ISRAEL JOURNAL OF MATHEMATICS **147** (2005), 371–379

RANDOM COMPLEX ZEROES, III. DECAY OF THE HOLE PROBABILITY

BY

MIKHAIL SODIN* AND BORIS TSIRELSON

School of Mathematics, Tel Aviv University
Tel Aviv 69978, Israel
e-mail: sodin@tau.ac.il, tsirel@tau.ac.il
www.tau.ac.il/~tsirel/

ABSTRACT

The 'hole probability' that a random entire function

$$\psi(z) = \sum_{k=0}^{\infty} \zeta_k \frac{z^k}{\sqrt{k!}},$$

where $\zeta_0, \zeta_1, \dots$ are Gaussian i.i.d. random variables, has no zeroes in the disc of radius r decays as $\exp(-cr^4)$ for large r.

We consider the (random) set of zeroes of a random entire function $\psi_\omega\colon \mathbb{C} \to \mathbb{C}$,

$$\psi(z,\omega) = \sum_{k=0}^{\infty} \zeta_k(\omega) \frac{z^k}{\sqrt{k!}}, \tag{0.1}$$

where ζ_k, $k = 0, 1, 2, \dots$ are independent standard complex-valued Gaussian random variables, that is the distribution $\mathcal{N}_{\mathbb{C}}(0,1)$ of each ζ_k has the density $\pi^{-1}\exp(-|w|^2)$ with respect to the Lebesgue measure m on $\mathbb{C}$. This model is distinguished by invariance of the distribution of zero points with respect to the motions of the complex plane

$$z \mapsto az + b, \quad |a| = 1, b \in \mathbb{C};$$

* Supported by the Israel Science Foundation of the Israel Academy of Sciences and Humanities.

Received January 21, 2004

see [6] for details and references.

Given large positive r, we are interested here in the 'hole probability' that ψ has no zeroes in the disc of radius r,

$$p(r) = \mathbb{P}(\psi(z,\cdot) \neq 0, |z| \leq r).$$

It is not difficult to show that $p(r) \leq \exp(-\operatorname{const} r^2)$; see the Offord-type estimate in [5]. Yuval Peres told one of us that the recent work [4] led to conjecture that the actual hole probability might have a faster decay. In this note, we confirm this conjecture.

The right order of decay of $p(r)$ can be guessed using one of the simple toy models discussed in [6]. Consider the random perturbation of the square lattice $S = \{\sqrt{\pi}(k+il) + \zeta_{k,l} : k,l \in \mathbb{Z}\}$, where the perturbations $\zeta_{k,l}$ are independent standard complex Gaussian random variables. Asymptotic similarity to the zero set of ψ was achieved by inventing special correlations between perturbations $\zeta_{k,l}$ but this is inessential for our purposes here. It is not difficult to see that the probability that $S \cap \{z : |z| \leq r\} = \emptyset$ decays like $\exp(-\operatorname{const} r^4)$. This prediction is correct:

THEOREM 1: $\exp(-Cr^4) \leq p(r) \leq \exp(-cr^4)$.

Throughout, by c and C we denote various positive numerical constants whose values can be different at each occurrence.

It would be interesting to check whether there exists the limit

$$\lim_{r\to\infty} \frac{\log^- p(r)}{r^4},$$

and (if it does) to find its value.

The lower bound in Theorem 1 will be obtained in Section 1 by a straightforward construction. The upper bound in Theorem 1 follows from a large deviation estimate which has independent interest.

THEOREM 2: *Let $n(r)$ be the number of random zeroes in the disc $\{|z| \leq r\}$. Then for any $\delta \in (0, \frac{1}{4}]$ and sufficiently large r,*

$$\mathbb{P}\left(\left|\frac{n(r)}{r^2} - 1\right| \geq \delta\right) \leq \exp(-c(\delta)r^4). \tag{0.2}$$

Throughout, by $c(\delta)$ we denote various positive constants which depend on δ only. Since our argument seems to be too crude to find a sharp constant $c(\delta)$ in (0.2), we freely change the values of $c(\delta)$ from line to line.

There is a fruitful analogy between random zero sets and one component Coulomb system which consists of charged particles of one sign in $\mathbb{R}^2$ embedded in a uniform background of the opposite sign (see [2] and references therein). Theorems 1 and 2 are consistent with the corresponding results for Coulomb systems [3].

ACKNOWLEDGEMENT: Yuval Peres brought our attention to the problem considered here. Fëdor Nazarov spotted an error in the first draft and suggested how to fix it. We thank both of them.

1. Proof of the lower bound in Theorem 1

In what follows, we frequently use two elementary facts: if ζ is a standard complex Gaussian variable, then

$$\mathbb{P}(|\zeta| \geq \lambda) = \frac{1}{\pi} \iint\limits_{|w|\geq\lambda} e^{-|w|^2} dm(w) = \int_{\lambda^2}^{\infty} e^{-t} dt = e^{-\lambda^2}, \tag{1.1}$$

and for $\lambda \leq 1$

$$\mathbb{P}(|\zeta| \leq \lambda) = 1 - e^{-\lambda^2} = \lambda^2 - \frac{\lambda^4}{2!} + \cdots \in \left[\frac{\lambda^2}{2}, \lambda^2\right]. \tag{1.2}$$

By Ω_r we denote the following event: (i) $|\zeta_0| \geq 2$; (ii) $|\zeta_k| \leq \exp(-2r^2)$ for $1 \leq k \leq 48r^2$; and (iii) $|\zeta_k| \leq 2^k$ for $k > 48r^2$. Since ζ_k are independent,

$$\mathbb{P}(\Omega_r) = \mathbb{P}(\mathrm{i}) \cdot \mathbb{P}(\mathrm{ii}) \cdot \mathbb{P}(\mathrm{iii}).$$

Evidently, the first and third factors on the RHS are $\geq$ const. By (1.2), the probability of the event $|\zeta_k| \leq \exp(-2r^2)$ is $\geq \frac{1}{2}\exp(-4r^2)$. Since the events within the second group are independent, the probability of all of them to happen is $\geq (\frac{1}{2}\exp(-4r^2))^{48r^2} = \exp(-192r^4 - Cr^2)$. Thus, $\mathbb{P}(\Omega_r) \geq \exp(-Cr^4)$.

Now, we show that for $\omega \in \Omega_r$ the function ψ does not vanish in the disc $\{|z| \leq r\}$. For such z and ω we have

$$|\psi(z)| \geq |\zeta_0| - \sum_{1\leq k\leq 48r^2} |\zeta_k| \frac{r^k}{\sqrt{k!}} - \sum_{k>48r^2} |\zeta_k| \frac{r^k}{\sqrt{k!}} = |\zeta_0| - \sum{}' - \sum{}''.$$

Then

$$\begin{aligned} \sum{}' &\overset{(\mathrm{ii})}{\leq} e^{-2r^2} \sum_{1\leq k\leq 48r^2} \frac{r^k}{\sqrt{k!}} \\ &\leq e^{-2r^2} \sqrt{48r^2} \cdot \sqrt{\sum_{1\leq k\leq 48r^2} \frac{r^{2k}}{k!}} < 7re^{-2r^2+0.5r^2} < e^{-r^2} < \frac{1}{2}, \end{aligned}$$

if r is sufficiently large. At the same time,

$$\sum\nolimits'' \overset{\text{(iii)}}{\leq} \sum_{k>48r^2} \frac{2^k}{\sqrt{k!}}\Big(\frac{k}{48}\Big)^{k/2} < \sum_{k>48r^2}\Big(\frac{k}{12}\cdot\frac{e}{k}\Big)^{k/2} < \sum_{k\geq 1} 2^{-k} = \frac{1}{2}$$

(we used inequality $k! > (k/e)^k$ which follows from Stirling's formula). Putting both estimates together, we get

$$|\psi(z)| \geq |\zeta_0| - 1 \overset{\text{(i)}}{\geq} 1, \quad |z| \leq r,$$

proving that ψ does not vanish in the closed disc $\{|z| \leq r\}$ for $\omega \in \Omega_r$.

2. Large deviations of $\log M(r,\psi) - r^2/2$

Let ψ be the random entire function (0.1) and let $M(r,\psi) = \max_{|z|\leq r}|\psi(z)|$. In this section we shall prove the following

LEMMA 1: *Given $\delta \in (0, \frac{1}{4}]$ and sufficiently large r,*

$$\mathbb{P}\Big(\Big|\frac{\log M(r,\psi)}{r^2} - \frac{1}{2}\Big| \geq \delta\Big) \leq \exp(-c(\delta)r^4).$$

The proof is naturally split into two parts. First we show that

$$\mathbb{P}\Big(\frac{\log M(r,\psi)}{r^2} \geq \frac{1}{2} + \delta\Big) \leq \exp(-c(\delta)r^4), \tag{2.1}$$

and then that

$$\mathbb{P}\Big(\frac{\log M(r,\psi)}{r^2} \leq \frac{1}{2} - \delta\Big) \leq \exp(-c(\delta)r^4). \tag{2.2}$$

Proof of (2.1): We use an argument similar to the one used in Section 1. We have

$$M(r,\psi) \leq \Big(\sum_{0\leq k<4er^2} + \sum_{k\geq 4er^2}\Big)|\zeta_k|\frac{r^k}{\sqrt{k!}} = \sum\nolimits_1 + \sum\nolimits_2 .$$

Consider the event A_r which consists of such ω's that (i) $|\zeta_k| \leq \exp(2\delta r^2/3)$ for $0 \leq k < 4er^2$; (ii) $|\zeta_k| \leq (\sqrt{2})^k$ for $k \geq 4er^2$. If A_r occurs and r is sufficiently large, then

$$\begin{aligned}\sum\nolimits_1^2 &\leq \Big(\sum_{0\leq k<4er^2}|\zeta_k|^2\Big)\cdot\Big(\sum_{0\leq k<4er^2}\frac{r^{2k}}{k!}\Big)\\ &\overset{\text{(i)}}{\leq} 4er^2\cdot\exp(4\delta r^2/3 + r^2) < \exp\Big(\Big(1+\frac{5}{3}\delta\Big)r^2\Big),\end{aligned}$$

and

$$\sum\nolimits_2 \overset{\text{(ii)}}{\leq} \sum_{k\geq 4er^2} |\zeta_k| \Big(\frac{k}{4e}\cdot\frac{e}{k}\Big)^{k/2} \leq \sum_{k\geq 4er^2} \frac{(\sqrt{2})^k}{2^k} \leq 1.$$

Thus

$$M(r,\psi) \leq \exp\Big(\Big(\frac{1}{2}+\delta\Big)r^2\Big).$$

It remains to estimate the probability of the complementary set $A_r^c = \Omega \setminus A_r$. If A_r^c occurs, then at least one of the following happens: $\exists k \in [0, 4er^2)$: $|\zeta_k| \geq \exp(\frac{2}{3}\delta r^2)$, or $\exists k \in [4er^2, \infty)$: $|\zeta_k| \geq (\sqrt{2})^k$. Therefore

$$\mathbb{P}(A_r^c) \leq 4er^2 \exp\Big(-\exp\Big(\frac{4}{3}\delta r^2\Big)\Big) + \sum_{k\geq 4er^2} \exp(-2^k) < \exp(-\exp(\delta r^2))$$

provided that $r \geq r_0(\delta)$. This is much stronger than (2.1). ∎

Proof of (2.2): Suppose that

$$\log M(r,\psi) \leq \Big(\frac{1}{2}-\delta\Big)r^2. \tag{2.3}$$

Then we use Cauchy's inequalities and Stirling's formula:

$$\begin{aligned}|\zeta_k| = \frac{|\psi^{(k)}(0)|}{\sqrt{k!}} &\leq \sqrt{k!}\frac{M(r,\psi)}{r^k} \\ &\leq Ck^{1/4}\exp\Big(\frac{k}{2}\log k - \frac{k}{2} + \Big(\frac{1}{2}-\delta\Big)r^2 - k\log r\Big).\end{aligned}$$

Observe that the exponent equals

$$\frac{k}{2}\Big((1-2\delta)\frac{r^2}{k} - \log\frac{r^2}{k} - 1\Big).$$

We note that

$$(1-2\delta)\frac{r^2}{k} - \log\frac{r^2}{k} - 1 < -\delta$$

when r^2/k is close enough to 1, whence for $(1-\epsilon)r^2 \leq k \leq r^2$,

$$|\zeta_k| \leq Ck^{1/4}\exp\Big(-\frac{k\delta}{2}\Big).$$

By (1.2), the probability of this event is $\leq \exp(-c(\delta)k)$. Since ζ_k are independent, multiplying these probabilities, we see that

$$\exp\Big(-c(\delta)\sum_{(1-\epsilon)r^2\leq k\leq r^2} k\Big) = \exp(-c_1(\delta)r^4)$$

is an upper bound for the probability that event (2.3) occurs. ∎

3. Mean lower bound for $\log|\psi(z)| - |z|^2/2$

Lemma 1 gives us a sharp upper bound for the 'random potential' $\log|\psi(z)| - \frac{1}{2}|z|^2$ when ω does not belong to an exceptional set in the probability space. Here, we give a mean lower bound for this potential.

LEMMA 2: *Given $\delta \in (0, \frac{1}{4}]$ and sufficiently large r,*

$$\mathbb{P}\left(\frac{1}{r^2}\int_{r\mathbb{T}} \log|\psi| d\mu \le \frac{1}{2} - \delta\right) \le \exp(-c(\delta)r^4).$$

Here, we denote by $r\mathbb{T}$ the circle $\{|z| = r\}$; μ is a normalized angular measure on $r\mathbb{T}$.

The proof uses the following

CLAIM 1: *Given $\delta \in (0, \frac{1}{4}]$, sufficiently large r, and z_0, $\frac{1}{2}r \le |z_0| \le r$, there exists $\zeta \in z_0 + \delta r\mathbb{D}$ such that*

$$\log|\psi(\zeta)| > \left(\frac{1}{2} - 3\delta\right)|z_0|^2,$$

unless ω belongs to an exceptional set of probability $\exp(-c(\delta)r^4)$.

Proof of the claim: The distribution (of probabilities) of the random potential $\log|\psi(z)| - \frac{1}{2}|z|^2$ is shift-invariant (see [6, Introduction]). Writing the lower bound (2.2) in Lemma 1 as

$$\mathbb{P}\left(\max_{z\in r\mathbb{D}}(\log|\psi(z)| - \frac{1}{2}|z|^2) \le -\delta r^2\right) \le \exp(-c(\delta)r^4)$$

we can apply it to the function $z \mapsto \log|\psi(z_0 + z)| - \frac{1}{2}|z_0 + z|^2$ on $\delta r\mathbb{D}$. We get

$$\mathbb{P}\left(\max_{z\in \delta r\mathbb{D}}(\log|\psi(z_0 + z)| - \frac{1}{2}|z_0 + z|^2) \le -\delta(\delta r)^2\right) \le \exp(-c(\delta)(\delta r)^4).$$

Assuming that ω does not belong to the exceptional set, we obtain $z \in \delta r\mathbb{D}$ such that

$$\log|\psi(z + z_0)| - \frac{1}{2}|z + z_0|^2 \ge -\delta^3 r^2.$$

Taking into account that $|z| \le 2\delta|z_0|$ we get $\frac{1}{2}|z_0 + z|^2 \ge \frac{1}{2}|z_0|^2(1 - 2\delta)^2$;

$$\begin{aligned}\log|\psi(z + z_0)| &\ge \frac{1}{2}|z_0|^2(1 - 2\delta)^2 - \delta^3 r^2 \\ &\ge \frac{1}{2}|z_0|^2 - 2\delta|z_0|^2 - \left(\frac{1}{4}\right)^2\delta(2|z_0|)^2 \ge \frac{1}{2}|z_0|^2 - 3\delta|z_0|^2,\end{aligned}$$

which yields the claim. ∎

Proof of Lemma 2: Now, we choose $\kappa = 1 - \delta^{1/4}$, take $N = [2\pi\delta^{-1}]$, and consider N discs (see Fig. 1)

$$z_j + \delta r\mathbb{D}, \quad z_j = \kappa r \exp\Big(\frac{2\pi i j}{N}\Big), \quad j = 0, 1, \dots, N-1.$$

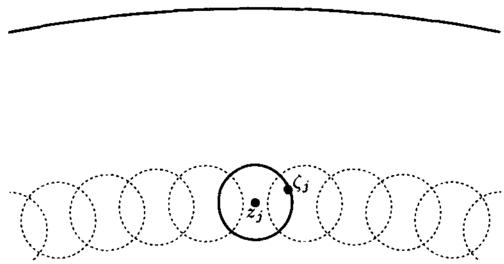

Figure 1. Small discs near the large circle

Claim 1 implies that if ω does not belong to an exceptional set of probability $N\exp(-c(\delta)r^4) = \exp(-c_1(\delta)r^4)$, then we can choose N points $\zeta_j \in z_j + \delta r\mathbb{D}$ such that

$$\log|\psi(\zeta_j)| \ge \Big(\frac{1}{2} - 3\delta\Big)|z_j|^2 \ge \Big(\frac{1}{2} - C\delta^{1/4}\Big)r^2.$$

Let $P(z,\zeta)$ be the Poisson kernel for the disc $r\mathbb{D}$, $|z| = r$, $|\zeta| < r$. We set $P_j(z) = P(z, \zeta_j)$. Then

$$\Big(\frac{1}{2} - C\delta^{1/4}\Big)r^2 \le \frac{1}{N}\sum_{j=0}^{N-1} \log|\psi(\zeta_j)| \le \int_{r\mathbb{T}} \Big(\frac{1}{N}\sum_{j=0}^{N-1} P_j\Big) \log|\psi| d\mu$$
$$= \int_{r\mathbb{T}} \log|\psi| d\mu + \int_{r\mathbb{T}} \Big(\frac{1}{N}\sum_{j=0}^{N-1} P_j - 1\Big) \log|\psi| d\mu.$$

We have

$$\int_{r\mathbb{T}} \Big(\frac{1}{N}\sum_{j=0}^{N-1} P_j - 1\Big) \log|\psi| d\mu \le \max_{z\in r\mathbb{T}} \Big|\frac{1}{N}\sum_{j=0}^{N-1} P_j - 1\Big| \cdot \int_{r\mathbb{T}} |\log|\psi|| d\mu.$$

The next two claims finish the job. ∎

CLAIM 2:

$$\max_{z\in r\mathbb{T}} \Big|\frac{1}{N}\sum_{j=0}^{N-1} P_j - 1\Big| \le C\delta^{1/2}.$$

CLAIM 3:

$$\int_{r\mathbb{T}} |\log|\psi||d\mu \le 10r^2$$

provided that $r \ge 1$, and ω does not belong to an exceptional set of probability $\exp(-cr^4)$.

Proof of Claim 2: We start with

$$\int_{\kappa r\mathbb{T}} P(z,\zeta)d\mu(\zeta) = 1,$$

and split the circle $\kappa r\mathbb{T}$ into a union of N disjoint arcs I_j of equal angular measure $\mu(I_j) = 1/N$ centered at z_j. Then

$$1 = \frac{1}{N}\sum_{j=0}^{N-1} P(z,\zeta_j) + \sum_{j=0}^{N-1}\int_{I_j}(P(z,\zeta) - P(z,\zeta_j))d\mu(\zeta),$$

and

$$\begin{aligned}|P(z,\zeta) - P(z,\zeta_j)| &\le \max_{\zeta\in I_j}|\zeta-\zeta_j|\cdot\max_{z,\zeta}|\nabla_\zeta P(z,\zeta)| \\ &\le C_1\delta r\cdot\frac{C_2 r}{(r-|\zeta|)^2} = \frac{C\delta}{\delta^{1/2}} = C\delta^{1/2},\end{aligned}$$

proving the claim. ∎

Proof of Claim 3: By Lemma 1, we know that unless ω belongs to an exceptional set of probability $\exp(-cr^4)$, there is a point $\zeta \in \frac{1}{2}r\mathbb{T}$ such that $\log|\psi(\zeta)| \ge 0$. Fix such a ζ. Then

$$0 \le \int_{r\mathbb{T}} P(z,\zeta)\log|\psi(z)|d\mu(z),$$

and hence

$$\int_{r\mathbb{T}} P(z,\zeta)\log^-|\psi(z)|d\mu(z) \le \int_{r\mathbb{T}} P(z,\zeta)\log^+|\psi(z)|d\mu(z).$$

It remains to recall that for $|z| = r$ and $|\zeta| = \frac{1}{2}r$,

$$\frac{1}{3} \le P(z,\zeta) \le 3,$$

and that

$$\int_{r\mathbb{T}} \log^+|\psi|d\mu \le \log M(r,\psi) \le r^2$$

(provided ω is non-exceptional). Hence

$$\int_{r\mathbb{T}} \log^-|\psi|d\mu \le 9r^2,$$

and

$$\int_{r\mathbb{T}} |\log|\psi||d\mu \leq 10r^2,$$

proving the claim. ∎

4. Proof of Theorem 2

We shall prove that

$$\mathbb{P}\Big(\frac{n(r)}{r^2} \geq 1+\delta\Big) \leq \exp(-c(\delta)r^4). \tag{4.1}$$

The proof of the lower bound for $n(r)$ is practically the same and is left to the reader.

Fix $\kappa = 1+\sqrt{\delta}$. Then by Jensen's formula [1, Chapter 5, Section 3.1]

$$n(r)\log\kappa \leq \int_r^{\kappa r} \frac{n(t)}{t}dt = \Big(\int_{\kappa r\mathbb{T}} - \int_{r\mathbb{T}}\Big)\log|\psi|d\mu,$$

whence by Lemmas 1 and 2

$$\frac{n(r)}{r^2} \leq \frac{1}{\log\kappa}\Big(\kappa^2\Big(\frac{1}{2}+\delta\Big) - \Big(\frac{1}{2}-\delta\Big)\Big) = \frac{1}{2}\frac{\kappa^2-1}{\log\kappa} + \delta\frac{\kappa^2+1}{\log\kappa} \leq 1 + C\sqrt{\delta},$$

provided that ω does not belong to an exceptional set of probability $\exp(-c(\delta)r^4)$. This proves estimate (4.1).

References

[1] L. V. Ahlfors, *Complex Analysis,* 3rd edition, McGraw-Hill, New York, 1979.

[2] P. G. Forrester and G. Honner, *Exact statistical properties of complex random polynomials,* Journal of Physics. A. Mathematical and General **32** (1999), 2961–2981.

[3] B. Jancovici, J. L. Lebowitz and G. Manificat, *Large charge fluctuations in classical Coulomb systems,* Journal of Statistical Physics **72** (1993), 773–787.

[4] Y. Peres and B. Virag, *Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process,* `arXiv: math.PR/0310297`; Acta Mathematica, to appear.

[5] M. Sodin, *Zeros of Gaussian analytic functions,* Mathematical Research Letters **7** (2000), 371–381.

[6] M. Sodin and B. Tsirelson, *Random complex zeroes, I. Asymptotic normality,* Israel Journal of Mathematics **144** (2004), 125–149.