambda_j$, $w_{\bar\lambda_j} > 0$ on $\overline{\partial'' \Omega_j}$. Estimate (72) and $w_{\bar\lambda_j} > 0$ on $\Sigma_{\bar\lambda_j}$ follow from the arguments following (38). So we only need to show that it is not possible to have $w_{\bar\lambda_j}(z) = 0$ for some z with $z_n = -T_j$. Indeed, were this to hold, we would have

$$0 \le \frac{\partial w_{\bar\lambda_j}}{\partial t}(z) = u_j(z_j)^{-\frac{n}{n-2}} h(u_j(z_j) v_j(z)) - \frac{\partial v_j^{\bar\lambda_j}}{\partial t}(z).$$

By (H2),

$$u_j(z_j)^{-\frac{n}{n-2}} h(u_j(z_j) v_j(z)) \le C v_j(z)^{\frac{n}{n-2}},$$

where C is some constant independent of j.

Thus

$$\frac{\partial v_j^{\bar\lambda_j}}{\partial t}(z) \le C v_j(z)^{\frac{n}{n-2}}.$$

This violates Lemma 7.3. □

It follows from Lemma 8.1 and the convergence of v_j to U that for every $x \in \mathbb{R}^n$,

$$U_{x,\lambda}(y) := \Big(\frac{\lambda}{|y - x|}\Big)^{n-2} U\Big(x + \frac{\lambda^2 (y - x)}{|y - x|^2}\Big) \le U(y), \qquad |y - x| \ge \lambda > 0.$$

By Lemma 11.2, $U \equiv U(0) = 1$. By (G2), we have for some positive constant a, $-\Delta v_j \geq a v_j^{\frac{n+2}{n-2}}$ in Ω_j. Let $j \to \infty$; we have $-\Delta U \geq a > 0$ in the distribution sense, a contradiction (since $U \equiv 1$). □

Reaching a contradiction in Case 2. Let $\hat{v}_j(y)$ be defined on $\hat{\Omega}_j$ as in Section 7. The equation of $\hat{v}_j(y)$ now becomes

$$\begin{cases} \Delta \hat{v}_j(y) + u_j(z_j)^{-\frac{n+2}{n-2}} g(u_j(z_j)\hat{v}_j(y)) = 0, & y \in \hat{\Omega}_j, \\ \dfrac{\partial \hat{v}_j}{\partial t}(y) = u_j(z_j)^{-\frac{n}{n-2}} h(u_j(z_j)\hat{v}_j(y)), & \text{on } \{t = 0\}, \\ \hat{v}_j(T_j e_n) = 1, \text{ and } \hat{v}_j(y) \leq 2^{\frac{n-2}{2}}, & \text{for } y \in \hat{\Omega}_j \text{ and } |y| \leq \gamma_j - T_j. \end{cases}$$

Estimate (74) still holds.

By (G1), (G2) and the fact that $u_j(z_j) \geq 1$, we know from the equation of v_j that

$$0 \leq -\Delta \hat{v}_j(y) = u_j(z_j)^{-\frac{n+2}{n-2}} g(u_j(z_j)\hat{v}_j(y)) \leq g(\hat{v}_j(y)) \leq C,$$

for $y \in \hat{\Omega}_j$ and $|y| \leq \gamma_j - T_j$. By (H1), (H2) and the fact that $u_j(z_j) \geq 1$, we have

$$-C \leq \frac{\partial \hat{v}_j}{\partial t}(y) = u_j(z_j)^{-\frac{n}{n-2}} h(u_j(z_j)\hat{v}_j(y)) \leq C \hat{v}_j(y)^{\frac{n}{n-2}} \leq C, \quad \text{on } \partial' \hat{\Omega}_j,$$

where $\partial' \hat{\Omega}_j = \partial \hat{\Omega}_j \cap \partial \mathbb{R}^n_+$.

By standard elliptic estimates and the fact that $\gamma_j \to \infty$, for $0 < \alpha < 1$ and $R > 1$,

$$\|\hat{v}_j\|_{C^\alpha(\overline{B_R^+})} \leq C(\alpha, R).$$

It follows that after we pass $\hat{v}_j$ to a subsequence, $\hat{v}_j$ converges to some $\hat{U}$ in C^α norm on compact subsets of $\overline{\mathbb{R}^n_+}$. In particular, $\hat{U}(Te_n) = 1$. Since v_j is super-harmonic in $\hat{\Omega}_j$, $\hat{U}$ is super-harmonic in $\mathbb{R}^n_+$, so $\hat{U}$ is positive in $\mathbb{R}^n_+$. Let $j \to \infty$; then either (if $u_j(z_j) \to \infty$) $\hat{U}$ satisfies

$$-\Delta \hat{U} = a \hat{U}^{\frac{n+2}{n-2}}, \qquad \text{in } \mathbb{R}^n_+,$$

for some $a > 0$, or (if $u_j(z_j) \to M \geq 1$)

$$-\Delta U_1 = M^{-\frac{n+2}{n-2}} g(M U_1), \qquad \text{in } \mathbb{R}^n_+.$$

Defining $\hat{v}^\lambda_{j,x}$, $\hat{\Sigma}_{\lambda,x}$ and $\bar{\lambda}_j(x)$ as in Section 7, we still have

Lemma 8.2. *For every $x \in \partial \mathbb{R}^n$, $\bar{\lambda}_j(x) \to \infty$ as $j \to \infty$.*

Proof. Without loss of generality, we take $x = 0$. Suppose, to the contrary, that along a subsequence $\bar{\lambda}_j \le C$. Let $w_\lambda = v_j - v_j^\lambda$. To reach a contradiction, we only need to show (75) and (76).

The equation of $\hat{v}_j^\lambda$ now is

$$\begin{cases} \Delta \hat{v}_j^\lambda(y) + \left(\left(\frac{|y|}{\lambda}\right)^{n-2} u_j(z_j)\right)^{-\frac{n+2}{n-2}} g\left(\left(\frac{|y|}{\lambda}\right)^{n-2} u_j(z_j)\hat{v}_j^\lambda(y)\right) = 0, & \text{in } \hat{\Sigma}_\lambda, \\ \frac{\partial \hat{v}_j^\lambda(y)}{\partial t} = \left(\left(\frac{|y|}{\lambda}\right)^{n-2} u_j(z_j)\right)^{-\frac{n}{n-2}} h\left(\left(\frac{|y|}{\lambda}\right)^{n-2} u_j(z_j)\hat{v}_j^\lambda(y)\right), & \text{on } \{t = 0\}. \end{cases}$$

Consider

$$O = \left\{ y \in \hat{\Sigma}_{\bar{\lambda}_j} \; ; \; v_j(y) < \left(\frac{|y|}{\bar{\lambda}_j}\right)^{n-2} v_j^{\bar{\lambda}_j}(y) \right\}.$$

As usual, we can show

$$\Delta(\hat{v}_j - \hat{v}_j^{\bar{\lambda}_j}) \le 0, \qquad \text{in } O,$$

and, for some function $c(x')$,

$$\frac{\partial}{\partial t}(\hat{v}_j - \hat{v}_j^{\bar{\lambda}_j}) \le c(x')(\hat{v}_j - \hat{v}_j^{\bar{\lambda}_j}), \qquad \text{on } \partial' O,$$

where $\partial' O = \partial O \cap \{t = 0\}$.

Since (77) still holds, $w_{\bar{\lambda}_j}$ is not identically zero. Thus we can apply the strong maximum principle, the Hopf lemma, and Lemma 10.1 in O the usual way to obtain (75) and (76).

We conclude that $\hat{U}$ depends only on t. Passing to the limit in the equation of $\hat{v}_j$, we see that either $\hat{U}$ satisfies

$$\hat{U}''(t) = a\hat{U}(t), \qquad t > 0,$$

for some $a > 0$, or for $M = \lim_{j\to\infty} u(z_j) < \infty$,

$$\hat{U}''(t) = M^{-\frac{n+2}{n-2}} g(M\hat{U}), \qquad t > 0.$$

This is impossible (see 2° in Remark 1.4). □

9 Energy estimate on half Euclidean balls, the second part of Theorem 1.7

In this section, we establish the energy estimate (16) in Theorem 1.7. We only need to prove (16) for $R = 1$. The general case follows by applying the result to

$v(\cdot) = R^{\frac{n-2}{2}}u(R\cdot)$. In order to prove (16) for $R = 1$, we analyze the interaction between large local maximum points of a solution u of

$$(78) \qquad \begin{cases} \Delta u + n(n-2)u^{\frac{n+2}{n-2}} = 0, & u > 0 \quad \text{in } B_3^+, \\ \dfrac{\partial u}{\partial t} = cu^{\frac{n}{n-2}}, & \text{on } \partial' B_3^+. \end{cases}$$

The following proposition indicates how the large local maximum points are determined; its proof is by a standard blow-up method based on the Liouville-type theorems of Caffarelli, Gidas and Spruck, and Li and Zhu. See [34] for a proof.

Proposition 9.1. *Suppose u is a solution of (78). Then for any $\epsilon \in (0,1)$, $R > 1$, there exist some positive constants $C_0^* = C_0^*(\epsilon, R, n)$, $C_1^* = C_1^*(\epsilon, R, n) > 1$ such that if $\max_{B_1^+} u > C_0^*$, there exists a set $Z = \{q_1, ..., q_k\} \subset \overline{B_2^+}$ of local maximum points of u such that for each $1 \le j \le k$, one of the two situations occurs:*

1. if $q_j \in \overline{B_2^+} \setminus \{t = 0\}$, we have

$$\left\| u(q_j)^{-1}u(u(q_j)^{-\frac{2}{n-2}}y + q_j) - \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}} \right\|_{C^2(B_R^{-T_j})} < \epsilon,$$

where $T_j = u(q_j)^{\frac{2}{n-2}}q_{jn}$; q_{jn} is the last component of q_j;

2. if $q_j \in \partial' B_2^+$, then

$$\left\| u(q_j)^{-1}u(u(q_j)^{-\frac{2}{n-2}}y + q_j) - \left(\frac{\lambda_c}{1+\lambda_c^2(|y'|^2 + |y_n - t_c|^2)}\right)^{\frac{n-2}{2}} \right\|_{C^2(B_R^+(0))} < \epsilon,$$

where $\lambda_c = 1 + (\frac{c}{n-2})^2$, $t_c = \frac{c}{(n-2)\lambda_c}$.

Moreover, for $r_j = u(q_j)^{-\frac{2}{n-2}}R$, we have

$$\begin{cases} \overline{B_{r_i}(q_i)} \cap \overline{B_{r_j}(q_j)} = \emptyset, & \textit{for } i \neq j, \\ |q_i - q_j|^{\frac{n-2}{2}}u(q_j) > C_0^*, & \textit{for } j > i, \\ u(q) \le C_1^* \textit{dist}(q, Z)^{-\frac{n-2}{2}}, & \textit{for all } q \in \overline{B_{3/2}^+}. \end{cases}$$

We deduce the energy estimate (16) in Theorem 1.7 from the following result, which says roughly that every two bubbles must be separated by a positive distance independent of u.

Proposition 9.2. *For suitably large R (depending only on n and c) and $0 < \epsilon \le e^{-R}$, there exists $d = d(R, \epsilon) > 0$ such that for all solutions u of (78) satisfying $\max_{\overline{B_1^+}} u \ge C_0^*$, we have*

$$\min\{\textit{dist}(q_i, q_j) : q_i, q_j \in Z \cap \overline{B_{3/2}^+}, i \neq j\} \ge d,$$

where C_0^* is the constant in Lemma 9.1, and Z is the set of large maximum points defined in Lemma 9.1 which is determined by ϵ, R and C_0^*.

Proposition 9.2 leads to (16) in Theorem 1.7. This is given towards the end of this section. Our main effort in this section is to establish Proposition 9.2.

We introduce the definition of isolated blow-up points and indicate some standard consequences.

Definition 9.1. Let $\{u_j\}$ be a sequence of solutions of (78). Suppose $\{x_j\}$ is a sequence of local maximum points of $\{u_j\}$ satisfying $x_j \to \bar{x} \in \overline{B_2^+}$. Then we say $x_j \to \bar{x}$ is an isolated blow-up point of $\{u_j\}$ if $\lim_{j\to\infty} u_j(x_j) = \infty$ and, for some $C > 0$ and $\bar{r} > 0$ (independent of j),

$$u_j(x)|x-\bar{x}|^{\frac{n-2}{2}} \le C \qquad \text{for } |x-\bar{x}| \le \bar{r}.$$

Proposition 9.3 and Remark 9.1 below can be found in [34] (see Proposition 1.3 and Proposition 1.4 there).

Proposition 9.3. *Let $x_j \to \bar{x} \in B_2^+ \cup \partial' B_2^+$ be an isolated blow-up point of $\{u_j\}$. Then for any sequence of positive numbers $R_j \to \infty, \epsilon_j \to 0$, there exists a subsequence of $\{u_j\}$ (still denoted as $\{u_j\}$) such that $r_j := R_j u_j^{-\frac{2}{n-2}}(x_j) \to 0$ and one of the following two assertions holds.*
1. If $x_j \in \mathbb{R}^n_+$, then

$$\left\| u_j(x_j)^{-1} u_j(u_j(x_j)^{-\frac{2}{n-2}} y + x_j) - \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}} \right\|_{C^2(B_{3R_j}^{-T_j}(x_j))} < \epsilon_j,$$

where $T_j = u_j(x_j)^{\frac{2}{n-2}} x_{jn}$.
2. If $x_j \in \partial' B_2^+$, then

$$\left\| u_j(x_j)^{-1} u_j(u_j(x_j)^{-\frac{2}{n-2}} y+x_j) - \left(\frac{\lambda_c}{1+\lambda_c^2(|y'|^2+|y_n-t_c|^2)}\right)^{\frac{n-2}{2}} \right\|_{C^2(B_{3R_j}^+(x_j))} < \epsilon_j,$$

where $\lambda_c = 1 + (\frac{c}{n-2})^2$ and $t_c = \frac{c}{(n-2)\lambda_c}$.
Moreover, there exists $r_1 \in (0, \bar{r})$ (independent of j) such that

$$u_j(x) \le C u_j(x_j)^{-1} |x-x_j|^{2-n}, \quad \textit{for } x \in \overline{B_{r_1}^+ \backslash B_{2r_j}},$$

where C is independent of j; $\bar{r}$ is the one in Definition 9.1.

Proposition 9.4. *Using the notation in Proposition 9.3, we have*

$$|D^k u_j(x)| \le \frac{C_k}{|x|^{n-2+k}} u_j(x_j)^{-1} \qquad \textit{for } 2r_j < |x-x_j| < r_1/2,\ x_n > 0, \tag{79}$$

where $D^k u_j$ is understood as all partial derivatives of u_j of order k.

Proof of Proposition 9.4. For any $4r_j \le |x - x_j| \le r_1/4$, set

$$v_j(y) = r^{n-2} u_j(x_j) u_j(ry),$$

where $r = |x - x_j|$ and $y \in \Omega := \{\frac{1}{2} < |y| < 2\ ;\ ry \in B_2^+\}$. By Proposition 9.3, $v_j \le C$ in Ω. By the equations of u_j, v_j satisfies

$$-\Delta v_j = n(n-2) r^{-2} u_j(x_j)^{-\frac{4}{n-2}} v_j^{\frac{n+2}{n-2}} \qquad \text{in } \Omega,$$

and, if $\partial'\Omega := \partial\Omega \cap \{y\ ;\ ry \in \partial' B_2^+\} \ne \emptyset$,

$$\frac{\partial v_j}{\partial t} = c r^{-1} u_j(x_j)^{-\frac{2}{n-2}} v_j^{\frac{n}{n-2}}, \qquad \text{on } \partial'\Omega.$$

Since $r \ge r_j$ and $R_j \to \infty$, the coefficients $r^{-2} u_j(x_j)^{-\frac{4}{n-2}}$ and $r^{-1} u_j(x_j)^{-\frac{2}{n-2}}$ tend to zero. By standard elliptic estimates,

$$|D^k v_j(y)| \le C_k, \qquad \text{for } k \ge 1 \text{ and } y \in \Omega \cap \partial B_1,$$

which implies (79). Proposition 9.4 is established. □

Remark 9.1. As a consequence of Proposition 9.3, for each isolated blow-up point $x_j \to \bar{x}$ of u_j, we have

$$u_j(x_j) u_j \to h \qquad \text{in } C^2_{loc}(\overline{B^+_{r_1}(\bar{x})} \backslash \{\bar{x}\})$$

for some $h \in C^2_{loc}(\overline{B^+_{r_1}(\bar{x})} \backslash \{\bar{x}\})$ satisfying

$$\begin{cases} \Delta h(x) = 0, & B^+_{r_1}(\bar{x}) \backslash \{\bar{x}\}, \\ h(x) \to \infty, & \text{as } x \to \bar{x}, \\ \dfrac{\partial h(x)}{\partial t} = 0, & x \in \partial' B_{r_1}(\bar{x}) \quad \text{if } \partial' B_{r_1}(\bar{x}) \ne \emptyset. \end{cases}$$

Remark 9.2. In fact, the domain of the harmonic function h and the convergence of $u_j(x_j) u_j$ to h can be extended to $\overline{B^+_{\bar{r}}(\bar{x})} \setminus (\{\bar{x}\} \cup \partial'' B^+_{\bar{r}}(\bar{x}))$.

Proof of Remark 9.2. This is rather standard. For the reader's convenience, we include a proof. It is enough to show that for any $r \in (0, r_1/4)$, $u_j(x_j) u_j$ converges in C^2 norm over $K = \overline{B^+_{\bar{r}-r}(\bar{x}) \setminus B^+_r(\bar{x})}$. It follows from Definition 9.1 that there exists $C = C(r) > 0$ such that

$$u_j \le C \quad \text{on } B^+_{\bar{r}-r/2}(\bar{x}) \setminus \overline{B^+_{r/2}(\bar{x})}.$$

Then u_j satisfies

$$\begin{cases} |\Delta u_j| \le C u_j & \text{in } K_1, \\ |\frac{\partial u_j}{\partial t}| \le C u_j & \text{on } \partial K_1 \cap \{t=0\}, \end{cases}$$

where $K_1 = B^+_{\bar r - r/2}(\bar x) \setminus \overline{B^+_{r/2}(\bar x)}$. By the Harnack inequality (see, e.g., Lemma A.1 in [34]), $\max_K u_j \le C \min_K u_j$. Then by Proposition 9.3, $u_j(x_j)\max_K u_j \le C u_j(x_j) \min_K u_j \le C$, i.e., $u_j(x_j)u_j$ is uniformly bounded over K. The equation satisfied by $u_j(x_j)u_j$ is

$$\begin{cases} \Delta(u_j(x_j)u_j) + n(n-2)u_j(x_j)^{-\frac{4}{n-2}}(u_j(x_j)u_j)^{\frac{n+2}{n-2}} = 0, & K_1, \\ \frac{\partial(u_j(x_j)u_j)}{\partial t} = c u_j(x_j)^{-\frac{2}{n-2}}(u_j(x_j)u_j)^{\frac{n}{n-2}}, & \partial K_1 \cap \{t=0\}. \end{cases}$$

Since $u_j(x_j) \to \infty$, $u_j(x_j)u_j$ converges to a harmonic function h over K. Remark 9.2 is established. □

We first prove Proposition 9.2; and towards the end of this section, we use Proposition 9.2 to establish (16) in Theorem 1.7.

The following two lemmas say that the magnitudes of two bubbles in set Z are comparable as long as they are not too close to $\partial'' B_3^+$. Note that in [34] two closest bubbles can be found because the solution is defined on the whole manifold. Here we do not have this advantage. The nature of our problem is purely local.

Lemma 9.1. *Let u be a solution of (78). Then there exists $R_0 = R_0(n,c) \ge 1$ such that for any $R \ge R_0$ and $0 < \epsilon \le e^{-R}$, we have*

$$u(q)u(x) \ge C^{-1}|x-q|^{2-n} \tag{80}$$

for any $q \in Z$ and $x \in \overline{B^+_{3/2}}$ satisfying $Ru(q)^{-\frac{2}{n-2}} \le |x-q| \le \frac{1}{4}$. Here Z is the set defined in Proposition 9.1 with respect to R and ϵ, and C is some constant depending only on R_0.

Proof of Lemma 9.1. Let $e_n = (0', 1)$,

$$\Omega = B(e_n, u(q)^{\frac{2}{n-2}}) \cap \{t > -u(q)^{\frac{2}{n-2}} q_n\},$$

and

$$v(y) = u(q)^{-1} u(u(q)^{-\frac{2}{n-2}} y + q), \qquad y \in \Omega.$$

It follows from Proposition 9.1 that

$$\left\| v(y) - \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}} \right\|_{C^2(\overline{B_R \cap \Omega})} < \epsilon$$

or

$$\left\| v(y) - \left(\frac{\lambda_c}{1 + \lambda_c^2(|y'|^2 + |y_n - t_c|^2)} \right)^{\frac{n-2}{2}} \right\|_{C^2(\overline{B_R \cap \Omega})} < \epsilon.$$

In either case, we have for some $\delta_1 = \delta_1(n, c) > 0$,

$$v(y) > \delta_1 |y|^{2-n}, \qquad y \in \overline{\Omega} \cap \partial B_R.$$

Here we have used the largeness of R_0.

To prove (80), we only need to show for some $\delta_2 = \delta_2(n, c) > 0$ that

$$v(y) \geq \delta_2 |y|^{2-n} \qquad \text{for } y \in (\overline{\Omega} \setminus B_R) \cap B(e_n, u(q)^{\frac{2}{n-2}}/2). \tag{81}$$

To see this, we set

$$\phi(y) = 2\delta_2(|y - e_n|^{2-n} - u(q)^{-2}),$$

where

$$\delta_2 = \min\left\{\delta_1/4, \frac{1}{2}\left(\frac{n-2}{|c|+1}\right)^{\frac{n-2}{2}}\right\}.$$

Clearly,

$$v(y) > \phi(y) \qquad \text{on } \partial B_R \cap \overline{\Omega},$$

and

$$v(y) > 0 = \phi(y) \qquad \text{for } |y - e_n| = u(q)^{\frac{2}{n-2}}.$$

By a direct computation,

$$\frac{\partial \phi(y)}{\partial t} \geq 2\delta_2(n-2)|y - e_n|^{-n} > |c|\phi(y)^{\frac{n}{n-2}}, \quad \text{on } \{t = -u(q)^{\frac{2}{n-2}} q_n\}.$$

It follows that for some $\xi \geq 0$,

$$\frac{\partial (v - \phi)}{\partial t} \leq |c| v^{\frac{n}{n-2}} - |c| \phi^{\frac{n}{n-2}} \leq \xi(v - \phi), \quad \text{on } \{t = -u(q)^{\frac{2}{n-2}} q_n\}.$$

Since $v - \phi$ is super-harmonic in $(\Omega \setminus \overline{B}_R) \cap B(e_n, u(q)^{\frac{2}{n-2}}/2)$, we apply the maximum principle to obtain that $v - \phi \geq 0$ on $(\Omega \setminus \overline{B}_R) \cap B(e_n, u(q)^{\frac{2}{n-2}}/2)$, from which (81) follows. □

Lemma 9.2. *For suitably large R and $0 < \epsilon \leq e^{-R}$, there exists $C = C(\epsilon, R, n)$ such that for any solution u of (78) and any $q \in Z \cap \overline{B_{3/2}^+}$, we have*

$$u(x) \leq C_1 u(q) \qquad \text{for } x \in \overline{B^+(q, 1/12)}, \tag{82}$$

where Z is the set of local maximum points of u defined in Proposition 9.1.

Proof of Lemma 9.2. By the Harnack-type inequality (Theorem 1.7),

$$\sup_{B^+(q,1/12)} u \inf_{B^+(q,1/6)} u \leq C.$$

Now u is well-approximated by standard bubbles; so, obviously, $u(x) \geq u(q)^{-1}$ for $|x-q| \leq Ru(q)^{-\frac{2}{n-2}}$. Thus, by (80), we have $\min_{B^+(q,1/6)} u \geq C^{-1}u(q)^{-1}$. Estimate (82) follows easily from above. Lemma 9.2 is established. □

Proof of Proposition 9.2. Suppose the contrary; then for some fixed large R_0 and $0 < \epsilon_0 \leq e^{-R_0}$, there is no such $d = d(\epsilon_0, R_0)$. Consequently, there exist a sequence of solutions $\{u_j\}$ to (78) and some $q_{1j} \in Z_j$ such that $\mathrm{dist}(q_{1j}, Z_j \backslash \{q_{1j}\}) \to 0$, where Z_j is the set of local maximum points of u_j defined in Proposition 9.1 with respect to ϵ_0 and R_0. Let q_{2j} be the local maximum of u_j in Z_j so that $\mathrm{dist}(q_{1j}, Z_j\backslash\{q_{1j}\}) = |q_{1j} - q_{2j}|$. Then we have

$$\sigma_j := |q_{1j} - q_{2j}| \to 0.$$

By Proposition 9.1, we have

$$u_j(y)\mathrm{dist}(y, Z_j)^{\frac{n-2}{2}} \leq C_1^*(\epsilon_0, R_0), \quad y \in \overline{B_2^+}. \tag{83}$$

Since $B(q_{1j}, u_j(q_{1j})^{-\frac{2}{n-2}}R_0)$ and $B(q_{2j}, u_j(q_{2j})^{-\frac{2}{n-2}}R_0)$ must be disjoint, we have $\sigma_j > u_j(q_{lj})^{-\frac{2}{n-2}}R_0$, $l = 1, 2$. Consequently,

$$u_j(q_{1j}), \ u_j(q_{2j}) \to \infty \quad \text{as } j \to \infty.$$

For the sake of simplicity, we write q_1 and q_2 instead of q_{1j} and q_{2j} later in this section. Still by Proposition 9.1, we have

$$\|u_j(q_l)^{-1}u_j(u_j(q_l)^{-\frac{2}{n-2}}y + q_l) - (1+|y|^2)^{-\frac{n-2}{2}}\|_{C^2(B_{R_0}^{-T_{jl}})} < \epsilon_0, \quad l = 1, 2$$

or

$$\left\|u_j(q_l)^{-1}u_j(u_j(q_l)^{-\frac{2}{n-2}}y + q_l) - \left(\frac{\lambda_c}{1+\lambda_c^2(|y'|^2 + |y_n - t_c|^2)}\right)^{\frac{n-2}{2}}\right\|_{C^2(B_{R_0}^+)} < \epsilon_0,$$
$$l = 1, 2,$$

where $T_{jl} = u_j(q_l)^{-\frac{2}{n-2}}q_{ln}$.

Lemma 9.3. *For any $N_j \to \infty$ and $0 < \epsilon_j \leq e^{-N_j}$, there exists a subsequence $\{u_{i_j}\}$ (still denoted as $\{u_j\}$) such that $\sigma_j > u_j(q_l)^{-\frac{2}{n-2}}N_j$ for $l = 1, 2$ and one of the following two assertions holds.*
1. If $q_l \in B_{3/2}^+$, we have

$$\|u_j(q_l)^{-1}u_j(u_j(q_l)^{-\frac{2}{n-2}}y + q_l) - (1+|y|^2)^{-\frac{n-2}{2}}\|_{C^2(B_{N_j}^{-T_{jl}})} < \epsilon_j. \tag{84}$$

2. If $q_l \in \{t=0\}$, *then*

(85)

$$\left\| u_j(q_l)^{-1} u_j(u_j(q_l)^{-\frac{2}{n-2}} y + q_l) - \left(\frac{\lambda_c}{1+\lambda_c^2(|y'|^2+|y_n-t_c|^2)}\right)^{\frac{n-2}{2}} \right\|_{C^2(B^+_{N_j})} < \epsilon_j.$$

Proof. Let

$$v_j(y) = u_j(q_1)^{-1} u_j(u_j(q_1)^{-\frac{2}{n-2}} y + q_1)$$

be defined in

$$\mathrm{Dom}(v_j) := B(0, \tfrac{1}{12} u_j(q_1)^{\frac{2}{n-2}}) \cap \{t > -u_j(q_1)^{\frac{2}{n-2}} q_{1n}\}.$$

By Lemma 9.2, v_j satisfies

$$\begin{cases} \Delta v_j(y) + n(n-2) v_j(y)^{\frac{n+2}{n-2}} = 0, & y \in \mathrm{Dom}(v_j), \\ \dfrac{\partial v_j(y)}{\partial t} = c v_j^{\frac{n}{n-2}}(y), & \text{on } \partial'\mathrm{Dom}(v_j), \\ v_j(0) = 1, \quad v_j(y) \le C, & \text{for } y \in \overline{\mathrm{Dom}(v_j)}, \end{cases}$$

where $\partial'\mathrm{Dom}(v_j) = \partial\mathrm{Dom}(v_j) \cap \{t = -u_j(q_1)^{\frac{2}{n-2}} q_{1n}\}$. Since $u_j(q_1) \to \infty$, v_j is uniformly bounded on any compact subset of $\{t \ge -\lim_{j\to\infty} u_j(q_1)^{\frac{2}{n-2}} q_{1n}\}$. After passing $\{v_j\}$ to a subsequence if necessary, (84) and (85) follow from the Liouville-type theorems of Caffarelli–Gidas–Spruck and Li–Zhu. Similarly, we have (84) and (85) after applying the same argument to q_2. Since q_2 is a local maximum point of u_j, (84) and (85) imply that $\sigma_j > u_j(q_1)^{-\frac{2}{n-2}} N_j$ because for $|y-q_1| \le u_j(q_1)^{-\frac{2}{n-2}} N_j$, q_1 is the only local maximum point of u_j. So we have $\sigma_j > u_j(q_1)^{-\frac{2}{n-2}} N_j$. Similarly, we also have $\sigma_j > u_j(q_2)^{-\frac{2}{n-2}} N_j$. Lemma 9.3 is established. □

It follows from Lemma 9.3 that $\{B(q_l, u_j(q_l)^{-\frac{2}{n-2}} N_j)\}_{l=1,2}$ are disjoint and $u_j(q_l)\sigma_j^{\frac{n-2}{2}} \to \infty$, $l = 1, 2$. We rescale u_j to w_j so that the distance between the two local maximum points corresponding to q_1 and q_2 becomes one. Indeed, let $w_j(y) = \sigma_j^{\frac{n-2}{2}} u_j(\sigma_j y + q_1)$. Then w_j satisfies

$$\begin{cases} \Delta w_j(y) + n(n-2) w_j(y)^{\frac{n+2}{n-2}} = 0, & y \in B^{-T_j}_{1/\sigma_j}, \\ \dfrac{\partial w_j(y)}{\partial t} = c w_j(y)^{\frac{n}{n-2}}, & y \in \partial B^{-T_j}_{1/\sigma_j} \cap \{t = -T_j\}, \\ w_j(0) \to \infty, \qquad w_j(e) \to \infty, \end{cases}$$

where $e = (q_2 - q_1)/\sigma_j$ and $T_j = \sigma_j^{-1} q_{1n}$.

By Lemma 9.1 (with $u = u_j$, $R = R_0$, and $\epsilon = \epsilon_0$), we have

$$(86)\quad w_j(z) \geq C^{-1} w_j(0)^{-1} |z|^{2-n}, \quad R_0 \sigma_j^{-1} u_j(q_1)^{-\frac{2}{n-2}} \leq |z| \leq \sigma_j^{-1}/4,\ z_n \geq -T_j,$$

and

(87)

$$w_j(z) \geq C^{-1} w_j(e)^{-1} |z-e|^{2-n}, \quad R_0 \sigma_j^{-1} u_j(q_2)^{-\frac{2}{n-2}} \leq |z-e| \leq \sigma_j^{-1}/4,\ z_n \geq -T_j,$$

where $C > 0$ is a positive constant depending on n only.

We also have

$$(88)\qquad R_0 \sigma_j^{-1} u_j(q_1)^{-\frac{2}{n-2}} \to 0 \quad \text{and} \quad R_0 \sigma^{-1} u_j(q_2)^{-\frac{2}{n-2}} \to 0.$$

In the rest of the proof, we analyze how $w_j(0) w_j$ approaches a harmonic function and employ the Pohozaev Identity to get a contradiction.

Lemma 9.4 and Remark 9.3 below are in correspondence with Proposition 9.3 and Lemma 9.2.

Lemma 9.4. *Let $D_j = w_j(0)^{\frac{2}{n-2}} T_j$. After passing to a subsequence, we have the following.*

1. If $T_j > 0$,

$$(89)\qquad \left\| w_j(0)^{-1} w_j(w_j(0)^{-\frac{2}{n-2}} y) - \left(\frac{1}{1+|y|^2} \right)^{\frac{n-2}{2}} \right\|_{C^2(B_{N_j}^{-D_j})} \leq \epsilon_j.$$

2. If $T_j = 0$ for all large j, then

$$(90)\quad \left\| w_j(0)^{-1} w_j(w_j(0)^{-\frac{2}{n-2}} y) - \left(\frac{\lambda_c}{1+\lambda_c^2(|y'|^2 + |y_n - t_c|^2)} \right)^{\frac{n-2}{2}} \right\|_{C^2(B_{3N_j}^{-D_j})} < \epsilon_j.$$

In either case, let $T = \lim_{j\to\infty} T_j \in [0,\infty]$; then there exists a harmonic function h defined on $B_1^{-T} \cup (\partial B_1^{-T} \cap \{t = -T\})$ such that

$$(91)\qquad \lim_{j\to\infty} \|w_j(0) w_j - h\|_{C^2(\overline{B_{1-\beta}^{-T_j} \setminus B_\beta})} = 0, \qquad 0 < \beta < \tfrac{1}{3},$$

where h satisfies

$$(92)\qquad \begin{cases} \Delta h(y) = 0, \quad h \geq 0, & \text{in } B_1^{-T} \setminus \{0\}, \\ h(y) \to \infty, & \text{as } y \to 0, \\ \dfrac{\partial h(y)}{\partial t} = 0, & y \in \partial B_1^T \cap \{t = -T\} \quad \text{if } \partial B_1^T \cap \{t = -T\} \neq \emptyset. \end{cases}$$

Proof of Lemma 9.4. Since

$$w_j(0)^{-1}w_j\left(w_j(0)^{-\frac{2}{n-2}}y\right) = u_j(q_1)^{-1}u_j\left(u_j(q_1)^{-\frac{2}{n-2}}y + q_1\right),$$

(89) and (90) are the same as (84) and (85) ($l = 1$). Let $\hat{Z}_j = \{\sigma_j^{-1}(q - q_1) : q \in Z_j\}$ be the set of large local maximum points of w_j, the rescaled version of Z_j for u_j. Since q_2 is the nearest point in Z_j to q_1, and $|q_2 - q_1| = \sigma_j$, for any compact subset K of $B_1^{-T} \cup \partial' B_1^{-T}$, there exists $C = C(K)$ such that

$$|y| \le C(K)\,\mathrm{dist}(y, \hat{Z}_j) \qquad \text{for all } y \in K.$$

Consequently, by (83),

$$w_j(y)|y|^{\frac{n-2}{2}} \le C(K) \quad \text{for } y \in K.$$

Therefore, 0 is an isolated blow up point of $\{w_j\}$; and (91) and (92) follow from Remark 9.1 (see also Remark 9.2). Lemma 9.4 is established. □

Remark 9.3. By Lemma 9.2 (with $u = u_j$, $R = R_0$ and $\epsilon = \epsilon_0$), and the fact that $\sigma_j = |q_1 - q_2| \to 0$, we have

$$C^{-1}w_j(0) \le w_j(e) \le Cw_j(0), \qquad e = \frac{q_2 - q_1}{|q_2 - q_1|}.$$

Remark 9.4. It is not hard to see that the harmonic function h in Lemma 9.4 is of the form $h(y) = a|y|^{2-n} + b(y)$, where $a > 0$ and b is harmonic on B_1^{-T}. Moreover, if $T = 0$, b satisfies $\partial b/\partial t = 0$ on $\partial' B_1^{+}$.

To complete the proof of Proposition 9.2, we need to rule out the following two cases.

Case 1: $T = \lim_{j\to\infty} T_j \in (0, \infty]$.

Case 2: $T = \lim_{j\to\infty} T_j = 0$.

We first

Rule out Case 1. Recall that $w_j(0)w_j(y) \to a|y|^{2-n} + b(y)$ on compact subsets of $B_1^{-T}\setminus\{0\}$. We show that $b > 0$ on B_1^{-T}.

Step 1: $b \ge 0$ on B_1^{-T}.

For $0 < \epsilon < a$, let

$$\phi_j(y) = (a - \epsilon)|y|^{2-n} - (a - \epsilon)\sigma_j^{n-2}.$$

We compare $w_j(0)w_j$ and ϕ_j in $B_{\sigma_j^{-1}}^{-T_j} \setminus B_{1/j}$. Since b is harmonic (and therefore bounded) near 0, we have for large j,

$$w_j(0)w_j(y) > \phi_j(y), \qquad |y| = 1/j \text{ or } |y| = \sigma_j^{-1}.$$

It is easy to see that for $y \in \{t = -T_j\}$,

$$\frac{\partial \phi_j}{\partial t} \geq (n-2)(a-\epsilon)^{\frac{2}{n-2}} T_j \phi_j^{\frac{n}{n-2}},$$

and

$$\begin{aligned} \frac{\partial (w_j(0)w_j(y))}{\partial t} &= c w_j(0)^{-\frac{2}{n-2}} (w_j(0)w_j(y))^{\frac{n}{n-2}} \\ &< (n-2)(a-\epsilon)^{\frac{2}{n-2}} T_j (w_j(0)w_j(y))^{\frac{n}{n-2}}. \end{aligned}$$

It follows, by the mean value theorem, that

$$\frac{\partial [w_j(0)w_j(y) - \phi_j(y)]}{\partial t} < \xi_j(y)[w_j(0)w_j(y) - \phi_j(y)], \quad y \in \{t = -T_j\},$$

where $\xi_j(y) > 0$. By the maximum principle,

$$w_j(0)w_j \geq \phi_j, \qquad \text{on } B_{\sigma_j^{-1}}^{-T_j} \setminus B_{1/j}.$$

Sending j to infinity, we obtain, for any compact subset ω of $B_1^{-T} \setminus \{0\}$,

$$a|y|^{2-n} + b(y) \geq (a-\epsilon)|y|^{2-n}, \qquad y \in \omega.$$

Letting $\epsilon \to 0^+$, we have $b \geq 0$ on $B_1^{-T} \setminus \{0\}$.

Step 2: $b > 0$ on B_1^{-T}.

For any compact subset ω of B_1 and j sufficiently large (which may depend on ω), we have by (87) and (88) that

$$w_j(y) \geq C^{-1} w_j(e)^{-1} |y - e|^{2-n} \quad \text{for } y \in \omega.$$

Letting $j \to \infty$, we have, by Remark 9.3,

$$a|y|^{2-n} + b(y) \geq C^{-1}|y - \bar{e}|^{2-n}, \qquad y \in B_1^{-T} \setminus \{0\},$$

where $\bar{e} = \lim_{j\to\infty} e$. It follows that $\lim_{y \in B_1^{-T}, y \to \bar{e}} b(y) = \infty$. In particular, $b(y) > 0$ for y in B_1 and y close to e. We already know that b is non-negative and harmonic in B_1^{-T}; so by the maximum principle, $b > 0$ on B_1^{-T}.

In order to reach a contradiction, we need the following Pohozaev Identity.

Lemma 9.5. *Let Ω be a piecewise smooth bounded domain in $\mathbb{R}^n$ and $u > 0$ a $C^2(\bar{\Omega})$ solution of*

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}}, \qquad \Omega.$$

Then

$$\int_{\partial\Omega} \left\{ x \cdot \nu \Big(\frac{(n-2)^2}{2} u^{\frac{2n}{n-2}} - \frac{|\nabla u|^2}{2} \Big) + \frac{\partial u}{\partial \nu}(x \cdot \nabla u) + \frac{n-2}{2} u \frac{\partial u}{\partial \nu} \right\} = 0, \tag{93}$$

where ν is the unit outer normal of $\partial\Omega$.

A proof of Lemma 9.5 can be found in [38].

Let $\Omega = B_r$ for $0 < r < 1$ and apply Lemma 9.5 to w_j. Multiply $w_j^2(0)$ on both sides and let $j \to \infty$. Elementary computation (see Proposition 1.1 in [38]) shows that the left hand side of (93) is negative for r sufficiently small, which is clearly a contradiction. Case 1 is ruled out.

Now we

Rule out Case 2. Recall that $w_j(0)w_j(y) \to a|y|^{2-n} + b(y)$ on all compact subsets of B_1^+, with $b(y)$ harmonic in B_1^+ and satisfying $\partial b(y)/\partial t = 0$ on $\partial' B_1^+$.

Let

$$\phi_j(y) := (a-\epsilon)|y - w_j(0)^{-\frac{1}{n-2}}e_n|^{2-n} - (a-\epsilon)(\sigma_j^{-1} - 1)^{2-n}.$$

We compare $w_j(0)w_j$ and ϕ_j on

$$\Omega_j = \{y \in B_{\sigma_j^{-1}}^{-T_j} : |y - w_j(0)^{-\frac{1}{n-2}}e_n| < \sigma_j^{-1} - 1\}.$$

It is clear that

$$w_j(0)w_j(y) > \phi_j(y) \qquad \text{for } |y| = 3w_j(0)^{-\frac{1}{n-2}} \text{ or } |y - w_j(0)^{-\frac{1}{n-2}}e_n| = \sigma_j^{-1} - 1.$$

Computation shows that for $y \in \{t = -T_j\}$,

$$\frac{\partial \phi_j(y)}{\partial t} = (n-2)(a-\epsilon)^{\frac{2}{n-2}}[w_j(0)^{-\frac{1}{n-2}} + T_j]\phi^{\frac{n}{n-2}}$$

and

$$\begin{aligned}\frac{\partial(w_j(0)w_j(y))}{\partial t} &= cw_j(0)^{-\frac{2}{n-2}}(w_j(0)w_j(y))^{\frac{n}{n-2}} \\ &< (n-2)(a-\epsilon)^{\frac{2}{n-2}}[w_j(0)^{-\frac{1}{n-2}} + T_j](w_j(0)w_j(y))^{\frac{n}{n-2}}.\end{aligned}$$

By the mean value theorem,

$$\frac{\partial[w_j(0)w_j(y) - \phi_j(y)]}{\partial t} < \xi_j(y)[w_j(0)w_j(y) - \phi_j(y)], \quad y \in \{t = -T_j\},$$

where $\xi_j(y) > 0$. By the maximum principle,

$$w_j(0)w_j - \phi_j > 0, \qquad \text{on } \Omega_j.$$

Letting $j \to \infty$, we have

$$a|y|^{2-n} + b(y) \geq (a-\epsilon)|y|^{2-n}, \qquad y \in B_1^+.$$

Sending ϵ to 0 shows that

$$b(y) \geq 0, \qquad y \in B_1^+.$$

Arguing as in Case 1, we have

$$b(y) > 0, \qquad y \in B_1^+.$$

Since $\frac{\partial b}{\partial t}(0) = 0$, by the Hopf Lemma, $b(0) > 0$. Apply Lemma 9.5 to w_j and let $\Omega = B_r^+$. Multiply $w_j^2(0)$ on both sides of (93) and let $j \to \infty$. Then $b(0) > 0$ makes the left hand side of (93) negative for r small, a contradiction. Case 2 is ruled out. □

Once Proposition 9.2 is established, we can finish the proof of (16) in Theorem 1.7 as follows.

Proof of (16) in Theorem 1.7. Clearly, we only need to establish it for $R = 1$. In fact, it is clear enough to show that $\int_{B_{1/2}^+}(|\nabla u|^2 + u^{\frac{2n}{n-2}})dx \le C(n,c)$. Suppose, to the contrary, that there exists a sequence of u_j satisfying (78) such that

$$\int_{B_{1/2}^+}(|\nabla u_j|^2 + u_j^{\frac{2n}{n-2}})dx \to \infty.$$

Then by standard elliptic estimates, $\max_{\overline{B_1^+}} u_j \to \infty$. Let ϵ and R be as in Proposition 9.2, and let Z_j be defined in terms of ϵ and R for u_j. By Proposition 9.2, any two points of $Z_j \cap \overline{B_{3/2}^+}$ are separated by a distance no less than $d(\epsilon, R) > 0$. In particular, the number of points in $Z_j \cap \overline{B_{3/2}^+}$ is bounded by a fixed number k. Since $\max_{\overline{B_1^+}} u_j \to \infty$, $\max_{z\in Z_j\cap\overline{B_{4/3}^+}} u_j(z) \to \infty$. For any fixed $r > 0$, $\{u_j\}$ is bounded on $\overline{B_{4/3}^+}\setminus\bigcup_{z\in Z_j} B_r(z)$; therefore, by the Harnack inequality, the maximum and the minimum of u_j on the set are comparable. So, by Proposition 9.3, the maximum of u_j on the set tends to zero; and by standard elliptic estimates,

$$\int_{\overline{B_1^+}\setminus\cup_{z\in Z_j} B_r(z)}(|\nabla u_j|^2 + u_j^{\frac{2n}{n-2}}) \to 0.$$

On the other hand, for every $z \in Z_j \cap \overline{B_{4/3}^+}$, we have, by Proposition 9.3 and Proposition 9.4, that

$$\int_{\overline{B^+(z,r)}}(|\nabla u_j|^2 + u_j^{\frac{2n}{n-2}})dx \le C.$$

Since $Z_j \cap \overline{B_{3/2}^+}$ has at most k points,

$$\int_{\overline{B_{4/3}^+}}(|\nabla u_j|^2 + u_j^{\frac{2n}{n-2}})dx \le C,$$

a contradiction. Estimate (16) in Theorem 1.7 is established. □

As pointed out in Remark 1.10, if $c \le 0$, estimate (16) in Theorem 1.7 can be established in a much simpler way. Indeed, we have

Proof of (16) in Theorem 1.7 for $c \le 0$**.** We first show

$$\int_{B_1^+} u^{\frac{2n}{n-2}} \le C(c,n). \tag{94}$$

For $y, \eta \in B_3$, let

$$G_1(y,\eta) = \begin{cases} \dfrac{1}{n(n-2)\omega_n}\left(|y-\eta|^{2-n} - \left(\dfrac{3}{|\eta|}\right)^{n-2}\left|\dfrac{9\eta}{|\eta|^2} - y\right|^{2-n}\right), & \eta \ne 0, \\ \dfrac{1}{n(n-2)\omega_n}(|y|^{2-n} - 3^{2-n}), \quad \eta = 0, \end{cases}$$

be the Green's function on B_3 with respect to the Dirichlet boundary condition. Here ω_n is the volume of the unit ball B_1. Set

$$G(y,\eta) = G_1(y,\eta) + G_1(y,\bar\eta), \qquad y,\eta \in \overline{B_3^+},$$

where $\bar\eta = (\eta', -\eta_n)$ is the reflection of η with respect to $\partial\mathbb{R}^n_+$.

Then from above it is immediate that for fixed $y \in B_3^+ \cup (\partial' B_3^+ \setminus \partial B_3^+)$, G satisfies

$$\begin{cases} -\Delta_\eta G(y,\eta) = \delta_y, & \eta \in B_3^+, \\ G(y,\eta) = 0, \quad \eta \in \partial'' B_3^+, \\ \dfrac{\partial G(y,\eta)}{\partial \nu} = 0, \quad \eta \in \partial' B_3^+, \\ \dfrac{\partial G(y,\eta)}{\partial \nu} < 0, \quad \eta \in \partial'' B_3^+. \end{cases}$$

It is also clear that

$$G(y,\eta) \ge C^{-1}, \quad y,\eta \in \overline{B_2^+},$$

for some $C \ge 1$.

Let $u(y) = \min_{\overline{B_2^+}} u$, $y \in \overline{B_2^+}$. By Green's formula, and as in [21],

$$\begin{aligned} u(y) &= \int_{B_3^+} G(y,\eta)(-\Delta u)d\eta + \int_{\partial B_3^+} G(y,s)\frac{\partial u}{\partial \nu}(y,s)ds - \int_{\partial B_3^+} \frac{\partial G(y,s)}{\partial \nu}u(s)ds \\ &\ge \int_{B_3^+} G(y,\eta)u(\eta)^{\frac{n+2}{n-2}}d\eta - \int_{\partial' B_3^+} G(y,\eta)cu(\eta)^{\frac{n}{n-2}}d\eta - \int_{\partial'' B_3^+} \frac{\partial G(y,s)}{\partial \nu}u(s)ds \\ &\ge C^{-1}\int_{B_1^+} u^{\frac{n+2}{n-2}}(\eta)d\eta. \end{aligned}$$

Therefore,

$$\int_{B_1^+} u^{\frac{2n}{n-2}} \le \max_{B_1^+} u \int_{B_1^+} u^{\frac{n+2}{n-2}} \le C(\max_{B_1^+} u)(\min_{B_2^+} u) \le C. \qquad \square$$

The derivation of

$$\int_{B_{1/2}^+} |\nabla u|^2 \le C \tag{95}$$

from (94) is as follows. Let $\phi \in C^\infty(\overline{B_3^+})$ such that

$$\phi(y) \equiv 1, \quad y \in \overline{B_{1/2}^+}, \qquad \phi(y) \equiv 0, \quad |y| \ge 1/\sqrt{2}.$$

First we multiply $\phi^2 u$ on (78) and integrate by parts to obtain

$$c\int_{\partial' B_3^+} \phi^2 u^{\frac{2n-2}{n-2}} + \int_{B_3^+} (\phi^2|\nabla u|^2 + 2\phi u \nabla\phi \cdot \nabla u - n(n-2)\phi^2 u^{\frac{2n}{n-2}}) = 0.$$

Then it follows by Hölder's inequality that

$$\int_{B_3^+} \phi^2|\nabla u|^2 \le C\Big(\int_{B_3^+} |\nabla\phi|^2 u^{\frac{2n}{n-2}} + \int_{B_3^+} \phi^2 u^2 + \int_{\partial' B_3^+} \phi^2 u^{\frac{2n-2}{n-2}}\Big).$$

To estimate the last term of the above, we have for $|x'| < 1/2$,

$$\begin{aligned}
\phi^2(x',0)u^{\frac{2n}{n-2}}(x',0) &= \Big|\int_0^{1/\sqrt{2}} \frac{d}{ds}(\phi^2(x',s)u^{\frac{2n-2}{n-2}}(x',s))ds\Big| \\
&= \Big|\int_0^{1/\sqrt{2}} 2\phi\frac{\partial\phi}{\partial x_n} u^{\frac{2n-2}{n-2}} ds + \int_0^{1/\sqrt{2}} \phi^2 \frac{2n-2}{n-2} u^{\frac{n}{n-2}} \frac{\partial u}{\partial x_n} ds\Big| \\
&\le C\int_0^{1/\sqrt{2}} u^{\frac{2n-2}{n-2}} ds + C\Big(\int_0^{1/\sqrt{2}} \phi^2(\frac{\partial u}{\partial x_n})^2 ds\Big)^{\frac{1}{2}}\Big(\int_0^{1/\sqrt{2}} \phi^2 u^{\frac{2n}{n-2}} ds\Big)^{\frac{1}{2}} \\
&\le C\int_0^{1/\sqrt{2}} u^{\frac{2n-2}{n-2}} ds + \epsilon\int_0^{1/\sqrt{2}} \phi^2\Big(\frac{\partial u}{\partial x_n}\Big)^2 ds + \frac{C}{\epsilon}\int_0^{1/\sqrt{2}} u^{\frac{2n}{n-2}} ds.
\end{aligned}$$

Integrating with respect to x' and choosing ϵ sufficiently small, we can derive (95) in view of (94). $\square$

Proof of (17). We only need to prove it for $R = 1$. Without loss of generality, we may assume that Λ_1 and Λ_2 are subsets of $\overline{B_{1/2}^+}$. We argue by contradiction. Suppose there is a sequence $\{u_j\}$ solving (78) such that

$$\inf_{\Lambda_1} u_j > j \inf_{\Lambda_2} u_j. \tag{96}$$

Then we must have $\max_{\overline{B_1^+}} u_j \to \infty$, since otherwise, by the Harnack inequality, u_j on $\overline{B_{1/2}^+}$ would be bounded below and above by positive constants and (96) would be impossible. Let Z_j be as in the proof of (16). We know that $Z_j \cap \overline{B_1^+}$ has at most k points with k independent of j, and the values of u_j on $Z_j \cap \overline{B_1^+}$ are comparable. So for $r > 0$ small, $\Lambda_1 \setminus \left(\bigcup_{z\in Z_j} B_r(z)\right)$ and $\Lambda_2 \setminus \left(\bigcup_{z\in Z_j} B_r(z)\right)$ are non-empty. The values of u_j on $\overline{B_1^+} \setminus (\bigcup_{z\in Z_j} B_r(z))$ are comparable; and by Proposition 9.3, are all bounded above by $C(r)u(z)^{-1}$ for $z \in Z_j \cap \overline{B_1^+}$. So in particular, $\inf_{\Lambda_1} u_j \leq C(r)u(z)^{-1}$ for $z \in Z_j \cap \overline{B_1^+}$. On the other hand, by Lemma 9.1, $\inf_{\Lambda_2} u_j \geq C(r)^{-1}u(z)^{-1}$ for $z \in Z_j \cap \overline{B_1^+}$. It follows that $\inf_{\Lambda_1} u_j \leq C(r) \inf_{\Lambda_2} u_j$, violating (96). This completes the proof of (17). □

10 Appendix A. A boundary lemma

In this section, we let Ω be a domain of $\mathbb{R}^n, n \geq 2$, with the origin 0 on its boundary. Assume that near 0 the boundary consists of two transversally intersecting C^2 hypersurfaces $\rho = 0$ and $\sigma = 0$. Also, suppose $\rho, \sigma > 0$ in Ω. Let $\nu(y)$ be the unit outer normal to the surface $\{\sigma = 0\} \cap \partial\Omega$ at y.

Let $\{b_i(y)\}$ be L^∞ functions, and let $\{a_{ij}(y)\}$ be a $n \times n$ matrix function satisfying, for some positive constant $\Lambda \geq 1$,

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j} a_{ij}(y)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \text{for } \xi \in \mathbb{R}^n, y \in \Omega.$$

Under this setting, we have

Lemma 10.1. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be positive in Ω, $u(0) = 0$, and suppose that for some positive constant A*

$$\begin{cases} \sum_{i,j=1}^n a_{ij}u_{ij} + \sum_{i=1}^n b_i u_i \leq Au, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} \geq -Au, & \text{on } \{\sigma = 0, \rho > 0\}, \end{cases}$$

where ν denotes the unit outer normal. Then

$$\frac{\partial u}{\partial \nu'}(0) > 0,$$

where ν' is any vector in the tangent space of $\{\sigma = 0\}$ that enters into $\{\rho > 0\}$.

Proof. Since the hypotheses and conclusions are invariant under change of coordinates, and of the choices of the particular ρ and σ representing the bounding hypersurfaces, we may assume without loss of generality that $\rho(y) \equiv y_1$ and

$\sigma(y) \equiv y_2$. By the Hopf lemma, $u > 0$ on $\{y_2 = 0, y_1 > 0\}$ (otherwise, by the boundary condition and the fact that $u > 0$ in Ω, $u = 0$ and $\partial u/\partial\nu = 0$ at a point on $\{y_2 = 0, y_1 > 0\}$, violating the Hopf lemma). So we may, as in [30], assume without loss of generality that $u > 0$ on $\overline{\Omega}\setminus\{0\}$, because we may replace $y_1 = 0$ by a sphere tangent to $y_1 = 0$ at the origin and then straighten the sphere to a hyperplane by a coordinate change and call the new hyperplane $y_1 = 0$.

Pick $\epsilon > 0$ so small that $\{y_1 > 0\} \cap \{y_2 > 0\} \cap B(0, 2\epsilon) \subset \Omega$. We wish to construct a function $\phi > 0$ in Ω such that

1. $\sum_{i,j} a_{ij}\phi_{ij} + \sum_i b_i\phi_i \geq A\phi$ in $\Omega \cap B(0,\epsilon)$,
2. $\phi = 0$ on $\{y_1 = 0\} \cap B(0,\epsilon)$,
3. $\dfrac{\partial\phi}{\partial\nu} \leq -A\phi$ on $\{y_2 = 0, y_1 > 0\} \cap B(0,\epsilon)$,
4. $\phi \leq u$ on $\partial B(0,\epsilon) \cap \overline{\Omega}$,
5. $\dfrac{\partial\phi}{\partial\nu'}(0) > 0$.

Once such ϕ is constructed, Lemma 10.1 can be proved as follows. Let $w = u - \phi$; then w satisfies

$$\begin{cases} \displaystyle\sum_{i,j} a_{ij}w_{ij} + \sum_i b_i w_i - Aw \leq 0, & \Omega \cap B(0,\epsilon), \\ w \geq 0, & \text{on } \{y_1 = 0\} \cap B(0,\epsilon) \text{ and } \partial B(0,\epsilon) \cap \Omega, \\ \dfrac{\partial w}{\partial\nu} + Aw \geq 0, & \text{on } \{y_2 = 0, y_1 > 0\} \cap B(0,\epsilon). \end{cases}$$

By the maximum principle, $w \geq 0$ on the closure of $B(0,\epsilon) \cap \Omega$; and therefore by $w(0) = 0$, we have

$$\frac{\partial w}{\partial\nu'}(0) \geq 0.$$

Consequently,

$$\frac{\partial u}{\partial\nu'}(0) \geq \frac{\partial\phi}{\partial\nu'}(0) > 0.$$

Such a ϕ can be given explicitly by setting

$$\phi(y) = \delta(e^{\alpha^2 y_1} - 1)e^{\alpha y_2}, \qquad y \in \Omega,$$

where $\alpha > 1$ is large and then $\delta > 0$ is chosen small.

By a direct calculation, we have for large α,

$$\sum_{i,j} a_{ij}\phi_{ij} + \sum_i b_i\phi_i \geq \delta\alpha^3(c\alpha - C)e^{\alpha^2 y_1}e^{\alpha y_2} \geq A\phi,$$

where c and C are generic positive constants.

On $\{y_2 = 0\}$, for large α,

$$\frac{\partial \phi}{\partial y_2} = \alpha\phi \geq A\phi,$$

i.e.,

$$\frac{\partial \phi}{\partial \nu} \leq -A\phi, \qquad \text{on } \{\sigma = 0\}.$$

Now fix the value of α. Since $u > 0$ on $\overline{\Omega} \setminus \{0\}$, choose $\delta > 0$ small enough such that

$$u > \phi \qquad \text{on } \partial B(0,\epsilon) \cap \overline{\Omega}.$$

Finally, it is immediate that $\frac{\partial \phi}{\partial y_1}(0) > 0$, so all the desired properties are satisfied. Lemma 10.1 is established. □

11 Appendix B. Some calculus lemmas

In this section, we present, for the reader's convenience, a few calculus lemmas and their proofs taken from [42] (see also [26]).

Lemma 11.1. *Let $f \in C^1(\mathbb{R}^n), n \geq 1, \nu > 0$. Suppose that for every $x \in \mathbb{R}^n$, there exists $\lambda(x) > 0$ such that*

$$(97) \qquad \Big(\frac{\lambda(x)}{|y-x|}\Big)^{\nu} f\Big(x + \frac{\lambda(x)^2(y-x)}{|y-x|^2}\Big) = f(y), \qquad y \in \mathbb{R}^n \backslash \{x\}.$$

Then for some $a \geq 0, d > 0, \bar{x} \in \mathbb{R}^n$,

$$f(x) = \pm\Big(\frac{a}{d + |x - \bar{x}|^2}\Big)^{\nu/2}.$$

Proof. It follows from (97) that

$$B := \lim_{|y|\to\infty} |y|^{\nu} f(y) = \lambda(x)^{\nu} f(x), \qquad x \in \mathbb{R}^n.$$

If $B = 0$, then $f \equiv 0$, and we are done. If $B \neq 0$, then $f(x)$ does not change sign. Without loss of generality we may assume that $B = 1$ and $f(x) > 0$. For large y, by making a Taylor expansion of the left hand side of (97) at 0 and x, we have

$$(98) \qquad f(y) = \Big(\frac{\lambda(0)}{|y|}\Big)^{\nu}\Big(f(0) + \frac{\partial f}{\partial y_i}(0)\frac{\lambda(0)^2 y_i}{|y|^2} + o\Big(\frac{1}{|y|}\Big)\Big)$$

and

$$(99) \qquad f(y) = \Big(\frac{\lambda(x)}{|y-x|}\Big)^{\nu}\Big(f(x) + \frac{\partial f}{\partial y_i}(x)\frac{\lambda(x)^2(y_i - x_i)}{|y-x|^2} + o\Big(\frac{1}{|y|}\Big)\Big),$$

where $\circ(\frac{1}{|y|})$ denotes some quantity satisfying $\lim_{|y|\to\infty} \circ(\frac{1}{|y|})/(\frac{1}{|y|}) = 0$. Combining (97), (98), (99), and our assumption $B = 1$, we have

$$f^{-1-2/\nu}(x)\frac{\partial f}{\partial y_i}(x) = f^{-1-2/\nu}(0)\frac{\partial f}{\partial y_i}(0) - \nu x_i.$$

It follows that for some $\bar{x} \in \mathbb{R}^n, d > 0$,

$$f^{-2/\nu}(y) = |y - \bar{x}|^2 + d.$$

□

Lemma 11.2. *Let $f \in C^1(R^n), n \geq 1, \nu > 0$. Assume that*

$$\left(\frac{\lambda}{|y-x|}\right)^\nu f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq f(y) \quad \text{for all } \lambda > 0,\ x \in \mathbb{R}^n, |y-x| \geq \lambda.$$

Then $f \equiv$ constant.

Proof. For $x \in \mathbb{R}^n$, $\lambda > 0$, set

$$g_{x,\lambda}(z) = f(x+z) - \left(\frac{\lambda}{|z|}\right)^\nu f\left(x + \frac{\lambda^2 z}{|z|^2}\right), \qquad |z| \geq \lambda.$$

It is easy to see that

$$\begin{cases} g_{x,|z|}(z) = 0, \\ g_{x,|z|}(rz) \geq 0, \quad r \geq 1. \end{cases}$$

It follows that

$$\frac{d}{dr}\{g_{x,|z|}(rz)\}|_{r=1} \geq 0.$$

A direct calculation yields

$$2\nabla f(z+x) \cdot z + \nu f(z+x) \geq 0.$$

Since z and x are arbitrary, by a change of variables, we have

$$2\nabla f(y) \cdot (y-x) + \nu f(y) \geq 0.$$

Dividing the above by $|x|$ and sending $|x|$ to infinity, we have $\nabla f(y) \cdot \theta \leq 0$ for all $x \in \mathbb{R}^n$ and $\theta \in \mathbb{S}^{n-1}$. It follows that $\nabla f \equiv 0$ in $\mathbb{R}^n$. □

Lemma 11.3. *Let $f \in C^1(\mathbb{R}^n_+), n \geq 2, \nu > 0$. Assume that*

$$\left(\frac{\lambda}{|y-x|}\right)^\nu f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq f(y), \quad \lambda > 0,\ x \in \partial\mathbb{R}^n_+, |y-x| \geq \lambda, y \in \mathbb{R}^n_+.$$

Then

$$f(x) = f(x', t) = f(0, t), \qquad x = (x', t) \in \mathbb{R}^n_+.$$

Proof. For $x \in \partial\mathbb{R}^n_+$, $\lambda > 0$, set

$$g_{x,\lambda}(z) = f(x+z) - \left(\frac{\lambda}{|z|}\right)^\nu f\left(x + \frac{\lambda^2 z}{|z|^2}\right), \qquad z \in \mathbb{R}^n_+,\ |z| \geq \lambda.$$

As in the proof of Lemma 11.2, we have

$$2\nabla f(z+x)\cdot z + \nu f(z+x) \geq 0 \qquad \text{for all } x \in \partial\mathbb{R}^n_+, z \in \mathbb{R}^n_+.$$

Making a change of variables, we have

$$2\partial_{y'} f(y',t)\cdot(y'-x') + 2\partial_t f(y',t)t + \nu f(y',t) \geq 0 \qquad \text{for all } x', y' \in \mathbb{R}^{n-1}, t > 0.$$

Dividing the above by $|x'|$ and sending $|x'|$ to infinity, we have $\partial_{y'} f(y',t)\cdot\theta \geq 0$ for all $(y't) \in \mathbb{R}^n_+$ and $\theta \in \mathbb{S}^{n-1}$. It follows that $\partial_{y'} f(y',t) \equiv 0$. □

12 Appendix C

In this appendix, we prove the following simple result which is needed for 2° in Remark 1.4.

Lemma 12.1. *Let g be a positive continuous function on $(0,\infty)$ satisfying*

$$\liminf_{s\to\infty} g(s) > 0.$$

Then

$$u''(t) + g(u(t)) = 0, \qquad 0 \leq t < \infty$$

does not have any positive solution u.

Proof. Let $v = u'$; then

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -g(u) \end{pmatrix}. \tag{100}$$

If $v(0) < 0$, we have from the second equation of (100) that $v(t) < v(0)$ for all $t > 0$. Then by the first equation, $u(t) < u(0) + v(0)t$. This is impossible for large t since u is positive. If $v(0) = 0$, then by the second equation, $v(t) < 0$ for $t > 0$. This is impossible by the above argument since the system is autonomous. So we only need to rule out the possibility that $v(t) > 0$ for all $t \geq 0$. In this case, by the first equation, $u(t) > u(0) > 0$ for all t; and therefore, by the hypothesis on g and the second equation, there exists some $\delta > 0$ such that $v'(t) < -\delta t$ for all t. This is impossible, since v is assumed to be positive all the time. □

13 Appendix D

In this appendix, we present a result which we could not find in the literature.

Theorem 13.1. *For $n \geq 1$ and $p_j \to p \in (1, \infty)$, let $\{g_j\}$ be a sequence of measurable functions on $(0, \infty)$ satisfying*

$$\sup_{j,\ 0<s<t} |g_j(s)| < \infty, \qquad t > 0,$$

and, for some $a < 0$,

$$\lim_{s\to\infty} \left(\sup_j \left| \frac{g_j(s)}{s^{p_j}} - a \right| \right) = 0. \tag{101}$$

Let $\{u_j\}$ be positive solutions (in the distribution sense) of

$$-\Delta u_j = g_j(u_j), \qquad \text{on } B_{2R} \subset \mathbb{R}^n.$$

Then we have

$$\limsup_{j\to\infty} (\sup_{B_R} u_j) < \infty. \tag{102}$$

Remark 13.1. If $1 < p < \frac{n+2}{n-2}, n \geq 3$, and $a > 0$, estimate (102) still holds. This can be seen easily from the proof, by using the result of Gidas and Spruck: For such p and n, $-\Delta u = u^p$ has no positive solution in $\mathbb{R}^n$.

Proof. It is easy to see that we only need to prove it for $a = -1$ and $R = 1$. Our proof is by contradiction. Suppose the contrary; we may assume, without loss of generality, that

$$u_j(0) \to \infty.$$

By Lemma 5.1 (with $a = \frac{2}{p_j-1}$), there exist $|x_j| < 1$ such that

$$u_j(x_j) \geq 2^{\frac{2}{1-p_j}} \sup_{B_{\sigma_j}(x_j)} u_j,$$

and

$$\sigma_j^{\frac{2}{p_j-1}} u_j(x_j) \geq 2^{\frac{2}{1-p_j}} u_j(0) \to \infty,$$

where $\sigma_j = (1 - |x_j|)/2$.

Consider

$$w_j(y) = \frac{1}{u_j(x_j)} u_j\Big(x_j + \frac{y}{u_j(x_j)^{(p_j-1)/2}}\Big), \qquad |y| < \sigma_j u_j(x_j)^{(p_j-1)/2} \to \infty.$$

Then w_j satisfies

$$-\Delta w_j(y) = \frac{g_j(u_j(x_j)w_j(y))}{u_j(x_j)^{p_j}}, \qquad |y| < \sigma_j u_j(x_j)^{(p_j-1)/2}.$$

By the hypothesis on g_j,

$$|g_j(s)| \le C(1 + |s|^{p_j}),$$

and therefore

$$|\Delta w_j| \le C.$$

After passing to a subsequence (still denoted as $\{w_j\}$, etc.), we have, by standard elliptic theories,

$$w_j \to w \ge 0 \qquad \text{in } C^1_{loc}(\mathbb{R}^n).$$

Sending j to ∞ in the equation of w_j, we have

$$\Delta w = w^p, \qquad \text{on } \mathbb{R}^n. \tag{103}$$

Indeed, if $u_j(x_j)w_j(y) \to \infty$, then by (101), $u_j(x_j)^{-p_j} g_j(u_j(x_j)w_j(y)) \to -w(y)^p$; if $u_j(x_j)w_j(y) \to 0$, $w_j(y) \to 0 = w(y)$, and then by the boundedness of $\{g_j\}$, $u_j(x_j)^{-p_j} g_j(u_j(x_j)w_j(y)) \to 0 = -w(y)^p$. Since $w_j(0) = 1$, $w(0) = 1$. By the strong maximum principle, w is a positive solution of (103), a contradiction (see Remark 1.2). □

Added in proof. We thank Yihong Du, who kindly informed us that Theorem 13.1 in Appendix D can be deduced from a result of J. B. Keller in: *On solutions of $\Delta u = f(u)$*, Comm. Pure Appl. Math. **10** (1957), 503–510.

REFERENCES

[1] H. Berestycki, L. Caffarelli and L. Nirenberg, *Symmetry for elliptic equations in a half space*, in *Boundary Value Problems for Partial Differential Equations and Applications*, RMA Res. Notes Appl. Math. 29, Masson, Paris, 1993, pp. 27–42.

[2] H. Berestycki, L. Caffarelli and L. Nirenberg, *Inequalities for second-order elliptic equations with applications to unbounded domains, I. A celebration of John F. Nash, Jr.*, Duke Math. J. **81** (1996), 467–494.

[3] H. Berestycki, L. Caffarelli and L. Nirenberg, *Monotonicity for elliptic equations in unbounded Lipschitz domains*, Comm. Pure Appl. Math. **50** (1997), 1089–1111.

[4] H. Berestycki and L. Nirenberg, *Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations*, J. Geom. Phys. **5** (1988), 237–275.

[5] H. Berestycki and L. Nirenberg, *Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains*, in *Analysis, et cetera*, Academic Press, Boston, MA, 1990, pp 115–164.

[6] H. Berestycki and L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. **22** (1991), 1–37.

[7] H. Berestycki and L. Nirenberg, *Traveling fronts in cylinders*, Ann. Inst. H. Poincaré Anal. Non Linéaire **9** (1992), 497–572.

[8] H. Berestycki, L. Nirenberg and S. R. S. Varadhan, *The principle eigenvalues for second order elliptic operators in general domains*, Comm. Pure Appl. Math. **47** (1994), 47–92.

[9] G. Bianchi, *Non-existence of positive solutions to semilinear elliptic equations on* $\mathbb{R}^n$ *or* $\mathbb{R}^n_+$ *through the method of moving planes*, Comm. Partial Differential Equations **22** (1997), 1671–1690.

[10] H. Brezis, *Semilinear equations in* $\mathbb{R}^N$ *without conditions at infinity*, Appl. Math. Optim. **12** (1984), 271–282.

[11] H. Brezis, Y. Y. Li and I. Shafrir, *A* $\sup + \inf$ *inequality for some nonlinear elliptic equations involving exponential nonlinearities*, J. Funct. Anal. **115** (1993), 344–358.

[12] L. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271–297.

[13] Sun-Yung A. Chang, M. Gursky and P. Yang, *The scalar curvature equation on* 2- *and* 3-*spheres*, Calc. Var. Partial Differential Equations **1** (1993), 205–229.

[14] S. Chanillo and M. K.-H. Kiessling, *Conformally invariant systems of nonlinear PDE of Liouville type*, Geom. Funct. Anal. **5** (1995), 924–947.

[15] C. C. Chen and C. S. Lin, *Local behavior of singular positive solutions of semilinear elliptic equations with Sobolev exponent*, Duke Math. J. **78** (1995), 315–334.

[16] C. C. Chen and C. S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes*, Comm. Pure Appl. Math. **50** (1997), 971–1017.

[17] C. C. Chen and C. S. Lin, *A sharp* $\sup + \inf$ *inequality for a nonlinear elliptic equation in* $\mathbb{R}^2$, Comm. Anal. Geom. **6** (1998), 1–19.

[18] C. C. Chen and C. S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes II*, J. Differential Geom. **49** (1998), 115–178.

[19] C. C. Chen and C. S. Lin, *On the asymptotics symmetry of singular solutions of the scalar curvature equations*, Math. Ann. **313** (1999), 229–245.

[20] C. C. Chen and C. S. Lin, *A Harnack inequality and local asymptotic symmetry for singular solutions of nonlinear elliptic equations*, preprint.

[21] C. C. Chen and C. S. Lin, *Prescribing scalar curvature on* S^n, *Part I: Apriori estimates*, J. Differential Geom. **57** (2001), 67–171.

[22] W. Chen and C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), 615–623.

[23] W. Chen and C. Li, *A priori estimates for prescribing scalar curvature equations*, Ann. of Math. **145** (1997), 547–564.

[24] W. Chen and C. Li, *Indefinite elliptic problems with critical exponents*, in *Advances in Nonlinear Partial Differential Equations and Related Areas (Beijing, 1997)*, World Sci. Publ., River Edge, NJ, 1998, pp. 67–79.

[25] M. Chipot, M. Chlebik, M. Fila and I. Shafrir, *Existence of positive solutions of a semilinear elliptic equation in* $\mathbb{R}^n_+$ *with a nonlinear boundary condition*, J. Math. Anal. Appl. **223** (1998), 429–471.

[26] M. Chipot, I. Shafrir and M. Fila, *On the solutions to some elliptic equations with nonlinear Neumann boundary conditions*, Adv. Differential Equations **1** (1996), 91–110.

[27] K. S. Chou and T. Y. H. Wan, *Asymptotic radial symmetry for solutions of* $\Delta u + e^u = 0$ *in a punctured disc*, Pacific J. Math. **163** (1994), 269–276.

[28] J. F. Escobar, *Uniqueness theorems on conformal deformation of metric, Sobolev inequalities, and an eigenvalue estimate*, Comm. Pure Appl. Math. **43** (1990), 857–883.

[29] N. Ghoussoub, C. Gui and M. Zhu, *On a singularly perturbed Neumann problem with critical exponent*, Comm. Partial Differential Equations **26** (2001), 1929–1946.

[30] B. Gidas, W. M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[31] B. Gidas, W. M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, Math. Anal. Appl. **7A** (1981), 369–402.

[32] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), 525–598.

[33] C. Gui and C. S. Lin, *Estimates for boundary-bubbling solutions to an elliptic Neumann problem*, J. Reine Angew. Math. **546** (2002), 201–235.

[34] Z. C. Han and Y. Y. Li, *The Yamabe problem on manifolds with boundaries: Existence and compactness results*, Duke Math. J. **99** (1999), 489–542.

[35] B. Hu, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*, Differential Integral Equations **7** (1994), 301–313.

[36] B. Hu and H.-M. Yin, *The profile near blowup time for solution of the heat equation with a nonlinear boundary condition*, Trans. Amer. Math. Soc. **346** (1994), 117–135.

[37] C. Li, *Local asymptotic symmetry of singular solutions to nonlinear elliptic equations*, Invent. Math. **123** (1996), 221–231.

[38] Y. Y. Li, *Prescribing scalar curvature on* S^n *and related problems, Part I*, J. Differential Equations **120** (1995), 319–410.

[39] Y. Y. Li, *The Nirenberg problem in a domain with boundary*, Topol. Methods Nonlinear Anal. **6** (1995), 309–329.

[40] Y. Y. Li, *Prescribing scalar curvature on* S^n *and related problems, Part II: Existence and compactness*, Comm. Pure Appl. Math. **49** (1996), 541–597.

[41] Y. Y. Li, *A Harnack type inequality: the method of moving planes*, Comm. Math. Phys. **200** (1999), 421–444.

[42] Y. Y. Li and M. Zhu, *Uniqueness theorems through the method of moving spheres*, Duke Math. J. **80** (1995), 383–417.

[43] Y. Y. Li and M. Zhu, *Sharp Sobolev trace inequalities on Riemannian manifolds with boundaries*, Comm. Pure Appl. Math. **50** (1997), 449–487.

[44] Y. Y. Li and M. Zhu, *Sharp Sobolev inequalities involving boundary terms*, Geom. Funct. Anal. **8** (1998), 59–87.

[45] Y. Y. Li and M. Zhu, *Yamabe type equations on three-dimensional Riemannian manifolds*, Comm. Contemp. Math. **1** (1999), 1–50.

[46] C. S. Lin, *Estimates of the scalar curvature equation via the method of moving planes. III*, Comm. Pure Appl. Math. **53** (2000), 611–646.

[47] C. S. Lin, *Locating the peaks of solutions via the maximum principle, I: the Neumann problem*, Comm. Pure Appl. Math. **54** (2001), 1065–1095.

[48] Y. Lou and M. Zhu, *Classifications of nonnegative solutions to some elliptic problems*, Differential Integral Equations **12** (1999), 601–612.

[49] M. Obata, *The conjecture on conformal transformations of Riemannian manifolds*, J. Differential Geom. **6** (1971), 247–258.

[50] B. Ou, *Positive harmonic functions on the upper half space satisfying a nonlinear boundary condition*, Differential Integral Equations **9** (1996), 1157–1164.

[51] B. Ou, *A uniqueness theorem for harmonic functions on the upper-half plane*, Conform. Geom. Dynam. **4** (2000), 120–125 (electronic).

[52] R. Schoen, Courses at Stanford University, 1988, and New York University, 1989.

[53] R. Schoen and D. Zhang, *Prescribed scalar curvature on the* n*-sphere*, Calc. Var. Partial Differential Equations **4** (1996), 1–25.

[54] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318.

[55] L. Zhang, *Classification of conformal metrics on $\mathbb{R}^2_+$ with constant Gauss curvature and geodesic curvature on the boundary under various integral finiteness assumptions*, Calc. Var. Partial Differential Equations, to appear.

[56] L. Zhang, *On Yamabe type equations with prescribed scalar curvature and boundary mean curvature*, preprint.

[57] M. Zhu, *Uniqueness results through a priori estimates, I. A three-dimensional Neumann problem*, J. Differential Equations **154** (1999), 284–317.

[58] M. Zhu, *Uniqueness results through a priori estimates, II. Dirichlet problem*, J. Math. Anal. Appl. **248** (2000), 156–172.

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