ISRAEL JOURNAL OF MATHEMATICS **128** (2002), 247–265

DISCRETENESS CRITERIA FOR $\mathcal{RP}$ GROUPS*

BY

E. KLIMENKO**

Department of Mathematics, Technion–Israel Institute of Technology
Haifa 32000, Israel
e-mail: klimenko@tx.technion.ac.il

AND

N. KOPTEVA

Department of Mathematics, Heriot-Watt University
Edinburgh EH14 4AS, Scotland
e-mail: natasha@ma.hw.ac.uk

ABSTRACT

Recently Gehring, Gilman, and Martin introduced an important class of two-generator groups with real parameters:

$$\{\Gamma = \langle f, g\rangle \mid f, g \in \mathrm{PSL}(2, \mathbf{C});\ \beta, \beta', \gamma \in \mathbf{R}\},$$

where $\beta = \mathrm{tr}^2 f - 4$, $\beta' = \mathrm{tr}^2 g - 4$, and $\gamma = \mathrm{tr}(fgf^{-1}g^{-1}) - 2$. The groups that belong to this class we call $\mathcal{RP}$ groups. We find criteria for discreteness of $\mathcal{RP}$ groups generated by a hyperbolic element and an elliptic one of even order with intersecting axes.

* The research was partially supported by the Russian Foundation for Basic Research (Grant 99-01-00630).

** *Current address*: Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208-2730, USA;
e-mail: klimenko@math.northwestern.edu
Received July 6, 2000

1. Introduction

The group of all Möbius transformations of the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is isomorphic to $\mathrm{PSL}(2, \mathbf{C}) = \mathrm{SL}(2, \mathbf{C})/\{\pm I\}$. The Poincaré extension gives the action of this group (as the group of all orientation preserving isometries) on hyperbolic 3-space

$$\mathbf{H}^3 = \{(z,t) |\, z \in \mathbf{C},\ t > 0\}$$

with the Poincaré metric

$$ds^2 = \frac{|dz|^2 + t^2}{t^2}.$$

Study of two-generator subgroups of $\mathrm{PSL}(2, \mathbf{C})$ and discreteness conditions for them has a rich history (see [2, 7, 8, 11], [15]–[20] and references therein). Criteria for discreteness are known for elementary groups (see [1, 25]) and for two-generator groups with invariant plane (see [6, 12, 13, 24, 28, 29, 30] for Fuchsian groups and [23] for groups containing elements reversing orientation of invariant plane).

As for non-elementary groups without invariant plane, in most papers either only necessary or only sufficient conditions for discreteness of such groups are given.

It is well known that as parameters for two-generator subgroup $\langle f, g\rangle$ of $\mathrm{PSL}(2, \mathbf{C})$ one can take

$$(\beta, \beta', \gamma) = (\beta(f), \beta(g), \gamma(f, g)),$$

where $\beta(f) = \mathrm{tr}^2 f - 4$, $\gamma(f, g) = \mathrm{tr}[f, g] - 2$. Further, if $\gamma \neq 0$ then $\langle f, g\rangle$ is uniquely determined by the parameters up to conjugacy [8]. In [7], Gehring, Gilman, and Martin suggest the investigation of the class of two-generator groups with real parameters:

$$\mathcal{RP} = \{\Gamma = \langle f, g\rangle |\, f, g \in \mathrm{PSL}(2, \mathbf{C});\ \beta, \beta', \gamma \in \mathbf{R}\}.$$

The groups that belong to this class we call $\mathcal{RP}$ **groups**. In [7] necessary conditions on the parameters for the discrete $\mathcal{RP}$ groups are obtained.

In Subsection 2.1 we obtain an exact geometric equivalent of the condition $(\beta, \beta', \gamma) \in \mathbf{R}^3$. Moreover, we characterize all non-elementary $\mathcal{RP}$ groups without invariant plane (Theorem 4). In Table 1 (Subsection 2.1) we distinguish 12 cases of such groups. Cases 1–6 were investigated earlier [17]–[20], and we include the list of parameters that correspond to the discrete groups in these cases (see Appendix: Table 2 and Remark 2).

Our main result is Theorem A in Section 3 which gives the complete description of the discrete $\mathcal{RP}$ groups in Case 7 for an even order elliptic generator. Case 7 with an elliptic generator of odd order is the topic of coming paper [21]. This will complete the full description of $\mathcal{RP}$ groups with non-π-loxodromic generators. (For the definition of π-loxodromic see the third paragraph in Subsection 2.1.)

ACKNOWLEDGEMENT: The authors would like to express their deep gratitude to the Department of Mathematics at the Technion–Israel Institute of Technology for support, and personally to Prof. R. Brooks and Prof. B. Wajnryb for such splendid hospitality during the preparation of this paper and related papers in the area. We thank Prof. A. Mednykh for his continued interest in this work. The second author thanks the Weizmann Institute of Science for inviting her to Summer Program 2000. The first author is indebted to Prof. M. Sakuma for useful discussions and to Prof. D. Derevnin who introduced her (in 1988) to the geometric meaning of the commutator of two elliptic elements f and g, and presented a beautiful geometric proof based on the consideration of the axes of f and $gf^{-1}g^{-1}$.

Both authors also thank the referee for many valuable suggestions.

2. Preliminaries

2.1 GEOMETRIC MEANING OF THE PARAMETERS. Let f and g be elements of $\mathrm{PSL}(2,\mathbf{C})$. Parameters $(\beta(f),\beta(g),\gamma(f,g))$ have a definite geometric meaning that we clarify in this section.

All theorems in the section can be easily proved and, perhaps, are known. However, we have not come across them and include the proofs for the reader's convenience.

Recall that an element $f \in \mathrm{PSL}(2,\mathbf{C})$ with real $\beta(f)$ is **elliptic**, **parabolic**, **hyperbolic**, or **π-loxodromic** according to whether $\beta(f) \in [-4,0)$, $\beta(f)=0$, $\beta(f) \in (0,+\infty)$, or $\beta(f) \in (-\infty,-4)$. If $\beta(f) \notin [-4,\infty)$, then f is called **strictly loxodromic**. Among all strictly loxodromic elements only π-loxodromics have real $\beta(f)$.

Let $f,g \in \mathrm{PSL}(2,\mathbf{C})$, $\beta(f)\neq 0$, $\beta(g)\neq 0$. Assume further that $\mathrm{Fix}f \neq \mathrm{Fix}g$ where $\mathrm{Fix}h$ denotes the fixed point set in $\overline{\mathbf{C}}$ of a transformation h. The condition $\beta(f)\neq 0$ (analogously, $\beta(g)\neq 0$) is equivalent to the fact that f (resp. g) has two fixed points in $\overline{\mathbf{C}}$. We normalize f and g (i.e., conjugate them by an appropriate element of $\mathrm{PSL}(2,\mathbf{C})$) so that 0 and ∞ are the fixed points of f in $\overline{\mathbf{C}}$; and g fixes

1 and $z = x + iy$, $z \neq 1$. Then (see [25]):

$$f = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{and} \quad g = \frac{1}{z-1}\begin{pmatrix} zs^{-1} - s & z(s - s^{-1}) \\ s^{-1} - s & zs - s^{-1} \end{pmatrix}.$$

We compute

$$\gamma(f,g) = \frac{z}{(z-1)^2}(t - t^{-1})^2(s - s^{-1})^2 = \frac{z}{(z-1)^2}\beta(f)\beta(g).$$

Thus we have proved the following

LEMMA 1: *Let $f, g \in \mathrm{PSL}(2, \mathbf{C})$, $\beta(f) \neq 0$, $\beta(g) \neq 0$, and $\mathrm{Fix} f \neq \mathrm{Fix} g$. Then*

$$\gamma(f,g) = \frac{z}{(z-1)^2}\beta(f)\beta(g), \tag{1}$$

where $z \in \mathbf{C}\backslash\{1\}$ is a fixed point of g when f and g are normalized as above.

The lemma above means, in particular, that if the axes of f and g are fixed, then $\gamma(f,g)/(\beta(f)\beta(g))$ is a constant (it does not depend on the type of elements f and g).

The next three theorems characterize the relative position of the axes or invariant planes of two elements with real $\beta(f)$ and $\beta(g)$. We start with non-parabolic elements.

THEOREM 1: *Let f and g be elements of $\mathrm{PSL}(2, \mathbf{C})$, and let $\beta(f)$ and $\beta(g)$ be non-zero real numbers. Then:*

(i) *$\gamma(f,g)$ is real if and only if the axes of f and g either lie in one hyperbolic plane or are mutually orthogonal skew lines.*

(ii) *$\gamma(f,g)$ is real and $\gamma(f,g)/(\beta(f)\beta(g)) \geq -\frac{1}{4}$ if and only if there exists a hyperbolic plane containing the axes of f and g. Moreover, if $\gamma(f,g)/(\beta(f)\beta(g)) > 0$ then the axes are disjoint, if $\gamma(f,g)/(\beta(f)\beta(g)) = 0$ then the axes of f and g are parallel or coincide, if $-1/4 < \gamma(f,g)/(\beta(f)\beta(g)) < 0$ then the axes intersect non-orthogonally, and if $\gamma(f,g)/(\beta(f)\beta(g)) = -1/4$ then they intersect orthogonally.*

(iii) *$\gamma(f,g)$ is real and $\gamma(f,g)/(\beta(f)\beta(g)) < -1/4$ if and only if the axes of f and g are mutually orthogonal skew lines.*

Proof: The case that f and g have a common fixed point in $\overline{\mathbf{C}}$ is equivalent to the condition $\mathrm{tr}[f,g] = 2$, or $\gamma(f,g) = 0$ (see [1], Theorem 4.3.5); and there is nothing to prove. Therefore, we assume that $\mathrm{Fix} f$ and $\mathrm{Fix} g$ are disjoint. Using Lemma 1 for normalized elements, we have

$$\gamma(f,g) = \frac{z}{(z-1)^2}\beta(f)\beta(g),$$

where $z \in \mathbf{C}$, $z \neq 1$.

Taking into account that $\beta(f)$ and $\beta(g)$ are non-zero real numbers, we see that $\gamma(f,g)$ is real if and only if $z/(z-1)^2 \in \mathbf{R} \iff y = 0$ or $|z| = 1$ $(z \neq 1)$.

Since $y = 0$ if and only if the axes of f and g lie in a hyperbolic plane, and $|z| = 1$ if and only if the axes of f and g are mutually orthogonal, we conclude the proof of (i).

It can easily be checked that $y = 0$ (i.e., $z = x$ is real) if and only if $z/(z-1)^2 \geq -1/4$. To prove (ii) we note that $x > 0$ ($x = 0$, $x < 0$, $x = -1$) means that the axes of f and g are disjoint (resp. parallel, intersecting, or intersecting orthogonally).

Furthermore, $z/(z-1)^2 < -1/4$ if and only if $|z| = 1$ and $z \neq \pm 1$. This completes the proof of the theorem. ∎

We next take up the case that one of two elements is parabolic.

THEOREM 2: *Let f and g be non-trivial elements of* $\mathrm{PSL}(2,\mathbf{C})$ *such that $\beta(f)$ is non-zero real number, $\beta(g) = 0$, and $\gamma(f,g) \neq 0$. Then:*

(i) *$\gamma(f,g)$ is real if and only if there is an invariant plane of g which either contains the axis of f or is orthogonal to the axis of f;*

(ii) *$\gamma(f,g)$ is real and $\gamma(f,g)/\beta(f) > 0$ if and only if the axis of f lies in an invariant plane of g;*

(iii) *$\gamma(f,g)$ is real and $\gamma(f,g)/\beta(f) < 0$ if and only if the axis of f is orthogonal to an invariant plane of g.*

Proof: The condition $\gamma(f,g) \neq 0$ means that f does not fix the fixed point of g. We can normalize f and g so that 0 and 1 are fixed points of f, and ∞ is the fixed point of g. Then we have

$$f = \begin{pmatrix} s & 0 \\ s - s^{-1} & s^{-1} \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

An easy computation now yields

$$\gamma(f,g) = t^2 \beta(f). \tag{2}$$

The rest of the proof is left to the reader. ∎

Finally, we consider the case where both elements are parabolic.

THEOREM 3: *Let f and g be two parabolic elements of* $\mathrm{PSL}(2,\mathbf{C})$, *that is, f and g are non-trivial and $\beta(f) = \beta(g) = 0$; and let $\gamma(f,g) \neq 0$. Then:*

(i) *$\gamma(f,g)$ is real if and only if either f and g have a common invariant plane or one of the invariant planes of f is orthogonal to all invariant planes of g. Moreover,*

(ii) *$\gamma(f,g)$ is a positive real number if and only if f and g have a common invariant plane;*

(iii) *$\gamma(f,g)$ is a negative real number if and only if g has an invariant plane that is orthogonal to all invariant planes of f.*

Remark 1: Conclusion (iii) implies that f has an invariant plane orthogonal to all invariant planes of g if and only if g has an invariant plane orthogonal to all invariant planes of f.

Proof: Since $\gamma(f,g) \neq 0$, f and g have different fixed points. Normalize f and g so that

$$f = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

where $t \in \mathbf{C}\backslash\{0\}$. Notice that ∞ is the fixed point of f, and 0 is that of g. Moreover, invariant planes for f are Euclidean half-planes that are parallel to the radius-vector with end point $z = t$, and invariant planes for g are the plane $\{\mathrm{Im} z = 0\}$ and all Euclidean hemispheres which are tangent to this plane at 0. We compute

$$\gamma(f,g) = t^2.$$

Hence $\gamma(f,g)$ is real if and only if t is real or $t = is$, $s \in \mathbf{R}$. If t is real then $\{\mathrm{Im} z = 0\}$ is a common invariant plane for f and g; if $t = is$ ($s \in \mathbf{R}$), then $\{\mathrm{Im} z = 0\}$ is orthogonal to all invariant planes of f. Moreover, if $t^2 \notin \mathbf{R}$ then f and g have no common invariant plane, but each invariant plane of f (except that passing through the fixed point of g) is orthogonal to only one invariant plane of g.

To conclude the proof it remains to note that $\gamma(f,g) > 0$ if and only if t is real, and $\gamma(f,g) < 0$ if and only if $t = is$, $s \in \mathbf{R}$. ∎

We now consider $\mathcal{RP}$ groups (two-generator groups with real parameters, see Section 1). Their generators are various combinations of elliptic, parabolic, hyperbolic, and π-loxodromic elements (real β and β' determine the type of the generators). Conclusions (i) in Theorems 1–3 show us what it means for γ also to be real.

We conclude this section with characterization of those $\mathcal{RP}$ groups that are "truly spatial" (i.e., non-elementary without invariant plane). One can easily obtain the complete list of such groups by analyzing conclusions (ii) and (iii) in

Theorems 1–3 for various types of generators. We distinguish 12 cases listed in Table 1.

Table 1. Non-elementary $\mathcal{RP}$ groups without invariant plane

	β	β'	γ
1	$(-4,0)$	$(-4,0)$	$(-\infty,-\frac{1}{4}\beta\beta')$
2	$(-4,0)$	0	$(-\infty,0)$
3	0	0	$(-\infty,0)$
4	0	$(0,+\infty)$	$(-\infty,0)$
5	$(0,+\infty)$	$(0,+\infty)$	$(-\infty,-\frac{1}{4}\beta\beta')$
6	$(-4,0)$	$(0,+\infty)$	$(-\infty,0)$
7	$(-4,0)$	$(0,+\infty)$	$(0,-\frac{1}{4}\beta\beta')$
8	$(-\infty,-4)$	0	$(0,+\infty)$
9	$(-\infty,-4)$	$(0,+\infty)$	$(-\frac{1}{4}\beta\beta',+\infty)$
10	$(-\infty,-4)$	$(-\infty,-4)$	$(-\infty,-\frac{1}{4}\beta\beta')$
11	$(-\infty,-4)$	$(-4,0)$	$(-\frac{1}{4}\beta\beta',0)$
12	$(-\infty,-4)$	$(-4,0)$	$(0,+\infty)$

An easy modification of the table yields the following.

THEOREM 4: *Let $\Gamma = \langle f,g\rangle$ be an $\mathcal{RP}$ group. Γ is a non-elementary group without invariant plane if and only if*

$$(-1)^k\gamma < (-1)^{k+1}\beta\beta'/4,\ \gamma\neq 0,\ \beta\neq -4,\ \text{and}\ \beta'\neq -4,$$

where $k\in\{0,1,2\}$ is the number of π-loxodromic elements among f and g.

2.2 POLYHEDRA AND LINKS. A plane divides $\mathbf{H}^3$ into two components; we will call the closure of either of them a **half-space** in $\mathbf{H}^3$.

A connected subset P of $\mathbf{H}^3$ with non-empty interior is said to be a **(convex) polyhedron** if it is the intersection of a family $\mathcal{H}$ of half-spaces with the property that each point of P has a neighborhood meeting at most a finite number of boundaries of elements of $\mathcal{H}$.

Definition: Let P be a polyhedron in $\mathbf{H}^3$ and let ∂P be its boundary in $\mathbf{H}^3$. In (1)–(3) below we define the link for different "boundary" points of P (cf. [3]).

(1) Let $p \in \partial P$. Let S be a sphere in $\mathbf{H}^3$ with center p, whose radius is chosen small enough so that it only meets faces of P which contain p. Such a sphere exists by the local finiteness property we claim in the definition of a polyhedron. There is a natural way to endow S with a spherical geometry identifying S with $\mathbf{S}^2$ as follows. Map conformally $\mathbf{H}^3$ onto the unit ball $\mathbf{B}^3 = \{x \in \mathbf{R}^3 \mid |x| < 1\}$ so that p goes to 0 and after that change the scale of the sphere to be of radius 1. The **link** of p in P is defined to be the image of $S \cap P$ under the above identification (it is well-defined up to isometry).

(2) Let $\overline{\partial P}$ be the closure of ∂P in $\overline{\mathbf{H}}^3 = \mathbf{H}^3 \cup \overline{\mathbf{C}}$. Suppose $\overline{\partial P} \backslash \partial P \neq \emptyset$, and let $p \in \overline{\partial P} \backslash \partial P$. Then $p \in \overline{\mathbf{C}}$ (i.e., it is an ideal point). Let S be a horosphere centered at p that only meets those faces of P whose closures in $\overline{\mathbf{H}}^3$ contain p. We can identify S with Euclidean plane $\mathbf{E}^2$ using an isometry of $\mathbf{H}^3$ that sends p to ∞. The image of $S \cap P$ under such identification is called the **link** of the ideal boundary point p in P. Note that such a link is defined up to similarity.

(3) Suppose that there exists a hyperbolic plane S orthogonal to some faces $F_1, \dots, F_t$ of P, and suppose that the other faces of P lie in the same open half-space which is bounded by S. If $t \geq 3$ then we say that S corresponds to an **imaginary vertex** p of P; and we define the **link** of p in P to be $S \cap P$.

Notice that the link of a proper (lying in $\mathbf{H}^3$), ideal, or imaginary vertex p in P is a spherical, Euclidean, or hyperbolic polygon, respectively.

The surface S in the definition of a link is orthogonal to all faces of P that meet S; hence the group generated by reflections in these faces keeps S invariant and can be considered as the group of reflections in sides of spherical, Euclidean, or hyperbolic polygon in accordance with the type of the vertex.

We denote the triangle with angles π/p, π/q, and π/r in any of spaces $\mathbf{S}^2$, $\mathbf{E}^2$ or $\mathbf{H}^2$ by (p, q, r).

3. Main Theorem

We give here a criterion for discreteness of the $\mathcal{RP}$ group $\langle f, g\rangle$, where f is an elliptic element of even order $n > 2$, g is a hyperbolic element and their axes intersect non-orthogonally (Theorem A).

It is easy to see that if f is a non-primitive elliptic element of order n, i.e., rotation through an angle of $2\pi q/n$ $(1 < q < n/2)$, then there exists an integer $r \geq 2$ such that f^r is a primitive elliptic element of the same order n (such an r satisfies the condition $rq = 1 \pmod n$ and exists because $(n, q) = 1$). It is clear that $\langle f, g\rangle = \langle f^r, g\rangle$. Therefore, we assume without loss of generality that f is primitive.

THEOREM A: *Let f be a primitive elliptic element of even order n $(n \geq 4)$, g be a hyperbolic element, and let the axes of f and g intersect non-orthogonally. Then:*

(1) *there exist elements $h_1, h_2 \in \mathrm{PSL}(2, \mathbf{C})$ such that $h_1^2 = gfg^{-1}f$, $h_2^2 = f^{n/2}g^{-1}fgf^{-n/2}gf^{-1}g^{-1}$, $(fh_1^{-1})^2 = 1$, $(h_2gfg^{-1})^2 = 1$; and*

(2) *$\Gamma = \langle f, g\rangle$ is discrete if and only if one of the following conditions is satisfied:*
   - (i) *h_1 is hyperbolic, parabolic or a primitive elliptic element of even order m $(1/n + 1/m < 1/2)$, and h_2 is hyperbolic, parabolic or a primitive elliptic element of order $l \geq 3$;*
   - (ii) *h_1 is a primitive elliptic element of odd order m $(1/n + 1/m < 1/2)$ and h_2h_1 is hyperbolic, parabolic or a primitive elliptic element of order $k \geq 3$;*
   - (iii) *$n = 4$, h_1 is a primitive elliptic element of odd order $m \geq 5$, and h_2h_1 is the square of a primitive elliptic element of the same order m.*

Proof: Our proof proceeds in three stages. In part 1, we construct a group Γ^* where Γ and Γ^* are simultaneously discrete or non-discrete; in part 2, we show that (i) implies the discreteness of Γ^*; and in part 3, we assume that (i) does not hold.

1. We start with construction of a group Γ^* containing Γ as a subgroup of finite index. Our distant goal is to work with a group generated by reflections in faces of some polyhedron. Of course, the group we find to work with should be discrete if and only if Γ is.

Let f be a primitive elliptic element of even order $n \geq 4$, g be a hyperbolic element, and let their axes intersect non-orthogonally. We denote elements and their axes by the same letters when it does not lead to any confusion. Let ω be a plane containing f and g, and let e be a half-turn with the axis which is orthogonal to ω and passes through the point of intersection of f and g (see Figure 1).

Let e_f and e_g be half-turns such that $f = e_fe$ and $g = e_ge$. Axes e_f and e lie in some plane, denote it by ε, and intersect at an angle of π/n; ε and ω

are mutually orthogonal; e_g is orthogonal to ω and intersects g, moreover, the distance between e_g and e is equal to half of the translation length of g.

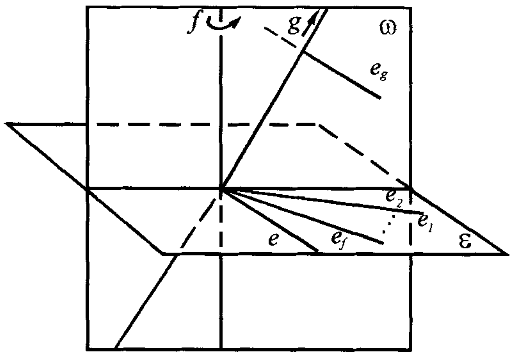

Figure 1.

Consider ε and $\langle e, e_f\rangle$ (see Figure 1). The group contains elements e, $e_f = fe$, f^2e, f^3e, ... Each element f^ke, $k = \overline{0,\infty}$ is a half-turn with axis lying in ε. Since n is even, in $\langle e, e_f\rangle$ there exist elements

$$e_1 = f^{n/2-1}e \quad \text{and} \quad e_2 = f^{n/2}e,$$

and the axis of e_2 coincides with the line of intersection of ω and ε (because the line is orthogonal to e).

Note that $f = R_\omega R_\alpha$, where α is the plane through f and e_1 (we denote the reflection in a plane κ by R_κ). It is clear that α intersects ω at an angle of π/n.

Define $\widetilde{\Gamma} = \langle f, g, e\rangle$ and $\Gamma^* = \langle f, g, e, R_\omega\rangle$. It is easy to show that $\widetilde{\Gamma} = \Gamma \cup \Gamma e$. If $e \in \Gamma$ then $\widetilde{\Gamma} = \Gamma$, and if $e \notin \Gamma$ then Γ is a subgroup of index 2 in $\widetilde{\Gamma}$. As we will see, both possibilities are realized. Since, moreover, $\widetilde{\Gamma}$ is the orientation preserving subgroup of index 2 in Γ^*, the groups Γ, $\widetilde{\Gamma}$, and Γ^* are either all discrete or all non-discrete.

Consider Γ^*. It is clear that

$$\Gamma^* = \langle f, g, e, R_\omega\rangle = \langle e_1, e_2, e_g, R_\omega\rangle = \langle e_g, R_\alpha, R_\varepsilon, R_\omega\rangle.$$

2. We now prove that (i) implies discreteness of Γ^* and, consequently, discreteness of Γ. More precisely, we first construct a polyhedron P which under some additional hypotheses is a fundamental polyhedron for Γ^*. Then we reformulate the hypotheses concerning P in terms of some conditions on elements of Γ.

It is easy to see that there exists a plane δ which is orthogonal to planes α, ω, and $e_g(\alpha)$. Such a plane passes through the common perpendicular to f and $e_g(f)$ orthogonally to ω. It is clear that $e_g \subset \delta$. Let P be a polyhedron bounded by α, ω, $e_g(\alpha)$, δ, and ε. Note that P can be compact or non-compact. Figure 2 shows P under the assumption that it is compact.

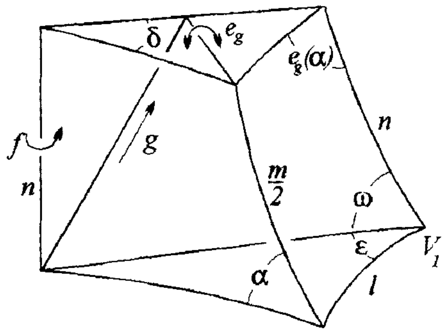

Figure 2.

If a polyhedron has a dihedral angle of π/p (p is not necessarily an integer), we label the corresponding edge by p in figures; if $p = 2$ we omit it. Our P has five right dihedral angles, two angles (formed by ω with α and $e_g(\alpha)$) of π/n, where n is the order of f. Planes α and $e_g(\alpha)$ as well as ε and $e_g(\alpha)$ can either intersect, or be parallel, or disjoint. Denote the angle between ε and $e_g(\alpha)$ by π/l, where $l \in (2, \infty) \cup \{\infty, \overline{\infty}\}$ (we use the notation π/∞ and $\pi/\overline{\infty}$ for parallel and disjoint planes, respectively). The angle between α and $e_g(\alpha)$ we denote by $2\pi/m$, where $m \in (2, \infty) \cup \{\infty, \overline{\infty}\}$, $1/n + 1/m < 1/2$. One can see that such a polyhedron P can be constructed in $\mathbf{H}^3$ for all values of m and l under consideration. Moreover, P is uniquely determined by its dihedral angles in case when $m \neq \overline{\infty}$, $l \neq \overline{\infty}$. Otherwise, if in addition we specify the distance between the disjoint planes corresponding to $m = \overline{\infty}$ or $l = \overline{\infty}$, we obtain uniqueness.

It is clear that *if l and $m/2$ are integers, ∞ or $\overline{\infty}$, then P and elements e_g, R_ω, R_α, R_ε, and $R'_\alpha = e_g R_\alpha e_g$ satisfy the conditions of the Poincaré Theorem* [3] *and Γ^* is discrete.*

We now seek to rewrite the above conditions as conditions on the generators of Γ. This must be carefully done. It might seem that, for example, the condition "$m/2$ is an integer" is equivalent to the condition "$R'_\alpha R_\alpha$ is a primitive elliptic element". However, this is not true. If the dihedral angle of P formed by α and

$e_g(\alpha)$ is equal to $(p-1)\pi/p$, then $R'_\alpha R_\alpha$ is a rotation through the angle $2\pi/p$; i.e., it is a primitive elliptic element, but $m/2 = p/(p-1)$ is not an integer.

Therefore, we proceed as follows. Instead of the element

$$R'_\alpha R_\alpha = R'_\alpha R_\omega R_\omega R_\alpha = f'f, \tag{3}$$

where

$$f' = R'_\alpha R_\omega = e_g R_\alpha e_g R_\omega = e_g R_\alpha R_\omega e_g = e_g f^{-1} e_g = e_g e f e e_g = g f g^{-1} \tag{4}$$

we consider the element $h_1 = R_\xi R_\alpha = R'_\alpha R_\xi$, where ξ is the bisector of α and $e_g(\alpha)$ which passes through e_g. Note that ξ is orthogonal to ω. Clearly,

$$h_1^2 = R'_\alpha R_\alpha \quad \text{and} \quad h_1 f^{-1} = R_\xi R_\omega. \tag{5}$$

From equations (3)–(5), it follows that h_1 satisfies two conditions:

$$h_1^2 = gfg^{-1}f \quad \text{and} \quad (h_1 f^{-1})^2 = 1. \tag{6}$$

Conversely, conditions (6) uniquely determine the element $h_1 \in \mathrm{PSL}(2, \mathbf{C})$ which maps $\alpha \cap P$ into $e_g(\alpha) \cap P$. Now h_1 is a primitive elliptic element of even order m $(1/n + 1/m < 1/2)$ if and only if the dihedral angle of P corresponding to the edge $\alpha \cap e_g(\alpha)$ is equal to $2\pi/m$, where $m/2$ is an integer; α and $e_g(\alpha)$ are parallel (disjoint) if and only if h_1 is parabolic (hyperbolic, respectively).

Consider the dihedral angle of P between $e_g(\alpha)$ and ε. Since the angle is acute (it is equal to π/l, where $l > 2$), the condition "l is an integer, ∞ or $\overline{\infty}$" is equivalent to the condition "$R_\varepsilon R'_\alpha$ is a primitive elliptic element". Denote $h_2 = R_\varepsilon R'_\alpha$ and find its relation to the generators of Γ.

$$\begin{aligned} h_2^2 =& (R_\varepsilon R'_\alpha)^2 = (R_\varepsilon R_\omega R_\omega R'_\alpha)^2 = (e_2 f'^{-1})^2 \\ =& f^{n/2} e g f^{-1} g^{-1} f^{n/2} e g f^{-1} g^{-1} = f^{n/2} g^{-1} f g f^{-n/2} g f^{-1} g^{-1}. \end{aligned}$$

Condition

$$h_2^2 = f^{n/2} g^{-1} f g f^{-n/2} g f^{-1} g^{-1} \tag{7}$$

determines h_2 non-uniquely. Note that $h_2 f' = R_\varepsilon R_\omega$, therefore,

$$(h_2 g f g^{-1})^2 = 1. \tag{8}$$

It is clear that the other square root (not h_2) from the right-hand side of (7) does not satisfy (8). Thus, h_2, which is responsible for the dihedral angle between $e_g(\alpha)$ and ε, is uniquely determined by (7) and (8).

The above shows that *P and elements e_g, R_ω, R_α, R'_α and R_ε satisfy the hypotheses of the Poincaré Theorem if condition (i) in item (2) of Theorem A holds. Therefore, discreteness of* Γ *follows from (i).*

Simultaneously we have proved the existence of h_1 and h_2; see conclusion (1) of the theorem.

3. Assume that condition (i) does not hold, but Γ (and Γ^*) is discrete. Then it suffices to investigate two cases:

(a) $m/2 \in \mathbf{Z} \cup \{\infty, \overline{\infty}\}$, where $1/n + 1/m < 1/2$, and l is fractional;

(b) $m/2$ is fractional ($1/n + 1/m < 1/2$);

and to select all the discrete groups which occur in each of these cases.

(a) Suppose that $m/2 \in \mathbf{Z} \cup \{\infty, \overline{\infty}\}$, $1/n + 1/m < 1/2$, and l is fractional. Since we suppose that Γ^* is discrete, each of its subgroups is also discrete. Hence $\langle R_\omega, R_\varepsilon, R'_\alpha \rangle$ is discrete. The intersection of ω, ε and $e_g(\alpha)$ forms a vertex V_1 of P (see Figure 2). Its link in P is either a spherical, Euclidean, or hyperbolic triangle $(2, n, l)$ according to whether V_1 is a proper, ideal, or imaginary vertex. Since the surface S (see the definition of link in Subsection 2.2) is invariant under $\langle R_\omega, R_\varepsilon, R'_\alpha \rangle$, we can consider the restriction of the action of this subgroup to S. Thus, $\langle R_\omega, R_\varepsilon, R'_\alpha \rangle$ acts as the group generated by three reflections in the sides of triangle $(2, n, l)$. But there is no such group with n even and l fractional that is discrete (for $\mathbf{E}^2$ it is a trivial exercise, for $\mathbf{H}^2$ and $\mathbf{S}^2$ see [24, 5]). Thus we arrive at a contradiction. Thus, *there are no discrete groups* Γ *in case* (a).

(b) From here on we assume that $m/2$ is fractional ($1/n + 1/m < 1/2$). Since Γ^* is discrete, its subgroup $\langle R_\omega, R_\alpha, R'_\alpha \rangle$ is also discrete. The latter acts as the group of reflections in the sides of hyperbolic triangle $(n, n, m/2)$, which is the upper face of P in Figure 2. From the list of all triangles with two primitive angles (an angle is said to be *primitive* if it is of the form π/p, where $p \in \mathbf{Z}$) that generate a discrete group [24], we have that $m/2$ is fractional if and only if m is odd.

Therefore, Γ^* contains the reflection R_ξ in ξ that bisects the dihedral angle of P at the edge $\alpha \cap e_g(\alpha)$. Moreover, ξ passes through e_g; since $e_g = R_\xi R_\delta$, R_δ also belongs to Γ^*. It is clear that Γ^* is generated by R_α, R_δ, R_ξ, R_ε, and R_ω.

Let $\widetilde{P}$ be the polyhedron bounded by α, δ, ξ, ε, and ω; π/k be the dihedral angle at the edge $\xi \cap \varepsilon$, $k \in (2, \infty) \cup \{\infty, \overline{\infty}\}$. The other angles of $\widetilde{P}$ are of the form π/p, where p is an integer (see Figure 3, where dashed lines can be lacking).

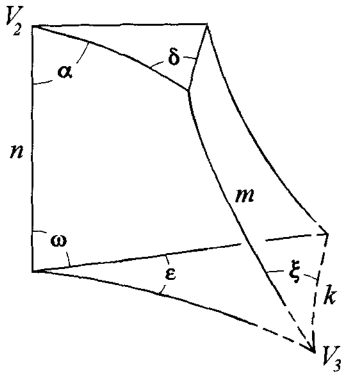

Figure 3.

If k is also an integer ($k \geq 3$), ∞ or $\overline{\infty}$, then Γ^* is actually discrete and $\widetilde{P}$ is its fundamental polyhedron. In other words, since $R_\varepsilon R_\xi = R_\varepsilon R'_\alpha R'_\alpha R_\xi = h_2 h_1$, *the discreteness of* Γ^* *follows from condition (ii) of the theorem.*

It remains to determine if there exist discrete Γ^*'s with k fractional.

If k is fractional then there are reflections of Γ^* in planes through the edge $\xi \cap \varepsilon$ which decompose $\widetilde{P}$.

Consider the face of $\widetilde{P}$ lying in ω. Planes δ, ξ, and ε are perpendicular to ω; the plane η through e and f is also perpendicular to ω. Reflections R_δ, R_ξ, R_ε, and R_η are elements of Γ^* ($R_\eta = f^{n/2} R_\omega$).

As above, the subgroup $\langle R_\delta, R_\xi, R_\varepsilon, R_\eta \rangle$ of Γ^* is discrete. Note that $\langle R_\delta, R_\xi, R_\varepsilon, R_\eta \rangle$ keeps ω invariant, thus reflections in the sides of the hyperbolic quadrilateral with angles $\pi/2$, $\pi/2$, $\pi/2$, and π/k must generate a discrete group. From [4] there exists a unique reflection line through the vertex with the acute angle of the quadrilateral which decomposes it into two symmetric triangles. Therefore, there exists a bisector ζ of the dihedral angle at $\xi \cap \varepsilon$ which is orthogonal to ω and passes through the vertex $V_2 = \alpha \cap \omega \cap \delta$.

The link of vertex V_3, which is formed by α, ξ, and ε, is a triangle $(2, m, k)$. From [24, 5], we could have two different possibilities for the link of V_3:

(P1) each of the triangles with $k = m/2$, where $m \geq 5$ is odd;

(P2) a triangle with $m = 3$ and $k = 5/2$.

However, case (P2) is impossible, because the reflection plane ζ cuts the triangle $(2, n, 5/2)$ off from the link of V_2 and since n is even, reflections in the sides of such a triangle generate a non-discrete group [5], i.e., $\langle R_\alpha, R_\omega, R_\zeta \rangle$ is a non-discrete subgroup of Γ^*. We have a contradiction.

Consider case (P1). One can see that the link of V_2 is divided by two reflection planes into three triangles $(2, 3, n)$, whence it follows that $n = 4$. Further, $\widetilde{P}$ is divided into three tetrahedra $T[2, 2, m; 2, 3, 4]$ (see Figure 4). Each of those tetrahedra can be taken as a fundamental polyhedron for Γ^*.

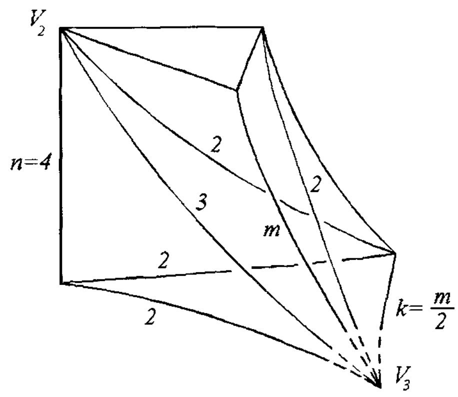

Figure 4.

Thus, *in the case that k is fractional we have a unique series of discrete groups corresponding to condition (iii) of the theorem.*

Since we have checked all possibilities for P and selected all the discrete groups, Theorem A is completely proved. ∎

In terms of parameters (β, β', γ), Theorem A can be reformulated as follows (see also Remark 2).

THEOREM B: *Let $f, g \in \mathrm{PSL}(2, \mathbf{C})$, $\beta = -4\sin^2(\pi/n)$, $n \geq 4$, $(n, 2) = 2$, $\beta' > 0$, and $0 < \gamma < -\frac{1}{4}\beta\beta'$.*

Then $\Gamma = \langle f, g\rangle$ is discrete if and only if one of the following conditions is satisfied:

(1) $\gamma = 2\left(\cos\frac{2\pi}{m} + \cos\frac{2\pi}{n}\right)$ *and* $\beta' = \frac{4\cos^2(\pi/l)}{\gamma} - \frac{4\gamma}{\beta}$, *where* $(m, 2) = 2$, $1/n + 1/m < 1/2$, $l \in \mathbf{Z}$, *and* $l \geq 3$;

(2) $\gamma = 2\left(\cos\frac{2\pi}{m} + \cos\frac{2\pi}{n}\right)$ *and* $\beta' \geq \frac{4}{\gamma} - \frac{4\gamma}{\beta}$, *where* $(m, 2) = 2$ *and* $1/n + 1/m < 1/2$;

(3) $\gamma \geq 2\left(1 + \cos\frac{2\pi}{n}\right)$ *and* $\beta' = \frac{4\cos^2(\pi/l)}{\gamma} - \frac{4\gamma}{\beta}$, *where* $l \in \mathbf{Z}$ *and* $l \geq 3$;

(4) $\gamma \geq 2\left(1 + \cos\frac{2\pi}{n}\right)$ *and* $\beta' \geq \frac{4}{\gamma} - \frac{4\gamma}{\beta}$;

(5) $\gamma = 2\left(\cos\frac{2\pi}{m} + \cos\frac{2\pi}{n}\right)$ *and* $\beta' = \frac{4(\gamma-\beta)\cos^2(\pi/k)}{\gamma} - \frac{4\gamma}{\beta}$, *where* $(m, 2) = 1$, $1/n + 1/m < 1/2$, $k \in \mathbf{Z}$, *and* $k \geq 3$;

(6) $\gamma = 2\left(\cos\frac{2\pi}{m} + \cos\frac{2\pi}{n}\right)$ *and* $\beta' \geq \frac{4(\gamma-\beta)}{\gamma} - \frac{4\gamma}{\beta}$, *where* $(m,2) = 1$ *and* $1/n + 1/m < 1/2$;

(7) $\beta = -2$, $\gamma = 2\cos(2\pi/m)$, *and* $\beta' = \gamma^2 + 4\gamma$, *where* $m \geq 5$ *and* $(m,2) = 1$.

Proof: To prove the theorem it suffices to obtain values of parameters (β, β', γ) corresponding to all discrete groups described in Theorem A. Recall that we know the form of a fundamental polyhedron for each discrete group.

Since f is a primitive elliptic element of order n, we have $\beta = \mathrm{tr}^2 f - 4 = -4\sin^2(\pi/n)$, where π/n is the dihedral angle between ω and α.

Then we calculate γ. Note that $\mathrm{tr}[f,g] = \mathrm{tr}(gf^{-1}g^{-1}f)$. Moreover, $f = R_\omega R_\alpha$ and from (5) it follows that $gf^{-1}g^{-1} = R_\omega R'_\alpha = R_{\alpha^*}R_\omega$, where $\alpha^* = R_\omega(e_g(\alpha))$. Note that α^* passes through f' and makes the angle π/n with ω symmetrically to how $e_g(\alpha)$ does. Therefore, $gf^{-1}g^{-1}f = R_{\alpha^*}R_\alpha$ is a hyperbolic element, and $\gamma = \mathrm{tr}[f,g] - 2 = 2(\cosh d - 1)$, where d is the distance between α and α^*, and can be measured in δ (we took $\mathrm{tr}[f,g] = +2\cosh d$, because $\gamma = \mathrm{tr}[f,g] - 2$ is positive in case 7, see Table 1 in Subsection 2.1). So, γ depends only on n and m.

Finally, we compute $\beta' = \mathrm{tr}^2 g - 4 = 4\sinh^2 T$, where T is the distance between e and e_g which can be measured in ω.

By straightforward calculation, using Figure 2, we obtain cases (1)–(4) from Theorem A(i); using Figure 3, we obtain (5)–(6) from Theorem A(ii); and finally, (7) follows from (iii) of Theorem A. ∎

4. Appendix

The main results of [17]–[20] are gathered together in Table 2.

Remark 2: For simplicity, in the formulation of Theorem B and in Table 2 all elliptic generators are assumed to be primitive. One can use Theorem 2.3 of [10] for non-primitive f. The theorem says that if f and g are Möbius transformations and f is not parabolic, then $\beta(f^r) = P_r(\beta(f))$ and $\gamma(f^r, g) = P_r(\beta(f))\gamma(f,g)/\beta(f)$, where $P_r(z)$ are shifted Chebyshev polynomials.

Applying this to our case, we see that when (β, β', γ) are parameters for a group $\Gamma = \langle f, g\rangle$, where $\gamma \neq 0$ and $\beta = -4\sin^2(q\pi/n)$ $(1 < q < n/2)$, then $(\widetilde{\beta}, \beta', \widetilde{\gamma})$ are the parameters for the same group Γ, where $\widetilde{\beta} = -4\sin^2(\pi/n)$ and $\widetilde{\gamma} = (\widetilde{\beta}/\beta)\gamma$ by Gehring and Martin.

Table 2. Non-elementary discrete $\mathcal{RP}$ groups without invariant plane; Cases 1–6. Here all parameters n, m, l, p are integers.

	β	β'	γ
1	$-4\sin^2\frac{\pi}{n}$, $n\geq 3$	$-4\sin^2\frac{\pi}{m}$, $m\geq 3$	$-4\cos^2\frac{\pi}{l}$, $\cos\frac{\pi}{l}>\sin\frac{\pi}{n}\sin\frac{\pi}{m}$
	$-4\sin^2\frac{\pi}{n}$, $n\geq 3$	$-4\sin^2\frac{\pi}{m}$, $m\geq 3$	$(-\infty,-4]$
	$-4\sin^2\frac{\pi}{n}$, $n\geq 7$, $(n,2)=1$	β	$-4-2\beta$
2	$-4\sin^2\frac{\pi}{n}$, $n\geq 3$	0	$-4\cos^2\frac{\pi}{l}$, $l\geq 3$
	$-4\sin^2\frac{\pi}{n}$, $n\geq 3$	0	$(-\infty,-4]$
3–5	$[0,+\infty)$	$[0,+\infty)$	$-4\cos^2\frac{\pi}{l}$, $l\geq 3$
	$[0,+\infty)$	$[0,+\infty)$	$(-\infty,-4]$
6	$-4\sin^2\frac{\pi}{n}$, $n\geq 3$	$(0,+\infty)$	$-4\cos^2\frac{\pi}{l}$, $l\geq 3$
	$-4\sin^2\frac{\pi}{n}$, $n\geq 3$	$(0,+\infty)$	$(-\infty,-4]$
	$-4\sin^2\frac{\pi}{n}$, $n\geq 5$, $(n,2)=1$	$4(\beta+4)\cos^2\frac{\pi}{p}-4$, $p\geq 4$	$-4-2\beta$
	$-4\sin^2\frac{\pi}{n}$, $n\geq 5$, $(n,2)=1$	$[4(\beta+3),+\infty)$	$-4-2\beta$
	-3	$\cos^2\frac{\pi}{p}/(1-\cos\frac{\pi}{5})^2-4$, $p\geq 3$	$-4\cos^2\frac{2\pi}{5}$
	-3	$[1/(1-\cos\frac{\pi}{5})^2-4,+\infty)$	$-4\cos^2\frac{2\pi}{5}$

References

[1] A. F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, New York–Heidelberg–Berlin, 1983.

[2] R. Brooks and J. P. Matelski, *The dynamics of 2-generator subgroups of* PSL(2, **C**), in *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference* (I. Kra and B. Maskit, eds), Annals of Mathematics Studies 97, Princeton University Press, Princeton, 1981, pp. 65–71.

[3] D. B. A. Epstein and C. Petronio, *An exposition of Poincaré's polyhedron theorem,* L'Enseignement Mathématique **40** (1994), 113–170.

[4] A. A. Felikson, *Coxeter decompositions of hyperbolic polygons,* European Journal of Combinatorics **19** (1998), 801–817.

[5] A. A. Felikson, *Coxeter decompositions of spherical tetrahedra,* preprint, 99–053, Bielefeld, 1999.

[6] B. Fine and G. Rosenberger, *Classification of generating pairs of all two generator Fuchsian groups,* in *Groups'93* (Galway/St.Andrews, Vol. 1, Galway, 1993), London Mathematical Society Lecture Note Series 211, Cambridge University Press, Cambridge, 1995, pp. 205–232.

[7] F. W. Gehring, J. P. Gilman, and G. J. Martin, *Kleinian groups with real parameters,* Communications in Contemporary Mathematics **3** (2001), 163–186.

[8] F. W. Gehring and G. J. Martin, *Stability and extremality in Jørgensen's inequality,* Complex Variables. Theory and Application **12** (1989), 277–282.

[9] F. W. Gehring and G. J. Martin, *Commutators, collars and the geometry of Möbius groups,* Journal d'Analyse Mathématique **63** (1994), 175–219.

[10] F. W. Gehring and G. J. Martin, *Chebyshev polynomials and discrete groups,* in *Proceedings of the Conference on Complex Analysis* (Tianjin, 1992), Conference Proceedings and Lecture Notes in Analysis, I, International Press, Cambridge, MA, 1994, pp. 114–125.

[11] J. Gilman, *A discreteness condition for subgroups of* PSL(2, **C**), in *Lipa's Legacy. Proceedings of the 1st Bers Colloquium* (New York, 1995), Contemporary Mathematics, Vol. 211, American Mathematical Society, Providence, RI, 1997, pp. 261–267.

[12] J. Gilman, *Two-generator discrete subgroups of* PSL(2, **R**), Memoirs of the American Mathematical Society **117** (1995), no. 561.

[13] J. Gilman and B. Maskit, *An algorithm for 2-generator Fuchsian groups,* The Michigan Mathematical Journal **38** (1991), 13–32.

[14] H. Hagelberg, C. Maclachlan, and G. Rosenberger, *On discrete generalized triangle groups,* Proceedings of the Edinburgh Mathematical Society **38** (1995), 397–412.

[15] T. Jørgensen, *On discrete groups of Möbius transformations,* American Journal of Mathematics **98** (1976), 739–749.

[16] T. Jørgensen, *A note on subgroups of* SL(2, **C**), The Quarterly Journal of Mathematics. Oxford (2) **28** (1977), 209–212.

[17] E. Klimenko, *Three-dimensional hyperbolic orbifolds and discrete groups of isometries of Lobachevsky space*, thesis, Novosibirsk, 1989.

[18] E. Klimenko, *Discrete groups in three-dimensional Lobachevsky space generated by two rotations*, Siberian Mathematical Journal **30** (1989), 95–100.

[19] E. Klimenko, *Some remarks on subgroups of* PSL(2, **C**), Questions and Answers in General Topology **8** (1990), 371–381.

[20] E. Klimenko, *Some examples of discrete groups and hyperbolic orbifolds of infinite volume*, Journal of Lie Theory **11** (2001), 491–503.

[21] E. Klimenko and N. Kopteva, *All discrete $\mathcal{RP}$ groups with non-π-loxodromic generators*, in preparation.

[22] E. Klimenko and N. Kopteva, *Two-generator hyperbolic orbifolds with real parameters*, in preparation.

[23] E. Klimenko and M. Sakuma, *Two-generator discrete subgroups of* $\mathrm{Isom}(\mathbf{H}^2)$ *containing orientation-reversing elements*, Geometriae Dedicata **72** (1998), 247–282.

[24] A. W. Knapp, *Doubly generated Fuchsian groups*, The Michigan Mathematical Journal **15** (1968), 289–304.

[25] B. Maskit, *Kleinian groups*, Grundlehren der Mathematischen Wissenschaften 287, Springer-Verlag, New York, 1988.

[26] B. Maskit, *Some special 2-generator Kleinian groups*, Proceedings of the American Mathematical Society **106** (1989), 175–186.

[27] B. Maskit and G. Swarup, *Two parabolic generator Kleinian groups*, Israel Journal of Mathematics **64** (1988), 257–266.

[28] J. P. Matelski, *The classification of discrete 2-generator subgroups of* PSL(2, **R**), Israel Journal of Mathematics **42** (1982), 309–317.

[29] N. Purzitsky, *All two-generator Fuchsian groups*, Mathematische Zeitschrift **147** (1976), 87–92.

[30] G. Rosenberger, *All generating pairs of all two-generator Fuchsian groups*, Archiv der Mathematik **46** (1986), 198–204.