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Introduction of « True Observables » into the Quantum Field Equations (*).

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Summary (+). — In connection with our work on the quantization of general relativity, we have investigated the equations of quantum electrodynamics, with the unrestricted gauge group, that is without specializing to Lorentz or Coulomb gauge. Even before quantization, we formulate the theory in terms of « true observables » only. We define these as dynamical variables generating canonical transformations leading from one permissible state to another; a permissible state, in turn, is a set of values of all canonical variables that obeys the gauge constraints. Similarly an observable in the quantized theory must be a Hermitian operator within the Hilbert space of permissible states; the latter are those obeying the gauge constraints. Transition to the true observables not only eliminates the longitudinal parts, of the vector potential and the scalar potential from the theory, but also the longitudinal components of the electric field strength. Likewise, the operators that create and annihilate charged particles are not themselves observables, but one can construct product combinations that are. Such (gauge-invariant) products involve the product of a creator by an annihilator at different space points, multiplied by a functional that depends on a line integral of the vector potential, the path of integration being any curve connecting the two space points. It turns out that the Hamiltonian can in fact be written in terms of these observables only, and that the infinite self-energy caused by non-transverse photons is eliminated automatically.

(+) *Editor's care.*

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In a previous paper (1) it was shown that in the presence of constraints in the Hamiltonian version of a theory the usual group of canonical transformations in phase space may be replaced advantageously by a different group, whose infinitesimal commutators are closely related to the so-called Dirac brackets (2). The new group resembles the group of canonical transformations insofar as it also reproduces the form of the canonical equations of motion, but differs in that it leaves the form of the constraints unchanged. In this transformation group one can also define generators, but in the presence of first-class constraints the relationship between infinitesimal transformations and generating dynamical variables is not one-to-one. Certain dynamical variables cannot serve as generators; on the other hand, there are different transformations belonging to the same generator. In fact, there exists a normal subgroup of infinitesimal transformations belonging to the generator zero. This subgroup represents a group of transformations with respect to which the theory is invariant, e.g. curvilinear coordinate transformations, gauge transformations and the like. It is characteristic for these groups that they depend on arbitrary functions of the time coordinate, unlike, for instance, the Lorentz group. If we form the factor group with respect to the normal subgroup, then we obtain a new (abstract) group that possesses a one-to-one relationship to the admissible generators. The admissible generators are those dynamical variables that are invariant under the normal subgroup. The transition to the factor group eliminates all those generators that are trivially zero, i.e. the constraints of the theory. We shall call these admissible, non-trivial variables « true observables ».

The true observables are the physically meaningful variables of a theory. Their values (at a given time) are independent of the choice of the frame of reference (including the gauge frame). Their values can be predicted from one time to another by integration of the canonical equations of motion (or canonical field equations, as the case may be). Any physical situation can be characterized uniquely in terms of the true observables.

The Lie algebra of the true observables affords an improved access to the task of quantizing theories with constraints. It would appear that particularly in theories possessing general covariance the determination of the true observables is a necessary preliminary to their quantization. Unfortunately this determination remains so far an unsolved problem. It was, however, considered a useful « practice » to test out the concept on a theory providing a less formidable challenge, i.e. electrodynamics. That is the topic of this paper.

(1) P. G. Bergmann and I. Goldberg: *Phys. Rev.*, **98**, 531, 544 (1955).
(2) P. A. M. Dirac: *Can. Journ. of Math.*, **2**, 129 (1950); **3**, 1 (1951).

1. – The True Observables of Electrodynamics.

We begin by identifying the true observables of classical (relativistic) electrodynamics in which the sources of the field are point charges. Ordinarily such a field is described in the canonical formalism by the four electromagnetic potentials φ and A_i ($i = 1, 2, 3$), by the four momentum densities π^4, π^s, and further by the coordinates and momentum components of the point charges $x_{(n)i}$, $p_{(n)i}$, where the subscript n identifies the n-th particle. These canonical coordinates are subject to two constraints at each space point,

$$\pi^4 = 0\,, \qquad \pi^s{}_{,s} + \frac{1}{c}\sum_n e_n \delta(\boldsymbol{x}_n, \boldsymbol{x}) = 0\,. \tag{1}$$

The symbol δ denotes here the three-dimensional δ-function. According to the rules derived earlier [1], and because all the constraints (1) are first-class constraints, the true observables are those dynamical variables that are left over after we have eliminated not only the constraints themselves, but also their canonical conjugates. More precisely, the true observables must be combinations of dynamical variables whose Poisson brackets with all constraints vanish. This requirement is equivalent to the one that the observables must be gauge invariant, because the constraints are actually the generators of infinitesimal gauge transformations. Under an infinitesimal gauge transformation the dynamical variables transform as follows:

$$\delta A_i = \xi_{,i}\,, \qquad \delta\varphi = -\frac{1}{c}\,\dot{\xi}\,, \qquad \delta x_{(n)i} = 0\,, \qquad \delta p_{(n)i} = \frac{e_n}{c}\,\xi(\boldsymbol{x}_n)_{,i}\,, \tag{2}$$

where ξ is a completely arbitrary function of the space and time coordinates and may even depend on the dynamical variables themselves. This transformation (2) is generated by the functional

$$\mathcal{G} = -\int \left\{\frac{1}{c}\,\pi^4\dot{\xi} + \left[\pi^s{}_{,s} + \frac{1}{c}\sum_n e_n\delta(\boldsymbol{x}_n, \boldsymbol{x})\right]\xi\right\} \mathrm{d}^3x\,. \tag{3}$$

It follows, then, that π^4, φ, and the longitudinal components of A_s and π^s must be eliminated from the formulation of the theory. Moreover, the canonical momentum components $p_{(n)i}$ must be replaced by the so-called kinetic momenta $p'_{(n)i}$,

$$\boldsymbol{p}'_{(n)} = \boldsymbol{p}_{(n)} - \frac{e}{c}\,\boldsymbol{A}(\boldsymbol{x}_n) = \left(1 - \frac{\dot{\boldsymbol{x}}^2_{(n)}}{c^2}\right)^{-\frac{1}{2}} m_n \dot{\boldsymbol{x}}_{(n)}\,, \tag{4}$$

which are gauge invariant.

We define the longitudinal and transverse components of a vector field as follows:

$$(5)\qquad \begin{cases} \boldsymbol{A} = \boldsymbol{A}_{(l)} + \boldsymbol{A}_{(t)}, \qquad \operatorname{curl} \boldsymbol{A}_{(l)} = 0, \qquad \operatorname{div} \boldsymbol{A}_{(t)} = 0, \\ \boldsymbol{A}_{(l)} = -\dfrac{1}{4\pi}\displaystyle\int \dfrac{1}{n} \operatorname{grad}' (\operatorname{div}' \boldsymbol{A}')\, \mathrm{d}^3x', \\ \boldsymbol{A}_{(t)} = \dfrac{1}{4\pi}\displaystyle\int \dfrac{1}{r} \operatorname{curl}' \operatorname{curl}' \boldsymbol{A}'\, \mathrm{d}^3x', \end{cases} \qquad r \equiv |\boldsymbol{x} - \boldsymbol{x}'|.$$

If we introduce these quantities into the customary Hamiltonian of the theory, we obtain the following expression:

$$(6)\qquad H = \frac{1}{2}\int\left[\frac{1}{4\pi}(\operatorname{curl} \boldsymbol{A}_{(t)})^2 + 4\pi c^2 \boldsymbol{\pi}^2_{(t)}\right]\mathrm{d}^3x + \\ + \sum_n \left(1 + \frac{\boldsymbol{p}'^2_{(n)}}{m_n^2 c^2}\right)^{\frac{1}{2}} m_n c^2 + \sum_{n<n'}\sum \frac{e_n e_{n'}}{|\boldsymbol{x}_{(n)} - \boldsymbol{x}_{(n')}|}.$$

Because of the introduction of the gauge-invariant quantities, the Hamiltonian appears as the sum of terms that refer either exclusively to the particle variables or to the (transverse) electromagnetic field. There is no «interaction» term. The actual interaction between field and particles is brought about by the circumstance that the Dirac brackets (identical in this case with ordinary Poisson brackets) between these two kinds of observables do not all vanish. We have, in fact:

$$(7)\qquad \{\pi_{(t)i}(\boldsymbol{x}), p'_{(n)j}\} = \frac{e_n}{c}\left[\delta_{ij}\,\delta(\boldsymbol{x}, \boldsymbol{x}_{(n)}) + \frac{1}{4\pi}\left(\frac{1}{r_n}\right)_{,ij}\right], \qquad r_n = |\boldsymbol{x}_n - \boldsymbol{x}|.$$

In the expression (6) we have also eliminated the self-energy terms (Coulomb energy of a particle with itself), which are infinite but c-numbers and, hence, without effect on the equations of motion of the system. It remains to be seen whether the expression (6) is Lorentz-invariant.

2. Lorentz Invariance of the Formulation.

In a consistently Hamiltonian formulation a four-dimensional notation is feasible only with the introduction of «parameters» ([3]), i.e. a super-many time formalism. Actually, Lorentz invariance does not depend on a four-

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([3]) P. G. Bergmann and J. H. M. Brunings: *Rev. Mod. Phys.*, **21**, 480 (1949).

dimensional notation, and particularly not in the present formulation, in which the quantities employed are not the components of four-vectors or four-tensors. In a Hamiltonian or quasi-Hamiltonian formulation of a theory it is much more appropriate to construct the generator of an infinitesimal Lorentz transformation (which is here a canonical or quasi-canonical transformation) and to show that the theory is invariant with respect to this transformation group. That means that the generator, or rather generators, must be constants of the motion. In view of the circumstance that these generators are physically interesting in themselves, representing the components of the total linear and angular momentum, the energy, and the motion of the center of mass of the whole system, this procedure of proof of Lorentz covariance is doubly advantageous.

We formulate the infinitesimal Lorentz transformation in terms of the coordinates of an arbitrary world point with the help of two constant vectors $\boldsymbol{\theta}$ and $\boldsymbol{v}$,

$$\delta\boldsymbol{x} = \boldsymbol{\theta x} - \boldsymbol{v}t\,, \qquad \delta t = -\frac{1}{c^2}\,\boldsymbol{v}\cdot\boldsymbol{x}\,. \tag{8}$$

The commutator of two such transformations (8), with different sets of values for the two vectors $\boldsymbol{\theta}$ and $\boldsymbol{v}$, is a law of the same form. For the group character of the Lorentz transformations, it is necessary to retain both of these vectors, even though the vector $\boldsymbol{\theta}$ describes a purely spatial rotation of the coordinate system. On the other hand, the inhomogeneous terms are not necessary to obtain a Lie group. In what follows, we shall omit reference to $\boldsymbol{\theta}$, as we need not make explicit use of the group character of the Lorentz transformation.

Without spatial rotation the infinitesimal transformation law for the coordinates of a particle, and for a fixed value of the time coordinate (not « at the same time »!) is

$$\bar{\delta}\boldsymbol{x}_n = -\boldsymbol{v}t - \dot{\boldsymbol{x}}_n\delta t = -\boldsymbol{v}t + \frac{(\boldsymbol{v}\cdot\boldsymbol{x}_n)}{c^2}\,\dot{\boldsymbol{x}}_n = \frac{(\boldsymbol{v}\cdot\boldsymbol{x}_n)}{m_n c^2}\,\frac{\boldsymbol{p}'_n}{\sqrt{1+(\boldsymbol{p}'^2_n/m_n^2c^2)}} - \boldsymbol{v}t\,. \tag{9}$$

It follows immediately, that the generator of an infinitesimal Lorentz transformation, $\mathcal{L}$, must contain terms of the form

$$\mathcal{L} = \ldots\,\frac{\boldsymbol{v}\cdot\boldsymbol{x}_n}{c}\sqrt{m_n^2c^2 + \boldsymbol{p}'^2_n} - (\boldsymbol{v}\cdot\boldsymbol{p})t\,. \tag{10}$$

In addition, we must consider the transformation law for the electromagnetic variables. We have for the electromagnetic potentials the transformation laws:

$$\delta\boldsymbol{A} = -\frac{\boldsymbol{v}}{c}\,\varphi\,, \qquad \delta\varphi = -\left(\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{A}\right). \tag{11}$$

For a point with fixed space and time coordinates (not the same world point), we obtain

$$(12)\qquad \begin{cases} \bar{\delta}\boldsymbol{B} = -\left(\dfrac{\boldsymbol{v}}{c}\cdot\boldsymbol{E}\right) + t(\boldsymbol{v}\cdot\nabla)\boldsymbol{B} - \left(\dfrac{\boldsymbol{v}}{c}\cdot\boldsymbol{x}\right)\operatorname{curl}\boldsymbol{E}\,, \\ \bar{\delta}\boldsymbol{\pi} = -\dfrac{1}{4\pi c}\left(\dfrac{\boldsymbol{v}}{c}\boldsymbol{B}\right) + t(\boldsymbol{v}\cdot\nabla)\boldsymbol{\pi} - \dfrac{1}{4\pi c}\left(\dfrac{\boldsymbol{v}}{c}\cdot\boldsymbol{x}\right)\left(\operatorname{curl}\boldsymbol{B} - \dfrac{4\pi}{c}\boldsymbol{j}\right), \\ \boldsymbol{j} = \sum\limits_n e_n\dot{\boldsymbol{x}}_n = \sum\limits_n e_n\dfrac{\boldsymbol{p}'_n}{m_n\sqrt{1+(\boldsymbol{p}'^2_n/m_n^2c^2)}}\,. \end{cases}$$

From these individual transformation laws for the mechanical and the field quantities one can derive the complete form of the generator of an infinitesimal Lorentz transformation. It is:

$$(13)\qquad \mathcal{L} = \boldsymbol{v}\cdot\left\{\int\left[\frac{\boldsymbol{x}}{c^2}\left(\frac{1}{8\pi}\boldsymbol{B}^2 + 2\pi c^2\boldsymbol{\pi}^2\right) - t(\boldsymbol{B}\boldsymbol{\pi})\right]\mathrm{d}^3x + \right.$$
$$\left. + \sum_n\left(\frac{\boldsymbol{x}_n}{c}\sqrt{m_n^2c^2 + \boldsymbol{p}'^2_n} - t\boldsymbol{p}'_n\right)\right\}.$$

This generator is gauge-invariant. It furnishes correctly the transformation laws of any gauge-invariant quantity, but not that of other variables, such as electromagnetic potentials. Actually, the law (11), for instance, is not unique. It is based on a convention as to what is meant by « the same gauge frame » in two different Lorentz frames. If we wish to produce a generator that yields also Eqs. (11), then we must add to the expression (13) terms which contain the constraints (1) as factors and which, therefore, vanish.

The generator (13) is gauge-invariant, but it does not as yet contain only true observables. In order to accomplish this purpose we must separate the vector field $\boldsymbol{\pi}$ into its longitudinal and its transverse parts. If we do so, we obtain the new expression:

$$(14)\qquad \mathcal{L} = \boldsymbol{v}\cdot\left\{\int\left[\frac{\boldsymbol{x}}{c^2}\left(\frac{1}{8\pi}\boldsymbol{B}^2 + 2\pi c^2\boldsymbol{\pi}_t^2\right) + t(\boldsymbol{\pi}_t\boldsymbol{B})\right]\mathrm{d}^3x + \right.$$
$$\left. + \sum_n\left(\frac{\boldsymbol{x}_n}{c}\sqrt{m_n^2c^2 + \boldsymbol{p}'^2_n} - t\boldsymbol{p}'_n\right) + \int U\left(\frac{2\pi}{c}\boldsymbol{x}\sigma - 4\pi\boldsymbol{\pi}_t - t\operatorname{curl}\boldsymbol{B}\right)\mathrm{d}^3x\,,\right.$$

$$U = \frac{1}{4\pi c}\sum_n\frac{e_n}{r_n}\,,\qquad \sigma = \sum_n e_n\,\delta(\boldsymbol{x}, \boldsymbol{x}_n)\,.$$

In this expression there is one obviously indefinite term: $\int U(2\pi/c)\boldsymbol{x}\sigma\mathrm{d}^3x$. Unlike in the expression for the Hamiltonian, Eq. (6), the « self-energy » term here is multiplied by the coordinate of the charge and will, therefore, make an (infinite) contribution to the Lorentz transformation. The reason is the following: If a charge is at rest, then its electric field strength has precisely the Coulomb type of infinity, which may be subtracted off simply by separating the transverse from the longitudinal electric field (the magnetic field is purely transverse to begin with). But when the point charge is in motion, then even the transverse electric field, as well as the magnetic field, possess singularities, which depend on the velocity of the point charge. Hence a Lorentz transformation changes the transverse fields by an amount which is infinite at the location of each point charge. Accordingly the generator must have a similarly infinite term of its own. It is conceivable that this singularity may be avoided if the total field is not simply separated into longitudinal and transverse part but in such a manner that the quasi-longitudinal part (which will now also have a magnetic component) subtracts off exactly the singularity of each point charge. The quasi-transverse part would still have a vanishing divergence, and the quasi-longitudinal part would be determined completely by the particle coordinates and kinetic momenta. This possibility has not yet been decided.

As for the Lorentz invariance of the Hamiltonian (6), it is sufficient to show that the functional (13) is a constant of the motion under the laws of motion determined by the Hamiltonian (6). This calculation is somewhat lengthy and will not be reproduced here; the result is, however, satisfactory.

3. – Dirac Electron Theory.

A more realistic version of quantum electrodynamics starts out not with point charges, but with expressions for current and charge densities that are furnished by the Dirac theory for a single electron. Field quantization of the particle wave functions as well as the electromagnetic field quantities then leads to the situation involving many electrons and the creation and annihilation of pairs.

From the point of view of a « true observables » formalism, the wave functions of charged particles are not gauge-invariant. Hence it becomes necessary to search for combinations that are. A complete theory has not yet been carried out. But instead of the creation and annihilation operators, we must introduce gauge-invariant « transition operators », which may be defined as follows:

$$\varrho(\boldsymbol{x}_2, \boldsymbol{x}_1) = \psi^*(\boldsymbol{x}_2) \exp\left[\frac{ie}{\hbar c}\int_{x_1}^{x_2}(\boldsymbol{A}\cdot\mathbf{d}\boldsymbol{l})\right]\psi(\boldsymbol{x}_1)\,. \tag{15}$$

They depend not only on the two end points but also on the path of integration that connects the two points x_1 and x_2. However, the ratio between two transition operators with the same end points but different connecting points is a purely electromagnetic and gauge-invariant quantity; it is an exponential function of the magnetic flux through the surface bounded by the two paths of integration. Naturally, the transition operator has two spinor indices, of which one transforms contragrediently to the other. If we interchange the two end points of a transition operator as well as ist two spinor indices with the same path of integration, then we obtain the hermitian conjugate of the original operator.

Regardless of whether the transition operator field $\varrho(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is to describe a Boson or a Fermion field, it always satisfies the same commutation (never anticommutation) relations. These are:

$$[\varrho(\boldsymbol{x}_2, \boldsymbol{x}_1), \varrho(\boldsymbol{x}_4, \boldsymbol{x}_3)] = \delta(\boldsymbol{x}_1, \boldsymbol{x}_4)\,\varrho(\boldsymbol{x}_2, \boldsymbol{x}_3) - \delta(\boldsymbol{x}_2, \boldsymbol{x}_3)\,\varrho(\boldsymbol{x}_4, \boldsymbol{x}_1)\,. \tag{16}$$

When we have to deal with a Fermion field, we have, however, the following additional relationship:

$$\varrho(\boldsymbol{x}_2, \boldsymbol{x}_1)\,\varrho(\boldsymbol{x}_2, \boldsymbol{x}_3) = \delta(\boldsymbol{x}_1, \boldsymbol{x}_2)\,\varrho(\boldsymbol{x}_2, \boldsymbol{x}_3)\,, \tag{17}$$

as well as its Hermitian conjugate,

$$\varrho(\boldsymbol{x}_2, \boldsymbol{x}_1)\,\varrho(\boldsymbol{x}_4, \boldsymbol{x}_1) = \delta(\boldsymbol{x}_4, \boldsymbol{x}_1)\,\varrho(\boldsymbol{x}_2, \boldsymbol{x}_1)\,. \tag{18}$$

It is not clear at present whether the relationships (17), (18) are sufficient to characterize a Fermion field completely. It is possible to set up an algebraic relationship that differentiates between Fermion and Boson fields, of which the relationships (17), (18) are special cases. Provided we normalize the path of integration so that it leads from one end point uniquely to some fixed reference point (say the origin of the coordinate system) and thence to the other end point, these relationships are:

$$\begin{aligned} &\varrho(\boldsymbol{x}_1, \boldsymbol{x}_2)\,\varrho(\boldsymbol{x}_3, \boldsymbol{x}_4) - \delta(\boldsymbol{x}_2, \boldsymbol{x}_3)\,\varrho(\boldsymbol{x}_1, \boldsymbol{x}_4) = \\ &\qquad = \pm\,[\varrho(\boldsymbol{x}_3, \boldsymbol{x}_2)\,\varrho(\boldsymbol{x}_1, \boldsymbol{x}_4) - \delta(\boldsymbol{x}_1, \boldsymbol{x}_2)\,\varrho(\boldsymbol{x}_3, \boldsymbol{x}_4)] = \\ &\qquad = \pm\,[\varrho(\boldsymbol{x}_1, \boldsymbol{x}_4)\,\varrho(\boldsymbol{x}_3, \boldsymbol{x}_2) - \delta(\boldsymbol{x}_3, \boldsymbol{x}_4)\,\varrho(\boldsymbol{x}_1, \boldsymbol{x}_2)]\,. \end{aligned} \tag{19}$$

The plus sign applies to Bosons, the minus sign to Fermions.

With the help of the electromagnetic field variables and the transition operators, the Hamiltonian of the electron-photon system can be formulated. This system of observables is closed insofar as the time derivative of each (in Heisenberg representation) is a function of these variables only.

RIASSUNTO (*)

In connessione col lavoro che stiamo svolgendo sulla quantizzazione della relatività generale, abbiamo investigato le equazioni della elettrodinamica quantistica col gruppo di gauge non ristretto, vale a dire senza specializzarsi al gauge di Lorentz e di Coulomb. Noi formuliamo la teoria, anche prima della quantizzazione, in termini solo di « vere osservabili ». Queste ultime sono definite come variabili dinamiche che generano trasformazioni canoniche che trasformano uno « stato permesso » in un altro, intendendosi per « stato permesso » un insieme di valori di tutte le variabili canoniche che rispettano i vincoli di gauge. Similmente, nella teoria quantizzata, un'osservabile dev'essere un operatore hermitiano nello spazio hilbertiano degli stati permessi. La transizione alle vere osservabili non solo elimina dalla teoria il potenziale scalare e le parti longitudinali del potenziale vettore, ma anche le componenti longitudinali del campo elettrico. Ovviamente, gli operatori che creano e distruggono particelle cariche non sono di per sè osservabili, ma se ne possono costruire combinazioni (in forma di prodotti) che lo sono. Tali prodotti (gauge-invarianti) involvono il prodotto di un creatore per un distruttore in punti differenti dello spazio, moltiplicati per un funzionale che dipende da un integrale di linea del potenziale vettore e calcolato su una qualsiasi curva che connette i due punti suddetti. Ne segue che l'hamiltoniana può essere espressa in termini soltanto di dette variabili e che la selfenergia infinita causata dai fotoni non trasversali è automaticamente eliminata.

(*) *A cura della Redazione.*