# SQUARE ROOTS OF ELLIPTIC SECOND ORDER DIVERGENCE OPERATORS ON STRONGLY LIPSCHITZ DOMAINS: $L^2$ THEORY

*By*

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**Abstract.** We prove the Kato conjecture for square roots of elliptic second order non-self-adjoint operators in divergence form $L = -\text{div}(A\nabla)$ on strongly Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, subject to Dirichlet or to Neumann boundary conditions. The method relies on a transference procedure from the recent positive result on $\mathbb{R}^n$ in [2].

## Introduction

Let $\Omega$ be an open subset of $\mathbb{R}^n$, $A$ a bounded uniformly elliptic complex matrix on $\Omega$, and $L = -\text{div}(A\nabla)$ the elliptic second order divergence operator defined as the maximal-accretive operator associated with a regularly accretive sesquilinear form on a closed subspace $V$ of $H^1(\Omega)$ containing $H_0^1(\Omega)$ (see Section 1 for precise definitions). The Kato conjecture amounts to showing that for any such $L$, the domain of the maximal-accretive square root $L^{1/2}$ of $L$ agrees with $V$ with equivalence of norms. One of Kato's questions was about perturbation theory for the square roots of real symmetric operators in order to study hyperbolic evolution equations with time-dependent coefficients. This conjecture is also related to other topics; see, e.g., [11].

In one dimension, the conjecture is now completely settled: for any $\Omega$, $V$ and $L$ as above, the domain of $L^{1/2}$ agrees with $V$. The first solution when $\Omega = \mathbb{R}$ was given by Coifman, McIntosh and Meyer [6]. Their argument relied on translation invariance, so other methods needed to be devised when $\Omega \neq \mathbb{R}$. We used *ad hoc* wavelets in [4], while Auscher, McIntosh and Nahmod used a reduction from the case $\Omega = \mathbb{R}$ via interpolation methods [3].

In higher dimensions, when $\Omega = \mathbb{R}^n$, we refer the reader to the introduction of our work [5] and the references therein for a discussion on progress over the years about this problem until 1998. Very recently, the conjecture has been established in arbitrary dimensions by Hofmann, Lacey and McIntosh along with us [2] after it was proved for $L^\infty$-perturbations of self-adjoint operators by Hofmann, Lewis and us [1].

When $\Omega \neq \mathbb{R}^n$, geometry at the boundary plays a role which prevents a straightforward generalisation of results and methods in $\mathbb{R}^n$. To our knowledge, not even the Kato conjecture for $L^\infty$-perturbations of the Laplacian is known. McIntosh proved it when the coefficients are in the space $MH^s(\Omega)$ of pointwise multipliers of the Sobolev space $H^s(\Omega)$ [10] for some $s > 0$ and $\Omega$ strongly Lipschitz. This seems to be the best result currently available on strongly Lipschitz domains.

The purpose of this paper is to establish the following result.

**Theorem 1.** *If $n \geq 2$, the Kato conjecture holds for any elliptic second order divergence operator $-div(A\nabla)$ subject to a Dirichlet or Neumann boundary condition on a strongly Lipschitz domain.*

The meaning of a Dirichlet and Neumann boundary condition will be explained in the notation section.

Although square roots are non-local operators, the proof of Theorem 1 follows procedures which are customary for boundary value problems: we transfer the result from $\mathbb{R}^n$ to $\mathbb{R}^n_+$ by a reflection principle; then to special Lipschitz domains by a bilipschitz change of variables; and eventually to general strongly Lipschitz domains by localisation. This last step relies upon a kind of "weak" comparison principle for solutions of complex elliptic operators.

Our method does not seem to work for more general boundary conditions (e.g., for mixed Dirichlet–Neumann conditions), which therefore are left as an open problem.

## 1 Notation

By a *strongly Lipschitz domain*, we mean an open connected set in $\mathbb{R}^n$ whose boundary is a finite union of parts of rotated graphs of Lipschitz maps, at most one of which parts is possibly infinite. These include special Lipschitz domains (the open set above a Lipschitz graph), bounded Lipschitz domains and exterior Lipschitz domains.

For an open set $\Omega$ of $\mathbb{R}^n$, $||f||_p$ or $||f||_{L^p(\Omega)}$ denotes the usual norm in the Lebesgue space $L^p(\Omega)$ equipped with Lebesgue measure. We write $H^1(\Omega)$ for the usual Sobolev space with norm $(||\nabla f||_2^2 + ||f||_2^2)^{1/2}$ and $H^1_0(\Omega)$ for the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

Denote by $\mathcal{A}$ the class of *elliptic matrices* in $L^\infty(\mathbb{R}^n, M_n(\mathcal{C}))$ with *ellipticity constants* $0 < \lambda, \Lambda < \infty$, that is, the best constants in the inequalities

$$\|A\|_\infty \leq \Lambda \quad \text{and} \qquad \forall \xi \in \mathcal{C}^n \quad \operatorname{Re} A(x)\xi \cdot \bar{\xi} \geq \lambda|\xi|^2, \quad \text{a.e. on } \mathbb{R}^n.$$

If $A$ is merely given on $\Omega$, we tacitly require $A$ to be the $\lambda$ times identity matrix elsewhere.

Given $A \in \mathcal{A}$, an open set $\Omega$ of $\mathbb{R}^n$ and a closed subspace $V$ of $H^1(\Omega)$ containing $H_0^1(\Omega)$, denote by $L$ the maximal-accretive operator on $L^2(\Omega)$, with largest domain $\mathcal{D}(L) \subset V$, such that

$$(1) \qquad \langle Lf, g\rangle = \int_\Omega A\nabla f \cdot \overline{\nabla g}, \quad f \in \mathcal{D}(L), \quad g \in V.$$

The domain of $L$ is characterized by the following condition. Let $f \in V$; then $f \in \mathcal{D}(L)$ if and only if there exists a constant $c$ such that for all $g \in V$,

$$(2) \qquad \left| \int_\Omega A\nabla f \cdot \overline{\nabla g} \right| \leq c\|g\|_2.$$

It is known that $\mathcal{D}(L)$ is dense in $V$ [8].

Set $-\text{div} = \nabla^*\colon L^2(\Omega, \mathcal{C}^n) \to V'$ the adjoint of $\nabla\colon V \to L^2(\Omega, \mathcal{C}^n)$. By density, we may extend $L$ continuously from $V$ to $V'$. We use the same letter to denote both $L$ or its extension depending on the context. Instead of the customary notation $-\text{div}(A\nabla)$, we prefer to write $L$ as the triplet $(A, \Omega, V)$ to indicate the matrix of coefficients $A$, the domain $\Omega$ and the boundary condition determined by the space $V$.

Any $L$ as above possesses a unique maximal-accretive square root $L^{1/2}$, given by Kato's representation

$$(3) \qquad L^{1/2}f = \frac{2}{\pi}\int_0^\infty (1 + t^2L)^{-1}tLf\,\frac{dt}{t}, \quad f \in \mathcal{D}(L).$$

For $f \in \mathcal{D}(L)$, we have

$$(4) \qquad (1 + t^2L)^{-1}Lf = L(1 + t^2L)^{-1}f = t^{-2}(f - (1 + t^2L)^{-1}f);$$

hence, $\|(1+t^2L)^{-1}Lf\|_2 \leq \inf(\|Lf\|_2, 2\|f\|_2\, t^{-2})$, since the resolvent is $L^2$-contractive. The above integral converges in $L^2(\Omega)$-norm. Observe that for each $t > 0$, $(1 + t^2L)^{-1}tL$ extends to a bounded operator on $L^2(\Omega)$ with

$$(5) \qquad \|(1 + t^2L)^{-1}tLf\|_2 \leq \frac{2}{t}\,\|f\|_2.$$

Note also that if $f \in V$, then $(1 + t^2L)^{-1}tLf \in V$.

To tackle the Kato conjecture, it is enough to prove one of the inequalities

$$\text{(K)} \qquad \|L^{1/2}f\|_2 \le c\|\nabla f\|_2, \quad f \in V,$$

$$\text{(K}_{\text{loc}}\text{)} \qquad \|L^{1/2}f\|_2 \le c(\|\nabla f\|_2 + \|f\|_2), \quad f \in V$$

(it suffices to do it *a priori* for $f \in \mathcal{D}(L)$). Indeed, it is well-known that (K) (resp. ($\text{K}_{\text{loc}}$)) for $L$ and its adjoint imply that the domain of $L^{1/2}$ is $V$ [9].

Here, Dirichlet boundary condition means $V = H_0^1(\Omega)$; Neumann: $V = H^1(\Omega)$. Assume $\Omega$ is strongly Lipschitz. In the first case, a function $f$ is in the domain of $L$ if $f \in H_0^1(\Omega)$ and the divergence of $A\nabla f$ in the distributional sense on $\Omega$ belongs to $L^2(\Omega)$. In the latter case, a function $f$ is in the domain of $L$ if $f \in H^1(\Omega)$, the divergence of $A\nabla f$ in the distributional sense on $\Omega$ belongs to $L^2(\Omega)$ and the conormal derivative of $f$ at the boundary vanishes.

One can think of (K) as a homogeneous or global inequality and ($\text{K}_{\text{loc}}$) as an inhomogeneous or local inequality. Proving the first one is harder.

When $\Omega$ is unbounded (e.g., special Lipschitz or an exterior domain), this does make a difference. In particular, we do not obtain (K) on an exterior domain while we expect it. This suggests finding a different argument.

In the case of bounded domains, there is no distinction between (K) and ($\text{K}_{\text{loc}}$). Indeed, when $\Omega$ is a bounded connected set with Lipschitz boundary, the Poincaré–Wirtinger inequality yields that $(\int_\Omega |\nabla f|^2)^{1/2}$ is a norm on $H_0^1(\Omega)$ or on the subspace of functions in $H^1(\Omega)$ with vanishing mean. Thus, (K) and ($\text{K}_{\text{loc}}$) are the same in the Dirichlet case. In the Neumann case, they are the same on functions with vanishing mean; this is harmless as $\nabla$ and $L$ annihilate constants and, in fact, $\mathcal{N}(\nabla) = \mathcal{C} = \mathcal{N}(L)$. Another way of saying this is by factoring out $\mathcal{C}$: write $L^2(\Omega) = L_0^2(\Omega) \oplus \mathcal{C}$, where $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ characterized by $\int_\Omega f = 0$; then the restriction of $L$ to $\mathcal{D}(L) \cap L_0^2(\Omega)$ is one-one, and so is the restriction of $\nabla$ to $H^1 \cap L_0^2(\Omega)$.

Thus, we can consider that $L$ has the form $D^*AD$ with $D$ being a one-one operator and the abstract nonsense material contained in Preliminaries of [5] applies.

**Remark.** As far as the Kato conjecture is çoncerned, adding lower order terms with bounded coefficients to such an operator is taken care of by a result in [4]: that is, Theorem 1 is valid for inhomogeneous second order elliptic operators in divergence form. See also Proposition 11 of Preliminaries in [5] for a more precise result with a proof which applies to higher order systems. Thus, we restrict ourselves to pure second order operators in this paper.

## 2 Proof of the main result

To prove Theorem 1, we establish (K) or ($K_{loc}$) (depending on $\Omega$) for any elliptic operator $L = (A, \Omega, V)$ as above: $A \in \mathcal{A}$, $\Omega$ is a strongly Lipschitz domain and $V$ is $H^1_0(\Omega)$ or $H^1(\Omega)$.

By [2], (K) holds for all elliptic operators of the form $(A, \mathbb{R}^n, H^1(\mathbb{R}^n))$. The argument to obtain the conclusion on any strongly Lipschitz domain contains four steps: localization, change of variables, multiplicative perturbations and the study on the upper half-space. We take them in reverse order.

**Step 1:** Study on the upper half-space.

Pick a coordinate system $(x_1, \dots, x_n)$ in $\mathbb{R}^n$. Let

$$\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,;\, x_n > 0\}.$$

Define the orthogonal symmetry $S$ of $\mathbb{R}^n$ across $\partial\mathbb{R}^n_+$ by

$$S(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, -x_n).$$

Denote by $I(f)(x) = f(x)$ the identity operator and by $J(f)(x) = f(Sx)$ the reflection operator for $f \colon \mathbb{R}^n \to \mathcal{C}$. The transformation $\mathcal{J}$ defined by

$$\mathcal{J}(f) = \frac{1}{\sqrt{2}}\left((I+J)(f)\,|_{\mathbb{R}^n_+}, (I-J)(f)\,|_{\mathbb{R}^n_+}\right)$$

is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n_+) \oplus L^2(\mathbb{R}^n_+)$ with

$$\int_{\mathbb{R}^n} |f|^2 = \frac{1}{2}\int_{\mathbb{R}^n_+} |(I+J)(f)|^2 + \frac{1}{2}\int_{\mathbb{R}^n_+} |(I-J)(f)|^2$$

and from $\dot{H}^1(\mathbb{R}^n)$ to $\dot{H}^1(\mathbb{R}^n_+) \oplus \dot{H}^1_0(\mathbb{R}^n_+)$ (homogeneous spaces) with

$$\int_{\mathbb{R}^n} |\nabla f|^2 = \frac{1}{2}\int_{\mathbb{R}^n_+} |\nabla(I+J)(f)|^2 + \frac{1}{2}\int_{\mathbb{R}^n_+} |\nabla(I-J)(f)|^2.$$

This map is also onto in both cases, and its inverse is given by

$$\mathcal{J}^{-1}(\phi, \psi) = \frac{1}{\sqrt{2}}(\phi_e + \psi_o),$$

where for $f \colon \mathbb{R}^n_+ \to \mathcal{C}$, $f_e$ (resp., $f_o$) is its even (resp., odd) extension to $\mathbb{R}^n$ defined by $f_e(x) = f(Sx)$ (resp., $f_o(x) = -f(Sx)$) if $x_n < 0$.

Given $A \in \mathcal{A}$, define $A^\sharp \in \mathcal{A}$ by $A^\sharp(x) = A(x)$ if $x_n \geq 0$ and $A^\sharp(x) = SA(Sx)S$ if $x_n < 0$. Let $L_D = (A, \mathbb{R}^n_+, H^1_0(\mathbb{R}^n_+))$, $L_N = (A, \mathbb{R}^n_+, H^1(\mathbb{R}^n_+))$ and

$L^\sharp = (A^\sharp, \mathbb{R}^n, H^1(\mathbb{R}^n))$; and let $Q_D$, $Q_N$ and $Q^\sharp$ be the associated sesquilinear forms as in (1). The operator $\mathcal{J}$ relates the forms by

$$Q^\sharp(f,g) = Q_N(f_N, g_N) + Q_D(f_D, g_D),$$

where $\mathcal{J}(f) = (f_N, f_D)$ and $\mathcal{J}(g) = (g_N, g_D)$. Using the characterization (2) of the domain of each operator, it is not difficult to show that

$$\mathcal{D}(L^\sharp) = \mathcal{J}^{-1}(\mathcal{D}(L_D) \oplus \mathcal{D}(L_N))$$

and that

$$L^\sharp = \mathcal{J}^{-1} \begin{pmatrix} L_N & 0 \\ 0 & L_D \end{pmatrix} \mathcal{J}.$$

It follows from the interpolation result of [9] that (K) holds for $L^\sharp$ if and only if it holds for both $L_N$ and $L_D$. Hence, we have proved that (K) holds for any $L = (A, \mathbb{R}^n_+, V)$.

**Step 2:** Perturbative multiplications.

Assume that $m$ is a positive real-valued function with $m, m^{-1} \in L^\infty(\mathbb{R}^n_+)$ and let $L = (A, \mathbb{R}^n_+, V)$. The operator $mL$ is well-defined on $\mathcal{D}(L)$ and has a square root. We have that (K) for $L$ is equivalent to (K) for $mL$. The proof of Lemma 14 in the Preliminaries of [5] given on $\mathbb{R}^n$ applies with the obvious changes.

**Step 3:** Bilipschitz change of variables.

Assume that $\Omega$ is a special Lipschitz domain: if $\Phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a defining Lipschitz function of $\partial\Omega$, the Lipschitz constant is, by definition, the quantity $\|\nabla\Phi\|_\infty$.

Choose $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ to be a bilipschitz change of variables with $\phi(\mathbb{R}^n_+) = \Omega$ and $\phi(\partial\mathbb{R}^n_+) = \partial\Omega$. Define $Tf = f \circ \phi$. Let $L = (A, \Omega, V)$. Then one has $\mathcal{D}(L) = T^{-1}(\mathcal{D}(mL_\phi))$ and

$$L = T^{-1}(mL_\phi)T,$$

where $L_\phi = (A_\phi, \mathbb{R}^n_+, T^{-1}(V))$ with, for $x \in \mathbb{R}^n_+$,

$$A_\phi(x) = |\det J_\phi(x)|\, {}^TJ_\phi^{-1}(x) A(\phi(x)) J_\phi^{-1}(x),$$

$J_\phi(x)$ being the jacobian matrix of $\phi$ at $x$, ${}^TJ_\phi(x)$ its transpose and $m(x) = |\det J_\phi(x)|^{-1}$. Note that $T^{-1}(V) = H^1(\mathbb{R}^n_+)$ if $V = H^1(\Omega)$ and $T^{-1}(V) = H^1_0(\mathbb{R}^n_+)$ if $V = H^1_0(\Omega)$.

From the first two steps, we deduce that (K) is valid for $L$.

**Step 4:** Localisation.

This relies on three lemmas, the first of which we only need for $k = 2$ being the key one. We stress that since the operators are complex, the usual comparison principles for weak solutions do not apply.

**Lemma 2.** *Let $\Omega$ be an open set of $\mathbb{R}^n$ and $A \in \mathcal{A}$. Let $V$ be a closed subspace of $H^1(\Omega)$ that contains $H^1_0(\Omega)$ such that $v \in V$ and $\eta \in C_0^\infty(\mathbb{R}^n)$ imply $v\eta_{|\Omega} \in V$. Let $P$ be an open set of $\mathbb{R}^n$ and, for $t > 0$, let $u_t \in V$ be such that*

$$\int_\Omega u_t \,\overline{v} + t^2 \int_\Omega A\nabla u_t \cdot \overline{\nabla v} = 0$$

*for all $v \in V$ such that* $\operatorname{supp} v \subset P$. *Let $O$ be an open set with positive distance to ${}^cP$ (in particular, $\overline{O} \subset P$). Then, for any $k \in \mathcal{N}^*$, we have*

$$\int_{O\cap\Omega} |u_t|^2 \le \frac{ct^{2k}}{d^{2k}} \int_{P\cap\Omega} |u_t|^2,$$

*where $d = d({}^cP, O) > 0$ and $c$ depends on $n$, $k$ and the ellipticity constants of $A$.*

**Proof.** The argument uses a Caccioppoli-type inequality. Let $\eta \in C_0^\infty(\mathbb{R}^n)$, where $\eta$ is real-valued with $\operatorname{supp}\eta \subset P$; then $v = u_t\eta^2_{|\Omega}$ is an appropriate test function in $V$. A calculation gives

$$\int_\Omega |u_t|^2\eta^2 + t^2 \int_\Omega A\nabla u_t \cdot \overline{\nabla u_t}\; \eta^2 = -2t^2 \int_\Omega A(\eta\nabla u_t)\cdot \overline{u_t\nabla\eta}.$$

Using ellipticity and $2|ab| \le \epsilon|a|^2 + \epsilon^{-1}|b|^2$ , we obtain for all $\epsilon > 0$

$$\int_\Omega |u_t|^2\eta^2 + \lambda t^2 \int_\Omega |\nabla u_t|^2\,\eta^2 \le \Lambda\epsilon t^2 \int_\Omega |\nabla u_t|^2\,\eta^2 + \Lambda\epsilon^{-1}t^2 \int_\Omega |u_t|^2|\nabla\eta|^2.$$

Choosing $\epsilon = \lambda/\Lambda$ leads to

$$\int_\Omega |u_t|^2\eta^2 \le \delta t^2 \int_\Omega |u_t|^2|\nabla\eta|^2.$$

We have set $\delta = \Lambda^2/\lambda$. Observe that this is valid for all $\eta$ as above. Hence, applying this inequality to $\eta^k$, $k$ integral, and iterating yields

$$\begin{aligned}
\int_\Omega |u_t|^2\eta^{2k} &\le \delta t^2 \int_\Omega |u_t|^2 k^2 \eta^{2(k-1)}|\nabla\eta|^2 \\
&\le k^2\delta\|\nabla\eta\|_\infty^2\, t^2 \int_\Omega |u_t|^2\eta^{2(k-1)} \\
&\le (k!)^2(\delta\|\nabla\eta\|_\infty^2\, t^2)^{k-1} \int_\Omega |u_t|^2\eta^2. \\
&\le (k!)^2(\delta\|\nabla\eta\|_\infty^2\, t^2)^{k} \int_{P\cap\Omega} |u_t|^2.
\end{aligned}$$

It remains to choose $\eta = 1$ on $O$ with $\|\nabla\eta\|_\infty \sim 1/d$ to conclude. □

**Remark 3.** Pursuing the argument of the lemma leads to the following off-diagonal estimate for the resolvent,

$$\|(1+t^2L)^{-1}f\|_{L^2(O\cap\Omega)} \le ce^{-\alpha d({}^cP,O)/t}\|f\|_{L^2({}^cP\cap\Omega)}, \quad \operatorname{Supp} f \subset {}^cP\cap\Omega,$$

where $c$ and $\alpha > 0$ depend only on $n$, ellipticity and $\Omega$. See [2], where this is done for $\Omega = \mathbb{R}^n$. Further considerations give us also the off-diagonal estimate for the semigroup

$$\|e^{-tL}f\|_{L^2(O\cap\Omega)} \le c'e^{-\alpha' d({}^cP,O)^2/t}\|f\|_{L^2({}^cP\cap\Omega)}, \quad \operatorname{Supp} f \subset {}^cP\cap\Omega.$$

This means that the well-known Gaffney Lemma [7] extends to any complex elliptic second order operator as above with the hypotheses on $V$ in the lemma. Of course, if $\Omega$ is strongly Lipschitz or $\mathbb{R}^n$ and $V = H_0^1(\Omega)$ or $H^1(\Omega)$, then the lemma applies.

The first consequence is the treatment of operators with coefficients that agree on an open set. We use a formulation that takes into account interior and boundary estimates in the same flow.

**Lemma 4 (*Comparison principle*).** *Assume both domains $\Omega_\alpha$ and $\Omega_\beta$ to be either $\mathbb{R}^n$ or strongly Lipschitz. Let $A_\alpha, A_\beta \in \mathcal{A}$ such that $L_\alpha = -\operatorname{div}(A_\alpha\nabla)$ and $L_\beta = -\operatorname{div}(A_\beta\nabla)$ are operators with $V_i = H_0^1(\Omega_i)$ (resp., $V_i = H^1(\Omega_i)$) for $i = \alpha, \beta$. Let $P$ be an open set of $\mathbb{R}^n$ such that $P\cap\partial\Omega_\alpha = P\cap\partial\Omega_\beta$, $P\cap\Omega_\beta \subset \Omega_\alpha$ and that $A_\alpha = A_\beta$ on $P\cap\Omega_\beta$. Then for any $\chi \in C_0^\infty(P)$ and for any open set $O$ of $\mathbb{R}^n$ such that $d = d({}^cP, O) > 0$, we have*

$$\int_0^\infty \left\|(1+t^2L_\beta)^{-1}tL_\beta(\chi f) - (1+t^2L_\alpha)^{-1}tL_\alpha(\chi f)\right\|_{L^2(O\cap\Omega_\beta)}\frac{dt}{t} \le \frac{c\,\|\chi f\|_{L^2(\Omega_\beta)}}{d}$$

*for all $f \in V_\beta$, where $c$ depends only on $n$ and the ellipticity constants of $A_\beta$.*

**Proof.** First, note that the assumptions on $f$ and $\chi$ insure that $\chi f \in V_\alpha\cap V_\beta$ (we are making a slight abuse of notation, as one should distinguish $f_\alpha = \chi_{|\Omega_\alpha} f$ from $f_\beta = \chi_{|\Omega_\beta} f$). Set $u_t^i = (1+t^2L_i)^{-1}tL_i(\chi f) \in V_i$, for $i = \alpha, \beta$, and $u_t = u_t^\beta - u_t^\alpha$. Since $\|u_t^i\|_{L^2(\Omega_i)} \le \frac{2}{t}\|\chi f\|_{L^2(\Omega_i)} = \frac{2}{t}\|\chi f\|_{L^2(\Omega_\beta)}$,

$$\int_d^\infty \|u_t^i\|_{L^2(\Omega_i)}\frac{dt}{t} \le \frac{2}{d}\,\|\chi f\|_{L^2(\Omega_\beta)},$$

so it is enough to prove

$$\int_0^d \|u_t\|_{L^2(O\cap\Omega_\beta)}\frac{dt}{t} \le \frac{c}{d}\,\|\chi f\|_{L^2(\Omega_\beta)}.$$

The variational formulation tells us that for all $v \in V_i$,

$$\int_{\Omega_i} u_t^i \,\overline{v} + t^2 \int_{\Omega_i} A_i \nabla u_t^i \cdot \overline{\nabla v} = -t \int_{\Omega_i} A_i \nabla(\chi f) \cdot \overline{\nabla v};$$

and since $A_\alpha = A_\beta$ on $P \cap \Omega_\beta$, we obtain

$$\int_{\Omega_\beta} u_t \,\overline{v} + t^2 \int_{\Omega_\beta} A_\beta \nabla u_t \cdot \overline{\nabla v} = 0$$

for all $v \in V_\beta$ such that $\operatorname{Supp} v \subset P$. We deduce from the previous lemma that

$$\|u_t\|_{L^2(O \cap \Omega_\beta)} \leq \frac{ct^2}{d^2} \|u_t\|_{L^2(P \cap \Omega_\beta)} \leq \frac{4ct}{d^2} \|\chi f\|_{L^2(\Omega_\beta)},$$

and the conclusion follows readily. □

Next, we can also obtain estimates taking care of non-local terms.

**Lemma 5 (*Off-diagonal estimates*).** *Let $\Omega$ be a strongly Lipschitz domain or $\mathbb{R}^n$ and $L = (A, \Omega, V)$ an elliptic operator on $\Omega$ with Dirichlet or Neumann boundary condition. Let $E, F$ be two closed subsets of $\mathbb{R}^n$ such that $d = d(E,F) > 0$ and $\chi \in C_0^\infty(E)$. Then*

$$\int_0^\infty \|(1+t^2L)^{-1} tL(\chi f)\|_{L^2(F\cap\Omega)} \frac{dt}{t} \leq \frac{c\|\chi f\|_{L^2(\Omega)}}{d}$$

*for all $f \in V$. The constant $c$ depends on $n$ and the ellipticity constants of $A$.*

**Proof.** Again, $\chi f$ should be interpreted as $\chi_{|\Omega} f$. Using (5), we have

$$\int_d^\infty \|(1+t^2L)^{-1} tL(\chi f)\|_{L^2(\Omega)} \frac{dt}{t} \leq \frac{2}{d} \|\chi f\|_{L^2(\Omega)}.$$

Next, using (4) and $(\operatorname{Supp}\chi) \cap F = \emptyset$, we obtain

$$(1+t^2L)^{-1} tL(\chi f) = -\frac{(1+t^2L)^{-1}(\chi f)}{t}, \quad \text{on } F \cap \Omega.$$

Hence, it suffices to prove

$$\int_0^d \frac{1}{t} \|(1+t^2L)^{-1}(\chi f)\|_{L^2(F\cap\Omega)} \frac{dt}{t} \leq \frac{c}{d} \|\chi f\|_{L^2(\Omega)}.$$

Setting $u_t = (1+t^2L)^{-1}(\chi f)$, we have for all $v \in V$ such that $v = 0$ on $F$

$$\int_\Omega u_t \overline{v} + t^2 \int_\Omega A \nabla u_t \cdot \overline{\nabla v} = 0.$$

By Lemma 2 applied with $O$ a neighborhood of $F$ and $P$ a neighborhood of $\overline{O}$ such that $d(E,P) = d(E,F)/2 > 0$, and the $L^2(\Omega)$-contractivity of the resolvent, we obtain

$$\|u_t\|_{L^2(F\cap\Omega)} \le \frac{ct^2}{d^2}\,\|u_t\|_{L^2(P\cap\Omega)} \le \frac{ct^2}{d^2}\,\|\chi f\|_{L^2(\Omega)}.$$

The conclusion follows at once. □

We are now ready to prove that ($\mathrm{K_{loc}}$) holds on all strongly Lipschitz domains. Let $L = (A,\Omega,V)$ be defined on the strongly Lipschitz domain $\Omega$ with boundary condition space given by $V$. Following [12], there exist an integer $s$, a number $d > 0$ and for $0 \le k \le s$, $C_0^\infty(\mathbb{R}^n)$ real-valued functions $\chi_k$ and $\eta_k$, and open sets $O_k, P_k, \Omega_k$ with the following properties:

1. $\sum_{0\le k\le s} \chi_k(x) = 1$, for $x$ in a neighborhood of $\Omega$;
2. $\Omega_0 = \mathbb{R}^n$, $\operatorname{Supp}\chi_0 \subset O_0 \subset \overline{O_0} \subset P_0 \subset \overline{P_0} \subset \Omega$;
3. for $k \ge 1$, $\Omega_k$ is the image of a special Lipschitz domain under an orthogonal transformation in $\mathbb{R}^n$ such that $\operatorname{Supp}\chi_k \cap \Omega \subset \Omega_k \cap \Omega$;
4. for $k \ge 1$, $O_k$ and $P_k$ are open neighborhoods of $\operatorname{Supp}\chi_k$ in $\mathbb{R}^n$ such that $\overline{O_k} \subset P_k$, $P_k \cap \Omega \subset \Omega_k \cap \Omega$ and $\partial\Omega \cap \overline{P_k} = \partial\Omega_k \cap \overline{P_k}$, at most one of the latter possibly infinite;
5. for $k \ge 0$, $\operatorname{Supp}\eta_k \subset P_k$, $\eta_k = 1$ on a neighborhood of $\overline{O_k}$, $\eta_k \ge 0$ and $\|\eta_k\|_\infty = 1$;
6. for $k \ge 0$, $d(O_k, {}^cP_k) \ge d$ and $d(\operatorname{Supp}\chi_k, {}^cO_k) \ge d$.

The Lipschitz constant of $\Omega$ is the infimum of $\max(M_1,\dots,M_s)$, where $M_k$ is the Lipschitz constant of $\Omega_k$, taken over all possible decompositions of $\Omega$ in this way. Roughly, there is one interior piece and $s$ boundary pieces to look at.

For $0 \le k \le s$, set $L_k = (A,\Omega_k,V_k)$, where $\Omega_0 = \mathbb{R}^n$, $V_0 = H^1(\mathbb{R}^n)$ and for $k \ge 1$, if $V = H_0^1(\Omega)$ (resp., $H^1(\Omega)$) then $V_k = H_0^1(\Omega_k)$ (resp., $H^1(\Omega_k)$). Note that if $f \in V$, then $\chi_k f \in V \cap V_k$, so that all operations make sense.

Now that these precautions are taken, fix $f \in \mathcal{D}(L) \subset V$; since $f = \sum \chi_k f$, we may write

$$L^{1/2}f = \sum_{0\le k\le s} \eta_k L_k^{1/2}(\chi_k f) + \sum_{0\le k\le s} \eta_k (L^{1/2} - L_k^{1/2})(\chi_k f) + \sum_{0\le k\le s} (1-\eta_k) L^{1/2}(\chi_k f).$$

By the result on $\mathbb{R}^n$ and on special Lipschitz domains together with rotational invariance, the inequality (K) holds for $L_k$; hence

$$\|\eta_k L_k^{1/2}(\chi_k f)\|_{L^2(\Omega)} \leq c_k \|\nabla(\chi_k f)\|_{L^2(\Omega)}.$$

Note that $c_k$ depends on $n$, $\lambda$, $\Lambda$ and also on $M_k$ if, in addition, $k \geq 1$.

Next, the comparison principle with $L_\alpha = L_k$, $L_\beta = L$, $P = P_k$, $O = O_k$, $\Omega_\alpha = \Omega_k$ and $\Omega_\beta = \Omega$ and the representation (3) for square roots yield

$$\|\eta_k (L^{1/2} - L_k^{1/2})(\chi_k f)\|_{L^2(\Omega)} \leq \frac{c'}{d} \|\chi_k f\|_{L^2(\Omega)}.$$

Finally, the off-diagonal estimates with $E = \operatorname{Supp} \chi_k$ and $F = {}^c\!O_k$ and (3) imply

$$\|(1 - \eta_k) L^{1/2}(\chi_k f)\|_{L^2(\Omega)} \leq \frac{c''}{d} \|\chi_k f\|_{L^2(\Omega)}.$$

Hence ($K_{\text{loc}}$) follows for $L$. This concludes the proof of Theorem 1. □

## REFERENCES


[1] P. Auscher, S. Hofmann, J. Lewis and Ph. Tchamitchian, *Extrapolation of Carleson measures and the analyticity of the Kato's square root operator*, Acta Math. **187** (2001), 161–190.

[2] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and Ph. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$, Ann. of Math. (2) **156** (2002), 633–654.

[3] P. Auscher, A. McIntosh and A. Nahmod, *The square root problem of Kato in one dimension, and first order systems*, Indiana Univ. Math. J. **46** (1997), 659–695.

[4] P. Auscher and Ph. Tchamitchian, *Conjecture de Kato sur les ouverts de* $\mathbb{R}$, Rev. Mat. Iberoamericana **8** (1992), 149–199.

[5] P. Auscher and Ph. Tchamitchian, *Square root problem for divergence operators and related topics*, Astérisque **249**, Soc. Math. France, 1998.

[6] R. Coifman, A. McIntosh and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur* $L^2(\mathbb{R})$ *pour les courbes lipschitziennes*, Ann. of Math. (2) **116** (1982), 361–387.

[7] M. P. Gaffney, *The conservation property for the heat equation on Riemannian manifold*, Comm. Pure Appl. Math. **12** (1959), 1–11.

[8] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.

[9] J.-L. Lions, *Espaces d'interpolation et domaines de puissances fractionnaires*, J. Math. Soc. Japan **14** (1962), 233–241.

[10] A. McIntosh, *Square roots of elliptic operators*, J. Funct. Anal. **61** (1985), 307–327.

[11] A. McIntosh, *The square root problem for elliptic operators*, in *Functional Analytic Methods for Partial Differential Equations*, Lecture Notes in Math. **1450**, Springer-Verlag, Berlin, 1990, pp. 122–140.

[12] E. M. Stein, *Singular Integrals and Differentiability of Functions*, Princeton University Press, 1970.

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