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ON A QUESTION OF N. BLACKBURN ABOUT FINITE 2-GROUPS

BY

ZDRAVKA BOŽIKOV

Faculty of Civil Engineering and Architecture, University of Split
21000 Split, Croatia
e-mail: Zdravka.Bozikov@gradst.hr

AND

ZVONIMIR JANKO

Mathematical Institute, University of Heidelberg
69120 Heidelberg, Germany
e-mail: janko@mathi.uni-heidelberg.de

ABSTRACT

We classify finite 2-groups G possessing an involution which is contained in a unique subgroup of order 4 in G. This answers a question of N. Blackburn about finite 2-groups. We show that the extended Blackburn's problem is reduced to the outstanding problem of p-group theory to classify 2-groups with exactly three involutions.

N. Blackburn proposed to classify finite 2-groups G that possess an involution which is contained in only one subgroup of G of order 4 (see [1, p. 2]). We have classified here all such 2-groups.

THEOREM 1: *Let G be a finite 2-group that has an involution z which is contained in at most one subgroup of G of order 4. Then one of the following holds:*

(a) *G is cyclic.*
(b) *$C_G(z) = \langle z \rangle \times C_{2^n}$, $n \geq 1$, and such groups are classified in [2].*
(c) *$C_G(z) = \langle z \rangle \times Q_{2^n}$, $n \geq 3$, and such groups are classified in [3].*
(d) $G = \langle a, b | a^{2^m} = b^4 = 1,\ m \geq 2,\ u = a^{2^{m-1}},\ b^2 = uz,\ z^2 = 1,\ a^b = a^{-1}z^{\epsilon},$ $\epsilon = 0, 1\rangle$.

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Here $Z(G) = \langle z, u\rangle \cong E_4$, $G/\langle z\rangle \cong Q_{2^{m+1}}, G' = \langle a^2 z^\epsilon\rangle$, *and* G *is metacyclic if and only if* $\epsilon = 0$.

Proof: Let G be a 2-group that has an involution z which is contained in only one subgroup of G of order 4. Set $Z = \langle z\rangle$ and consider the subgroup $H = C_G(Z)$. Our assumption implies that H/Z is either cyclic or generalized quaternion.

Suppose that $Z \not\leq \Phi(H)$. Then $H = Z \times H_0$, where H_0 is either cyclic or generalized quaternion. If $|H_0| > 1$, then we have obtained the cases (b) and (c) of our theorem. If $|H_0| = 1$, then $Z = H = G$ and G is cyclic (case (a) of the theorem).

Suppose that $Z \leq \Phi(H)$. If H/Z is cyclic, then H is also cyclic. In that case $N_G(H)$ centralizes Z and so $H = G$ is cyclic. This is the case (a) of our theorem.

It remains to consider the case $Z \leq \Phi(H)$ and $H/Z \cong Q_{2^n}, n \geq 3$. Let T/Z be a cyclic subgroup of index 2 in H/Z so that T is abelian. If T is cyclic, then H must be of maximal class. But in that case H does not have a proper homomorphic image which is isomorphic to a generalized quaternion group. Thus T is noncyclic and so we may set $T = \langle z\rangle \times \langle a\rangle$, where the element a is of order $2^m, m \geq 2$.

If $|H'| \geq \langle z\rangle$, then $|H : H'| = 4$ and so, by a very well known result of O. Taussky, H is of maximal class, a contradiction. Hence $H' = \langle a^2 z^\epsilon\rangle$ with $\epsilon = 0, 1$ and setting $u = a^{2^{m-1}}$, we get $Z(H) = \langle z, u\rangle$ and $\Phi(T) = \langle a^2\rangle \geq \langle u\rangle$. Since $\Phi(H) \geq \langle z\rangle$, there is $b \in H - T$ such that $b^2 \in T - \langle z\rangle$ and so $b^2 = uz$ and $a^b = a^{-1}z^\epsilon$. If $\epsilon = 0$, then $\langle a\rangle$ is normal in H and H is metacyclic (since $H = \langle a\rangle\langle b\rangle$). Since $H/\langle z\rangle \cong Q_{2^{m+1}}$, there is no $x \in H - T$ such that $x^2 = z$. We have $Z(H) = \langle z, u\rangle$ and both u and uz are squares in H whereas z is not a square in H. Thus, $\langle z\rangle$ is characteristic in H which implies that $N_G(H)$ centralizes $\langle z\rangle$. This forces $H = G$ and so the structure of G is completely determined as stated in the case (d) of our theorem. ∎

THEOREM 2: *Let G be a finite 2-group that has an involution z which is contained in exactly one four-subgroup V. Then one of the following holds:*

(a) *G is dihedral or semidihedral.*

(b) *$C_G(z) = \langle z\rangle \times Q$, where Q is either cyclic or generalized quaternion.*

(c) *V is normal in G.*

Proof: If G is of maximal class, then it is dihedral or semidihedral. Next, we assume that G is not of maximal class. Then G possesses a normal subgroup R

of type $(2,2)$. Set $Z = \langle z \rangle$ and assume that $V \neq R$. Then $H = RZ$ is dihedral of order 8 so Z is not contained in $T = C_G(R)$. It follows that $|G : T| = 2$ so $G = TZ$. By the modular law, $C_G(Z) = Z \times Q$, where $Q = C_G(T)$. By the hypothesis, Q has no abelian subgroup of type $(2,2)$. It follows that Q is either cyclic or generalized quaternion and we are done. ∎

Another version of N. Blackburn's problem is to classify the 2-groups G containing an involution z such that z is contained in exactly one four-subgroup V of G. According to Theorem 2, it suffices to consider the case where V is normal in G. This problem is surprisingly complicated. Indeed, all 2-groups with exactly three involutions satisfy this hypothesis; the classification of the last groups is one of the outstanding problems of p-group theory. As the following theorem shows, a solution of this extended Blackburn's problem follows from the classification of 2-groups with exactly three involutions.

THEOREM 3: *Let G be a finite 2-group that has an involution z which is contained in exactly one four-subgroup V. Suppose that V is normal in G (see Theorem 2). Then one of the following holds:*

(a) *G has exactly three involutions.*

(b) *G possesses an involution t such that $C_G(t) = \langle t \rangle \times Q$, where Q is either cyclic or generalized quaternion. (Such groups are classified in [2, 3].)*

Proof: Set $T = C_G(V)$. Then T contains exactly three involutions and all of them lie in V; in particular, V is the unique abelian subgroup of type $(2,2)$ in T. Assume that there is an involution $t \in G - T$. By the modular law, $C_G(t) = \langle t \rangle \times Q$, where $Q = C_T(t)$. By hypothesis, V is not contained in Q. It follows that Q has no abelian subgroup of type $(2,2)$ and so Q is either cyclic or generalized quaternion. The proof is complete. ∎

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