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QUADRATIC BASE CHANGE OF θ_{10}

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ABSTRACT

In case of GL_n over p-adic fields, it is known that Shintani base change is well behaved. However, things are not so simple for general reductive groups. In the first part of this paper, we present a counterexample to the existence of quadratic base change descent for some Galois invariant representations. These are representations of type θ_{10}. In the second part, we compute the local L-factor of θ_{10}. Unlike many other supercuspidal representations, we find that the L-factor of θ_{10} has two poles. Finally, we discuss these two results in relation to the local Langlands correspondence.

Introduction

Let k_0 be a p-adic field with odd residue characteristic and let k be a cyclic Galois extension of k_0. Let $\mathrm{Gal}(k/k_0)$ be its Galois group generated by σ. Let $\mathbf{G}$ be a connected reductive algebraic group defined over k_0 and G_{k_0} (resp. G_k) be its k_0-rational (resp. k-rational) points. Let $\widehat{G}_{k_0}$ be the set of irreducible

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admissible representations π of G_{k_0} and let $\widehat{G}_k^\sigma$ be the set of irreducible admissible representations Π of G_k which are σ-invariant, that is, $\Pi \simeq \Pi \circ \sigma$.

In general, the conjectural Shintani lifting describes a (surjective) map from $\widehat{G}_{k_0} \big/ \sim$ to $\widehat{G}_k^\sigma$ defined via a twisted character formula where for $\pi, \pi' \in \widehat{G}_{k_0}$, $\pi \sim \pi'$ if and only if $\pi \simeq \pi' \otimes \chi$ for a character χ of $k_0^\times$ which is trivial on the image of the norm map N_{k/k_0}. More precisely, this map can be defined as follows:

Definition [AC, La]*:* Let π and Π be irreducible, admissible representations of G_{k_0} and G_k respectively. Suppose that Π is Galois invariant. Then we can extend Π to a representation of the semi-direct product $G_k \rtimes \langle\sigma\rangle$. We say that Π is a **(base change) lift** or **Shintani ascent** of π if for any $g \in G_k$ such that $N_{k/k_0}(g)$ is regular and for some extended representation $\widetilde{\Pi}$, we have

$$(*) \qquad \chi_\pi(N_{k/k_0} g) = \chi_{\widetilde{\Pi}}(\sigma \cdot g).$$

Here χ_π and $\chi_{\widetilde{\Pi}}$ are the characters of π and $\widetilde{\Pi}$. We will also call π a **(base change) descent** or **Shintani descent** of Π in this case.

Here characters are represented by functions which are locally integrable and locally constant on the set of regular semisimple elements [HC, Cl] and N_{k/k_0}: $G_k \to G_{k_0}$ is a norm map. If $\mathbf{G} = \mathbf{GL}$, N_{k/k_0} is well defined up to conjugacy [AC]. However, for general $\mathbf{G}$, since conjugacy classes are not stable with respect to field extensions [Ko], a norm map is not always well defined. Hence for the left hand side of $(*)$ to be well defined, χ_π should be constant on stable conjugacy classes.

For the case $\mathbf{G} = \mathbf{GL}$, it is known that the Shintani lifting is surjective [AC, La] and it also coincides with Langlands functorial lift. However, as the examples of this paper show, in general, σ-invariant representations do not necessarily have Shintani descents to G_{k_0}. More precisely, we consider some representations of $GSp_4(k)$ of type θ_{10} (defined in §0.2) associated to a two dimensional algebra K over k. These are analogous to θ_{10} of $Sp_4(k)$ [As, Sr]. Assuming that K/k_0 is a cyclic extension of fields (then K/k_0 is unramified or totally ramified), we prove that these representations of type θ_{10} are σ-invariant; however, they cannot be lifted from any admissible irreducible representation of $GSp_4(k_0)$ in the sense of Shintani base change. In the first part (I), we prove this by showing that $\chi_{\widetilde{\theta}_{10}}$, the right hand side of $(*)$, vanishes in a small neighborhood of σ while the left hand side of $(*)$ never vanishes in any small neighborhood of the identity.

In the second part (II), we compute the L-factor [PS] of θ_{10} associated to a quadratic unramified extension K of k. In general, L-functions of supercuspidal representations are trivial. However, we show that the L-function (defined in [PS]) of θ_{10} has two poles while this representation is still supercuspidal. This was already predicted in [PS]. However, the computation has not appeared anywhere and we will produce it here.

In the third part (III), we discuss these two results in relation to Langlands parameters and functoriality according to the following picture:

$$(*)\qquad \begin{array}{ccccc} GSp_4(k_0)^\frown & & \longrightarrow & & \mathrm{Hom}(W_{k_0}, {}^LGSp_4(k_0)) \\ \Big\downarrow \text{Langlands lift} & & & & \Big\downarrow \text{res} \\ GSp_4(k)^\frown & \ni \theta_{10}^k & \longrightarrow & {}^L\theta_{10}^k \in & \mathrm{Hom}(W_k, {}^LGSp_4(k)) \end{array}$$

Here θ_{10}^k denotes the θ_{10} for $GSp_4(k)$ and ${}^L\theta_{10}^k$: $W_k \longrightarrow {}^LGSp_4(k) = GSp_4(\mathbb{C}) \rtimes W_k$ denotes the Langlands parameter of θ_{10}^k. Here, W_k and W_{k_0} denote the Weil groups for k and k_0 respectively [De, T]. To find ${}^L\theta_{10}^k$, we consider the following maps:

$$(**)\qquad \begin{array}{ccccc} & GSp_4(k_0)^\frown & \xrightarrow{l_1} & GL_4(k_0)^\frown & \\ & {\scriptstyle L_3}\Big\downarrow & & & \\ \theta_{10}^k \in & GSp_4(k)^\frown & \xrightarrow{L_2} & GL_4(k)^\frown & \ni \Pi \end{array}$$

where vertical arrows are defined via base change $(*)$ and horizontal arrows are defined via the functoriality associated to an embedding $GSp_4(\mathbb{C}) \longrightarrow GL_4(\mathbb{C})$. Using results in part (II) and [PS], we find $\Pi = L_2(\theta_{10}^k)$ and hence we can also find the Langlands parameter ${}^L\theta_{10}^k$ of θ_{10}^k. Using this parameter, we show that when k/k_0 is unramified, θ_{10}^k does not have a descent $L_3^{-1}(\theta_{10}^k)$ via Langlands correspondence L_3 over k/k_0 while $L_4^{-1}(\Pi)$ does. This phenomenon is also related to the fact that the L-packet of θ_{10} has more than one element. In fact, its L-packet has two elements and it is conjectured [Re] that the other element is the unique Iwahori spherical non-Steinberg discrete series of GSp_4.

However, if k/k_0 is ramified, we show that θ_{10} has a descent $L_3^{-1}(\theta_{10}^k) = \theta_{10}^{k_0}$, making the diagram $(**)$ commutative, that is, $\Pi = L_2(\theta_{10}^k) = L_2 \circ L_3(\theta_{10}^{k_0}) = L_1 \circ L_4(\theta_{10}^{k_0})$.

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0. Preliminaries

§1. NOTATION AND CONVENTIONS. Let k_0 be a p-adic field with odd residue characteristic. Let $\langle\ ,\ \rangle$ be a skew symmetric form defined on k_0^4 by

(0.1.1) $$\langle v, w\rangle = vJ^tw, \quad \text{for } v, w \in {k_0}^4$$

where

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Let $\mathbf{G}^0 = \mathbf{Sp_4}$ (resp. $\mathbf{G} = \mathbf{GSp_4}$) be the symplectic group (resp. the similitude group) preserving $\langle\ ,\ \rangle$ (up to constant). That is,

(0.1.2) $$\begin{aligned} &\mathbf{G^0} = \mathbf{Sp_4} = \{g \in \mathbf{GL_4} | {}^tgJg = J\}, \\ &\mathbf{G} = \mathbf{GSp_4} = \{g \in \mathbf{GL_4} | {}^tgJg = \eta(g)\cdot J \quad \text{for some } \eta(g) \in \mathbf{G}_m\}, \end{aligned}$$

where $\mathbf{G}_m$ is the multiplicative group. For any algebraic extension E of k_0, let $\overline{E}$, $\mathcal{O}_E$ and $\mathfrak{p}_E$ be its residue field, its ring of integers and the maximal ideal in $\mathcal{O}_E$ with its generator ϖ_E, respectively. We also let G_E and G_E^0 denote the E-rational points of $\mathbf{G}$ and $\mathbf{G}^0$, respectively. Let

(0.1.3) $$\eta\colon G_E \longrightarrow E^\times$$

be defined as follows: For $g \in G_E$, $\eta(g) \in E^\times$ is the similitude of g, that is, ${}^tgJg = \eta(g)\cdot J$.

Let k be a quadratic extension over k_0 with its Galois group $\mathrm{Gal}(k/k_0) = \langle\sigma\rangle$. Let K be a quadratic extension of k with its Galois group $\mathrm{Gal}(K/k) = \langle\tau\rangle$. We also fix an extension of σ to K and denote it also by σ.

Let ψ be a fixed σ-invariant additive character of k with conductor $\mathfrak{p}_k$.

§2. REPRESENTATIONS OF TYPE θ_{10}.

0.2.1. Representations of type θ_{10} are representations of GSp_4 which are lifted from the sign characters of two dimensional similitude orthogonal groups via the Howe correspondence [MVW]. More precisely, let K be a 2-dimensional semisimple algebra over k with nontrivial involution τ. Then $K = k(\sqrt{\rho})$ for

$\rho \notin (k^\times)^2$ or $K = k \oplus k$. In the first case, τ is given by the nontrivial Galois action. In the second case, k is embedded diagonally and

$$\tau\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\beta\\ \alpha\end{pmatrix} \quad \text{for } \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in K = k \oplus k.$$

Define a k-linear symmetric form on K given by

$$\mathfrak{f}_K(x,y) = \frac{1}{2}(x \cdot y^\tau + x^\tau \cdot y).$$

Let $GO(\mathfrak{f}_K)$ be the group of similitudes on K with respect to $\mathfrak{f}_K$. Consider a dual pair $(GSp_4(k), GO(\mathfrak{f}_K))$. For details about such dual pairs, we refer to [Ro, HK]. Let sgn be a quadratic character of $GO(\mathfrak{f}_K)$ which is trivial on the connected component containing the unit element of $GO(\mathfrak{f}_K)$. Then a **representation of type θ_{10}** is an irreducible representation of $GSp_4(k)$ which is a Howe-lift of the character sgn of $GO(\mathfrak{f}_K)$.

In particular, when K/k is an unramified quadratic extension, this coincides with the unipotent supercuspidal representation which extends θ_{10} [As, HPS] of $Sp_4(k)$.

0.2.2. In this section, when K/k is a quadratic extension, we realize representations of type θ_{10} explicitly.

Let $O(\mathfrak{f}_K)$ be the group of isometries on K with respect to $\mathfrak{f}_K$ and let $SO(\mathfrak{f}_K)$ be the connected component of $O(\mathfrak{f}_K)$ containing the unit element. We first define an irreducible representation θ_{10}^0 of $Sp_4(k)$ as an $O(\mathfrak{f}_K)$-isotypic component in $C_c^\infty((k \oplus k) \otimes K) = C_c^\infty(K \oplus K)$ where $O(\mathfrak{f}_K)$ acts as its unique nontrivial quadratic character sgn, that is, it is a Howe-lift of the character sgn of $O(\mathfrak{f}_K)$ [As, HPS]. More precisely, θ_{10}^0 can be realized on the complex vector space given by

(0.2.1)
$$V_{\theta_{10}^0} = V_0 = \left\{ f \in C_c^\infty(K \oplus K) \,\middle|\, \begin{matrix} f(x,y) = -f(x^\tau, y^\tau),\ f(ux,uy) = f(x,y) \\ \text{for } u \in SO(\mathfrak{f}_K) \end{matrix} \right\}.$$

Let

$$(0.2.2) \qquad m(A) = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \quad u(S) = \begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

for $A \in GL_2(k)$ and $S \in M(2,k)$ with $S = {}^tS$. Then these are elements of G_k^0

and they generate G_k^0. They act on V_0 through the operators

(0.2.3)

$$\theta_{10}^0(m(A))f(x,y) = \mathrm{sgn}_K(\det A)|\det(A)|_k f((x,y)A) \quad \text{for } (x,y) \in K \oplus K,$$
$$\theta_{10}^0(u(S))f(x,y) = \psi\Big(\frac{1}{2}\operatorname{tr}_{K/k}((x,y)S\,{}^t(x^\tau, y^\tau))\Big)f(x,y),$$
$$\theta_{10}^0(W)f(x,y) = \zeta^2 \widehat{f}(x,y) = \zeta^2 c^2 \int_{K\oplus K} f(v)\psi\Big(\frac{1}{2}\operatorname{tr}_{K/k}(v\cdot {}^t(x^\tau, y^\tau))\Big)\,dv.$$

Here dx is the Haar measure with $\mu(\mathcal{O}_K \times \mathcal{O}_K) = 1$, sgn_K is the unique nontrivial character of $k^\times/N_{K/k}(K^\times)$, c is the positive number making $f \to \widehat{f}$ unitary and ζ is a constant of modulus 1. In our case, $c^2 = \mu(\mathfrak{p}_K \times \mathfrak{p}_K)^{-\frac{1}{2}} = \sharp\overline{K}$ and $\zeta^4 = 1$.

To extend θ_{10}^0 to a representation θ_{10} of $G = G_k$, we first let H be the stabilizer of θ_{10} in G, that is, $H = \{g \in G|\ \theta_{10}^0 \circ \operatorname{Ad} g \simeq \theta_{10}^0\}$. Then we can find H as follows [As]:

$$H = \mathrm{Stab}_G(\theta_{10}) = \{g \in G|\eta(g) \in \mathrm{Im}(N_{K/k})\}. \tag{0.2.4}$$

We first extend θ_{10} to a representation of H irreducibly as follows: for $\lambda(b) = \mathrm{diag}(b,b,1,1) \in H$ with $b = N_{K/k}(\tilde{b})$ for some $\tilde{b} \in K$, and for $f \in V_0$, define

(0.2.5)

$$\theta_{10}(\lambda(b))f(x,y) = \mathrm{sgn}_K(b)|b|_k f(x\tilde{b}, y\tilde{b}) = |b|_k f(x\tilde{b}, y\tilde{b}), \quad (x,y) \in K \oplus K.$$

Then we can easily check that

$$\begin{aligned} \theta_{10}(\lambda(b))\theta_{10}(m(A)) &= \theta_{10}(m(A))\theta_{10}(\lambda(b)),\\ \theta_{10}(\lambda(b))\theta_{10}(u(S)) &= \theta_{10}(u(bS))\theta_{10}(\lambda(b)),\\ \theta_{10}(\lambda(b))\theta_{10}(W) &= \theta_{10}(W)\theta_{10}(\lambda(b))\theta_{10}(m(b^{-1}I_2)). \end{aligned} \tag{0.2.6}$$

In addition, since such $\lambda(b)$'s and $m(A)$, $u(S)$, W in (1.2.2) generate H, we get a representation of H. Note that H contains the center Z_G of G and the central character of θ_{10} is trivial. Now, we extend it to G by induction, $\operatorname{Ind}_H^G \theta_{10}$. We see that this representation is irreducible by Mackey decomposition and its representation space $V_{\theta_{10}} = V$ is given as follows:

$$V_{\theta_{10}} = V = \{\widetilde{f}\colon G \to V_0|\ \widetilde{f}(hg) = \theta_{10}(h)\widetilde{f}(g)\}.$$

To simplify notation, we denote this representation still by θ_{10}. We have $[k^\times : N_{K/k}(K^\times)] = 2$ and $[G : H] = 2$. Moreover, in this case, $\widetilde{f} \in V$ is determined by its value on I_4 and $\lambda(\varepsilon)$ where $\varepsilon \notin N_{K/k}(K^\times)$. For the simplicity of notation, we fix ε as follows:

$$\varepsilon = \begin{cases} \omega = \omega_k & \text{if } K/k \text{ is unramified},\\ \varepsilon_0 & \text{if } K/k \text{ is ramified}, \end{cases}$$

where ε_0 is a non-square unit element, that is, $\varepsilon_0 \in \mathcal{O}_k^\times \setminus (\mathcal{O}_k^\times \cap (k^\times)^2)$. For $m \in \mathbb{Z}$ and $v \in K \oplus K \setminus \mathfrak{p}_K^m \times \mathfrak{p}_K^m$ such that

$$(SO_2(\mathfrak{f}_K) \cdot (v + \mathfrak{p}_K^m \times \mathfrak{p}_K^m)) \cap (SO_2(\mathfrak{f}_K) \cdot (v^\tau + \mathfrak{p}_K^m \times \mathfrak{p}_K^m)) = \emptyset,$$

let $f_{(v,m)} \in V_0$ be locally constant on the cosets of $\mathfrak{p}_K^m \times \mathfrak{p}_K^m$ and be supported on the $O(\mathfrak{f}_K)$-orbit of $v + \mathfrak{p}_K^m \times \mathfrak{p}_K^m$ with $f_{(v,m)}(v) = 1$. That is,

$$f_{(v,m)}(Y) = \begin{cases} 1 & \text{if } Y \in SO_2(\mathfrak{f}_K) \cdot (v + \mathfrak{p}_K^m \times \mathfrak{p}_K^m), \\ -1 & \text{if } Y \in SO_2(\mathfrak{f}_K) \cdot (v^\tau + \mathfrak{p}_K^m \times \mathfrak{p}_K^m), \\ 0 & \text{otherwise.} \end{cases}$$

Then we observe V_0 is linearly spanned by such $f_{(v,m)}$'s:

$$V_0 = \langle f_{(v,m)} \in V_0 |\, v \in K \oplus K,\ m \in \mathbb{Z} \rangle. \tag{0.2.7}$$

For $\lambda = 1$ or $\lambda(\varepsilon)$ and $f_{(v,m)} \in V$, let $\widetilde{f} = \widetilde{f}_{(\lambda,v,m)}$ be defined as $\widetilde{f}(\lambda) = f_{(v,m)}$ with $\mathrm{supp}(\widetilde{f}) = H\lambda$. Then $V = V_{\theta_{10}}$ is linearly spanned by all $\widetilde{f}_{(\lambda,v,m)}$:

$$V = V_{\theta_{10}} = \langle \widetilde{f}_{(\lambda,v,m)} \in V |\ \lambda = 1 \text{ or } \lambda(\varepsilon),\ f_{(v,m)} \in V_0 \rangle. \tag{0.2.8}$$

0.2.3. Remark: Global case. Let F be a number field and let F' be a quadratic extension of F. Then the norm map $F' \to F$ induces a two dimensional orthogonal form on $\mathbb{A}_{F'}$ over F. Let GO_2 be its similitude group on $(\mathbb{A}_{F'}, N_{F'/F})$ and let sgn be a representation of GO_2 defined as follows:

$$\mathrm{sgn} = \prod_{\mathfrak{p}} \mathrm{sgn}_{\mathfrak{p}}^{\delta_{\mathfrak{p}}}$$

where $\mathrm{sgn}_{\mathfrak{p}}$ is the quadratic character of $GO_2(F_{\mathfrak{p}})$ defined in §0.2.1. Here, $\delta_{\mathfrak{p}}$ is 0 or 1 and it is zero for all but a finite number of places $\mathfrak{p}$ with $\sum \delta_{\mathfrak{p}}$ even. Then after fixing an additive character of $\mathbb{A}_F/F$, a representation of type θ_{10} for $GSp_4(\mathbb{A}_F)$ is defined as a Howe-lift of sgn.

I. Shintani descent of θ_{10}

Let $k_0 \subset k \subset K$ and σ, τ be as in §0.1. In this section, we assume that K/k_0 is cyclic Galois. Then K/k_0 is unramified or totally ramified. In both cases, we choose σ such that $\sigma^2 = \tau$.

§I.1. $\widetilde{\theta}_{10}$ and its Character $\widetilde{\Theta}_{10}$.

1.1.1. Here, we extend θ_{10} to a representation $\widetilde{\theta}_{10}$ of $\widetilde{G}_k = G_k \rtimes \mathrm{Gal}(k/k_0)$ on the same vector space $V_{\theta_{10}}$. Consider the following action of σ on V:

$$(\sigma\widetilde{f})(g)(x,y) = i\widetilde{f}(g^\sigma)(x^\sigma, y^\sigma)$$

where $i^2 = -1$. Then since

$$\sigma\theta_{10}(h^\sigma)\widetilde{f}(g)(x,y) = i\widetilde{f}(g^\sigma h^\sigma)(x^\sigma, y^\sigma) = \theta_{10}(h)\sigma\widetilde{f}(g)(x,y),$$

we have $\theta_{10}^\sigma(h) = \theta_{10}(h^\sigma)$. Hence, A_σ: $(\theta_{10}^\sigma, V) \to (\theta_{10}, V)$ is an isomorphism with $A_\sigma^2 = 1$ and it defines an extension of θ_{10} to a representation of $\widetilde{G}_k$.

1.1.2. [HC, Cl] Let $\widetilde{\Theta}_{10}$ be the character distribution defined by $\widetilde{\theta}_{10}$: For any $\Phi \in C_c^\infty(\widetilde{G}_k)$, if Φ is constant on each double coset of an open compact subgroup $U \subset G_k$, then the operator $\int_{\widetilde{G}_k} \Phi(g)\widetilde{\theta}_{10}(g)dg$ is well defined on V^U and

$$\widetilde{\Theta}_{10}(\Phi) = \mathrm{Tr}_{V^U}\left(\int_{\widetilde{G}_k} \Phi(g)\widetilde{\theta}_{10}(g)dg\right).$$

This invariant distribution is represented by a locally integrable function $\chi_{\widetilde{\theta}_{10}}$ on $\widetilde{G}_k$ which is also locally constant on the set $\widetilde{G}'_k$ of regular elements in $\widetilde{G}_k$, i.e., $\widetilde{\Theta}_{10}(\Phi) = \int_{\widetilde{G}_k} \chi_{\widetilde{\theta}_{10}}(g)\Phi(g)dg$. Moreover, for $x, y \in \widetilde{G}_k$ with x regular, we have

$$\begin{aligned} \mathrm{Ad}\, y(\chi_{\widetilde{\theta}_{10}})(x) = \chi_{\widetilde{\theta}_{10}}(y^{-1}xy) = \chi_{\widetilde{\theta}_{10}}(x), \\ \chi_{\widetilde{\theta}_{10}}(x) = \overline{\chi}_{\widetilde{\theta}_{10}}(x^{-1}), \end{aligned} \tag{1.1.3}$$

where $\overline{\chi}_{\widetilde{\theta}_{10}}$ is the complex conjugation of $\chi_{\widetilde{\theta}_{10}}$.

§I.2. THE SHINTANI DESCENT OF θ_{10}. We keep the notation from the previous section.

THEOREM 1: *Suppose that K/k_0 is cyclic. Then the representation $\theta_{10} = \theta_{10}^k$ is Galois stable, but the Shintani descent of θ_{10} does not exist.*

Here, the Shintani descent of θ_{10} is defined as in the introduction. As we mentioned in the introduction, we will prove this theorem by proving the following proposition:

PROPOSITION 1: *There is a small neighborhood Ω of $1 \rtimes \sigma \in \widetilde{G}_k$ such that $\widetilde{\Theta}_{10}|C_c^\infty(\Omega) = 0$.*

The above proposition implies that $\chi_{\widetilde{\theta}_{10}} = \widetilde{\Theta}_{10}$ on the right hand side of $(*)$ in Definition (see Introduction) vanishes in some small neighborhood of σ while the left hand side of $(*)$ never vanishes in any small neighborhood of the identity for any $\pi \in \widehat{G}_{k_0}$. Hence this will prove Theorem 1.

To prove Proposition 1, we first find a neighborhood $\Omega = \mathcal{V} \rtimes \sigma \subset G_k \rtimes \sigma$ of σ such that $\mathcal{V}$ is a neighborhood of I_4 in G_k where each element $g \in \mathcal{V}$ is σ-conjugate to an element $g' \in G_{k_0}$. For this, we need the following lemma:

1.2.1. LEMMA: *The map φ: $G_k \times G_{k_0} \to G_k$ defined by $\varphi(g,h) = g^\sigma h g^{-1}$ is submersive in some neighborhood of $(I_4, I_4) \in G_k \times G_{k_0}$.*

Proof: For $X \in \mathrm{Lie}(G_k)$ and $Y \in \mathrm{Lie}(G_{k_0})$, we have

$$\begin{aligned} d\varphi_{(g,h)}(X,Y) &= \varphi(g,h)^{-1} g^\sigma h(Y + \mathrm{Ad}\, h^{-1}(X^\sigma) - X)g^{-1} \\ &= \mathrm{Ad}\, g(Y + \mathrm{Ad}\, h^{-1} X^\sigma - X). \end{aligned}$$

For $h = I_4$, if $X = X_1 + \beta X_2$ with $X_i \in \mathrm{Lie}(G_{k_0})$ and $\beta \in k$ with $\mathrm{Tr}_{k/k_0}(\beta) = 0$, we have $X^\sigma - X = -2\beta X_2$, and hence $d\varphi$ is surjective onto $\mathrm{Lie}(G_k)$ at (g, I_4). Let $\phi_{(g,h)}$ be the restriction of $d\varphi_{(g,h)}$ to the space $\beta\,\mathrm{Lie}(G_{k_0}) \times \mathrm{Lie}(G_{k_0})$. Note that $\phi_{(I_4,I_4)}$ is bijective and thus $\det \phi_{(I_4,I_4)} \neq 0$. Put $\Phi(g,h) = \det(\phi_{(g,h)} \circ \phi^{-1}_{(I_4,I_4)})$. Since Φ is continuous and we have $\Phi(I_4, I_4) = 1$, in some neighborhood $\mathcal{U}$ of $(I_4, I_4) \in G_k \times G_{k_0}$, we have $\Phi(g,h) \neq 0$ and $\phi_{(g,h)}$ is bijective for $(g,h) \in \mathcal{U}$. Hence $d\varphi_{(g,h)}$ is submersive in $\mathcal{U}$. ∎

1.2.2. Since φ is submersive in the neighborhood $\mathcal{U}$ of (I_4, I_4), $\mathrm{Im}(\varphi)$ contains a neighborhood K_s of $I_4 \in G_k$ where K_s is the s-th principal congruence subgroup of G_k with $s \geq 1$. Let $\mathcal{V} = \varphi(\varphi^{-1}(K_s) \cap (K_0 \times (G_{k_0} \cap K_s)))$. Then $I_4 \in \mathcal{V}$ and each element of $\mathcal{V}$ is σ-conjugated to an element $g' \in G_{k_0} \cap K_s$. Moreover, each element in $\mathcal{V}$ can be conjugated by an element in K_0 to an element in $G_{k_0} \cap K_s$ from the choice of $\mathcal{V}$. Moreover, since $\mathcal{V} \subset K_1$, the Cayley transformation $\mathbf{c}$ defined by $\mathbf{c}(x) = (1+x)(1-x)^{-1}$ induces a homeomorphism of $\mathcal{V}$ onto its image. Let $\Omega = \sigma\mathcal{V}$ and let $\mathfrak{D} = C_c^\infty(\Omega)$.

1.2.3. Now we will show $\widetilde{\Theta}_{10}|\mathfrak{D} \equiv 0$, which will prove Proposition 1. For $X \in \mathfrak{D}$, there is $t \geq s$ such that X is constant on the double cosets of $K_t \subset K_s \cap \mathcal{V}$. By linearity, we may assume $X = X_{K_t \sigma g K_t}$, the characteristic function supported on $K_t \sigma g K_t \subset \Omega$. For simplicity of notation, we denote $X = X_{K_t\sigma g K_t}$ by $X_{(g,t)}$. Since σg is conjugate to $\sigma g'$ for some $g' \in G_{k_0} \cap K_s$ by an element in K_0 from the choice of Ω, we may assume $g \in G_{k_0} \cap K_s$. Moreover, for any $g \in G_{k_0} \cap K_s$, $\eta(g) \equiv 1 \pmod{\mathfrak{p}_{k_0}}$ and $\eta(g) = \gamma^2$ for some $\gamma \in k_0$ by Hensel's Lemma. Then $\eta(\gamma I_4) = \gamma^2$ and $\gamma^{-1} g \in G^0_{k_0} \cap K_s$. Since Z_G acts trivially, we have $\widetilde{\theta}_{10}(\sigma g) = \widetilde{\theta}_{10}(\sigma\gamma^{-1} g)$. Hence we may even further assume that $g \in G^0_{k_0} \cap K_s$.

1.2.4. LEMMA: *Let E be either a finite field or a p-adic field. Any $g \in G^0_E$ is conjugate to g^{-1} by an element in G_E.*

Proof: Case 1. g is regular semisimple.
This follows from Proposition 4.I.2 in [MVW], that is, they are conjugate by some $\Gamma \in GSp_4(E)$ with $\eta(\Gamma) = -1$.

Case 2. g is non regular and semisimple.
In this case, g is conjugate to

$$g_0 = \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1} & \\ & & & b^{-1} \end{pmatrix}$$

with $a, b \in E^\times$, or

$$\begin{pmatrix} a' & 0 & b' & 0 \\ 0 & a & 0 & b \\ \epsilon' b' & 0 & a' & 0 \\ 0 & \epsilon b & 0 & a \end{pmatrix}$$

where ϵ and ϵ' are nonsquare elements in E and $a'^2 - \epsilon' b'^2 = 1$, $a^2 - \epsilon b^2 = 1$. In the former case, W (see (0.2.2)) conjugates g_0 and g_0^{-1}, and in the latter case, $d = \operatorname{diag}(-1, -1, 1, 1)$ conjugates g_0 and g_0^{-1}.

Case 3. g is unipotent.
Let $\mathfrak{G}_E$ be the Lie algebra of G_E. In this case, since the Cayley transformation $\mathbf{c}$ is a well defined map from the set of nilpotent elements in $\mathfrak{G}_E$ onto the set of unipotent elements in G_E and since $(\mathbf{c}(Y))^{-1} = \mathbf{c}(-Y)$, it is enough to show that for a nilpotent $Y \in \mathfrak{G}_E$, Y and $-Y$ are conjugate in G_E up to outer conjugation by $d = \operatorname{diag}(-1, -1, 1, 1)$. Now since Y is conjugate to one of the following forms in G_E, we assume that Y is one of them:

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a, b, c \in E$. In any case, Y and $-Y$ are conjugate by d.

Case 4. Other cases.
In this case, g is conjugate to $g_0 = -u$ with u unipotent of the form in Case 3, or

$$\begin{pmatrix} \pm 1 & 0 & a & 0 \\ 0 & \mp 1 & 0 & b \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mp 1 \end{pmatrix} \text{ with } a, b \in E, \text{ or } \begin{pmatrix} \pm 1 & 0 & c & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \pm 1 & 0 \\ 0 & \epsilon b & 0 & a \end{pmatrix} \text{ where } c \in E$$

and $a^2 - \epsilon b^2 = 1$ with ϵ a nonsquare in E, or

$$\begin{pmatrix} \pm 1 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix}$$

with $a \in E^\times$ and $b \in E$. In the first three cases, d conjugates g_0 and g_0^{-1}. In the last case, g_0 and g_0^{-1} are conjugate via

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \qquad \blacksquare$$

By the above Lemma, for any $g \in G_{k_0}^0$, σg is conjugate to $(\sigma g)^{-1} = g^{-1}\sigma = \sigma g^{-1}$ by an element of G_{k_0}. Hence, we have $\chi_{\widetilde{\theta}_{10}}(\sigma g) = \chi_{\widetilde{\theta}_{10}}((\sigma g)^{-1})$ from (1.1.3) and thus $\chi_{\widetilde{\theta}_{10}}(\sigma g)$ is real valued (recall that $\chi_{\widetilde{\theta}_{10}}$ is defined in (1.1.2)). Moreover, since both $X_{(g,t)}$ and $\chi_{\widetilde{\theta}_{10}}$ are real valued on $\mathcal{V}$, so is $\widetilde{\Theta}_{10}(X_{(g,t)}) = \int_{\widetilde{G}_k} \chi_{\widetilde{\theta}_{10}}(g')X_{(g,t)}(g')dg'$. Now we claim that $\widetilde{\Theta}_{10}(X_{(g,t)})$ is also purely imaginary. Then we see $\widetilde{\Theta}_{10}(X_{(g,t)})$ is both real and pure imaginary and thus 0 for any $X_{(g,t)} \in \mathfrak{D}$. Hence it follows that $\widetilde{\Theta}_{10}|\mathfrak{D} = 0$, which will prove Proposition 1.

1.2.5. To prove the claim, we decompose V into $V^+ \oplus V^-$ as vector spaces where $V^\pm$ is ± 1 σ-eigenspace of V. This can be done since for any $\widetilde{f} \in V$, we have

$$\widetilde{f} = \frac{\widetilde{f} + \sigma\widetilde{f}}{2} + \frac{\widetilde{f} - \sigma\widetilde{f}}{2} \quad \text{with } \frac{\widetilde{f} + \sigma\widetilde{f}}{2} \in V^+, \quad \frac{\widetilde{f} - \sigma\widetilde{f}}{2} \in V^-.$$

Since σ and G_{k_0} commute and both V^+, V^- are G_{k_0}-stable, they are G_{k_0}-modules. We will show that V^+ and V^- are dual to each other as G_{k_0}-modules.

First we note that the space V_0 of (0.2.1) carries a natural Hermitian structure $(\, , \,)$ coming from the L^2 structure on $C_c^\infty(K \oplus K) \subset L^2(K \oplus K)$, that is, for $f, f' \in V_0$,

$$(f, f') = \int_{K \oplus K} f(x)\overline{f'(x)}\, dx.$$

Moreover, $(\, , \,)$ is invariant under the action of H defined by (0.2.3) and (0.2.5). The induced representation V then has a G_k-invariant Hermitian structure defined by

$$\langle \widetilde{f}, \widetilde{f}' \rangle = (\widetilde{f}(e), \widetilde{f}'(e)) + (\widetilde{f}(\lambda(\varepsilon)), \widetilde{f}'(\lambda(\varepsilon))).$$

Hence V is unitary and its complex conjugate is naturally isomorphic to its contragredient. Let $\alpha \in K$ with $N_{K/k}(\alpha) = -1$. Such an α exists from the assumption that K/k_0 is cyclic Galois (if K/k_0 is ramified, it follows that K/k_0 is cyclic Galois if and only if k_0 contains a square root of unity). Define a $\mathbb{C}$-antilinear map $r\colon V \longrightarrow V$ as

$$r(\widetilde{f})(g)(v) = \overline{\widetilde{f}(g)(\alpha v)}.$$

It can be easily checked using elements of the form (0.2.2) that r is well defined and G_k-equivariant. Hence V is self-contragredient. Moreover, $r(V^{\pm}) = V^{\mp}$ and hence V^+, V^- are dual to each other as $G^0_{k_0}$ modules.

Denote $\mathrm{Tr}(\widetilde{\theta}_{10}(\cdot)|V^{K_t})$ by $\widetilde{\chi}(\cdot)$. Let $V_{\pm}^{K_t} = V^{\pm} \cap V^{K_t}$ and let $\chi_{\pm}(g)$ be $\mathrm{Tr}(\theta_{10}(g)|V_{\pm}^{K_t})$. Then via the map r, $V_+^{K_t}$ and $V_-^{K_t}$ are dual to each other as $G^0_{k_0} \cap K_0$ modules and hence we have $\chi_+(g) = \overline{\chi}_-(g)$. Now we have

$$\begin{aligned}\widetilde{\Theta}_{10}(X_{K_t\sigma gK_t}) &= \chi_+(g) - \chi_-(g)\\ &= \chi_+(g) - \bar{\chi}_+(g),\end{aligned}$$

which implies $\widetilde{\Theta}_{10}(X_{(g,t)})$ is pure imaginary.

1.2.6. Conclusion. Combining §1.2.4 and §1.2.5, we see $\chi_{\widetilde{\theta}_{10}}$ is both real and pure imaginary valued on Ω. Then the character distribution $\widetilde{\Theta}_{10}$ represented by $\chi_{\widetilde{\theta}_{10}}$ vanishes on Ω, that is, $\widetilde{\Theta}_{10}|\mathfrak{D} = 0$ where $\mathfrak{D} = C_c^\infty(\Omega)$. Hence Proposition 1 and Theorem 1 are proved.

1.2.7. Remark: Finite field case. In this case, since the norm map $k^\times \to k_0^\times$ is surjective, $H = G$ and θ^0_{10} extends irreducibly to θ_{10}. We can also prove that θ_{10} does not have a descent. More precisely, we can directly compute $\mathrm{Tr}(\widetilde{\theta}_{10}(\sigma, 1)) = 0$. On the other hand, since $\mathrm{Tr}(\pi(1)) = \dim(\pi) > 0$ for any $\pi \in GSp_4(k_0)^\frown$, it implies that θ_{10} does not have a descent. It can be also proved that θ_{10} does not have a Shintani ascent [Gy].

II. L-factor of θ_{10}

§II.1. Preliminaries on L-functions. In §II.1.A and §II.1.B, we will introduce generalized Whittaker models and L-functions for representations of GSp_4 defined by the second author [PS, PSS]. All the results in these sections can be found in [PS] and [PSS]. We also refer most of notation and definitions to [PS], [PSS] and we will repeat only what we need here.

§II.1.A. Generalized Whittaker model of θ_{10}. We have the following subgroups of G:

$$\begin{aligned} S &= \left\{ u(s) = \begin{pmatrix} I_2 & s \\ 0 & I_2 \end{pmatrix} \;\middle|\; \begin{matrix} s \in M(2,k) \\ s = {}^t s \end{matrix} \right\}, \\ M &= \left\{ \begin{pmatrix} A & 0 \\ 0 & x\,{}^tA^{-1} \end{pmatrix} \;\middle|\; \begin{matrix} A \in GL(2,k) \\ x \in k^\times \end{matrix} \right\}, \qquad (2.1.1) \\ P &= MS. \end{aligned}$$

Then P is a parabolic subgroup of G with reductive part M and unipotent radical S. Since S is abelian, the application $u(s) \to \psi(\mathrm{tr}(\phi s))$, where ${}^t\phi = \phi \in M(2,k)$, defines a character ψ_ϕ of S. All characters of S can be obtained in this way. In particular, if $\phi \in GL(2,k)$, we call ψ_ϕ nondegenerate.

Let ψ_ϕ be nondegenerate and let $\widetilde{D}$ be the stabilizer of ψ_ϕ in M. There exists a unique semisimple algebra K over k, with $(K:k) = 2$ such that $\widetilde{D} = K^\times \cdot Z_2$. Denote by D the connected component of $\widetilde{D}$; then $D \simeq K^\times$. K is either a quadratic extension of k, $K = K_1 = k(\sqrt{\rho})$ with $\rho \notin (k^\times)^2$ or $K = K_2 = k \oplus k$ with k embedded diagonally. We take in the first case

$$\phi = \phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -\rho \end{pmatrix}$$

and in the second case

$$\phi = \phi_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

In both cases the isomorphism $K_i^\times \simeq D_i = D$ is given by

$$r \to \begin{pmatrix} f_i(r) & 0 \\ 0 & \det f_i(r) \cdot {}^t f_i(r)^{-1} \end{pmatrix}$$

where f_i is the following embedding of $K_i^\times$ in $GL(2,k)$:

$$f_1(x + y\sqrt{\rho}) = \begin{pmatrix} x & y\rho \\ y & x \end{pmatrix}, \quad f_2(x,y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Denote $\psi_i = \psi_{\phi_i}$, $i = 1, 2$ and let $R_i = D_i S$. Each character ν of $K_i^\times$ defines, together with ψ_i, a character of R_i, which we denote by $\nu \otimes \psi_i$.

2.1.2. Theorem ([PS]): *Let k be a local field and $i = 1, 2$. Let π be an irreducible admissible pre-unitary representation of G. Then up to a scalar there exists at most one nonzero linear functional $l: V_\pi \to \mathbb{C}$ satisfying*

$$l(\pi(r)v) = (\nu \otimes \psi_i)(r) l(v), \quad \textit{for } r \in R_i,\ v \in V_\pi. \tag{2.1.3}$$

A functional satisfying (2.1.3) is called a **generalized Whittaker functional** with respect to (ν, ψ_i).

Let π have a nonzero generalized Whittaker functional l with respect to (ν, ψ_i) and let $v \in V_\pi$. Let w_v be the function on G defined by

$$w_v(g) = l(\pi(g)v).$$

Then we note that $w_v(rg) = (\nu \otimes \psi_i)(r) w_v(g)$ for $r \in R_i$, $g \in G$; w_v is called the **generalized Whittaker function of** v. Denote by W_π^{ν,ψ_i} the space of all these functions. G acts on W_π^{ν,ψ_i} by right translations, and the representation of G in W_π^{ν,ψ_i} is equivalent to π. W_π^{ν,ψ_i} is called the **generalized Whittaker model** of π with respect to (ν,ψ_i).

§II.1.B. Definition of the L-function. Denote by $^-$ the unique nontrivial k automorphism of K_i. Put $\mathrm{Tr}=\mathrm{Tr}_{K_i/k}$ and $N=N_{K_i/k}$. Let $V_i=K_i\oplus K_i$. We write vectors in V_i in a row form. Define $\tau_i(x,y)=\frac{1}{2}\mathrm{Tr}(x_1\bar{y}_2-x_2\bar{y}_1)$ for $x=(x_1,x_2)$, $y=(y_1,y_2)$ in V_i. Then τ_i is a nondegenerate antisymmetric form on V_i. Regard V_i as a 4-dimensional vector space over k. Let

(2.1.4)
$$GSp(\tau_i)=\left\{g\in GL(4,k)|\tau_i(xg,yg)=\eta(g)\tau_i(x,y);\ x,y\in V_i,\ \eta(g)\in k^\times\right\}.$$

Consider the group $G_i=\{g\in GL(2,K_i)|\det g\in k^\times\}$. G_i acts on V_i from the right, preserving τ_i up to a scalar, and so we get a natural embedding $G_i\subset GSp(\tau_i)$. Let $N_i=\{u(s)\in S|\mathrm{Tr}(\phi_i s)=0\}$. There exists an isomorphism $\varphi_i\colon GSp_4(\tau_i)\to G$ such that $\varphi_i(G_i)\cap R_i=D_iN_i$. Let

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in G_i$$

(for $i=1$, $a=a_1+a_2\sqrt{\rho}$ etc., and for $i=2$, $a=(a_1,a_2)$ etc.). Then

$$(2.1.5)\qquad \varphi_1\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a_1 & a_2\rho & b_1 & b_2\\ a_2 & a_1 & b_2 & \frac{b_1}{\rho}\\ c_1 & c_2\rho & d_1 & d_2\\ c_2\rho & c_1\rho & d_2\rho & d_1\end{pmatrix},$$

$$\varphi_2\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a_1 & 0 & b_1 & 0\\ 0 & a_2 & 0 & b_2\\ c_1 & 0 & d_1 & 0\\ 0 & c_2 & 0 & d_2\end{pmatrix}.$$

From now on, we shall identify G_i with $\varphi_i(G_i)$. Note that $\varphi_i(U_i)=N_i$ where

$$U_i=\left\{\begin{pmatrix} 1 & z\\ 0 & 1\end{pmatrix}\mathrm{St}\, z\in K_i\right\}.$$

Let π have a unique generalized Whittaker functional with respect to (ν,ψ_i). Let μ be a character of $k^\times$. Define for $\Phi\in S(V_i)$, the Schwartz–Bruhat functions on V_i, $w\in W_\pi^{\nu,\psi_i}$, $s\in\mathbb{C}$,

$$(2.1.6)\qquad L_\nu^i(w,\Phi,\mu,s)=\int_{N_i\backslash G_i} w(g)\Phi((0,1)g)\mu(\det g)|\det g|^{s+\frac{1}{2}}dg.$$

The integral in (2.1.6) converges in a half plane $\mathrm{Re}(s) > s_0$ and has a meromorphic continuation to the whole plane [PS]. There is an Euler factor $L^i_\nu(\pi,\mu,s)$ such that $\frac{L^i_\nu(W,\phi,\mu,s)}{L^i_\nu(\pi,\mu,s)}$ is entire for all W, Φ. It is easy to see that for a fixed i, $L^i_\nu(\pi,\mu,s)$ does not depend on ψ. In many cases, $L^i_\nu(\pi,\mu,s)$ does not depend on i and ν. $L^i_\nu(\pi,\mu,s)$ is called the L-**factor** associated to (π,μ). From now on, we drop $_\nu$ and ^{i} for simplicity of notation.

Let $S_0(K\oplus K) = \{\Phi \in S(K\oplus K) | \Phi((0,0)) = 0\}$. Then we divide the poles of $L(\pi,\mu,s)$ into two types. We call a pole of $L(\pi,\mu,s)$ **regular** if it is a pole of some $L(w,\Phi,\mu,s)$ with $\Phi\in S_0(K\oplus K)$. A pole of $L(\pi,\mu,s)$ is called **exceptional** if it is not a pole of any $L(w,\Phi,\mu,s)$ with $\Phi \in S_0(K\oplus K)$.

§II.2. L-FUNCTION OF θ_{10} ON G. In this section, we assume that K/k is an unramified quadratic extension and let θ_{10} be the representation associated to K/k. This is the unipotent supercuspidal representation extending the one on Sp_4 [As, PS].

Let l be the linear functional defined on V as

$$l(\widetilde{f}) = \widetilde{f}(1)(1,\sqrt{\rho}) \quad \text{for } \widetilde{f}\in V. \tag{2.2.1}$$

Then l is a generalized Whittaker functional with respect to (ν,ψ_1), where $\nu = 1$ and ψ_1 is as in §1.1. We can define the generalized Whittaker model as

$$w_{\widetilde{f}}(g) = \widetilde{f}(g)(1,\sqrt{\rho}). \tag{2.2.2}$$

THEOREM 2: *If μ is unramified, we have*

$$L(\theta_{10},\mu,s) = \frac{1}{1-\mu(\omega_k^2)q^{-2s-1}} = \frac{1}{(1-\mu(\omega_k)q^{-s-\frac12})}\frac{1}{(1+\mu(\omega_k)q^{-s-\frac12})}.$$

If μ is ramified,

$$L(\theta_{10},\mu,s) = 1.$$

Since θ_{10} is not generic, by Theorem 2.3 and its Corollary in [PSS], we have only exceptional poles, that is, poles are coming from $L(w,\Phi,\mu,s)$ for $\Phi\notin S_0(K\oplus K)$. Hence we may assume that $\Phi = X_{\mathfrak{p}^n_K\times\mathfrak{p}^m_K}$. We may further assume that $m=n$, since if $m<n$, $X_{\mathfrak{p}^n_K\times\mathfrak{p}^m_K} = \sum X_{(0,x)+\mathfrak{p}^n_K\times\mathfrak{p}^n_K}$ with $\sum$ over $x\in\mathfrak{p}^m \pmod{\mathfrak{p}^n}$ and unless $x\equiv 0$, $X_{(0,x)+\mathfrak{p}^n_K\times\mathfrak{p}^n_K}\in S_0(K\oplus K)$ and they do not contribute to poles. Here X_Z with $Z\subset K\oplus K$ is a characteristic function supported on Z. Combining this with (0.2.8), in computing $L(\theta_{10},\mu,s)$, it is enough to consider $L(w_{\widetilde{f}},\Phi,\mu,s)$ with

$$\widetilde{f} = \widetilde{f}_{(\lambda,v,m)},\quad \Phi = X_{\mathfrak{p}^n_K\times\mathfrak{p}^n_K}.$$

We compute only the case $\lambda = 1$. The other case is similar. Note that the Borel subgroup $\mathcal{B}$ of G_i is given by

$$\mathcal{B} = \left\{ B = \begin{pmatrix} a & \\ & \overline{a}b \end{pmatrix} \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix} \;\middle|\; a \in K^\times,\ b \in k^\times,\ c \in K \right\}.$$

We have the following notations for the computation:

(1) $v = (x, y)$, $\quad f = f_{(v,m)}$,

(2) $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1/\rho & & \\ & & 1 & \\ & & & \rho \end{pmatrix} \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} = \varrho \cdot W,$

(3) $\operatorname{supp}(f) = O(\mathfrak{f}_K)\text{-orbit of } \left(v + \mathfrak{p}_K^m \times \mathfrak{p}_K^m\right)$

$$= \overset{\circ}{\bigcup_i} \left(u_i(x,y) + \mathfrak{p}_K^m \times \mathfrak{p}_K^m \cup \overline{u}_i(\overline{x},\overline{y}) + \mathfrak{p}_K^m \times \mathfrak{p}_K^m\right)$$

for some finite number of u_i's in $\ker(N_{K/k})$.

In (2),

$$w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the longest Weyl element in GL_2 and $w_0 = \varrho \cdot W$ is its expression as an element of $GSp(4)$ where

$$\varrho = \begin{pmatrix} 1 & & & \\ & 1/\rho & & \\ & & 1 & \\ & & & \rho \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}.$$

We first consider the cases where μ is ramified. In these cases, we can use Bruhat decomposition for computation.

$$L(w_{\widetilde{f}}, \Phi, \mu, s) = \int_{N_1 \backslash G_1} w_{\widetilde{f}}(g) \Phi((0,1)g) \mu(\det g) |\det g|_k^{s+\frac{1}{2}}\, dg$$

$$\overset{(1)}{=} \int_{\mathcal{B}} \widetilde{f}(w_0 B)(1, \sqrt{\rho}) \Phi((1,0) w_0 B) \mu(\det B) |\det B|_k^{s+\frac{1}{2}}\, dB$$

$$= \iiint_{\substack{b \in \operatorname{Im}(N_{K/k}) \\ a \in K^\times,\ c \in K}} \left|\tfrac{a}{\overline{a}b}\right|_K^{\frac{1}{2}} \widetilde{f}\left(\begin{pmatrix} \overline{a}b & \\ & a \end{pmatrix} w_0 \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix} \right) (1, \sqrt{\rho}) \cdot \Phi(a, ac) \mu(a\overline{a}b)\, |a\overline{a}b|_k^{s+\frac{1}{2}}\; d_K^\times a\; d_k^\times b\; d_K^+ c$$

$$= \iiint_{\substack{a \in \mathfrak{p}_K^n,\ c \in a^{-1}\mathfrak{p}_K^n, \\ b \in \operatorname{Im}(N_{K/k}),\ N_{K/k}(\tilde{b}) = b}} \frac{1}{|b|_K^{\frac{1}{2}}} |b|_k \widetilde{f}\left(W \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix} \right) ((1, \sqrt{\rho}) \tilde{b} \varrho) \cdot \Phi(a, ac) \mu(a\overline{a}b)\, |a\overline{a}b|_k^{s+\frac{1}{2}}\; d_K^\times a\; d_k^\times b\; d_K^+ c$$

$$\overset{(2)}{=}\gamma\cdot\iiiint\limits_{\substack{a\in\mathfrak{p}_K^n,\ c\in a^{-1}\mathfrak{p}_K^n,\\ \tilde b\in K^\times,\ z\in\mathrm{supp}(f)}} \begin{array}{c}\psi(zc^t\overline{z})f(z)\psi((1,\frac{1}{\sqrt{\rho}})\tilde b^t\overline{z})\\ \Phi(a,ac)\mu(a\overline{a}\tilde b\overline{\tilde b})|a\overline{a}\tilde b\overline{\tilde b}|_k^{s+\frac12}\ dz\ d_K^\times a\ d_k^\times b\ d_K^+c\end{array}$$

$$\overset{(3)}{=}\gamma\cdot\iiiint\limits_{\substack{b\in\mathrm{Im}(N_{K/k}),\ N_{K/k}(\tilde b)=b,\\ z\in\mathrm{supp}(f)\\ a\in\mathfrak{p}_K^n,\ c\in a^{-1}\mathfrak{p}_K^n,\ \psi(zc^t\overline{z})=1}} \cdots\cdots\cdots$$

$$\overset{(4)}{=}\gamma\cdot\iint\limits_{\substack{\tilde b\in K^\times,\\ \tilde b\overline{\tilde b}=b\\ z\in\mathrm{supp}(f)}}\ \iint\limits_{\substack{a\in\mathfrak{p}_K^n,\\ c\in a^{-1}\mathfrak{p}_K^n,\\ \psi(zc^t\overline{z})=1}} \begin{array}{c}\psi(zc^t\overline{z})\Phi(a,ac)\mu(a\overline{a})|a\overline{a}|^{s+\frac12}\ d_K^\times a d_K^+c\\ \cdot f(z)\psi((1,\frac{1}{\sqrt{\rho}})\tilde b^t\overline{z})\mu(b)|b|_k^{s+\frac12}\ d^\times b\ d^+z\end{array}$$

$$=\gamma\cdot\iint\limits_{\substack{a\in\mathfrak{p}^n\\ c\in a^{-1}\mathfrak{p}_K^n,\psi(zc^t\overline{z})=1}}\mu(a\overline{a})|a\overline{a}|_k^{s+\frac12}\ d_K^\times a d_K^+c$$

$$\cdot\iint\limits_{\substack{\tilde b\in K^\times,\ \tilde b\overline{\tilde b}=b\\ z\in\mathrm{supp}(f)}} f(z)\psi((1,\frac{1}{\sqrt{\rho}})\tilde b^t\overline{z})\mu(b)|b|_k^{s+\frac12}\ d^\times b\ d^+z.$$

(1) follows from the decomposition $G_1=\mathcal{B}\cup N_1w_0\mathcal{B}$ and that $N_1w_0\mathcal{B}$ is a big cell. For (2),

$$\tilde f\left(W\begin{pmatrix}1 & c\\ & 1\end{pmatrix}\right)((1,\sqrt{\rho})\tilde b\varrho)=\gamma\cdot\int_{z\in\mathrm{supp}(f)\subset K\oplus K}\psi(zc^t\overline{z})f(z)\psi((1,\frac{1}{\sqrt{\rho}})\tilde b^t\overline{z})\ dz,$$

where $\gamma=\zeta^2c^2$ is a constant which comes from the action of $\theta_{10}(W)$ in (0.2.3). (3) follows from the observation that if $\psi(zc^t\overline{z})\not\equiv 1$, $\psi(zc^t\overline{z})$ becomes a nontrivial additive character for $c\in a^{-1}\mathfrak{p}_K^n$ and the integral over c gives 0. When μ is ramified, we have

$$\iint\limits_{\substack{a\in\mathfrak{p}^n\\ c\in a^{-1}\mathfrak{p}_K^n,\ \psi(zc^t\overline{z})=1}}\mu(a\overline{a})|a\overline{a}|_k^{s+\frac12}\ d_K^\times a d_K^+c=0.$$

Hence

$$L(\theta_{10},\mu,s)=1.$$

Now, we consider the case when μ is unramified. We will use Iwasawa decomposition, that is,

$$G_1=N_1AK_0$$

where

$$A = \left\{ \begin{pmatrix} \omega^i & 0 \\ 0 & \omega^j \end{pmatrix} \;\middle|\; i,j \in \mathbb{Z} \right\}$$

and K_0 is the maximal compact subgroup of G_1.

$$\begin{aligned}
L(w_{\widetilde{f}}, \Phi, \mu, s) &= \int_{N_1\backslash G_1} w_{\widetilde{f}}(g)\Phi((0,1)g)\mu(\det g)|\det g|^{s+\frac{1}{2}} dg \\
&= \sum_{i,j\in\mathbb{Z}} \int_{K_0} w_{\widetilde{f}}\left(\begin{pmatrix} \omega^i & 0 \\ 0 & \omega^j \end{pmatrix} k\right) \Phi((1,0)\begin{pmatrix} \omega^i & 0 \\ 0 & \omega^j \end{pmatrix} k) \\
&\qquad\qquad \cdot \mu(\det k)\mu(\omega^{i+j})|\omega^{i+j}|^{s+\frac{1}{2}}\, dk \\
&\overset{(1)}{=} \sum_{i,j\in\mathbb{Z}} \int_{K_0} w_{\widetilde{f}}\left(\begin{pmatrix} \omega^{i-j} & 0 \\ 0 & 1 \end{pmatrix} k\right) \Phi((1,0)\begin{pmatrix} \omega^i & 0 \\ 0 & \omega^j \end{pmatrix} k)\mu(\omega^{i+j})|\omega^{i+j}|^{s+\frac{1}{2}}\, dk \\
&\overset{(2)}{=} \sum_{\substack{i,j\in\mathbb{Z} \\ j\geq n}} \int_{K_0} w_{\widetilde{f}}\left(\begin{pmatrix} \omega^{i-j} & 0 \\ 0 & 1 \end{pmatrix} k\right) \mu(\omega^{i+j})|\omega^{i+j}|^{s+\frac{1}{2}}\, dk \\
&\overset{(3)}{=} \sum_{\substack{i,j\in\mathbb{Z} \\ j\geq n}} \int_{K_0} w_{\widetilde{f}}\left(\begin{pmatrix} \omega^{i} & 0 \\ 0 & 1 \end{pmatrix} k\right) \mu(\omega^{i+2j})|\omega^{i+2j}|^{s+\frac{1}{2}}\, dk \\
&\overset{(4)}{=} \sum_{j\in\mathbb{Z},\, j\geq n} \mu(\omega^{2j})|\omega^{2j}|^{s+\frac{1}{2}} \cdot \sum_{i\in\mathbb{Z}} \int_{K_0} w_{\widetilde{f}}\left(\begin{pmatrix} \omega^{i} & 0 \\ 0 & 1 \end{pmatrix} k\right) \mu(\omega^{i})|\omega^{i}|^{s+\frac{1}{2}}\, dk \\
&\overset{(5)}{=} (\mu(\omega^{2n})|\omega^{2n}|^{s+\frac{1}{2}} + \mu(\omega^{2n+2})|\omega^{2n+2}|^{s+\frac{1}{2}} \\
&\qquad\qquad + \mu(\omega^{2n+4})|\omega^{2n+4}|^{s+\frac{1}{2}} + \cdots) \cdot \sum_{i\in\mathbb{Z}} J(i,s) \\
&\qquad \left(\text{Here, } J(i,s) = \int_{K_0} w_{\widetilde{f}}\left(\begin{pmatrix} \omega^{i} & 0 \\ 0 & 1 \end{pmatrix} k\right) \mu(\omega^{i})|\omega^{i}|^{s+\frac{1}{2}}\, dk\right) \\
&\overset{(6)}{=} \mu(\omega^{2n})|\omega^{2n}|^{s+\frac{1}{2}} \cdot \frac{1}{1-\mu(\omega^2)|\omega^2|^{s+\frac{1}{2}}} \cdot \sum_{i\in\mathbb{Z}} J(i,s).
\end{aligned}$$

Since Z_G acts trivially, (1) follows. Since

$$(1,0)\begin{pmatrix} \omega^i & 0 \\ 0 & \omega^j \end{pmatrix} k \in \operatorname{supp}(\Phi)$$

if and only if $j \geq n$, (2) follows. Since $\widetilde{f}$ is compactly supported,

$$w_{\widetilde{f}}\begin{pmatrix} w^i & 0 \\ 0 & 1 \end{pmatrix} k \neq 0$$

for only finitely many $i \in \mathbb{Z}$. Hence, $\sum_i J(i,s)$ is holomorphic in s. Hence $L(\theta_{10}, \mu, s)^{-1}$ divides $1 - \mu(\omega_k^2)|\omega|^{2s+1}$.

Especially, if we take $\Phi = X_{\mathcal{O}_K \times \mathcal{O}_K}$ and $\widetilde{f} = \sum_v \widetilde{f}_{(1,v,1)}$ with

$$v \in \{(\alpha, \alpha\sqrt{\rho}) |\ \alpha \in \mathcal{O}_K^\times \pmod{1+\mathfrak{p}_K}\},$$

then we can directly compute

$$L(w_{\tilde{f}}, \Phi, \mu, s) = \gamma \cdot \mathrm{vol}(K_1)\sharp(K_0/I_0)^{-1}(q_K^2+1)\frac{1}{1-\mu(\omega_k^2)|\omega_k|^{2s+1}},$$

where K_1 is the subgroup of K_0 which is trivial mod $\mathfrak{p}_K$ and I_0 is an Iwahori subgroup of G_1, that is, the subgroup of K_0 projected to upper triangular matrices mod $\mathfrak{p}_K$. Here, q_K denotes the cardinality of the residue field of K. Hence,

$$\begin{aligned} L(\theta_{10}, \mu, s) &= \frac{1}{1-\mu(\omega_k^2)|\omega_k|^{2s+1}} = \frac{1}{1-\mu(\omega_k^2)q^{-2s-1}} \\ &= \frac{1}{(1-\mu(\omega_k)q^{-s-\frac{1}{2}})}\frac{1}{(1+\mu(\omega_k)q^{-s-\frac{1}{2}})}. \end{aligned}$$ ∎

Corollary: *Let θ_0 be a representation of $GSp_4(k)$ which is a Howe-lift of the trivial character of GO_2 (here, GO_2 is associated to $(K, N_{K/k})$ as before). Then*

$$L(\theta_0, \mu, s) = L(\theta_{10}, \mu, s).$$

Note that the generalized Whittaker model of θ_0 can be realized in a similar way as θ_{10} (see (2.2.1)–(2.2.2)). Moreover, computation of L-factors is the same except for sign changes. We will give another proof using global L-functions in Section III below.

III. Langlands correspondence

In this section, we assume k/k_0 is a (unramified or ramified) quadratic extension and K is the unramified quadratic extension of k. Let $\theta_{10}^{k_0}$ and θ_{10}^{k} denote representations of $GSp_4(k_0)$ and $GSp_4(k)$, respectively, constructed as in §0.2. Let W_{k_0} and W_k be Weil groups for k_0 and k, respectively. Consider the following Langlands liftings:

(**)
$$\begin{array}{ccccc} \theta_{10}^{k_0} \in GSp_4(k_0)^\frown & \xrightarrow{L_1} & GL_4(k_0)^\frown & \xrightarrow{L} & \widehat{W}_{k_0} \\ \Big\downarrow L_3 & & \Big\downarrow L_4 & & \Big\downarrow \\ \theta_{10}^{k} \in GSp_4(k)^\frown & \xrightarrow{L_2} & GL_4(k)^\frown & \xrightarrow{L} & \widehat{W}_{k} \end{array}$$

where $^\frown$ means the set of admissible irreducible representations. Here, down arrows come from $W_k \hookrightarrow W_{k_0}$ by the restriction map and right arrows come from

$GSp_4(\mathbb{C}) \hookrightarrow GL_4(\mathbb{C})$. For any $\tau \in G^{\wedge}$, we denote its Langlands parameter by ${}^L\tau$.

We first consider the behavior of θ_{10} under these liftings.

§1. COMPUTING $L_2(\theta_{10}^k)$, $L_1(\theta_{10}^{k_0})$. It is known that θ_{10} has the same L-factor as the following representation π_K of $GL_4(k)$ [PS]: Let σ_0 be the special representation of $GL_2(k)$ which is the uniquely defined subrepresentation of $\mathrm{ind}_{B_0}^{GL_2} \alpha$, where B_0 is the standard Borel subgroup of GL_2 and

$$\alpha \begin{pmatrix} b_1 & x \\ 0 & b_2 \end{pmatrix} = \left| \frac{b_1}{b_2} \right|^{\frac{1}{2}}.$$

Denote by $P_{2,2}$ the parabolic subgroup

$$P_{2,2} = \left\{ \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} \;\middle|\; g_{ij} \in M_2(k) \right\} \cap GL_4(k).$$

Then $\pi_K = \mathrm{ind}_{P_{2,2}}^{GL_4}(\sigma_0 \otimes (\sigma_0 \otimes \beta_K(\det g)))$, where β_K is the character of $k^\times$ given by

$$\beta_K(x) = \begin{cases} 1 & \text{if } x \in N_{K/k}(K^\times), \\ -1 & \text{if } x \notin N_{K/k}(K^\times). \end{cases}$$

One can prove that π_K is the only generic unitary representation of $GL_4(k)$ with the same L-function as θ_{10}. One can also prove, by using other properties of the conjectured Langlands correspondence, that for any cuspidal representation τ of $PGSp_4(k)$, the representation $L_2(\tau)$ of $GL_4(k)$ must be generic and unitary. Hence if $L_2(\theta_{10}^k)$ exists, it should be equal to π_K [PS]. Similarly, let K_0 be the quadratic unramified extension of k_0. Then, we have $L_1(\theta_{10}^{k_0}) = \pi_{K_0}$ where π_{K_0} is the representation of $GL_4(k_0)$ constructed similarly as π_K. Now, we are ready to give another proof for the Corollary of Theorem 2.

Another Proof for the Corollary of Theorem 2: Let F and F' be number fields with $F \subset F'$ and $(F' : F) = 2$. Let $\mathbb{A}_F = \prod F_\mathfrak{p}$ and $\mathbb{A}_{F'} = \prod F'_{\mathfrak{p}'}$. Assume that $F_{\mathfrak{p}_0} = k$ and $F'_{\mathfrak{p}'_0} = K$ for some places $\mathfrak{p}_0$ and $\mathfrak{p}'_0$. Let $GO_2(\mathbb{A}_F)$ be defined as in (0.2.9). Let $\mathrm{sgn}_1 = \prod_\mathfrak{p} \mathrm{sgn}_\mathfrak{p}^{\delta_\mathfrak{p}^1}$ be a character of $GO_2(\mathbb{A}_F)$ as in (0.2.9) such that $\delta_\mathfrak{p}^1 = 1$ at more than three finite places, say, $\mathfrak{p}_0, \mathfrak{p}_1, \ldots$ with $O_2(F_{\mathfrak{p}_i})$ compact. In particular, note that we assume $\delta_{\mathfrak{p}_0}^1 = 1$. Let $\mathrm{sgn}_2 = \prod_\mathfrak{p} \mathrm{sgn}_\mathfrak{p}^{\delta_\mathfrak{p}^2}$ be a character of $GO_2(\mathbb{A}_F)$ with

$$\delta_\mathfrak{p}^2 = \begin{cases} 0 & \text{if } \mathfrak{p} = \mathfrak{p}_0, \mathfrak{p}_1, \\ \delta_\mathfrak{p}^1 & \text{otherwise.} \end{cases}$$

Let Θ_i be the Howe-lift of sgn_i with $i = 1, 2$. Then

$$\Theta_i = \prod_{\mathfrak{p}} \Theta_{\mathfrak{p}}(\mathrm{sgn}_{\mathfrak{p}}^{\delta_{\mathfrak{p}}^i})$$

where $\Theta_{\mathfrak{p}}(\mathrm{sgn}_{\mathfrak{p}}^{\delta_{\mathfrak{p}}^i})$ is the Howe-lift of $\mathrm{sgn}_{\mathfrak{p}}^{\delta_{\mathfrak{p}}^i}$ at $\mathfrak{p}$. Especially, when $\delta_{\mathfrak{p}}^i = 1$, $\Theta_{\mathfrak{p}}(\mathrm{sgn}_{\mathfrak{p}}^{\delta_{\mathfrak{p}}^i})$ is θ_{10} for $GSp_4(F_{\mathfrak{p}})$. Considering the functoriality

$$GSp_4(\mathbb{A}_F)^\frown \xrightarrow{L_2} GL_4(\mathbb{A}_F)^\frown$$

as in (**), we have

$$L_2(\Theta_i) = \prod_{\mathfrak{p}} L_2(\Theta_{\mathfrak{p}}(\mathrm{sgn}_{\mathfrak{p}}^{\delta_{\mathfrak{p}}^i})).$$

Since $L_2(\Theta_1)$ and $L_2(\Theta_2)$ are different at two places, from Strong multiplicity one on $GL_4(\mathbb{A}_F)$, we have $L_2(\Theta_1) \simeq L_2(\Theta_2)$ and this implies L-factors of $L_2(\Theta_1)$ and $L_2(\Theta_2)$ are the same. Hence the L-factors of Θ_1 and Θ_2 are also the same. Thus

$$\prod_{\mathfrak{p}} L_{\mathfrak{p}}(\Theta_{\mathfrak{p}}(\mathrm{sgn}_{\mathfrak{p}}^{\delta_{\mathfrak{p}}^1}), \mu_{\mathfrak{p}}, s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\Theta_{\mathfrak{p}}(\mathrm{sgn}_{\mathfrak{p}}^{\delta_{\mathfrak{p}}^2}), \mu_{\mathfrak{p}}, s),$$

where $L_{\mathfrak{p}}$ denotes the local L-factor. By cancelling same factors, we have

$$L_{\mathfrak{p}_0}(\theta_{10}^k, \mu_{\mathfrak{p}_0}, s) L_{\mathfrak{p}_1}(\theta_{10}^{F_{\mathfrak{p}_1}}, \mu_{\mathfrak{p}_1}, s) = L_{\mathfrak{p}_0}(\theta_0^k, \mu_{\mathfrak{p}_0}, s) L_{\mathfrak{p}_1}(\theta_0^{F_{\mathfrak{p}_1}}, \mu_{\mathfrak{p}_1}, s)$$

for all μ and s. Hence $L(\theta_{10}^k, \mu_{\mathfrak{p}_0}, s) = L(\theta_0^k, \mu_{\mathfrak{p}_0}, s)$ and the corollary follows. ∎

§2. Computing ${}^L\pi_K$, ${}^L\pi_{K_0}$. We use the results in [De] to compute ${}^L\pi_K$ and ${}^L\pi_{K_0}$. We still assume that K/k is unramified. Let us use the same notation β_K for a quadratic character of W_k corresponding to β_K above. Then we have

$$\beta_K \colon W_k \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2 \simeq \{\pm 1\}.$$

Let τ_k and σ_k be the 2-dimensional representation of W_k given by

$$\begin{pmatrix} 1 & 0 \\ 0 & \beta_K \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} |\,|^{\frac{1}{2}} & 0 \\ 0 & |\,|^{-\frac{1}{2}} \end{pmatrix}.$$

Then

$${}^L\pi_K = \tau_k \otimes \sigma_k = \sigma_k \oplus (\sigma_k \otimes \beta_K).$$

Moreover, since τ_k is orthogonal and σ_k is symplectic, $\tau_k \otimes \sigma_k$ is a representation into $Sp_4(\mathbb{C})$. Hence

$${}^L\theta_{10}^k = \tau_k \otimes \sigma_k = \sigma_k \oplus (\sigma_k \otimes \beta_K).$$

Case 1. k/k_0 is ramified.

Let τ_{k_0}, σ_{k_0} and β_{K_0} be the representations of W_{k_0} constructed similarly. Then note that $\beta_K|k_0 = \beta_{K_0}$ and

$$\begin{aligned} {}^L\pi_{K_0} &= \tau_{k_0} \otimes \sigma_{k_0} = \sigma_{k_0} \oplus (\sigma_{k_0} \otimes \beta_{K_0}), \\ {}^L\theta_{10}^{k_0} &= \tau_{k_0} \otimes \sigma_{k_0} = \sigma_{k_0} \oplus (\sigma_{k_0} \otimes \beta_{K_0}). \end{aligned}$$

Note $\tau_{k_0} \otimes \sigma_{k_0}|W_k = \tau_k \otimes \sigma_k$. Then from the commutativity of the second square in $(**)$, we have

$$L_4(\pi_{K_0}) = \pi_K.$$

Then combining this with the conjectural commutativity of the first square in $(**)$, we can conclude that

$$L_3(\theta_{10}^{k_0}) = L_2^{-1}(L_4 \circ L_1(\theta_{10}^{k_0})) = L_2^{-1}(L_4(\pi_{K_0})) = L_2^{-1}(\pi_K) = \theta_{10}^k.$$

Moreover, we note that

$$L(\theta_{10}^k, \mu \circ N_{k/k_0}, s) = \prod_{\zeta \in \Xi} L(\theta_{10}^{k_0}, \zeta \otimes \mu, s) = L(\theta_{10}^{k_0}, \mu, s),$$

where Ξ is the set of characters for k/k_0.

Case 2. k/k_0 is unramified.

In this case, we have $k = K_0$. Let γ be a character of W_{k_0} such that $\gamma|W_k = \beta_K$. Then γ is of order 4 and we have

$$\begin{array}{llcccccc} \beta_K\colon & & W_k & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \simeq \{\pm 1\} \\ & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\ \gamma\colon & & W_{k_0} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/4 \simeq \langle \sqrt[4]{1} \rangle \end{array}$$

Then the representation

$$\tau_k = \begin{pmatrix} 1 & 0 \\ 0 & \beta_K \end{pmatrix}$$

of W_k can be extended to a representation τ of W_{k_0}:

$$\tau \in \left\{ \begin{pmatrix} 1 & \\ & \gamma \end{pmatrix}, \begin{pmatrix} \beta_{K_0} & \\ & \gamma \end{pmatrix}, \begin{pmatrix} 1 & \\ & \beta_{K_0}\gamma \end{pmatrix}, \begin{pmatrix} \beta_{K_0} & \\ & \beta_{K_0}\gamma \end{pmatrix} \right\}.$$

However, none of them is orthogonal. Moreover, for any τ as above, $\tau \otimes \sigma_{k_0}$ extends $\tau_k \otimes \sigma_k$ while none of them respects symplectic form. Hence, θ_{10}^k cannot be in the image of L_3.

§3. REMARKS. In general, if k/k_0 is a cyclic extension, L_4 should coincide with Shintani's lifting [AC, La] and its images are $\mathrm{Gal}(k/k_0)$ invariant representations. However, for homomorphisms $W_k \longrightarrow GSp_4(\mathbb{C})$, $\mathrm{Gal}(k/k_0)$-invariance is not sufficient to extend it to $W_{k_0} \longrightarrow GSp_4(\mathbb{C})$, while it is enough for the GL_4 case.

In case 1, that is, when k/k_0 is a ramified quadratic extension, we observe that θ_{10}^k has the Langlands descent $\theta_{10}^{k_0}$. Then we may expect that the L-packet of θ_{10}^k can be descended to that of $\theta_{10}^{k_0}$.

In case 2 where k/k_0 is an unramified quadratic extension, θ_{10}^k has neither a Shintani descent nor a Langlands descent. Following the general philosophy, it seems unlikely that the L-packet of θ_{10}^k would have a base change descent via trace formula.

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