IL NUOVO CIMENTO VOL. XXVII, N. 3 1° Febbraio 1963

Schwarzschild Field in n Dimensions and the Dimensionality of Space Problem.

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(ricevuto l'11 Agosto 1962)

Summary. — The fact that our present laws of physics admit of a formal extension to spaces of an arbitrary number of dimensions suggests that there must be some principle (or principles) operative which in conjunction with these laws entails the observed specificity of spatial dimensionality, $n = 3$. Generalizing from an approach suggested by the work of EHRENFEST (and independently by G. J. WHITROW) on the Newtonian Keplerian problem in n dimensions, it is proposed that this principle may be tentatively summarized in the postulate that there shall be stable bound orbits or « states » for the equations of motion governing the interaction of bodies (considered as « material points »). This postulate is applied to the geodesic equations of motion obtained from a generalization of the Schwarzschild field to static systems with hyper-spherical symmetry, and it is shown that the bound state postulate uniquely entails the spatial dimensionality. This result is not entirely peculiar to general relativity because it also holds for Newtonian theory (Ehrenfest-Whitrow) if one also introduces an asymptotic condition to exclude cases $n < 3$. The Schrödinger hydrogen atom in n dimensions is also briefly considered for which the postulate also excludes $n > 3$, and in conjunction with the asymptotic condition $n < 3$. An attempt is made to understand the logical origin of this postulate and it is argued that if one assumes the basic representatives of a dynamics with a metric to be material points, one needs such a postulate to construct Einstein's « practically rigid rods », since point bodies in themselves do not provide us with a measure of distance. Some brief qualitative applications of these ideas are made to quantum electrodynamics.

1. – Introduction.

It has been frequently observed that in the statements and mathematics describing the laws of nature there is a greater generality regarding the dimensionality of space than space itself exhibits (1). This is readily seen upon examination of Newton's laws of motion, the Lagrangian and Hamiltonian formalisms, the two principles underlying special relativity, the principle of equivalence, the principle of general covariance, the geodesic principle, and the principles of quantum mechanics. In none of the above-cited cases do either the statements of the principles or the mathematical machinery restrict us to three dimensions.

Because of this rather general and indeed remarkable property of our physical principles on the one hand, and the apparent specificity implied by the three dimensionality of space on the other, there has developed, broadly speaking, two major trends of thought concerning the dimensionality of space problem: One trend consists in the attempt to enlarge the dimensionality of space, such as in the multidimensional unified field theories, while the other trend concentrates on attempting to explain why space is three-dimensional.

Although the trend towards enlarging the dimensionality of space has led to many important mathematical developments, it invariably encounters a stumbling block in the apparent lack of generality displayed by nature in this problem. For it is clear that even if we assume that somehow we have been deceived and that space is not three-dimensional, but k-dimensional, the proposition that space is k-dimensional introduces a new specificity which must be explained (2) and this explanation must also account for the « apparent » three-dimensional specificity as well.

We are therefore led to the second trend of thought which seeks to find a principle (or principles) from which the specificity of the spatial dimensionality may be deduced in conjunction with other principles. It is primarily to this latter trend of thought that the ideas presented in this paper belong. In the next section we shall briefly review some important contributions to this problem which will also serve to suggest an avenue of approach within the framework of general relativity.

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(1) In this paper we shall not enter into the dimensionality of time problem. We shall therefore assume the dimensionality of the space-time manifold to be $r=1+n$, where n is the number of spatial dimensions, and is assumed to be an integer. In terms of a many-particle formalism for which one assigns co-ordinates $(t_i, x_i^1, x_i^2, x_i^3)$ per i-th particle, the dimensionality of space problem is: Why does one assign three spatial co-ordinates per particle?

(2) Except possibly if $k=\infty$, since nature would then admit of as much generality as is compatible with the assumed countability of the number of dimensions.

2. – Historical note: the bound state postulate; relation to Mach's principle; summary.

The line of thought which we shall follow in this paper (3) originates with Kant's observation that the three-dimensionality of space may be in some way related to Newton's inverse square law (4). This of course became clear from Laplace's equation and Gauss's theorem by means of which Kant's remark may be reformulated into the elementary proposition: If the force intensities of a field are deriveable from a potential, which in empty space satisfies a generalized Laplace's equation, and if this force obeys an inverse square-law, space must be 3-dimensional (5). *The importance of Kant's observation is that it leads one to study the dimensionality of space problem from the standpoint of force laws and their effects on the motion of bodies.* This was done by EHRENFEST (6) in a fundamental paper which does not seem to have attracted much attention (7), although it contains a key-contribution to the problem. An essential feature of Ehrenfest's idea has been recently rediscovered by WHITROW (8).

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(3) Outside the scope of this paper are the well-known considerations of H. WEYL, based on demanding an invariance of the Lagrangian under $g_{\mu\nu} \to \lambda g_{\mu\nu'}$, since the theory contains serious difficulties as first pointed out by Einstein and fully recognized by Weyl himself.

(4) See, for example, the very interesting monograph by R. WEITZENBÖCK: *Der Vierdimensionale Raum* (Braunschweig, 1929).

(5) A slightly different statement of this proposition is customarily attributed to J. ÜBERWEG: *System der Logik* (various editions, Bonn, 1857-1882), although it was probably known to G. GREEN and other mathematicians who studied problems in n dimensions somewhat earlier.

(6) P. EHRENFEST: *Proc. Amsterdam Acad.*, **20**, 200 (1917); *Ann. Physik*, **61**, 440 (1920). It need not be emphasized that abstract force laws and their effects on the equations of motion were of course studied for hundreds of years previously, *e.g.* Cotes' spirals (1722). But what is missing in these earlier investigations is an observation that the results can be used to arrive at a physical principle that may explain the three-dimensionality of space. For a discussion and useful references to the earlier work of LEGENDRE, STADER, KORTWEG, GREENHILL and BERTRAND see E. J. ROUTH: *Dynamics of a Particle* (Cambridge, 1898). Sections **356-367**, **428-429**.

(7) For example, no reference is made to Ehrenfest's work in the recent historical treatment of K. JAMMER: *Concepts of Space* (Cambridge, 1954); nor in the discussion of H. WEYL: *Philosophy of Mathematics and Natural Science* (Princeton, 1949), p. 136. However, Weyl makes the following important observation with respect to this problem: « The best chances for success seem to me to lie in theoretical physical construction. » In addition to the monograph of Weitzenböck, reference is to be found in H. WEYL: *Raum, Zeit, Materie* (Berlin, 1923), p. 331.

(8) G. J. WHITROW: *The Structure and Evolution of the Universe* (New York, 1959) Appendix. WHITROW uses a classical gravitational argument similar to Ehrenfest's for $n > 3$, but for $n < 3$ invokes biological arguments based on the interesting topological problems in designing nervous systems so that arbitrary numbers of « cells » can be connected in pairs without intersection of the connecting « nerves ». This rules

EHRENFEST notes that for the Newtonian-Keplerian problem, generalized to n dimensions, one obtains stable, bound non-colliding orbits if and only if $n = 2, 3$. If one requires that the potential should vanish at infinity, the case $n = 2$ (and incidentally, $n = 1$) is excluded, so that with two conditions imposed on the Keplerian problem, one can deduce the dimensionality of space. In addition to these two conditions, in the discussion of the Keplerian problem one introduces the idea of bodies regarded as material points. The basic assumptions may be summarized as

A) The « bodies » used in formulating the principles of mechanics may be treated as material points.

B) The fields produced by bodies asymptotically approach a constant value at « large distances » (Asymptotic condition).

C) There shall exist stable bound orbits or « states » for bodies interacting via these fields.

Assumptions *A*) and *B*) are used so frequently that we may regard them as part of our present axiomatic structure (9). On the other hand, *C*) is customarily something we look for in a dynamical theory; it is more in the nature of a postulate. We shall call it the « bound state postulate » (10).

On the basis of the work of EHRENFEST and WHITROW, it follows that if we do not assume the dimensionality of space, but append the bound state postulate *C*) to the principles of n-dimensional Newtonian dynamics and gravitational theory, inclusive of axioms *A*) and *B*), then the proposition that space is three-dimensional becomes a theorem, rather than an axiom, and the observation of this specificity in nature an experimental verification of the theory!

Since we know that Newtonian gravitational theory must be replaced by general relativity, the question arises as to whether *C*) also leads to the dimensionality of space within this broader framework. There is the well-known observation that if the space-time manifold has dimensionality $n+1<4$, there is no gravitational field for matter that satifies *A*), *i.e.*, $G_{\mu\nu} = 0$, implies the Riemann tensor $R_{\lambda\mu\nu\varrho} = 0$. Hence if $n < 3$, we cannot satisfy *C*). We also

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out $n = 2,1$. However, the argument breaks down if one assumes that the « cells » are located, say, on multiply-connected surfaces, and one then needs additional assumptions. The fact that Whitrow finds it necessary to introduce a new kind of argument for $n < 3$, is tacit recognition of a profound difference between the cases $n > 3$ and $n < 3$.

(9) It is perhaps interesting to recall that Newton went to great pains to establish that axiom *A*) held in his gravitational theory, see also footnote (17). Our statement of axiom *B*) is not very « strong » since we do not need a strong statement in this paper.

(10) Our use of this postulate will be confined to classical central forces and the binding between two bodies.

know that in the weak-field approximation Newtonian theory holds, hence the Ehrenfest-Whitrow result certainly holds approximately in general relativity. As we shall see it actually holds rigorously as well.

Indeed, if it were not possible to deduce the dimensionality of space in general relativity on the basis of a principle such as C), there would be a serious inconsistency with Mach's principle as formulated by EINSTEIN. According to this principle we expect that the properties of matter should not only determine the geometry of the space-time manifold, but its topological properties as well ([11]), in particular, its dimensionality. If this were not the case, space would have absolute properties. The issue is therefore an extremely fundamental one.

Our approach to the problem in general relativity in the next sections is quite straightforward. We generalize the Schwarzschild field to n dimensions and examine the generalized Keplerian orbital equation. It is clear that as $c \to \infty$, we should (and do) obtain the generalized Newtonian orbital equation, and hence the Ehrenfest-Whitrow result, and the only question is whether the general relativistic correction alters the conclusion. As we indicated above, it does not.

As a matter of curiosity we have also generalized the Reissner-Nordström solution, although it must be kept in mind that this solution is not on the same footing as the Schwarzschild solution, since it is invalid at the origin without a compensating energy-stress tensor for which we still do not have a generally accepted theory ([12]).

For comparison, we have also briefly, studied the Schrödinger hydrogen atom in n dimensions to see whether it also entails the dimensionality of space as a consequence of C) and as one might expect from the analogy with the classical Keplerian problem (inclusive of axioms A) and B)), $n = 3$ is the only admissible dimension. EHRENFEST also considered this problem using Bohr-quantization arguments.

Some additional results will be stated in the text and we shall attempt to analyse the basis of C) somewhat further in the concluding section.

([11]) The modern topological theory of dimensionality begins with Poincaré's essay (1912), *Pourquoi l'Espace a Trois Dimensions*, Dernières Pensées (Paris, 1926). For later developments see K. MENGER: *Dimension Theorie* (Leipzig, 1926) Chapter II. W. HUREWICZ and H. WALLMAN: *Dimension Theory* (Princeton, 1941). Of interest to the physicist is perhaps the *relativity of dimensionality* implied by the inductive definition given in the above treatments. This definition requires the fixing of the dimensionality of one set inductively connected to all spaces in order to be able to assign absolute values of dimensionality. This set is taken to be the null set and is given dimensionality -1. Our discussion relies heavily on the metrical properties of space in fixing the dimensionality. Thus, the dimensionality of the space-time manifold is trace $(g_\nu{}^\mu) \equiv \delta_\mu{}^\mu = 1 + n$.

([12]) To be published in *Nuovo Cimento*.

3. – Field equations for static systems with hyper-spherical symmetry.

We shall adopt a co-ordinate system that is an obvious generalization of the one used customarily in treating the Schwarzschild field; we hav efor the line element (we set $c=1$)

$$\mathrm{d}s^2 = e^\nu \mathrm{d}t^2 - e^\lambda \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2 , \tag{3.1}$$

where $g_{00} = e^\nu$, $g_{11} = -e^\lambda$ and

$$\mathrm{d}\Omega^2 = \mathrm{d}\chi_2^2 + \sin^2\chi_2 \mathrm{d}\chi_3^2 + \dots + \prod_{l=2}^{n-1} \sin^2\chi_l \mathrm{d}\chi_n^2 . \tag{3.2}$$

After a standard calculation, the field equations in mixed form $G_\nu{}^\mu = -\varkappa T_\nu{}^\mu$ reduce to

$$\left\{\begin{aligned}
&(n-1)\frac{\exp[-\lambda]}{2}\left[-\frac{\lambda'}{r}+\frac{n-2}{r^2}\right]-\frac{(n-1)(n-2)}{2r^2}=-\varkappa T_0^0 ,\\
&(n-1)\frac{\exp[-\lambda]}{2}\left[\frac{\nu'}{r}+\frac{n-2}{r^2}\right]-\frac{(n-1)(n-2)}{2r^2}=-\varkappa T_1^1 ,\\
&\frac{\exp[-\lambda]}{2}\left[\nu''+\frac{\nu'^2}{2}-\frac{\lambda'\nu'}{2}+\frac{(n-3)(n-2)}{r^2}+\frac{(n-2)(\lambda'-\nu')}{r}\right]-\\
&\qquad\qquad -\frac{(n-2)(n-3)}{2r^2}=-\varkappa T_2^2 ,
\end{aligned}\right. \tag{3.3}$$

and we have $T_2^2 = T_3^3 = \dots = T_n^n$ because of spherical symmetry. As a « check » we note that the equations reduce to well-known expressions if $n=3$. We also have from the contracted Bianchi identities

$$\frac{1}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}(r^{n-1}T_1^1) - \frac{n-1}{r}T_2^2 = \frac{\nu'}{2}(T_0^0 - T_1^1) . \tag{3.4}$$

Although our primary interest lies in a generalization of the Schwarzschild field, for greater generality we consider energy-stress tensors for which $T_0^0 = T_1^1$, as this includes the « vacuum » as a special case. We then find, upon adopting a suitable normalization $g_{00}g_{11} = -1$, and introducing the scalar potential U, that (3.3) reduces to the following pair of dependent linear equations

$$\left\{\begin{aligned}
&\frac{n-1}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}(r^{n-2}U) = -\varkappa T_0{}^0 = -\varkappa T_1{}^1 ,\\
&\nabla^2 U + \frac{n-3}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}(r^{n-2}U) = -\varkappa T_2{}^2 .
\end{aligned}\right. \tag{3.5}$$

The stress-equilibrium (3.4) reduces to

$$\frac{1}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}(r^{n-1}T_1^1)-\frac{n-1}{r}T_2^2=0\,. \tag{3.6}$$

Thus, the simple linear character noted previously for $n=3$ continues to hold in spaces of higher dimension. This is to be expected on very general grounds (principle of equivalence, general covariance) because of the dimension-invariant character of the discussion we have given elsewhere ([13]).

We can use these equations to construct a generalization of the point-mass tensor ([14]). One finds readily

$$P_\nu^{\ \mu}=\operatorname{diag}\left(1,1,-\frac{1}{n-1},\dots,-\frac{1}{n-1}\right)\frac{m\varepsilon'}{\omega_n r^{n-1}}\,, \tag{3.7}$$

where ε' is the (double) radial δ-function $\int_0^\infty \varepsilon'\,\mathrm{d}r=1$, and ω_n is the area of the unit sphere in n dimensions, and is given by

$$\omega_n=\frac{2\pi^{n/2}}{\Gamma(n/2)}\,. \tag{3.8}$$

The point mass has identically vanishing spatial trace $T_i^i=0$; there is a companion tensor to the point mass tensor which also has this property. Upon introducing the requirement $T_i^i=0$ into (3.6), we obtain

$$D_\nu^{\ \mu}=\operatorname{diag}\left(1,1,-\frac{1}{n-1},\dots,-\frac{1}{n-1}\right)\frac{m}{\omega_n r^n}, \tag{3.9}$$

where the normalization is chosen for comparison. This tensor, however, produces logarithmic divergences.

For a tensor, $O_\nu^{\ \mu}$, with vanishing total trace (by analogy with the Maxwell energy-stress tensor for $n=3$), we find as a solution to (3.6)

$$O_\nu^{\ \mu}=\operatorname{diag}\left(1,1,-\frac{2}{n-1},\dots,-\frac{2}{n-1}\right)\frac{K}{r^{n+1}}, \tag{3.10}$$

where K is an integration constant. In both (3.9) and (3.10) we have ignored the singular behavior at the origin.

([13]) See Section **2** of the above paper, where it will be seen that the argument depends only on invariance under transformation of the form $t'=t'(r,t)$, $r'=r'(r,t)$ and « spherical » simmetry. On the other hand, our discussion of radial parity, Section **4**, requires some qualification when n is even.

([14]) F. R. TANGHERLINI: *Phys. Rev. Lett.*, **6**, 147 (1961).

Although $O_\nu{}^\mu$ is proportional to the Maxwell tensor for $n=3$, this is not the case in general. Maxwell's equations in nonrationalized units in polar co-ordinates take the form

$$\frac{1}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}(r^{n-1}F^{01}) = \omega_n j^0 , \tag{3.11}$$

and by setting $j^0 = e\varepsilon'/\omega_n r^{n-1}$, we have

$$F^{01} = e/r^{n-1} . \tag{3.12}$$

Hence, since the Maxwell tensor $M_\nu{}^\mu$ is proportional to $(F^{01})^2$, or $M_\nu{}^\mu \propto r^{2-2n}$, we find $M_\nu{}^\mu \propto O_\nu{}^\mu$, if and only if,

$$2n-2 = n+1 \tag{3.13}$$

or $n=3$. More generally, the Maxwell tensor

$$\omega_n M_\nu{}^\mu = F^{\mu\alpha}F_{\alpha\nu} + \frac{1}{4}\delta_\nu^\mu F^{\alpha\beta}F_{\alpha\beta} . \tag{3.14}$$

has identically-vanishing trace only for $n=3$, *i.e.*,

$$\omega_n M = F^{\alpha\mu}F_{\alpha\mu}\left(\frac{n+1}{4} - 1\right) , \tag{3.15}$$

Another approach to the dimensionality of the space problem might conceivably be based on this observation.

On the other hand, we have implicitly assumed in eq. (3.11) that the relation $T_0^0 = T_1^1$ holds for the Maxwell tensor of a static spherically symmetric charge distribution independently of dimension. This is in fact the case, since

$$2\omega_n M_\nu{}^\mu = \mathrm{diag}[1, 1, -1, \ldots, -1](F^{01})^2 , \tag{3.16}$$

and hence $M_0^0 = M_1^1$.

It is perhaps interesting to observe that for a two dimensional electron, the trace of the spatial stresses vanishes $M_i^i = 0$, and hence the classical electron has vanishing self-stress, although of course nonvanishing self-force.

Let us now write down the general solution to (3.5), but for simplicity, we shall discard singular terms at the origin. It proves convenient, for purposes of comparison, to rewrite Einstein's gravitational constant $\varkappa$ in a form which introduces the Newtonian constant γ. Since the left-hand side of the field equations has a factor that vanishes for $n=1$, we can insure a similar vanishing of the right-hand side, if we set $\varkappa=(n-1)\omega_n\gamma$; this yields the usual $\varkappa=8\pi\gamma$ for $n=3$. With this definition of $\varkappa$ one finds

$$(3.17)\qquad U=-\frac{\gamma}{r^{n-2}}\int\limits_0^r T_0^0\omega_n r^{n-1}\,\mathrm{d}r\,.$$

An interesting feature of the field equations is that unlike Newtonian gravitational theory, we do not find the logarithmic potential for $n=2$, for a « point » mass. This is due to the fact that while $\nabla^2 U=0$, $r>0$, admits of two solutions $U=A\ln r+B\varepsilon(r)$, the fact that we must also have from the equation for T_0^0, $U_{,r}=0$, $r>0$, requires that we set $A=0$. Such a behavior is to be expected, because for $r>0$, the space must be flat, and for the solution $U=B$ (setting $\varepsilon(r)=1$, $r>0$), the line element can be brought into pseudo-Euclidean form by changes of scale of t, r, and χ. The logarithmic solution of course can be obtained, but it belongs to the point-mass companion energy-stress tensor (3.9).

Returning now to (3.17), let us obtain the n-dimensional generalization of the Schwarzschild Reissner-Nordström solution. We find, introducing (3.7) and (3.16) into (3.17) with F^{01} in (3.16) given by (3.12)

$$(3.18)\qquad U=-\frac{\gamma m}{r^{n-2}}+\frac{\gamma e^2}{2r^{2n-4}}\,,\qquad r>0\,,\quad n\geqslant 3\,,$$

where n is restricted to three or greater because of the logarithm that occurs for $n=2$, in the electromagnetic term, and we have assumed a compensation that cancels the singularity obtained from integrating $M_\nu{}^\mu$ at the origin. We note, incidentally, that if we use relativistic units for which $\gamma=1$, we see that the dimensions of mass and charge go as $[m]=L^{n-2}$, $[e^2]=L^{2n-4}$, and hence $[e^2/m]=L^{n-2}$, thus only for $n=3$, is the classical electron « radius » a length. It also follows that the fine structure constant is not dimensionless $n\neq 3$, since the definition of $\hbar$ is such that $\hbar/m$ always has the dimensions of a length.

One can readily superimpose a cosmological term onto (3.18), although there is some ambiguity with regard to the dimension factor. We believe it is preferable to adopt a definition based on the form of the Riemann tensor for a space of constant curvature, *i.e.*, $R_{\lambda\mu\nu\varrho}=-\Lambda'(g_{\lambda\nu}g_{\mu\varrho}-g_{\lambda\varrho}g_{\mu\nu})$. Upon contraction, this expression yields

$$(3.19)\qquad G_\nu{}^\mu=-\frac{n(n-1)}{2}\Lambda'\delta_\nu{}^\mu\,.$$

The above expression agrees to within a factor of 3 with the usual definition $G_\nu{}^\mu = -\Lambda\delta_\nu{}^\mu$ for a three-dimensional matter-free space. However, (3.19) has the advantage of automatically vanishing for $n=0, 1$, just as the left-hand side of the field equations. The expression (3.18) with the de Sitter term becomes

$$U = -\frac{\gamma m}{r^{n-1}} + \frac{\gamma e^2}{2r^{2n-4}} - \frac{\Lambda' r^2}{2} . \tag{3.20}$$

In the discussion that follows in the next section, we shall set both e, and Λ' equal to zero, and consider only the case of a pure Schwarzschild field.

4. – Orbital equation for the generalized Schwarzschild field.

In this section, we shall show that there do not exist stable bound orbits in the Schwarzschild field for $n > 3$. We shall temporarily ignore the Schwarzschild singularity and return to it below.

The geodesic equations of motion are obtained from $\delta\int \mathrm{d}s = 0$, where

$$\mathrm{d}s^2 = \left(1 - \frac{2\gamma m}{r^{n-2}}\right)\mathrm{d}t^2 - g_{00}^{-1}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 , \tag{4.1}$$

in which we have set $\mathrm{d}\chi_2 = \mathrm{d}\theta$, $\mathrm{d}\chi_3 = ... = \mathrm{d}\chi_n = 0$. We introduce the energy, and angular momentum constants defined by

$$g_{00}\frac{\mathrm{d}t}{\mathrm{d}s} = k_0 , \qquad r^2\frac{\mathrm{d}\theta}{\mathrm{d}s} = k_\theta . \tag{4.2}$$

Upon introducing $u = 1/r$, we may rewrite (4.1) as

$$\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{1}{2}u^2 - \frac{\gamma m}{k_\theta^2}u^{n-2} - \gamma m u^n = \frac{k_0^2 - 1}{2k_\theta^2} , \tag{4.3}$$

for comparison with the Newtonian expressions. From (4.3), we obtain a simple generalization of the usual Schwarschild orbital equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{\gamma m}{k_\theta^2}(n-1)u^{n-3} + n\gamma m u^{n-1} . \tag{4.4}$$

To study the question of stable bound orbits, it is actually more convenient to use (4.3), and to consider the effective orbital potential defined by

$$V = \tfrac{1}{2}u^2 - \gamma m u^{n-2}k_\theta^{-2} - \gamma m u^n . \tag{4.5}$$

We shall assume $n > 3$, since the case $n = 3$ has been discussed in detail. The location of stable points may be obtained directly from a plot of V; however, the analysis is readily carried out. We set $\partial V/\partial u = 0$, and find for $u \neq 0$,

$$(n-2)\gamma m u_0^{n-4} k_\theta^{-2} + n\gamma m u_0^{n-2} = 1 , \qquad (n \geqslant 4). \tag{4.6}$$

We note the equation has only one positive root ([15]) and that it is a point of instability since

$$\left(\frac{\partial^2 V}{\partial u^2}\right)_{u_0} = 1 - (n-3) - 2n\gamma m u_0^{n-2} < 0 . \tag{4.7}$$

Thus, the essential results of the Ehrenfest-Whitrow investigation are unchanged, and hence the condition that there be stable bound orbits for the Keplerian problem is sufficient to exclude $n > 3$ in general relativity as well as in Newtonian theory.

In the above discussion we have ignored for simplicity the Schwarszchild singularity. The question naturally arises as to whether the unstable point is inside or outside the radius of the singularity, *i.e.*, whether $r_0^{n-2} > r_s^{n-2}$, where $r_s^{n-2} = 2\gamma m$. We have from (4.6) that

$$n\gamma m u_0^{n-2} \leqslant 1 , \tag{4.8}$$

the equality sign holds for the case of the unstable state of a light ray (or system with zero rest mass) travelling around on a circular orbit. It follows that $r_0^{n-2} \geqslant n\gamma m$ and hence $r_0^{n-2} > r_s^{n-2}$.

It is also tacit in the above discussion that the predictions of the geodesic equation coincide with those of a determination of the equations of motion from the field equations and the contracted Bianchi identities. Since we know from the Einstein-Infeld-Hoffmann method (or that of Fock) that this is certainly the case under reasonable assumptions for the case $n = 3$, the only question is whether or not these methods generalize to $n > 3$. It is easily verified that they do in the Newtonian approximation, which as we have already seen is sufficient, to fix the dimensionality.

It is a curious feature of the perturbation method, however, that if we apply it to the case $n = 2$, in the absence of axiom *B*), then we can be led to an infinite series for $g_{\mu\nu}$ even though we know that if axiom *A*) is fulfilled, the space-time manifold outside the bodies is flat independently of whether one's choice of co-ordinates fulfills *B*).

([15]) The case $n = 4$ exhibits three possibilities as in Newtonian theory, for one of these cases there is a positive root, for the other two cases $\partial V/\partial u \neq 0$, unless $u = 0$.

5. – Schrödinger « hydrogen atom » in n dimensions.

In his treatment of the generalized Keplerian problem, EHRENFEST did not confine himself solely to classical theory, but also applied Bohr quantization to the circular orbits of « generalized hydrogen ». As one might expect from effective potential considerations, he obtained a spectrum for $n>5$ for which the energy increases to infinity for increasing quantum numbers and for which the orbits draw closer and closer to the nucleus ([16]). A difficulty is of course that these force-equilibrium orbits are not classical stable states: Under small variations, the electron spirals into the proton or spirals off to infinity. Let us therefore briefly consider the problem from the standpoint of wave mechanics. For simplicity we shall ignore relativistic corrections except as noted below. We shall also assume $n>3$ because of our previous remarks and on the basis of the asymptotic condition B). After separating out the center-of-mass motion for the proton and electron, we have the eigenvalue equation

$$-\frac{\hbar^2}{2m}\nabla^2_{(n)}\psi - eV\Psi = E\Psi\,, \tag{5.1}$$

where $V=e/(n-2)r^{n-2}$, $\nabla^2_{(n)}$ is the Euclidean Laplacian in n-dimensions and the other quantities have their usual interpretation ([17]).

If we now transform to n-dimensional polar co-ordinates, introduce n-dimensional spherical harmonics and factor out the angular dependence ([18]), the radial wave equation takes the form

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{n-1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}+\frac{2m}{\hbar^2}\left[E-\frac{\hbar^2}{2m}\frac{l(l+n-2)}{r^2}+\frac{e^2}{(n-2)r^{n-2}}\right]R=0\,, \tag{5.2}$$

where l is the angular momentum eigenvalue, and we note the generalization of the term $l(l+1)$. It is immediately clear that for the cases $n>5$ the energy levels have a point of accumulation at minus infinity: *i.e.*, $r=0$ is not a regular point; and hence there are no stable bound states. The case $n=4$, can also be excluded by standard arguments. Alternatively, if we use

([16]) The radii of the Ehrenfest orbits are given by $r=(me^2/\tau^2\hbar^2)^{1/(n-4)}$ $(n>2,\ n\neq 4)$, and hence draw closer for increasing angular-momentum quantum number τ, $n>5$. For the energy levels see Ehrenfest's paper—to be found in his *Collected Scientific Papers* (Amsterdam, 1959), p. 400.

([17]) Because of Born's interpretation of ψ as the amplitude for finding the electron at a point, we have not violated axiom A). It would be very interesting if one could deduce Born's interpretation of ψ from axiom A).

([18]) See, *e.g.*, A. SOMMERFELD: *Partial Differential Equations in Physics* (New York, 1949), Appendix IV.

the relativistic Schrödinger equation, we have a radial equation of the form

$$(5.3)\qquad \frac{d^2R}{dr^2}+\frac{n-1}{r}\frac{dR}{dr}+ \\ +\left[\frac{E^2+m^2}{\hbar^2}-\frac{l(l+n-2)}{r^2}+\frac{2Ee^2}{\hbar^2(n-2)r^{n-2}}+\frac{e^4}{\hbar^2(n-2)^2r^{2n-4}}\right]R=0\,,$$

from which it is clear that already for $n=4$, $r=0$ is not a regular point. Thus the bound state postulate applied to the Keplerian problem serves to exclude spaces in quantum mechanics as well as in classical orbit theory ([19]).

In conjunction with this section it is appropriate to take account of possible general relativistic effects as discussed by CALLAWAY ([20]) and PERES ([21]) which might at first appear to rule out $n=3$!. We confine ourselves to the example of PERES. Consider the quantum mechanical problem for $n=3$ of a neutral particle interacting with a « heavy » point source of mass m by means of a generally covariant form of the Schrödinger or Dirac equation. One finds the origin is not a regular point and the particle « falls ». This result is to be expected from eq. (4.3), since we see that there is the quantum-mechanical possibility for the particle to tunnel through the centrifugal barrier and get into the u^3 region and fall. However, such quantum-mechanical considerations based on a nonrecoiling point nucleus are reasonable only if, crudely, the Compton wave length of the nucleus is smaller than the Schwarzschild radius, *i.e.*, $\hbar/m \leqslant 2\gamma m$, so that the nucleus would have to possess a mass $m \geqslant (\hbar/2\gamma)^{\frac{1}{2}} \approx$ $\approx 10^{-5}$ g—a familiar quantity in such considerations. Since all the « particles » that we know of with such a mass are highly composite, one should conclude that the bound state postulate serves to impose some restrictions on the specifities of other physical parameters besides the dimensionality of space. A more thorough investigation should investigate the appropriateness of the class of representations which one now uses in attempts to discuss general relativistic corrections to quantum mechanics—and in particular, discuss critically the remarks of É. CARTAN ([22]).

([19]) It may be of interest to supplement these investigations with a similar analysis for spinor wave equations in n dimensions as given by R. BRAUER and H. WEYL: *Am. Journ. Mat.*, **57**, 425 (1935). However, if we « square » the generalized Dirac equation we have terms such as (5.3) plus the spin interaction with the generalized electro-magnetic field, and it is difficult to see how there could be any cancellation which makes the equation regular at the origin, since the square of the potential is more singular than its gradient for $n > 3$.

([20]) J. CALLAWAY: *Phys. Rev.*, **112**, 290 (1958).

([21]) A. PERES: *Phys. Rev.*, **120**, 1044 (1960).

([22]) É. CARTAN: *Oeuvres Complètes* (Paris, 1952), part I, vol. I, p. 112.

6. – Concluding remarks.

The preceding analysis shows that the requirement that there be stable bound orbits for the Keplerian problem is sufficient to fix the dimensionality of space within the framework of general relativity, under the restrictions and assumptions noted in the text. However, we have also seen that this result is not merely a feature of general relativity but is reproduced in classical theory as well as in quantum mechanics (in conjunction with axiom B)). If we ask what general mathematical feature is common to all these theories, we see that it is the notion of distance based on a differential quadratic form ([23]). If we adopt the viewpoint of Mach's principle, this notion is not something independent of bodies but is based on relations between bodies considered as material points. Moreover, these relations possess the important property of being numerical, and hence the distance relation between body α and body β can be compared with that between body α and body γ—at least in principle. However, if we ask how a comparison could ever be made, or « observed », we recognize that there must be the possibility of point bodies maintaining distance relations that are invariable (in a given frame), since point bodies in themselves do not provide us with a measure of distance. Thus we are led to the conclusion that the bound state requirement is necessary in order that a comparison of relative distances between (point) bodies be physically possible, and hence a metrical dynamics constructed with such bodies as representatives, self-consistent. Given a set of bodies that fulfill C) one can construct Einstein's « pratically-rigid » rods within the theory ([24]).

It is perhaps of interest to illustrate briefly how we contemplate the bound state postulate may be used as a « building-up principle » in arriving at other specificities in nature. For example, an important prediction of quantum electrodynamics is that positronium is unstable. It would appear that we have reached a contradiction with C), and that C) must be given up. However, if we stick to the postulate, we are led to conclude that quantum electrodynamics does not provide us with a complete set of « bodies » or representatives that interact with the electrodynamic field. Most simply, there must be positively (negatively) charged bodies that form stable bound states with electrons (positrons). We know experimentally that there are such bodies, *i.e.*, protons ([25]), although, to be sure, the postulate in its present form is mathematically too qualitative to assure us that its « predicted » protons are actual protons.

([23]) Ehrenfest emphasizes the importance of investigating the quadratic nature of ds^2 in the concluding paragraph of his second article.

([24]) A. EINSTEIN: *Geometry and Experience*, to be found, for example in *Ideas and Opinions* (New York, 1954).

([25]) At least to within the limits set by the proton stability experiment of C. C. GIAMITI and F. REINES: *Phys. Rev.*, **126**, 2178 (1962).

As a further justification for the postulate, we observe that it is quite impossible to carry out the type of experiments envisaged by BOHR and ROSENFELD solely with the electrons, positrons and photons of quantum electrodynamics—as BOHR has emphasized one needs « massive » classical apparatus. This in turn requires, if axiom *A*) holds for the bodies composing the apparatus as well as the measured quantity, that the bound state postulate also holds. Thus, in its present form, quantum electrodynamics does not contain a subset of bodies and their « motions » that can play the role of measuring apparatus while another set of bodies and their motions play the role of quantities to be measured.

Perhaps part of the difficulty lies in the fact that the asymptotic condition in quantum field theory is too strong since it excludes bound states ([26]). On the other hand, the renormalization criterion of quantum electrodynamics appears to be too weak, for although it excludes spaces with dimensionality $n > 3$, as is well-known, it admits spaces with dimensionality $n < 3$. Thus if we denote the number of vertices by v, external electron lines by E_e and external photon lines by P_e, by a purely formal generalization, the degree of divergence is

$$D = \left(\frac{1+n}{4} - 1\right) 2v - \frac{n}{2} E - \frac{n-1}{2} P_e + (1+n)\,. \tag{6.1}$$

and hence $\Delta D/\Delta v > 0$, $n > 3$, and $\Delta D/\Delta v < 0$, $n < 3$. However, since the coefficient of v is linear and vanishes for $n = 3$, there is some hope of establishing the dimensionality of space uniquely within this framework, although it is clear that the renormalization criterion itself is not quite sufficient ([27]).

In conclusion, the above remarks suggest that it is logically advantageous to regard the dimensionality of space as a specificity to be derived from physical principles rather than simply inserted into the theory from the beginning. With further work, we may come to regard $n = 3$ as an eigenvalue.

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The author wishes to thank Professor E. R. CAIANIELLO and Professor F. R. HALPERN for some provocative discussions on the dimensionality-of-space problem a few years ago at the Istituto di Fisica Teorica in Naples.

([26]) H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **1**, 1425 (1955). For a recent discussion see W. THIRRING: *Phys. Rev.*, **126**, 1209 (1962).

([27]) Note that the same remark applies to the bound state postulate if we do not stay within the framework of the exact field equations of general relativity, that is, as we have seen to eliminate the cases $n < 3$ in Newtonian theory we require axiom *B*). It would be very interesting to know to what extent axiomatic field theory determines the dimensionality of space.

RIASSUNTO (*)

Il fatto che le attuali leggi fisiche ammettono una estensione formale a spazi con un numero arbitrario di dimensioni, suggerisce che deve esistere qualche principio (o alcuni principi) operativi che in unione con queste leggi implichino l'osservata specificità della dimensionalità spaziale, $n = 3$. Generalizzando uno spunto suggerito dal lavoro di EHRENFEST (ed indipendentemente da G. J. WHITROW) sul problema di Keplero n dimensionale, si propone di riassumere in via di tentativo, questo principio nel postulato che devono esistere delle orbite o « stati » legati stabili nelle equazioni del moto che governano l'interazione dei corpi considerati come punti materiali. Si applica questo postulato alle equazioni geodetiche del moto ottenute da una generalizzazione del campo di Schwarzschild a sistemi statici con simmetria ipersferica e si dimostra che il postulato degli stati legati implica unicamente la dimensionalità spaziale. Questo risultato non è affatto peculiare della relatività generale perchè vale anche per la teoria newtoniana (Ehrenfest-Whitrow) se si introduce una condizione asintotica che escluda i casi con $n < 3$. Si considera brevemente anche l'atomo di idrogeno di Schrödinger in n dimensioni, per cui il postulato esclude $n > 3$ e, in unione con la condizione asintotica, $n < 3$. Si cerca di comprendere l'origine logica di questo postulato e si suggerisce che, se si suppone che i rappresentanti fondamentali di una dinamica avente una metrica siano punti materiali, si ha bisogno di un tale postulato per costruire le « sbarre praticamente rigide » di Einstein, in quanto i corpi puntiformi di per sè non ci forniscono una misura della distanza. Si fanno alcune brevi applicazioni qualitative di queste idee alla elettrodinamica quantistica.

(*) *Traduzione a cura della Redazione.*