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VALUE DISTRIBUTION OF THE PAINLEVÉ TRANSCENDENTS

BY

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ABSTRACT

We consider the solutions of the First Painlevé Differential Equation $w'' = z + 6w^2$, commonly known as First Painlevé Transcendents. Our main results are the sharp order estimate $\lambda(w) \le 5/2$, actually an equality, and sharp estimates for the spherical derivatives of w and $f(z) = z^{-1}w(z^2)$, respectively: $w^{\#}(z) = O(|z|^{3/4})$ and $f^{\#}(z) = O(|z|^{3/2})$. We also determine in some detail the local asymptotic distribution of poles, zeros and a-points. The methods also apply to Painlevé's Equations II and IV.

1. Introduction

The Painlevé transcendents are solutions of Painlevé's differential equations (I), (II) and (IV). Every local solution admits unrestricted analytic continuation, and hence is a meromorphic function in the complex plane. Recent proofs of this Painlevé result [14, 15] can be found in the paper [9] by Hinkkanen and Laine for equations (I), (II), and, by different methods, in the author's paper [23] for equations (I), (II) and (IV). We will present our results in detail only for Painlevé's first equation

$$w'' = z + 6w^2, \tag{I}$$

but note that the methods apply in cases (II) and (IV) as well; see Section 6 for more details and hints.

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In a remarkable paper which has appeared in two parts, P. Boutroux [3] has given a detailed description of the value distribution, and, in particular, of the distribution of poles of the Painlevé transcendents. In a former paper [2] he had established estimates for the order of growth of the first and second transcendents. His papers, however, are hard to read, one reason certainly being that they were written in the typical style of that time, the beginning of the twentieth century. They also contain a lot of reasoning, which can hardly be understood. Thus, there is reasonable doubt about Boutroux's methods; details can be found in Appendix B.

It is thus small wonder that several authors* made attempts to prove, for example, an estimate like $\lambda(w) \le 3$ for the order of growth, although Boutroux had stated that the exact order is 5/2**. We mention the book of Hille [8], the paper(s) of Schubart and Wittich [21], and Wittich [24]; see also Laine's book [11]. All these attempts failed by different reasons. What is definitely known for the first transcendents is the lower estimate $\lambda(w) \ge 5/2$, which can be found in the paper [12] due to Mues and Redheffer.

Our paper is organized as follows. In Section 2 we describe the main tool, and in Section 3 the order estimate $\lambda(w) \le 5/2$ is established. The methods developed there yield a lot of results about the value distribution of the first Painlevé transcendents and information on their spherical derivatives $w^{\#}$ in Section 4. In Section 5 asymptotic distribution of zeros and poles is discussed. In Section 6 we describe how to adapt the methods for the requirements of equations (II) and (IV). In Appendix A we discuss the differential equation $y'^2 = Q(y)$ for the convenience the reader, while Appendix B is devoted to Boutroux's papers.

2. The main tool

In the present section we will describe our main tool. It is always assumed, without further notice, that w is some fixed solution of Painlevé's first equation (I). If $z_0 \neq 0$ is a regular point of w, then we define a **local scale** with unit

$$\mathsf{r}(z_0) = \min\{|w(z_0)|^{-1/2}, |w'(z_0)|^{-1/3}, |z_0|^{-1/4}\} \tag{1}$$

* Caution: A Zentralblatt check (July 4, 2000) for the period 1990–1999 yielded 355 items having Painlevé's name in the title.

** Note, however, that Boutroux had no definition at hand for the order of a **meromorphic** function. The order was defined only 20 years later in R. Nevanlinna's fundamental paper *Zur Theorie der meromorphen Funktionen,* Acta Mathematica **46** (1925). Boutroux himself only made statements about the order of certain entire functions, e.g., he showed that E in $w = -(E'/E)'$, w a first transcendent, has order of growth $\lambda = 5/2$.

($0^{-1/2} = 0^{-1/3} = +\infty$). Then what we need is the following form of the theorem on analytic dependence on parameters and initial values:

MAIN LEMMA: *Suppose (z_n) is a sequence of regular points with $z_n \to \infty$ and sequence of scaling units (r_n), $r_n = \mathsf{r}(z_n)$, such that the limits*

$$(2) \qquad y_0 = \lim_{n\to\infty} r_n^2 w(z_n), \; y_0' = \lim_{n\to\infty} r_n^3 w'(z_n) \quad \text{and} \quad a = \lim_{n\to\infty} r_n^4 z_n$$

exist. Then, if y denotes the unique solution of the initial value problem

$$(3) \qquad y'' = 6y^2 + a, \; y(0) = y_0, \; y'(0) = y_0',$$

we have, with respect to the spherical metric,

$$(4) \qquad r_n^2 w(z_n + r_n z) \to y(z)$$

as $n \to \infty$, locally uniformly in $\mathbb{C}$.

Remark: We note the trivial but important fact that the limit function y cannot vanish identically, since $\max\{|y(0)|, |y'(0)|, |a|\} = 1$. Also, the rate of convergence is $O(|y_n(0) - y(0)| + |y_n'(0) - y'(0)| + |a - r_n^4 z_n| + r_n^5)$.

Proof: Set

$$(5) \qquad y_n(z) = r_n^2 w(z_n + r_n z),$$

so that

$$(6) \qquad y_n'' = 6y_n^2 + z_n r_n^4 + r_n^5 z$$

and $\max\{|y_n(0)|, |y_n'(0)|, |z_n r_n^4|\} = 1$ hold. By analytic dependence on initial values and parameters we have

$$y_n(z) \to y(z)$$

in a neighbourhood of $z = 0$, and locally uniformly in $\mathbb{C}$ by Poincaré's result; see, e.g., Bieberbach [1], p. 14, or the books of Golubew [4], Hille [7, 8] and Ince [10]. ∎

Remark: The "Ansatz" (5) is strongly suggestive of the Pang version of the so-called Zalcman or re-scaling Lemma (see [26], [16], [17]). It will, however, turn out that the limit function y may be a non-zero constant, which is excluded by the Zalcman–Pang Lemma. There is thus reasonable doubt whether the Zalcman–Pang method applies in this case, but there is no doubt that I was inspired by this method. It should also be mentioned that perhaps the first most successful application of re-scaling methods was performed by Painlevé himself when classifying the second-order differential equations without movable singularities other than poles!

3. The order of growth

We start with a technical lemma.

LEMMA 1: *Let (q_n) be a sequence of zeros of w, and (z_n) be any other sequence such that $|z_n - q_n| = o(\mathsf{r}(z_n))$. Then $\mathsf{r}(z_n) = O(\mathsf{r}(q_n))$.*

Proof: We apply the Main Lemma to the sequence (q_n) with scaling unit $\rho_n = \mathsf{r}(q_n)$, where we assume that the limits $\tilde{y}_0 = \lim_{n\to\infty} \rho_n^2 w(q_n) = 0$, $\tilde{y}_0' = \lim_{n\to\infty} \rho_n^3 w'(q_n)$ and $\tilde{a} = \lim_{n\to\infty} \rho_n^4 q_n$ exist (choose a subsequence, if necessary). Then, besides the asymptotics (4) with $r_n = \mathsf{r}(z_n)$, we also have the asymptotic relation $\rho_n^2 w(q_n + \rho_n z) \to \tilde{y}(z)$. Note that $\tilde{y}$ is non-constant, this following from $\tilde{y}(0) = 0$. This is also true for y, since with $\epsilon_n = (q_n - z_n)/r_n \to 0$ we have $y(0) = \lim_{n\to\infty} r_n^2 w(z_n) = \lim_{n\to\infty} r_n^2 w(z_n + r_n\epsilon_n) = 0$ by uniform convergence. Since both functions are non-constant, they have poles ω and τ, say (see Appendix A), of smallest modulus, and so by Hurwitz' Theorem w has poles at $z_n + r_n\omega_n$ and $q_n + \rho_n\tau_n$, $\omega_n \to \omega$ and $\tau_n \to \tau$, respectively. Thus,

$$\begin{aligned} r_n|\omega| \le& |z_n - (q_n + \rho_n\tau_n)| + o(r_n) + o(\rho_n) \\ \le& \rho_n|\tau| + o(r_n) + o(\rho_n), \end{aligned}$$

and hence $\mathsf{r}(z_n) \le 2\frac{|\tau|}{|\omega|}\mathsf{r}(q_n)$, $n \ge n_0$, follows. ∎

Let (q_n) denote the sequence of zeros of w, and let $\epsilon > 0$ be any positive number. Then $\mathsf{Q}(\epsilon)$ will denote the neighbourhood

$$\mathsf{Q}(\epsilon) = \bigcup_n \{z : \ |z - q_n| < \epsilon\mathsf{r}(q_n)\}$$

of the sequence (q_n). Our first application of the Main Lemma will be

PROPOSITION 1: *Let w be a solution of (I). Then given $\epsilon > 0$, the expression $|w'(z)||w(z)|^{-3/2}$ is bounded outside $\mathsf{Q}(\epsilon)$. In particular, the spherical derivative $w^{\#}(z) = |w'(z)|(1 + |w(z)|^2)^{-1}$ is bounded there.*

Proof: Suppose (z_n) is any sequence such that $0 < |z_n| \to \infty$ and

$$2 < |w'(z_n)||w(z_n)|^{-3/2} \to \infty$$

as $n \to \infty$. We note that z_n is not a pole of w. Then, choosing a subsequence of (z_n), still denoted (z_n), we may assume that the limits (2) exist. Then the Main Lemma applies with $y_0 = 0$, this following from $r_n^2 w(z_n) = o((|w'(z_n)|r_n^3)^{2/3}) = o(1)$, and hence, by Hurwitz' Theorem, there exists $z_n' \to 0$ with $w(z_n + r_n z_n') = 0$. We thus obtain for the zero $\tilde{q}_n = z_n + r_n z_n'$ of w that $|z_n - \tilde{q}_n| = o(r_n) = o(\mathsf{r}(\tilde{q}_n))$ by Lemma 1. This proves Proposition 1, since the statement on the spherical derivative follows immediately. ∎

Remark: The Main Lemma works like a search algorithm. If a certain inequality or asymptotic relation has to be proved, which should be valid outside some (possibly unknown) exceptional set, one chooses a sequence (z_n), $z_n \to \infty$, such that the inequality is not valid on (z_n). Then, applying the Main Lemma, it turns out that the sequence (z_n) approaches automatically the exceptional set. The inequality thus proves valid outside this set, which, moreover, is detected by the **test-sequence** (z_n).

One more example for this method is

PROPOSITION 2: *Given $\epsilon > 0$ we have*

$$|z| = O(|w(z)|^2) \quad \text{and} \quad |w'(z)||w(z)|^{-3} = O(|z|^{-3/4})$$

as $z \to \infty$ outside $\mathsf{Q}(\epsilon)$.

Proof: The second statement follows from the first one and Proposition 1. To prove the first assertion we again consider a sequence (z_n), $0 < |z_n| \to \infty$, such that $|w(z_n)| = o(|z_n|^{1/2})$, hence $|w(z_n)| = o(\mathsf{r}(z_n)^{-2})$. We may assume, as we always do, that with $r_n = \mathsf{r}(z_n)$ the corresponding limits (2) exist. In particular we again have $y_0 = 0$, so that, as was the case in the proof of Proposition 1, $|z_n - \tilde{q}_n| = o(r_n) = o(\mathsf{r}(\tilde{q}_n))$ holds for some sequence $(\tilde{q}_n)$ of zeros of w. ∎

Before we come to our next proposition, which plays the central role, some comments are in order. Equation (I) has a first integral

$$w'^2 = 2zw + 4w^3 - 2U \quad \text{with } U' = w. \tag{7}$$

At a pole p, w has the Laurent expansion

$$w(z) = \frac{1}{(z-p)^2} - \frac{p}{10}(z-p)^2 - \frac{1}{6}(z-p)^3 + h(z-p)^4 + \cdots. \tag{8}$$

The coefficient h cannot be determined from (I). The subsequent coefficients are (universal) polynomials in p and h. It turns out that $14h$ is the **local constant**

of integration,

$$(9)\qquad U(z) = \frac{-1}{z-p} + 14h - \frac{p}{30}(z-p)^3 - \frac{1}{24}(z-p)^4 + \frac{h}{5}(z-p)^5 + \cdots,$$

and that the value of

$$(10)\qquad V = 2U - w'/w$$

at $z = p$ is the crucial number $V(p) = 28h$.

PROPOSITION 3: *Given $\sigma > 0$ and $\epsilon > 0$ there exists $K > 0$, such that for $z \in \mathbb{C} \setminus \mathsf{Q}(\epsilon)$ either $|V(z)||w(z)|^{-2} \le K|z|^{1/2}$ or else $|w(z)|^{-2} \le \sigma|z|^{-1}$ holds. In any case we have*

$$\frac{|V(z)|}{|w(z)|^2} \le \sigma\frac{|V(z)|}{|z|} + K|z|^{1/2}$$

outside $\mathsf{Q}(\epsilon)$.

Remark: It is indispensable that the exceptional set $\mathsf{Q}(\epsilon)$ is independent of the choice of $\sigma > 0$. It is just K which depends on σ (and ϵ).

Proof: Suppose (z_n) is any sequence with $0 < |z_n| \to \infty$ and $z_n \notin \mathsf{Q}(\epsilon)$, such that

$$0 < \frac{|V(z_n)|}{|w(z_n)|^2|z_n|^{1/2}} \to +\infty$$

as $n \to \infty$. Note that z_n is not a pole of w. Also, $|z_n| = O(|w(z_n)|^2)$ holds by Proposition 2. From

$$w'^2 = 2zw + 4w^3 - V - w'/w$$

it then follows that

$$\frac{|V(z_n)|}{|w(z_n)|^2|z_n|^{1/2}} \le \frac{|w'(z_n)|^2}{|w(z_n)|^2|z_n|^{1/2}} + \frac{4|w(z_n)|}{|z_n|^{1/2}} + \frac{2|z_n|^{1/2}}{|w(z_n)|} + \frac{|w'(z_n)|}{|w(z_n)|^3|z_n|^{1/2}}.$$

The right hand side of this inequality is $O(|w(z_n)||z_n|^{-1/2})$ by Propositions 1 and 2, and hence we have $|z_n||w(z_n)|^{-2} \to 0$ as $n \to \infty$. This proves Proposition 3. ∎

So far our arguments were completely local. But it is obvious that one cannot prove any global property of a single solution by purely local considerations—this was one reason why former attempts failed. These local results have to be connected, and one possible link is the linear differential equation for V,

$$V' = \frac{z}{w(z)} - \frac{w'(z)}{w(z)^3} - \frac{1}{w(z)^2}V = a(z) + b(z)V,$$

and the resulting integral equation

$$V(z) = V(z_0) + \int_\Gamma \big(a(t) + b(t)V(t)\big)dt, \tag{11}$$

where Γ is any path of integration joining z_0 with z and avoiding the zeros of w. By Propositions 2 and 3 from (11) then follows

$$|V(z)| \le |V(z_0)| + \int_\Gamma \left(\sigma\frac{|V(t)|}{|t|} + K|t|^{1/2}\right)|dt|,$$

$\sigma > 0$ arbitrary and $K = K(\sigma, \epsilon) > 0$, whenever Γ is a path of integration joining z_0 with z outside $\mathsf{Q}(\epsilon)$.

To proceed further we will ignore the exceptional set $\mathsf{Q}(\epsilon)$ for one moment and show by a Gronwall-like argument that $|V(z)||z|^{-3/2}$ is bounded outside $\mathsf{Q}(\epsilon)$, which we ignore. Suppose that N is a bound for $|V(z_0)|$ on $|z_0| = R_0 > 1$, and set $M(R) = \max\{|V(z)||z|^{-3/2}\colon R_0 \le |z| \le R, z \notin \mathsf{Q}(\epsilon)\}$ for $R > R_0$. Assume also that the maximum is attained at $z_1 = \rho e^{i\alpha}$, $R_0 < \rho \le R$. Choose $\sigma = 1$ and assume that the straight line segment Γ joining $z_0 = R_0e^{i\alpha}$ and z_1 does not meet $\mathsf{Q}(\epsilon)$. Then on Γ we have $|V(t)||t|^{-1} \le M(R)|t|^{1/2}$. From this it immediately follows that

$$M(R) \le N + (M(R) + K)|z_1|^{-3/2}\int_\Gamma |t|^{1/2}|dt| \le N + \frac{2}{3}(M(R) + K),$$

and hence $M(R) \le 3N + 2K$.

This is the manner in which we will prove Proposition 4 below. We have, however, to ensure that the disks $|z - q_n| < \epsilon\mathsf{r}(q_n)$ do not form barriers like the **Great Barrier Reef**. This will be done in Lemma 2 below by a slight modification of the exceptional set $\mathsf{Q}(\epsilon)$.

LEMMA 2: *Given any set $\mathsf{Q}(\epsilon)$, $\epsilon > 0$ sufficiently small, there exists $R_0 > 1$ and an open set $\Delta(\epsilon)$, such that $\mathsf{Q}(\epsilon) \cap \{z\colon |z| > R_0\} \subset \Delta(\epsilon)$. The connected components of $\Delta(\epsilon)$ are some of the disks*

$$\Delta_n(\epsilon) = \{z : \ |z - q_n| < \epsilon\theta_n\mathsf{r}(q_n)\}, \ 1 \le \theta_n \le 3.$$

Proof: First we will show that there exists $\delta > 0$ with the following property: given three zeros of w, then at least one of them has distance at least δ from the other ones, distance measured in local scale—in other words, the set $\mathsf{Q}(\epsilon)$ consists of atoms and 2-molecules only, provided ϵ is sufficiently small. For suppose to the contrary that there exists some sequence of disks $D_n = \{z\colon |z - z_n| < \eta_n\mathsf{r}(z_n)\}$, $0 < \eta_n \to 0$, such that D_n contains three (or more) zeros of w, one being z_n itself.

Then the Main Lemma applies to the sequence (z_n), and by Hurwitz' Theorem the limit function y has at least a triple zero at $z = 0$, which is impossible for $y(z) \not\equiv 0$.

We increase the radius of each disk $E_n = \{z : |z - q_n| < \epsilon r(q_n)\}$ by the factor 3 and set $E_n' = \{z : |z - q_n| < 3\epsilon r(q_n)\}$. If we first take ϵ sufficiently small and then n sufficiently large, we see that each E_n' intersects at most one disk E_m', $m \neq n$. In case $E_n \cap E_m = \emptyset$ for $m \neq n$ we set $\Delta_n(\epsilon) = E_n$. Now suppose $E_n \cap E_m \neq \emptyset$ and note that m is uniquely determined. Then if $r_n \geq r_m$ we set $\Delta_n(\epsilon) = E_n'$, and $\Delta_n(\epsilon) = E_m'$ otherwise. Then, obviously, Lemma 2 is true with $\Delta(\epsilon) = \bigcup_n \Delta_n(\epsilon)$. ∎

PROPOSITION 4: *Given $\epsilon > 0$ sufficiently small we have $V(z) = O(|z|^{3/2})$ outside $\Delta(\epsilon)$ and, in particular,*

$$V(p_n) = O(|p_n|^{3/2}) \quad \text{as} \quad n \to \infty, \tag{12}$$

where (p_n) denotes the sequence of poles of w.

Proof: We proceed as before where we ignored the set $Q(\epsilon)$. Let $R_0 > 1$ and the set $\Delta(\epsilon)$ be chosen as in Lemma 2, and define $M(R)$ for $R > R_0$ as was done above, but now taking into consideration the exceptional set $\Delta(\epsilon) \supset Q(\epsilon)$:

$$M(R) = \max\{|V(z)||z|^{-3/2} \colon R_0 \leq |z| \leq R,\ z \notin \Delta(\epsilon)\}.$$

This maximum is attained at $z_1 = \rho e^{i\alpha}$, $R_0 < \rho \leq R$. Let L be the radial line segment joining $z_0 = R_0 e^{i\alpha}$ with z_1. If L intersects some disk $\Delta_n(\epsilon)$, we replace part of L by part of the boundary of $\Delta_n(\epsilon)$. Also, if z_0 is in some disk $\Delta_m(\epsilon)$, we replace z_0 by the point of $L \cap \overline{\Delta_m(\epsilon)}$ with largest absolute value. In this way we obtain the path of integration Γ. It is obvious that $\int_\Gamma |t|^{1/2}|dt| < \pi |z_1|^{3/2}$. Also, for $|V(z_0)|$ there exists an upper bound N which is independent of R. Since on Γ we have $|V(t)||t|^{-1} \leq M(R)|t|^{1/2}$, we finally obtain, choosing $\pi\sigma = 1/2$ and proceeding as before, the inequality $M(R) \leq 2(N + K\pi)$, and hence

$$V(z) = O(|z|^{3/2})$$

as $z \to \infty$ outside $\Delta(\epsilon)$.

The centers of the disks $\Delta_n(\epsilon)$ are zeros q_n of w, and the Main Lemma applies to (a sub-sequence of) the sequence (q_n), yielding a limit function $y(z)$ with $y(0) = 0$. Thus y is non-constant, and the poles of y have euclidian distance

$r > 0$ from the origin, and so, for n sufficiently large and ϵ sufficiently small, the disks $\Delta_n(\epsilon)$ contain no poles of w, i.e., (12) is valid. ∎

More or less a corollary of Proposition 4 is

THEOREM 1: *Let w be any Painlevé I transcendent with sequence of poles (p_n). Then*

$$\sum_{0<|p_n|\le r} |p_n|^{-1/2} = O(r^2) \quad \textit{as } r \to \infty,$$

and the Nevanlinna functions satisfy

$$N(r,w) = O(r^{5/2}), \ T(r,w) = O(r^{5/2}) \quad \text{and} \quad m(r,w) = O(\log r).$$

Remark: For notation in Nevanlinna Theory see the monographs of Nevanlinna [13] or Hayman [6].

Proof: Let p be a pole of w, $|p| > 1$, say. Then w has the Laurent series expansion (8) with $28h = V(p)$, and hence $|h| = O(|p|^{3/2})$ as $p \to \infty$. Also, the coefficients $c_n = c_n(p,h)$ satisfy the recurrence relation

$$(n^2 - n - 12)c_n = 6\sum_{\nu=2}^{n-4} c_\nu c_{n-\nu-2}$$

for $n \ge 5$, from which it is easily deduced that $|c_n| \le M^{n+2}$ for $n \ge 5$, where $M = \max\{|p|^{1/4}, |h|^{1/6}\} = O(|p|^{1/4})$. Thus the series for w converges in $|z-p| < c|p|^{-1/4}$, where c an absolute constant. Hence the disks $|z-p_n| < \frac{1}{2}c|p_n|^{-1/4}$ are mutually disjoint, and a simple geometric argument then gives the first assertion of Theorem 1. From this the inequalities $n(r,w)$ = counting function of poles $= O(r^{5/2})$ and $N(r,w) = O(r^{5/2})$ follow. Since $m(r,w)$ grows like the Nevanlinna error term, see Wittich [25, p. 80], we finally obtain $T(r,w) = O(r^{5/2})$ and $m(r,w) = O(\log r)$. ∎

Combining Theorem 1 with the result of Mues and Redheffer [12] we obtain

THEOREM 2: *The first Painlevé transcendents have order of growth*

$$\lambda(w) = \limsup_{r\to\infty} \frac{\log T(r,w)}{\log r} = 5/2.$$

Remark: From (8) it follows that w has the form $w = -(E'/E)'$, where E is an entire function with simple zeros at poles of w. Since the canonical product P having these zeros has order of growth $5/2$, and since $E = Pe^Q$, Q entire, it follows easily from $w = -(P'/P + Q')'$ that the Nevanlinna proximity function of Q'' is $m(r, Q'') = O(\log r)$, and hence Q is a polynomial. In fact it was shown in [12] that the degree of Q is at most $[\lambda(w)] = 2$; see also Boutroux [3]. Hence $\lambda(E) = 5/2$ and

$$w(z) = \sum_{n=1}^{\infty} \left[(z - p_n)^{-2} - p_n^{-2}\right] + c_0,$$

provided $z = 0$ is not a pole of w; otherwise the term z^{-2} has to be added. Similarly, U has the partial fraction expansion

$$U(z) = -\sum_{n=1}^{\infty} \left[(z - p_n)^{-1} + z p_n^{-2} + p_n^{-1}\right] + c_0 z + c_1.$$

4. Value distribution of w and U

In Proposition 1 we have shown that the spherical derivative $w^{\#}$ is bounded outside small disks $\{z : |z - q_n| < \epsilon \mathsf{r}(q_n)\}$ about the zeros of w. We will now complete this result by proving a sharp estimate for $w^{\#}$ inside these disks. This estimate shows that a sharp order estimate cannot be obtained by just estimating $w^{\#}$.

Before estimating $w^{\#}$ we extend the Main Lemma to sequences of poles. Noting that $\min\{|p|^{-1/4}, |h|^{-1/6}\}$ is a (good) lower bound for the radius of convergence of the series (8), it would be quite natural to define a local scale at a pole p by $\mathsf{r}(p) = \min\{|p|^{-1/4}, |h|^{-1/6}\}$. Since, however, by Proposition 4, $|h| = O(|p|^{3/2})$ holds, we (may as well and) will define $\mathsf{r}(p) = |p|^{-1/4}$. The Main Lemma is valid in this case, too:

Main Lemma for Poles: *Let (p_n) be a sequence of poles, such that for $r_n = |p_n|^{-1/4}$ the limits $a = \lim_{n\to\infty} p_n r_n^4$ and $\eta = \lim_{n\to\infty} h_n r_n^6$ exist (note that the sequence $(h_n r_n^6)$ is bounded). Then*

$$r_n^2 w(p_n + r_n z) \to y(z)$$

as $n \to \infty$, where y is the unique solution of

$$y'' = 6y^2 + a, \ |a| = 1,$$

with pole at $z = 0$ and Laurent series expansion

$$y(z) = z^{-2} - \frac{a}{10} z^2 + \eta z^4 + c_6 z^6 + \cdots.$$

Remark: Note that, in general, η is a free parameter! The first integral of the differential equation above is

$$y'^2 = 4y^3 + 2ay - 28\eta.$$

From the above result it is easy to deduce the following technical lemma:

LEMMA 3: *Let (q_n) denote the sequence of zeros of w. Then $\mathsf{r}(q_n) \asymp |q_n|^{-1/4}$.*

Remark: Here and in the sequel $a_n \asymp b_n$ means $a_n = O(|b_n|)$ and $b_n = O(|a_n|)$ as $n \to \infty$.

Proof: Applying the Main Lemma to (a subsequence of) the sequence (q_n) with $r_n = \mathsf{r}(q_n)$ we obtain the asymptotic relation

$$r_n^2 w(q_n + r_n z) \to y(z), \quad y(0) = y_0 = 0.$$

Then y has a pole $z = \omega$, say, of smallest absolute value, which by Hurwitz' Theorem gives rise to a pole $p_n = q_n + r_n\omega_n$ of w, $\omega_n \to \omega$. Now the modified Main Lemma applies to the sequence (p_n) with local scaling unit $\rho_n = |p_n|^{-1/4}$ and limit function $\tilde{y}(z) = z^{-2} + \cdots$ satisfying $\tilde{y}'' = 6\tilde{y}^2 + \tilde{a}$, $|\tilde{a}| = 1$. This $\tilde{y}$ has a zero $z = \tau$ of smallest absolute value (see Appendix A), giving rise to a zero $q_n' = p_n + \rho_n\tau_n$ of w, $\tau_n \to \tau$. We thus may conclude that, with $\rho_n = |p_n|^{-1/4} \sim |q_n|^{-1/4}$,

$$\begin{aligned}|\tau|\rho_n \le& |q_n' - p_n| + o(\rho_n) \le |q_n - p_n| + o(\rho_n) + o(\mathsf{r}(q_n))\\ =& \mathsf{r}(q_n)|\omega| + o(\rho_n) + o(\mathsf{r}(q_n)),\end{aligned}$$

which gives

$$\mathsf{r}(q_n) \ge \frac{1}{2}\frac{|\tau|}{|\omega|}|q_n|^{-1/4} \quad \text{for } n \ge n_0.$$

On the other hand we have $\mathsf{r}(q_n) \le |q_n|^{-1/4}$ by definition, and so $\mathsf{r}(q_n) \asymp |q_n|^{-1/4}$ follows. ∎

Remark: One can prove a similar result for any sequence (z_n), provided (z_n) stays away from the sequence of poles in a certain way; see Proposition 5 below.

Lemma 3 and the fact that $\mathsf{r}(p_n) = |p_n|^{-1/4}$ lead us to introduce the Riemannian metric

$$ds = |z|^{1/4}|dz|;$$

the distance $\mathsf{d}(a,b)$ between a and b then is the infimum of the lengths $\int_\gamma ds$, where γ ranges over all paths joining a and b, and the disks

$$\mathsf{D}(a,r)=\{z\colon \mathsf{d}(z,a)<r\}$$

are comparable with the euclidian disks $|z-a|<r|a|^{-1/4}$, provided $|a|$ is large compared with r.

Besides the sequences (p_n) and (q_n) of poles and zeros of w and the sets P and Q, we introduce two more sets and sequences:

S:	set of zeros of w'	(s_n):	sequence of zeros of w'
T:	set of zeros of U	(t_n):	sequence of zeros of U

Then the distances between different poles of w are bounded away from zero, and the distance $\mathsf{d}(\mathsf{P},\mathsf{Q})$ between P and Q is positive. Also, given $\epsilon>0$, the set $\mathsf{Q}(\epsilon)$, say, is approximately the ϵ-neighbourhood of Q with respect to the metric d. We thus **re-define**

$$\mathsf{Q}(\epsilon)=\{z:\ \mathsf{d}(z,\mathsf{Q})<\epsilon\},$$

and similarly for other sets like $\mathsf{P},\mathsf{S},\mathsf{T}$.

Before proceeding further we will complete Proposition 2 and Lemma 3 in a certain way.

PROPOSITION 5: *Given $\epsilon>0$ we have $|w(z)|\asymp|z|^{1/2}$ outside $\mathsf{Q}(\epsilon)\cup\mathsf{P}(\epsilon)$, and $|w'(z)|=O(|z|^{3/4})$ outside $\mathsf{P}(\epsilon)$.*

Proof: We have already shown in Proposition 2 that $|z|=O(|w(z)|^2)$ as $z\to\infty$ outside $\mathsf{Q}(\epsilon)$. Now suppose (z_n) is a sequence outside $\mathsf{P}(\epsilon)$, such that $z_n\to\infty$ and $r_n=\mathsf{r}(z_n)=o(|z_n|^{-1/4})$. Then the usual procedure yields $r_n^2w(z_n+r_nz)\to y(z)$, where y solves $y''=6y^2$. Since $y\not\equiv 0$, y is non-constant, and thus has a pole $z=\omega$, say (see Appendix A). By Hurwitz' Theorem, w has a pole $p_n=z_n+r_n\omega+o(r_n)$, so that $|z_n-p_n|=|\omega|r_n+o(r_n)=o(|z_n|^{-1/4})=o(|p_n|^{-1/4})$, which contradicts our hypothesis $z_n\notin\mathsf{P}(\epsilon)$. We thus have $r_n\asymp|z_n|^{-1/4}$, and so $|w(z_n)|=O(|z_n|^{1/2})$, thus $|w(z_n)|\asymp|z_n|^{1/2}$, and $|w'(z_n)|=O(|z_n|^{3/4})$ by definition of $\mathsf{r}(z_n)$. ∎

Remark: Note that we may not conclude $|w'(z)|\asymp|z|^{3/4}$ outside $\mathsf{S}(\epsilon)\cup\mathsf{P}(\epsilon)$ by the following reason: if we suppose $|w'(z_n)|=o(|z_n|^{3/4})$ on some sequence (z_n), then the Main Lemma applies, leading to $y''=6y^2+a$, $y'(0)=y_0'=0$. If Hurwitz' Theorem would apply we would, of course, obtain $\mathsf{d}(z_n,\mathsf{S})\to 0$ as $n\to\infty$. The limit function y, however, could be a non-zero constant, so that Hurwitz' Theorem would not apply to the sequence of derivatives (y_n').

The result already announced is now

THEOREM 3: *Given $\epsilon > 0$, the spherical derivative of w satisfies $w^\#(z) = O(|z|^{3/4})$ inside $\mathsf{Q}(\epsilon)$, and $w^\#(z) = O(|z|^{-1/4})$ outside.*

Proof: In Proposition 1 we have already proved boundedness of $w^\#$ outside $\mathsf{Q}(\epsilon)$. Moreover, from Proposition 5 it easily follows that $w^\#(z) = O(|z|^{-1/4})$ outside $\mathsf{Q}(\epsilon) \cup \mathsf{P}(\epsilon)$. It is, however, easily seen that the neighbourhood $\mathsf{P}(\epsilon)$ of poles is not really exceptional for that inequality. Finally, from Proposition 5 it follows $w^\#(z) \le |w'(z)| = O(|z|^{3/4})$ for $\mathsf{d}(z, q_n) < \epsilon$. ∎

Remark: The exponent 3/4 is sharp, except when we are in the quadratic case (see Appendix A): $w^\#(q) = |w'(q)| \asymp |q|^{3/4}$ holds at any **simple** zero q of w. In the quadratic case, however, the zeros of w occur in pairs very close to zeros of w', so that the effect on $w^\#$ could be $w^\#(q) = o(|q|^{3/4})$.

It is obvious that the estimate given in Theorem 3 leads to $T(r, w) = o(r^{7/2})$ in the best case. On the other hand, it is not hard to see that $f(z) = z^{-1}w(z^2)$ has spherical derivative $f^\#(z) = O(|z|^{3/2})$, which gives $T(r^2, w) = T(r, f) + O(\log r) = O(r^5)$, but only **a posteriori**.

We now prove a result on the spherical derivative of U which is similar to Proposition 1.

PROPOSITION 6: *Given $\epsilon > 0$, the spherical derivative $U^\#$ is bounded outside the d-neighbourhood $\mathsf{T}(\epsilon)$ of the sequence of zeros of U.*

Proof: Let z_n be any sequence such that $U^\#(z_n) \to \infty$. Then from $U^\#(z) = |w(z)|(1+|U(z)|^2)^{-1}$ it follows that $|w(z_n)| \to \infty$ and $|U(z_n)| = o(|w(z_n)|^{1/2})$. As before, the Main Lemma applies to the sequence (z_n), and from $w'^2 - 4w^3 - 2zw = -2U$ and $y_n(z) = r_n^2 w(z_n + r_n z)$ it then follows that $y_n'^2 - 4y_n^3 - 2ay_n \to 0$ as $n \to \infty$. Also,

$$r_n U(z_n) = o(r_n|w(z_n)|^{1/2}) = o(|y_n(0)|^{1/2}) \to 0$$

as $n \to \infty$, this giving

$$r_n U(z_n + r_n z) \to y_0 z + y_0' z^2/2 + \cdots,$$

and so $U(z_n + r_n z)$ has a zero z_n', $z_n' \to 0$ as $n \to \infty$. Thus, for $t_n = z_n + r_n z_n'$ there follows $\mathsf{d}(z_n, t_n) \asymp |z_n|^{1/4}|z_n - t_n| = |z_n|^{1/4} r_n |z_n'| \le |z_n'|$, this showing that $U^\#$ is bounded outside the union of disks $\mathsf{D}(t_n, \epsilon)$. ∎

To identify the zeros of U, we have to discuss two different cases.

PROPOSITION 7: *Let p denote any pole of U with $|p|$ sufficiently large and associated number h. Then the following is true:*

1. *There exists $\sigma > 0$, such that for $|p| \le \sigma|h|^4$ the image under U of the disk $T = \{z : |z - (p + (14h)^{-1})| < (10|h|)^{-1}\}$, say (the radius is less than $\sqrt[4]{\sigma}|p|^{-1/4}$), covers the disk $|u| < 8|h|$. In particular, U has a zero in T.*
2. *If $|h| \le C|p|^{1/4}$ for some $C > 0$, then there exist $c_2 > c_1 > 0$, depending only on C, and a zero t of U with $c_1 < \mathsf{d}(t,p) < c_2$. Also, given $\epsilon > 0$, the image under U of the disk $T = \{z\colon |z - t| < \epsilon|p|^{-1/4}\}$ covers some disk $\{u\colon |u| < \rho(\epsilon, C)|p|^{1/4}\}$.*

Proof: Both assertions will be proved by contradiction. We assume that for some sequence of poles (p_ν) with associated sequence (h_ν) the assertion is false, and extract an appropriate sub-sequence, again denoted (p_ν), such that, in the second case, the respective limits $c = 14\lim_{\nu\to\infty} h_\nu|p_\nu|^{-1/4}$ and $a = \lim_{\nu\to\infty} p_\nu/|p_\nu|$ exist.

In the first case we consider the re-scaled function $z \mapsto h_\nu^{-1}U(p_\nu + h_\nu^{-1}z)$, which is close to $M(z) = 14 - 1/z$ for ν, and hence $|h_\nu|$, large. This Möbius transformation M maps the disk $|z - \frac{1}{14}| < \frac{1}{10}$ to the complement of the disk $|u - \frac{343}{12}| \le \frac{245}{12}$. Since $\frac{343}{12} > \frac{245}{12} + 8$, U maps the disk $|z - (p_\nu + (14h_\nu)^{-1})| < (10|h_\nu|)^{-1}$ onto a domain which covers the disk $|u| < 8|h_\nu|$, provided $\sigma_\nu = |p_\nu||h_\nu|^{-4}$ is sufficiently small. The proof of the first part is thus finished.

To prove the second part we re-scale by the factor $r_\nu = |p_\nu|^{-1/4}$ to obtain, as $\nu \to \infty$, the limit function $-\zeta(z) + c = z^{-1} + c + c_1 z + \cdots$, where $|c| \le 14C$. The derivative of this limit is some Weierstrass P-function satisfying $\wp'' = 6\wp^2 + a$, and from $|h_\nu| = O(|p_\nu|^{1/4}) = o(|p_\nu|^{3/2})$ it follows that $\wp$ also solves $\wp'^2 = 4\wp^3 + 2a\wp$; see Appendix A. Thus $\wp$ with associated Zeta-function ζ has a quadratic period lattice $\eta(\mathbb{Z} \times \mathbb{Z})$. Its mesh-size $|\eta|$ depends only on a, and hence from $|a| = 1$ it follows that it is uniformly bounded and bounded away from zero. Now $-\zeta(z) + c$ has a zero $z = \tau$ of smallest absolute value, depending only on the period lattice and on c, and τ corresponds to a zero $t_\nu \sim p_\nu + r_\nu\tau$ of U. This proves existence of a lower bound c_1 and an upper bound c_2 for $\mathsf{d}(t_\nu, p_\nu)$, these bounds depending only on C. Finally, the image under $z \mapsto -\zeta(z) + c$ of the disk $\{z\colon |z - \tau| < \epsilon\}$ covers some disk $\{u\colon |u| < 2\rho\}$, where ρ has a lower bound $\rho(\epsilon, C)$ only depending on ϵ and C. Hence the disk T is mapped by U onto some domain which covers the disk $|u| < |p_\nu|^{1/4}\rho$, provided ν is sufficiently large. ∎

Remark: In the second case the position of the zero t depends on the constant c. It is asymptotically associated with the pole p in the same sense as is the zero

τ of $-\zeta(z)+c$ with the pole $z=0$.

We do not know whether both alternatives in Proposition 7 actually may occur, and, if not both, which one will occur in general. The first one indicates that value distribution of U takes place very close to the poles of w (and U). In the second case nothing can be said about the positions of zeros. Of course, there are many zeros, this following from $N(r,1/U)\sim T(r,U)\sim \frac{1}{2}T(r,w)$.

By analogy with the Weierstrass Zeta function the first case should occur, as it does in the degenerate case. One more reason to believe this is that in the second case $|h|$ has to be very small compared with $|p|$.

From Propositions 6 and 7 there easily follows

THEOREM 4: *Given $\epsilon>0$, the spherical derivative of U is bounded outside $\mathsf{T}(\epsilon)$, and satisfies $U^{\#}(z)=O(|z|^3)$ inside. If the zero t is bounded away from the set of poles, i.e., if $\mathsf{d}(t,\mathsf{P})\geq\delta$ (this is the second case in Proposition 7), then we have $U^{\#}(z)=O(|z|^{1/2})$ in $\mathsf{D}(t,\epsilon)$.*

Proof: Boundedness outside $\mathsf{T}(\epsilon)$ was already proved. If the zero t_n is close to some pole p_n, i.e., if $t_n=p_n+1/(14h_n)+o(|h_n|^{-1})$, we then have $\mathsf{r}(t_n)^{-1}=\max\{|w(t_n)|^{1/2},|w'(t_n)|^{1/3},|t_n|^{1/4}\}=O(|h_n|)=O(|p_n|^{3/2})$, and the Main Lemma then yields $U^{\#}(z)\leq|w(z)|=O(\mathsf{r}(t_n)^{-2})=O(|p_n|^3)=O(|z|^3)$. Otherwise, for $\mathsf{d}(t_n,\mathsf{P})\geq\delta$, from $\mathsf{r}(t_n)^{-1}\asymp|t_n|^{1/4}$ then $U^{\#}(z)=O(|z|^{1/2})$ follows. ∎

5. Asymptotic distribution of poles and zeros

It is not hard to draw a **local picture** of the asymptotic distribution of poles and zeros by using a variant of the Main Lemma. We note that it is not necessary for the scaling factor r_n to be real and positive. Suppose (p_n) is a sequence of poles with associated sequence (h_n). If we re-scale by the factor $\rho_n=(-\frac{2}{3}p_n)^{-1/4}$, say,

$$y_n(z)=\rho_n^2w(p_n+\rho_nz),$$

and assume that $\eta=28\lim_{n\to\infty}\rho_n^6h_n$ exists, we always arrive at

$$y''=6y^2-3/2,\quad y'^2=4y^3-3y-\eta$$

and

$$y(z)=z^{-2}+\frac{3}{20}z^2+\frac{\eta}{28}z^4+\cdots.$$

Hence, in any neighbourhood $\mathsf{d}(z,p_n) < R$ of p_n, n large, the distribution of poles and zeros of w is approximately the image of the distribution of poles and zeros of y about $\zeta = 0$, under the map

$$\ell(\zeta, p_n) = p_n + \left(-\frac{2}{3}p_n\right)^{-1/4}\zeta.$$

To be more precise we have to consider two cases: $\Delta = 27(1-\eta^3) \neq 0$ and $\Delta = 0$, i.e., $\eta^3 = 1$; see Appendix A. In the first case let L denote the period lattice of y (including the trivial period 0), while in the second case L denotes the integer multiples of a primitive period of y. Then we have

PROPOSITION 8: *Given any radius $R > 0$ and any tolerance $\sigma > 0$, there exist $K > 0$ with the following property: if p is any pole of w with $|p| > K$, then for $\mathsf{D} = \mathsf{D}(p, R)$ the Hausdorff distance (measured with respect to the metric d) between $\mathsf{P} \cap \mathsf{D}$ and $\ell(L,p) \cap \mathsf{D}$, for some suitably chosen L, is less than σ. The corresponding result is also true for the set Q of zeros of w, and for the set S of zeros of w'.*

Remark: We note that the a-points of w are contained in small disks $\mathsf{D}(q,\epsilon)$ about the zeros. Thus, value distribution of w takes place in $\mathsf{Q}(\epsilon)$, as is also indicated by Theorem 3. The poles, however, are separated from the a-points. We would of course have obtained the same result if we would have started with a sequence of zeros. If, in some large unbounded domain S, the limit $\eta = 28\lim_{n\to\infty} h_n\rho_n^6$ were to exist, the above local picture would asymptotically be a global one in S. In particular, for $\eta = 0$, the poles of w in S then would asymptotically form a square grid, distances measured with respect to the metric d.

What happens in regions which contain no zeros or poles? A complete answer can be given for large regions.

PROPOSITION 9: *As $z \to \infty$ and $\mathsf{d}(z, \mathsf{P}) \to \infty$ the following asymptotic relations hold, for some suitably chosen branch of $z \mapsto z^{-1/4}$:*

$$w(z)z^{-1/2} \to i\sqrt{1/6}, \quad w'(z)z^{-3/4} \to 0 \quad \text{and} \quad U(z)z^{-3/2} \to i\sqrt{2/27}.$$

Remark: We do not know whether large pole-free (or zero-free) regions exist, this probably depending on initial values. A similar remark applies to the degenerate case—asymptotic distribution of zeros and poles along lines.

Proof: Let (z_n) be any sequence with $\mathsf{d}(z_n, \mathsf{P}) \to \infty$. We then re-scale by the complex factor $\rho_n = (-\frac{1}{6}z_n)^{-1/4}$ and obtain a limit function y satisfying

$y'' = 6y^2 - 6$. If y were non-constant it would have a pole $z = \omega$, and so w would have a pole $p_n \sim z_n + \rho_n\omega$ with $\mathsf{d}(z_n, p_n) \asymp |\omega|$, which is against our hypothesis. Thus y is constant, hence $y^2 = 1$ and $w(z_n)^2 z_n^{-1} \to -1/6$ and $w'(z_n)^2 z_n^{-3/2} = o(|\rho_n|^{-6}|z_n|^{-3/2}) \to 0$ as $n \to \infty$ follow. Finally, the relation $-2U = w'^2 - 4w^3 - 2zw$ gives $U(z_n)z_n^{-3/2} \to \frac{2}{3}\sqrt{-1/6}$. ∎

We conclude this section with several remarks, which also throw some light on our method and its connection with Boutroux's.

1. To describe the asymptotic distribution of poles and zeros it is also convenient to use the Riemannian metric $ds = |z|^{1/4}|dz|$. Denote by $\mathbb{H}$ the half-plane $\operatorname{Re}\zeta > 0$, say. Then $\phi(\zeta) = e^{i\alpha}\left(\frac{5}{4}\zeta\right)^{4/5}$ maps $\mathbb{H}$ conformally onto some sector S of angle width $4\pi/5$, the position of S depending on α. The geodesics in S are the images under ϕ of straight line segments in $\mathbb{H}$. Then ϕ: $(\mathbb{H}, |.|) \longrightarrow (S, \mathsf{d})$ is a conformal isometry, and the same is true if $\mathbb{H}$ is replaced by an euclidian disk $D = \{\zeta\colon |\zeta - \zeta_0| < R\} \subset \mathbb{H}$ (or any convex domain) with image domain $\phi(D) = \mathsf{D}(\phi(\zeta_0), R)$. Then, in the situation of Proposition 8, we place the set L to the ζ-plane. Given a pole $p = \phi(p')$ of w with sufficiently large absolute value, the euclidian disk $D(p', R)$ is mapped by ϕ onto the disk $\mathsf{D}(p, R)$, and the image of some $L \cap D$, $p' \in L$, coincides approximately with the set $\mathsf{P} \cap \mathsf{D}$.

We consider the sector $S = \{z\colon |\arg z| < 2\pi/5\}$, say, which is the image of $\mathbb{H}$ under $z = \phi(\zeta) = (\frac{5}{4}\zeta)^{4/5}$. Since $|z|^{-1/2}w(z)$ is bounded outside $\mathsf{P}(\varepsilon)$, it is quite natural to introduce new coordinates $\zeta = \frac{4}{5}z^{5/4}$ and $W(\zeta) = z^{-1/2}w(z)$. This is actually what Boutroux did, the result in $\mathbb{H}$ being the differential equation

$$W'' = 6W^2 + 1 + \frac{24}{25}\frac{W(\zeta)}{\zeta^2} - \frac{W'(\zeta)}{\zeta}.$$

Since W and W' are bounded outside the set $\tilde{\mathsf{P}}(\epsilon) = \phi^{-1}(\mathsf{P}(\epsilon) \cap S)$, we may regard

$$A(\zeta) = \frac{24}{25}\frac{W(\zeta)}{\zeta^2} - \frac{W'(\zeta)}{\zeta}$$

as a known function of ζ, a coefficient which satisfies

$$A(\zeta) = O(|\zeta|^{-1})$$

as $\zeta \to \infty$ in $\mathbb{H}$ outside $\tilde{\mathsf{P}}(\epsilon)$. The euclidian distance between any two poles of W is bounded away from zero. With this a priori knowledge it will probably be easier to verify Boutroux's arguments (see also Appendix B).

Suppose (ζ_n) is a sequence in $\mathbb{H}$, $|\zeta_n| \to \infty$. We set $z_n = \phi(\zeta_n)$ and assume that either $z_n = p_n$ is a pole or else $\mathsf{d}(z_n, \mathsf{P}) > \delta > 0$. Then, for $|z| < R$, say, we have

$$\begin{aligned} W(\zeta_n + z) =& \big(z_n^{-1/2} - (\frac{5}{64})^{1/4} z_n^{-7/4} z + \cdots\big) w\big(z_n + (\frac{4}{5} z_n)^{-1/4} z + \cdots\big) \\ \sim& z_n^{-1/2} w\big(z_n + (\frac{4}{5} z_n)^{-1/4} z\big), \end{aligned}$$

i.e., $W(\zeta_n + z)$ is approximately w re-scaled. We finally remark that $W^\#$ is bounded in $\mathbb{H}$.

2. The class of first Painlevé transcendents is invariant under the transformation $w(z) \mapsto \sigma^2 w(\sigma z)$, $\sigma^5 = 1$. There are also one-dimensional invariant subclasses: one of them consists of solutions $w_0(z,\mu)$ with $w_0(0,\mu) = 0$ and $w_0'(0,\mu) = \mu$, and the other one is formed by solutions $w_\infty(z,h) = z^{-2} - z^3/6 + hz^4 + \cdots$. Of particular interest should be the fix-points of the σ-transformation, namely $w_0(z,0)$ and $w_\infty(z,0)$.

3. The value distribution of the Painlevé transcendents from the viewpoint of Nevanlinna Theory is almost completely understood, due to work of Wittich and Schubart, see [19, 20, 24, 25], and also [22] for equation (IV). One problem, however, is still open: Are there any solutions of (I) with finite ramified values? All is known is that the ramification index satisfies $\vartheta(a) \le 1/6$ for every $a \ne \infty$. We can't give any answer to that problem, but note the following: suppose (z_n) is any sequence tending to infinity, such that $(w(z_n))$ and $(w'(z_n))$ are bounded (this is, in particular, the case if w has a finite ramified value). Then the usual procedure leads to the following situation: the limit case is $y'' = 6y^2 + 1$, say, with $y(0) = y'(0) = 0$, so that we are in the quadratic case $y'^2 = 4y^3 + 2y$. The point z_n is approximately the center of a square, whose vertices $z_n \pm \tau_n \pm i\tau_n$ are poles of w, and w' has zeros approximately at z_n, $z_n \pm \tau_n$ and $z_n \pm i\tau_n$, where $\tau_n \sim const. |z_n|^{-1/4}$. Of course, our method cannot distinguish between doubly a-points of w and points where w and w' are simultaneously uniformly bounded.

6. The second and fourth transcendents

The methods in Sections 3 to 5 also apply to the second and fourth transcendents. Most details are left to the reader; we just note the changes which have to be made to adapt the method. We start with Painlevé's second equation

$$\text{(I)} \qquad w'' = \alpha + zw + 2w^3,$$

with first integral

$$w'^2 = w^4 + zw^2 + 2\alpha w - U, \quad \text{where } U' = w^2.$$

The local unit of scale at a regular point $z = z_0$ now is

$$\mathsf{r}(z_0) = \min\{|w(z_0)|^{-1}, |w'(z_0)|^{-1/2}, |z_0|^{-1/2}\}.$$

The re-scaling "Ansatz"

$$y_n(z) = r_n w(z_n + r_n z),$$

$r_n = \mathsf{r}(z_n)$, leads to the differential equation

$$y_n'' = 2y_n^3 + r_n^2 z_n y_n + r_n^3 z y_n + r_n^3 \alpha,$$

with limit equation

$$y'' = 2y^3 + ay, \quad y'(0) = y_0, \quad y'(0) = y_0',$$

$$a = \lim_{n\to\infty} r_n^2 z_n, \ y_0 = \lim_{n\to\infty} r_n w(z_n) \text{ and } y_0' = \lim_{n\to\infty} r_n^2 w'(z_n),$$

and $\max\{|a|, |y_0|, |y_0'|\} = 1$.

At a pole p, w has the Laurent series expansion

$$w(z) = \epsilon(z-p)^{-1} - \frac{\epsilon p}{6}(z-p) - \frac{\alpha+\epsilon}{4}(z-p)^2 + h(z-p)^3 + \cdots,$$

$\epsilon = \pm 1$; the coefficient h remains undetermined.

Our function V is now

$$V = U - w'/w$$

with corresponding linear differential equation

$$V' = \frac{\alpha}{w} - \frac{w'}{w^3} - \frac{V}{w^2}$$

and

$$V(p) = 10\epsilon h - \frac{7}{36}p^2.$$

The scaling unit at a pole p is $\min\{|p|^{-1/2}, |h|^{-1/4}\} \asymp |p|^{-1/2}$, provided the goal $|V(p)| = O(|p|^2)$, i.e., $|h| = O(|p|^2)$, is reached. Then

$$\sum_{0<|p_n|\le r} |p_n|^{-1} = O(r^2),$$

and hence $T(r, w) = O(r^3)$. To reach this goal one has to show that $|z| = O(|w|^2)$ and $|V||w|^{-2} \le \sigma|V||z|^{-1} + K|z|$ outside $\mathsf{Q}(\epsilon)$. The proof is the same as in I-case.

There are, however, particular solutions of order 3/2, see Schubart and Wittich [20], namely solutions which also satisfy one of the Riccati equations

$$w' = \frac{z}{2} + w^2 \quad \text{and} \quad w' = -\frac{z}{2} - w^2.$$

These solutions have order of growth 3/2 and solve (II) with $\alpha = \pm 1/2$.

Painlevé's fourth equation is commonly written as

$$\text{(IV)} \qquad 2ww'' = w'^2 + 3w^4 + 8zw^3 + 4(z^2 - \alpha)w^2 + 2\beta;$$

α and β are fixed parameters. Since $w = 0$ is a singularity of (IV), it is advisable to work with $w(z) + z$ rather than w itself. This is also indicated by the Laurent series expansion at a pole of w. Set

$$r_n = \mathsf{r}(z_n) = \min\{|z_n|^{-1}, |w(z_n)|^{-1}, |w'(z_n)|^{-1/2}\}$$

and

$$y_n(z) = r_n w(z_n + r_n z) + r_n z_n + r_n^2 z$$

to obtain, in the limit with $a = \lim_{n\to\infty} r_n z_n$, $y_0 = \lim_{n\to\infty} r_n w(z_n) + a$ and $y_0' = \lim_{n\to\infty} r_n^2 w'(z_n)$, the differential equation

$$2(y-a)y'' = y'^2 + 3(y-a)^4 + 8a(y-a)^3 + 4a^2(y-a)^2.$$

It is obvious (and important) that y does not vanish identically, since we have $\max\{|a|, |y_0|, |y_0'|\} > 0$. Differentiating, dividing by $2(y-a)$ and integrating again finally yields, after re-arranging terms,

$$y'' = 2y^3 - 2a^2 y + b, \quad y(0) = y_0,\ y'(0) = y_0',\ y \not\equiv 0,$$

where b is some constant of integration.

Equation (IV) has first integral

$$w'^2 = w^4 + 4zw^3 + 4(z^2 - \alpha)w^2 - 2\beta - 4wU$$

with $U' = w^2 + 2zw$. At a pole $z = p$ we have

$$w(z) + z = \epsilon(z-p)^{-1} + \frac{1}{3}(\epsilon p^2 + 2\epsilon\alpha - 1)(z-p) + h(z-p)^2 + \cdots,$$

$\epsilon = \pm 1$, and again h is unknown resp. free. The value h occurs in $V(z) = U(z) - w'(z)/(w(z) + z)$ at $z = p$,

$$V(p) = 2\alpha p + 2h - 2\epsilon p.$$

The differential equation for V is, almost surely,

$$V' = \frac{4zw'}{(w+z)^3} - \frac{2w' - z^4 + 4\alpha z^2 + 2\beta}{(w+z)^2} + \frac{w' + 4\alpha z}{w+z} - z^2 - \frac{2w - 2z}{(w+z)^2}V.$$

The goal now is to prove the estimate $|V(p)| = O(|p|^3)$, this showing that the series expansion for $w(z) + z$ about $z = p$ has radius of convergence at least $c|p|^{-1}$, $c > 0$ an absolute constant. From this it then easily follows that

$$\sum_{0<|p_n|\le r} |p_n|^{-2} = O(r^2),$$

and hence $n(r) = O(r^4)$, $N(r,w) = O(r^4)$ and $T(r,w) = O(r^4)$. To reach the goal one has to prove, similar to case I, that $|z| = O(|w+z|)$, $|w'| = O(|w+z|^2)$ and

$$\frac{|2w - 2z|}{|w+z|^2}|V| \le \sigma\frac{|V|}{|z|} + K|z|^2$$

as $z \to \infty$ outside the ϵ-neighbourhood $\mathrm{Q}(\epsilon)$ of the zeros of $w + z$.

We note, however, that (IV) has also solutions of order $\lambda = 2$; see [22]. One example is given by the solutions of the Riccati equations

$$w' = 2zw + w^2 \quad \text{or} \quad w' = -2zw - w^2,$$

which have order of growth $\lambda = 2$, and which solve equation (IV) with parameters $\alpha = \mp 1$ and $\beta = 0$. A second example is due to Gromak [5]: every solution of

$$w'^2 + 4w' - w^4 - 4w^3 - 4(z^2 - \alpha)w^2 + 4 = 0$$

solves also (IV) with parameters $\beta = 2$, $\alpha \neq 0$, and has order of growth $\lambda = 2$.

Appendix A: The differential equation $y'^2 = Q(y)$

We will give a short course on the differential equation

$$y'^2 = Q(y), \quad Q \text{ a polynomial of degree 3 or 4},$$

for the convenience of the reader. By a simple transformation of type $v(z) = M(y(az))$, M a suitably chosen Möbius transformation and a some appropriate constant, it may be brought to the Weierstrass normal form $v'^2 = 4v^3 - g_2v - g_3$, ignoring the cases (equivalent to) $v'^2 = v$ and $v'^2 = 1 - v^2$.

We remark that local solutions clearly exist by the Picard Existence Theorem, and that they admit unrestricted analytic continuation by a theorem of Painlevé, so that the whole theory of elliptic functions can be built upon certain algebraic differential equations; see Rellich [18].

The solutions are constants, or non-constant rational, simply periodic or doubly periodic, this depending on whether the discriminant $\Delta = g_2^3 - 27g_3^2$ vanishes or not. We briefly discuss these cases now:

1. **Constant** solutions are of course zeros of the polynomial $Q(c) = 4c^3 - g_2c - g_3$.
2. **Rational** non-constant solutions occur if and only if $g_2 = g_3 = 0$. They have the form $v(z) = (z - z_0)^{-2}$.
3. In the **degenerate** case $\Delta = 0$, but $g_2g_3 \neq 0$, simply periodic non-constant solutions occur. For $v'^2 = 4(v+2c)(v-c)^2$, $c \neq 0$, the general non-constant solution is $v(z) = -2c - 3c\tan^2(z_0 + \sqrt{-3c}\, z)$.
4. In the **generic** case, the solutions are elliptic functions, Weierstrass $\wp$-functions; they occur for $\Delta \neq 0$. We mention two cases of particular interest:
   (a) The **quadratic** case $g_3 = 0$. The periods (poles) of every solution form a square grid. Given any period parallelogram ("period square") with vertices $0, 2\tau, 2(1+i)\tau, 2i\tau$, say, then v' has zeros at $\tau, i\tau$ and $(1+i)\tau$, which is also a (doubly) zero of v itself.
   (b) The **hexagonal** case $g_2 = 0$. The poles may be viewed as the vertices of a triangulation of the plane by equilateral triangles. Each pole is surrounded by six poles which form the vertices of a regular hexagon.

In any case, at the pole $z = 0$, say, v has the Laurent series expansion

$$v(z) = z^{-2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + c_6z^6 + \cdots.$$

We remark that we have spent much work in Section 3 to show that the hexagonal case cannot occur as limit case. The estimate $\lambda(w) \leq 5/2$ depends on the estimate $|h| = O(|p|^{3/2})$, while the hexagonal case occurs as a limit if and only if $|p| = o(|h|^{2/3})$.

Appendix B: The Boutroux papers

The paper [2] (127 pages) consists essentially of four parts, only the first two being of interest for us. In the first part Boutroux develops a theory of entire functions of finite order, based on Hadamard's Theorem; in the second part he considers logarithmic derivatives of these functions, with applications to the first and second Painlevé equation. He shows that for $w = -(E'/E)'$ in the first case, E has genus 2 and order 5/2, with an estimate of the counting function of zeros of E as follows: $r^{5/2}/\log r \le n(r) \le r^{5/2}\theta(r)$, with $\theta(r)$ unbounded and slowly increasing. This part is not easy to read, e.g., when Boutroux derives estimates of type $|f(z)| < \varepsilon\mu^3 r_1^{3/2}$, where " ... *$\mu$ un nombre qui sera fixe dans cette couronne, mais qui deviendra infiniment grand en même temps que r_1.*"

The paper [3] appeared in two parts; of particular interest for us is the first one (122 pages, second part 61 pages), where Boutroux develops some kind of asymptotic integration for non-linear differential equations. Painlevé's equation $y'' = 6y^2 - 6x$ (we use Boutroux's terminology) is transformed by $X = \frac{4}{5}x^{5/4}$, $Y = x^{-1/2}y$ into

$$Y'' + \frac{Y'}{X} - \frac{4}{25}\frac{Y}{X^2} = 6Y^2 - 6.$$

For $|X|$ large, the solution is compared with an appropriate solution of the equation $Y_0'' = 6Y_0^2 - 6$. Considering rather complicated asymptotic integration methods via analytic continuation of local inverse functions, and with a considerable amount of technique, Boutroux is able to show that the poles of Y are asymptotically distributed in the same way as are the poles of Y_0. This distribution can be pulled back to the z-plane. Since the poles of Y_0 form a lattice or are equally spaced on a line, Boutroux's conclusion is that the counting function of poles of Y is $O(r^2)$, and hence the counting function of poles of y is $O(r^{2\times(5/4)}) = O(r^{5/2})$.

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