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GOLDIE CONDITIONS FOR CONSTANTS OF ALGEBRAIC DERIVATIONS OF SEMIPRIME ALGEBRAS

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ABSTRACT

Relations between Goldie conditions of a semiprime algebra R and its subalgebra R^d of constants under an algebraic derivation are studied. The results obtained are applied to actions of finite dimensional solvable Lie algebras on associative algebras with no non-zero nilpotent elements.

Introduction

Let R be an associative algebra over a field F and L be a finite dimensional Lie algebra over F. Let us recall that the action of L on R means a homomorphism $\Phi: L \to \mathrm{Der}_F R$. The subalgebra of constants of L on R is the subset $R^L = \{r \in R|\ d(r) = 0 \text{ for all } d \in \Phi(L)\}$. For a single derivation d of R we will write $R^d = \{r \in R|\ d(r) = 0\}$. In [7], [8], the relations between finiteness conditions of R and R^L have been studied in the case R is a prime algebra of non-zero

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characteristic and L is a restricted Lie algebra acting by outer derivations. This means, in particular, that L acts by algebraic derivations d such that no power of d is X–inner. In [2] Bergen considered similar problems for R being without non–zero nilpotent elements, L solvable and restricted. In particular, he proved that R is Goldie if and only if R^L is Goldie and the Goldie localization of R can be obtained by inverting the regular elements of R^L.

In the paper we will consider the action of a single algebraic derivation d on a semiprime algebra R of arbitrary characteristic. We will show that the investigations can be reduced to two cases, namely when d is either nilpotent or separable. Using this aproach we prove, in particular, that R has a finite Goldie rank if and only if R^d has a finite Goldie rank and R is semisimple Artinian provided R^d is Artinian. This enable us to extend the result of Bergen [2, Theorem 3.4] to the case of the action of finite dimensional solvable Lie algebra acting by algebraic derivations.

By an ideal I of R we will mean a two–sided ideal. We will say I is a d–ideal of R if $d(I) \subseteq I$. For subsets A, B of R $\mathrm{l.ann}_A B$ ($\mathrm{r.ann}_A B$) will denote the left (resp. right) annihilator of B in A, i.e. $\mathrm{l.ann}_A B = \{a \in A \mid aB = 0\}$ and $\mathrm{r.ann}_A B = \{a \in A \mid Ba = 0\}$. For a right R–module M $\operatorname{rank} M_R$ will denote the Goldie rank of M.

1. Nilpotent derivations

In this part d will stand for a nilpotent derivation of a ring R, $n(R)$ will denote the index of nilpotency of d on R.

For any $k \geq 0$ define $R_k = \{x \in R \mid d^k(x) = 0\}$. Clearly $R_0 = 0$, $R_1 = R^d$ and $R_k = R$ for all $k \geq n(R)$. It is easy to see that the additive groups R_k define a filtration of R, i.e., $R_i \subseteq R_{i+1}$ and $R_i R_j \subseteq R_{i+j}$ for all $i, j \geq 0$.

For any $1 \leq k \leq n(R)$ let d_k denote the restriction of d to R_k. Then $d_k\colon R_k \to R_{k-1}$ is a homomorphism of R^d–bimodules.

Let Ω be a module property which is closed with respect to taking submodules and extensions, i.e.

(1) if $N \subseteq M$ and $M \in \Omega$, then $N \in \Omega$,

(2) if $N \subseteq M$ and N, $M/N \in \Omega$, then $M \in \Omega$.

Ω can mean, for example, one of the following properties: to be Artinian, Noetherian, to have finite Goldie rank, to have Krull dimension and so on.

By inductive argument, using the filtration of R defined above together with homomorphisms d_k: $(R_k)_{R^d} \to (R_{k-1})_{R^d}$, one can easily prove the following

PROPOSITION 1.1: *If $R^d_{R^d} \in \Omega$, then $R_{R^d} \in \Omega$. In particular;*

(i) *If R^d is right Artinian (Noetherian), then R is right Artinian (Noetherian).*

(ii) *If* $\operatorname{rank} R^d_{R^d}$ *is finite, then* $\operatorname{rank} R_R$ *is finite and*

$$\operatorname{rank} R_R \leq \operatorname{rank} R_{R^d} \leq n(R) \cdot \operatorname{rank} R^d_{R^d}$$

The following example shows that the converse implication does not hold in general.

Example 1: Let $L = Fx + Fy$ be a two-dimensional non-abelian Lie algebra over a field F of characteristic 0; $[x, y] = x$. Define $R = U(L)/(x^2)$, where $U(L)$ is the universal enveloping algebra of L. Let $X, Y \in R$ denote the natural images of x, y and d be the inner derivation of R adjoint to X. Since $X^2 = 0$, d is nilpotent with $n(R) = 3$. Notice that R is Noetherian as a homomorphic image of an enveloping algebra. Thus R is of finite Goldie rank. In fact one can verify that $\operatorname{rank} R_R = 1$.

We will show that $\operatorname{rank} R^d_{R^d}$ is infinite. Using relations $XY - YX = X$ and $X^2 = 0$ it is easy to see that every element from R is of the form $f(Y) + Xg(Y)$ for some suitable $f(Y), g(Y) \in F[Y]$ and $XYX = 0$. Moreover by inductive argument, $XY^mX = 0$ for all $m \geq 1$. The above yields $R^d = F + XF[Y]$ and $(XF[Y])^2 = 0$. Therefore $\operatorname{rank} R^d_{R^d}$ is infinite, as required. ∎

In the sequel we will show that a similar example can not be constructed when R is a semiprime ring. For doing so, some preparation is needed.

Henceforth we will assume that the ring R is semiprime. $\mathcal{F}_R$ will denoted the filter of all two-sided ideals of R with zero right annihilator. Since R is semiprime, $\mathcal{F}_R$ consists of all ideals which are essential as right ideals. $m(R)$ will stand for the smallest natural number m such that for any non-zero d-ideal I of R $\operatorname{l.ann}_I d^m(R) \neq 0$. We say that the ring R is **homogeneous** (cf. [5]) if for any non-zero d-ideal I of R $m(I) = m(R)$ and $n(I) = n(R)$. A d-ideal I of R is called **homogeneous** if I is homogeneous as a ring.

The following lemma will allow us to reduce our considerations to homogeneous rings.

LEMMA 1.2:

(1) *If R is homogeneous, then every non-zero d-ideal I of R is homogeneous and $m(I) = m(R)$.*

(2) *There is a family $\{I_\alpha\}_{\alpha\in A}$ of homogeneous d-ideals of R such that:*

(i) $\bigoplus_{\alpha\in A} I_\alpha \in \mathcal{F}_R$,

(ii) *each I_α is either $\mathbf{Z}$-torsion free or $pI_\alpha = 0$ for some prime number p, i.e.* $\operatorname{char} I_\alpha = 0$ *or* $\operatorname{char} I_\alpha = p$.

Proof: See [5, Lemma 2 and Proposition 3]. ∎

LEMMA 1.3: *Let R be a homogeneous ring which does not contain an infinite direct sum of ideals. Then for any non-zero right d-ideal K of R $d^{m-1}(K) \neq 0$, where $m = m(R)$.*

Proof: Let K be a non-zero right d-ideal of R. Assume $d^{m-1}(K) = 0$. By the definition of $m = m(R)$, there is a non-zero d-ideal I_1 of R such that $\text{l.ann}_{I_1} d^{m-1}(R) = 0$. Define $J_1 = \text{r.ann}_R I_1$ and $K_1 = J_1 \cap K$. Then $I_1 \cap J_1 = 0$, since R is semiprime, and $K_1 \neq 0$. To see that $K_1 \neq 0$, take $0 \neq x \in K^d$. Then $0 = d^{m-1}(xR) = xd^{m-1}(R)$. Hence $L \cap K \neq 0$, where $L = \text{l.ann}_R d^{m-1}(R)$. Clearly $I_1 L \subseteq I_1 \cap L = 0$. It means that $L \subseteq J_1$ and proves that $K_1 \neq 0$.

We have constructed non-zero d-ideals I_1, J_1 of R such that $I_1 \cap J_1 = 0$ and a non-zero right d-ideal K_1 of J_1 satisfying $d^{m-1}(K_1) = 0$. By Lemma 1.2(1), J_1 is a homogeneous ring with $m(J_1) = m$. Therefore we can apply the above procedure to J_1 and K_1, instead of R and K. Continuing this process we can construct a sequence of non-zero d-ideals $R = J_0 \triangleright J_1 \triangleright J_2 \triangleright \cdots$ and non-zero d-ideals I_j of J_{j-1}, $j > 0$, such that $I_j \cap J_j = 0$ for all j. Since R is semiprime, every I_j contains a non-zero ideal $\overline{I}_j$ of R. The above implies that R contains an infinite direct sum $\bigoplus_{j=1}^{\infty} \overline{I}_j$ of ideals. This contradicts our assumption on R and the thesis follows. ∎

The formulation of the following proposition is fairly complicaded, however the result itself is very important. It provides a construction of a map which behaves like a non-degenerate trace map.

PROPOSITION 1.4: *Suppose R does not contain an infinite direct sum of ideals. Let I be a non-zero homogeneous d-ideal of R with either* $\operatorname{char} I = 0$ *or* $\operatorname{char} I = p$. *Then there exist a d-ideal J of R such that $J \subseteq I$, $J \in \mathcal{F}_I$ and a homomorphism t: $J \to I^d$ of R^d-bimodules. The homomorphism t has the* *following property: for any right d–ideal K of R we have $t(K \cap J) \neq 0$ provided $K \cap J \neq 0$.*

Proof: For defining t we will make use of an over–ring S of I. The definition of S depends on the characteristic of I. Let $m = m(I)$.

CASE 1: $\operatorname{char} I$ does not divide m. In this case we set S equal to the symmetric Martindale ring of quotients of I. By [5, Theorem 5], there is $x \in S$ such that $x^m = 0$ and $d(r) = xr - rx$ for all $r \in I$. The construction of S provides us a d–ideal J of I such that $J \in \mathcal{F}_I$ and $x^k J x^l \subseteq I$ for $0 \leq k, l \leq m$. Replacing J by IJI we may assume that J is a d–ideal of R.

CASE 2: $\operatorname{char} I = p > 0$ and $m = p^k l$, where $k \geq 1$ and the prime number p does not divide l. Consider the ring of differential polynomials $I^1[X; d]$, where I^1 denotes the natural extension of I to a unital ring. Define $S = I^1[X; d]/(X^m)$. Applying the same arguments as in [5] we will prove that $(X^m) \cap I = 0$. Indeed, it is easy to see that every polynomial in (X^m) has a free coefficient of the form $\sum d^m(a_i)b_i$, where $a_i, b_i \in I^1$. Hence $I \cap (X^m) \subseteq d^m(I)I^1$ and

$$\mathrm{l.ann}_{I\cap(X^m)}(d^m(I)) \cdot (I \cap (X^m)) = 0.$$

Now the definition of $m(I)$ and the semiprimeness of I imply immediately that $I \cap (X^m) = 0$. Thus I can be treated as a subring of S. Let x be the canonical image of X in S. Clearly $x^m = 0$ and $d(r) = xr - rx$ for all $r \in I$. It was shown in [6, Lemma 3] that in the above situation $\mathrm{r.ann}_I x^{p^k}$ intersects every non–zero d–ideal of I non–trivially. Therefore $J = I \cdot \mathrm{r.ann}_I x^{p^k} \in \mathcal{F}_I$ and $x^{p^k} J,\ J x^{p^k} \subseteq J$. Obviously J is a d–ideal of I. As in the previous case, we may assume that J is a d–ideal of R such that $x^{ip^k} J x^{jp^k} \subseteq I$ for $0 \leq i, j \leq l$.

In both cases we define t: $J \to I^d$ by the formula $t(r) = x^{m-1}r + x^{m-2}rx + \cdots + xrx^{m-2} + rx^{m-1}$. Using the identity $xr = rx + d(r)$, for $r \in J$, we can write $t(r)$ in the following form

$$t(r) = \sum_{i=0}^{m-1} (-1)^i \binom{m}{i+1} x^{m-i-1} d^i(r), \tag{1}$$

for $r \in J$.

In case $\operatorname{char} I = p$ and $m = p^k l$ the above formula reduces to (see [6])

$$t(r) = \sum_{i=1}^{l} (-1)^{ip^k} \binom{l}{i} x^{p^k(l-i)} d^{ip^k-1}(r), \tag{2}$$

for $r \in J$.

Since $xt(r) - t(r)x = 0$ for $r \in J$, the choice of J and x together with formulas (1), (2) yield that t really acts into I^d.

Now we will show that t is a homomorphism of R^d-bimodules. The presented proof covers the case $\operatorname{char} I = 0$. If $\operatorname{char} I = p > 0$, the proof is the same if we replace x by x^{p^k} and formula (1) by (2). Suppose $\operatorname{char} I = 0$. Notice that, since S is an over-ring of I not of R, we do not know a priori that $(xj)r = x(jr)$ for $j \in J$ and $r \in R$. However working inside S we have

$$(x(jr))a = x((jr)a) = x(j(ra)) = ((xj)r)a$$

for any $j \in J$, $a \in I$, $r \in R$. It means that $((xj)r - x(jr))I = 0$ in the ring I, and the semiprimeness of I give us $(xj)r = x(jr)$ for any $j \in J$ and $r \in R$. Using this and the formula (1) it is clear that t is a homomorphism of right R^d-modules. Similarly, expressing $t(r)$ in terms of combinations of $d^i(r)$ with coefficients x^j on the right hand side, one can prove that t is a homomorphism of left R^d-modules, as well.

Now let K be a right d-ideal of R such that $K \cap I \neq 0$. Because J is essential in I, we may additionally assume that $K \subseteq J$. Notice also that the condition imposed on R inherits on I. Thus we can apply Lemma 1.3 to I. Assume $t(K) = 0$. Then, depending on $\operatorname{char} I$, either

$$0 = x^{m-1}t(K) = x^{m-1}d^{m-1}(K)$$

or

$$0 = x^{p^k(l-1)}t(K) = x^{p^k(l-1)}d^{m-1}(K),$$

respectively. By Lemma 1.3, $d^{m-1}(K) \neq 0$, so $K_1 = \operatorname{r.ann}_I x^{m-1} \cap K \neq 0$ ($K_1 = \operatorname{r.ann}_I x^{p^k(l-1)} \cap K \neq 0$, respectively). Clearly $t(K_1) = 0$. Repeating similar procedure enough times, we can construct a non-zero right d-ideal $\overline{K}$ of I such that $t(\overline{K}) = 0$ and $x\overline{K} = 0$ ($x^{p^k}\overline{K} = 0$, respectively). Hence, by (1) and (2), $d^{m-1}(\overline{K}) = 0$, which contradicts Lemma 1.3. Therefore $t(K) \neq 0$. ∎

LEMMA 1.5: *Let R be a homogeneous ring with either* $\operatorname{char} R = 0$ *or* $\operatorname{char} R = p$. *If* $\operatorname{rank} R_R$ *is finite, then* $\operatorname{rank} R^d_{R^d}$ *is finite and*

$$\operatorname{rank} R^d_{R^d} \leq \operatorname{rank} R_R.$$

Proof: Applying Proposition 1.4 to $I = R$ we obtain an essential d–ideal J of R and a homomorphism $t\colon J \to R^d$ of R^d–bimodules such that $t(K) \neq 0$ for any non–zero right d–ideal of R contained in J.

Let K_i, $1 \leq i \leq s$, be non-zero right ideals of R^d such that the sum $\sum_{i=1}^{s} K_i$ is direct. Since R is semiprime and $J \in \mathcal{F}_R$, the d–invariant right ideal K_iJ is non–zero for every $1 \leq i \leq s$. Let $K = K_iJ \cap \sum_{j\neq i} K_jJ$. Then $t(K) \subseteq t(K_iJ) \cap t(\sum_{j\neq i} K_jJ) \subseteq K_it(J) \cap \sum_{j\neq i} K_jt(J) \subseteq K_i \cap \sum_{j\neq i} K_j = 0$, i.e., $t(K) = 0$. This implies $K = 0$, so the sum $\sum_{i=1}^{s} K_iJ$ is direct and the thesis follows. ∎

Let us remark that in the above lemma we proved that $\operatorname{rank} R^d$ is equal to d–invariant Goldie rank of R, i.e., the Goldie rank with respect to right d–ideals.

LEMMA 1.6: *Let R be a semiprime ring which does not contain an infinite direct sum of ideals. Then for every essential right d–ideal K of R, $K^d = K \cap R^d$ is an essential right ideal of R^d.*

Proof: By Lemma 1.2, there exist homogeneous d–ideals $I_1, \ldots, I_s$ of R such that $E = \bigoplus_{i=1}^{s} I_i \in \mathcal{F}_R$ and the charcteristic of I_i is either zero or prime for any $1 \leq i \leq s$. Moreover, by Proposition 1.4, there are non–zero d–ideals J_i of R contained in I_i, $J_i \in \mathcal{F}_{I_i}$ and non–trivial homomorphisms $t_i\colon J_i \to I_i^d$ of R^d–bimodules, $i = 1, \ldots, s$. Notice that $J = \bigoplus_{i=1}^{s} J_i \in \mathcal{F}_R$, since $E \in \mathcal{F}_R$ and $J_i \in \mathcal{F}_{I_i}$ for $1 \leq i \leq s$.

Let K be an essential right d–ideal of R and A a non–zero right ideal of R^d. We will show that $K^d \cap A \neq 0$, i.e. K^d is essential in R^d. Because R is semiprime and $J \in \mathcal{F}_R$, $AJ \neq 0$. It means that for some $1 \leq j \leq s$ AJ_j is a non–zero right d–ideal of R. Thus $AJ_j \cap K$ is a non–zero right d–ideal contained in J_j. Take $0 \neq a \in (AJ_j \cap K)^d$. By the semiprimeness of J_j aJ_j is a non–zero right d–ideal, so Proposition 1.4 gives

$$0 \neq t_j(aJ_j) = at_j(J_j) \subseteq aI_j^d \subseteq K^dI_j^d \subseteq K^d.$$

On the other hand $aJ_j \subseteq AJ_j^2 \subseteq AJ_j$. Since t_j is a homomorphism of R^d–bimodules,

$$0 \neq t_j(aJ_j) \subseteq t_j(AJ_j) = At_j(J_j) \subseteq AI_j^d \subseteq A.$$

Consequently $A \cap K^d \neq 0$, as required. ∎

Now we are in position to prove the main result of this section.

THEOREM 1.7: *Let R be a semiprime ring with a nilpotent derivation d. Then* $\operatorname{rank} R_R$ *is finite if and only if* $\operatorname{rank} R^d_{R^d}$ *is finite and*

$$\operatorname{rank} R^d_{R^d} \leq \operatorname{rank} R_R \leq n(R) \cdot \operatorname{rank} R^d_{R^d}.$$

Proof: Suppose $\operatorname{rank} R_R$ is finite. Let $I_1, \ldots, I_s$ and E be ideals of R defined in the proof of Lemma 1.6. We have $\operatorname{rank}(I_j^d)_{R^d} \leq \operatorname{rank}(I_j^d)_{E^d} = \operatorname{rank}(I_j^d)_{I_j^d}$, so applying the above lemma to $K = E$

$$\operatorname{rank} R^d_{R^d} = \operatorname{rank} E^d_{R^d} \leq \sum_{j=1}^{s} \operatorname{rank}(I_j^d)_{I_j^d}.$$

Moreover, since $E \in \mathcal{F}_R$ and R is semiprime, similar considerations give us $\operatorname{rank} R_R = \sum_{j=1}^{s} \operatorname{rank}(I_j)_{I_j}$. Now Lemma 1.5 applied to homogeneous rings I_j, $1 \leq j \leq s$ implies that $\operatorname{rank} R^d_{R^d}$ is finite and $\operatorname{rank} R^d_{R^d} \leq \operatorname{rank} R_R$. The converse implication and the second inequality are given by Proposition 1.1. ∎

2. Algebraic derivations

In this part we will show how to apply results of the previous section to algebraic derivations of semiprime algebras. Henceforth R will be denote a semiprime algebra over a field F and d will stand for an algebraic derivation of R, i.e., d is algebraic over F as an element of $\operatorname{End}_F(R^+)$. Thus there are $k, n \geq 0$ and elements $a_0, \ldots, a_k \in F$ such that

$$a_0 d^n + a_1 d^{n+1} + \cdots + a_k d^{n+k} = 0. \tag{3}$$

Clearly we may assume $a_0 = 1$. Notice that $n \geq 1$, since otherwise $R^d = 0$ and by [1, Theorem 1.3], R would be nilpotent. From the same reasons $K^d \neq 0$ for every non-zero right d-ideal of R. Let us remark that if I is an ideal of R, then $d^s(I^{n+k}) \subseteq I$ for any $s \geq 0$. This and the semiprimeness of R yield that any non-zero ideal I of R contains a non-zero d-ideal $\widehat{I}$. Thus, in particular, $I \cap R^d \neq 0$. We will frequently use the above remarks.

Let $R_d = \{x \in R | \ \exists j \geq 0 \ \ d^j(x) = 0\}$. It is standard to see that R_d is a subalgebra of R and $R_d = \{x \in R \mid d^n(x) = 0\}$, where n is as in (3). Obviously $R^d \subseteq R_d$ and d is nilpotent on R_d. Moreover, it is known ([6]) that R_d is semiprime. It means that we will able to apply results of the first section to the extension $R^d \subseteq R_d$.

Let us consider an F-linear map $f\colon R \to R_d$ given by the formula $f(x) = x + a_1 d(x) + \cdots + a_k d^k(x)$. Remark that f is a homomorphism of R^d-bimodules. In the following lemma we collect basic properties of the map f; A denotes the kernel of f.

LEMMA 2.1:

(1) $f(R_d) = R_d$.

(2) $(R_d A + A) \cap R_d = 0$.

(3) $R = R_d \oplus A$ *as* R^d*-bimodules.*

(4) *If* I *is a right ideal of* R_d*, then* $(I + IR) \cap R_d = I$.

(5) *If* K *is a non-zero right ideal of* R*, then* $f(K) \neq 0$.

Proof: Consider the natural filtration of R_d defined at the beginning of the first section, i.e. $(R_d)_k = \{x \in R_d \mid d^k(x) = 0\}$, for $k \geq 0$.

(1) Clearly $0 = (R_d)_0 \subseteq f(R_d)$. Assume that $(R_d)_i \subseteq f(R_d)$ for some $i \geq 0$. Let $x \in (R_d)_{i+1}$. Then $v = a_1 d(x) + \cdots + a_k d^k(x) \in (R_d)_i$. By assumption, there is $y \in R_d$ such that $f(y) = v$. Now $x - y \in R_d$ and $f(x - y) = x$. This shows that $(R_d)_{i+1} \subseteq f(R_d)$ and yields $R_d = (R_d)_n \subseteq f(R_d)$.

(2) Let $i > 0$ and $x \in (R_d)_i$, $a \in A$. Using the definition of f it is easy to see that

$$f(xa) \in xf(a) + \operatorname{span}_F\{d^l(x)d^s(a) \mid l \geq 1,\ s \geq 0\} \subseteq$$

$$xf(a) + (R_d)_{i-1}A \subseteq (R_d)_{i-1}A,$$

as $a \in A = \ker f$. Hence $f^n(R_d A + A) \subseteq (R_d)_0 A = 0$. $(R_d A + A) \cap R_d$ is a d-invariant subspace of R_d, thus for $x \in (R_d A + A) \cap R^d$ we have $x = f(x) = f^n(x) = 0$. Therefore $(R_d A + A) \cap R_d = 0$.

(3) This statement is an easy consequence of (1) and (2).

(4) Let I be a right ideal of R_d. Using (3) and (2) one obtains $I \subseteq (I + IR) \cap R_d \subseteq (I + IA) \cap R_d \subseteq I + (IA \cap R_d) = I$.

(5) Let K be a right ideal of R. Assume $f(K) = 0$. Since f and d commute, $f(\sum_{i \geq 0} d^i(K)) = 0$. Therefore, eventually replacing K by $\sum_{i \geq 0} d^i(K)$, we may assume $d(K) \subseteq K$ and $\delta = d_{|K}$ is a derivation of K satisfying $1 + a_1\delta + \cdots + a_k\delta^k = 0$. This implies $K^\delta = 0$, so $K = 0$ and the thesis follows. ∎

Now we can extend Theorem 1.7 to algebraic derivations.

THEOREM 2.2: *Let R be a semiprime algebra and d be an algebraic derivation of R. The following conditions are equivalent:*

(1) $\operatorname{rank} R_R$ *is finite,*

(2) $\operatorname{rank}(R_d)_{R_d}$ *is finite,*

(3) $\operatorname{rank} R^d{}_{R^d}$ *is finite.*

Proof: In virtue of Theorem 1.7, conditions (2) and (3) are equivalent.

(1)⇒(2). Suppose $\operatorname{rank} R_R$ is finite. If $\operatorname{rank}(R_d)_{R_d}$ is infinite, then R_d contains an infinite direct sum of right d-ideals $I_1 \oplus I_2 \oplus \dots$ ([4, Theorem 3.1]). Then for $i \geq 1$

$$f(I_i^d R \cap \sum_{j\neq i} I_j^d R) \subseteq f(I_i^d R) \cap f(\sum_{j\neq i} I_j^d R)$$
$$\subseteq I_i^d R_d \cap \sum_{j\neq i} I_i^d R_d \subseteq I_i \cap \sum_{j\neq i} I_j = 0.$$

Therefore, by Lemma 2.1(5), $I_i^d R \cap \sum_{j\neq i} I_j^d R = 0$ for any $i \geq 1$. This shows that the sum $\sum_{i\geq 1} I_i^d R$ is direct and yields the thesis.

(3)⇒(1). If $\operatorname{rank} R_R$ is infinite then R contains an infinite direct sum $K_1 \oplus K_2 \oplus \cdots$ of right d-ideals ([4, Theorem 3.1]). Then $\sum_{i\geq 1} K_i^d$ is an infinite direct sum of right ideals of R^d. Therefore $\operatorname{rank} R_R$ is finite provided $\operatorname{rank} R^d_{R^d}$ is finite. ∎

For the algebra R, $Z(R)$ will denote the right singular ideal of R. Recall that R is said to be right non-singular if $Z(R) = 0$.

LEMMA 2.3: *Let K be a right d-ideal of R.*

(1) *If $R^d = R_d$, then $K_d = K \cap R_d$ is essential in R_d provided K is d-essential in R, i.e. for every non-zero right d-ideal K' of R $K \cap K' \neq 0$;*

(2) *If R is right non-singular then K_d essential in R_d provided K is essential in R.*

Proof: (1) Suppose $R_d = R^d$. Let $a \in R^d$. Then aR is a right d-ideal of R and $f(K \cap aR) \subseteq K_d \cap aR_d$. Now Lemma 2.1(5) provides the thesis in this case.

(2) Let R be right non-singular. Assume K_d is not essential in R_d. Take $0 \neq a \in R_d$ such that $K_d \cap aR_d = 0$ and define $K_j = \{x \in K \mid d^j(a)x \in K\}$ for $j \geq 0$. Since K is essential, each K_j is essential and hence $\widehat{K} = K_0 \cap K_1 \cap \cdots \cap K_{n+k}$ is an essential right ideal of R. Moreover, since d is algebraic of degree $n+k$ and K is d-invariant, standard calculations show that $\widehat{K}$ is also d-invariant. Let $\widehat{K}_d = \widehat{K} \cap R_d$. Notice that $a\widehat{K}_d \subseteq K_d \cap aR_d = 0$ and $d^j(a)\widehat{K}_d = 0$ for $j \geq 0$

follows, because $\widehat{K}_d$ is d–invariant. Let $m \geq 0$ be such that $0 \neq b = d^m(a) \in R^d$. Using the above, one gets

$$f(b\widehat{K}) = bf(\widehat{K}) \subseteq b\widehat{K}_d = 0$$

Applying Lemma 2.1(5) we obtain $b\widehat{K} = 0$. It means that $0 \neq b \in Z(R)$. This contradicts our assumption on $Z(R)$. Consequently K_d is essential in R_d. ∎

PROPOSITION 2.4: *For the algebra R the following conditions are equivalent:*

(1) *R is right Artinian.*

(2) *R_d is right Artinian.*

Proof: The implication (1)⇒(2) is a direct consequence of Lemma 2.1(4).

(2)⇒(1). We know that R_d is semiprime. Suppose R_d is right Artinian. Then R_d is an unital algebra with a unity e. We will show that e is the unity of R. Consider the right ideal $K = \{x - ex \mid x \in R\}$ of R. Noticing that $e \in R^d$ and f is a homomorphism of R^d–bimodules one easily gets $f(K) = 0$. Therefore, by Lemma 2.1(5), $K = 0$. It means that e is the left unit of R. Now $L = \{x - xe \mid x \in R\}$ is a left ideal of a semiprime algebra R such that $L^2 = 0$. Thus $L = 0$. This shows that R is a unital algebra with the unity $1 = e$.

Observe that R is right non–singular. Indeed, if $Z(R) \neq 0$ then $Z(R)$, as a non–zero two–sided ideal of R, contains a non–zero d–ideal. Therefore $Z(R) \cap R_d$ is a non–zero ideal of a semisimple artinian algebra. This yields that $Z(R)$ contains a non–trivial idempotent, which is impossible.

Now we will divide the proof into two cases depending on char F. Let $\operatorname{char} F = 0$. By [4, Corollary 4.4] every essential right ideal K of R contains a d–invariant essential right ideal $\widehat{K}$. By Lemma 2.3(2) $\widehat{K}_d$ is essential in R_d, so $1 \in \widehat{K}_d \subseteq K$ because R_d is semisimple Artinian. This means that R does not contain proper essential right ideals. Moreover, by Theorem 2.2, R_R is of finite Goldie rank, so R is semisimple Artinian.

Let $\operatorname{char} F = p > 0$ and $m \geq 0$ be such that $p^m \geq n$ (recall that n is such that $R_d = \{r \in R \mid d^n(x) = 0\}$. Then the algebraic derivation $\delta = d^{p^m}$ satisfies $R_d = R_\delta = R^\delta$. By Lemma 2.3(1) R has no proper δ–essential right ideals. Thus R_R is completely reducible with respect to right δ–ideals, i.e., for every proper right δ–ideal K there exists a right δ–ideal K' such that $K \oplus K' = R$. This together the fact that $\operatorname{rank} R_R$ is finite yields that R_R has a finite length with respect to right δ–ideals. By making use of [4, Theorem 3.1] we get that R_R has a finite length, so is Artinian. ∎

As a direct consequence of the above proposition and Proposition 1.1 we get

THEOREM 2.5: *Let d be an algebraic derivation of a semiprime algebra R. If R^d is right Artinian, then R is semisimple Artinian.*

Now we are in position to prove the following result.

PROPOSITION 2.6: *Suppose R^d is semiprime. Then:*

(1) *R is right Artinian if and only if R^d is right Artinian,*

(2) *R is right Goldie if and only if R^d is right Goldie.*

Moreover, in the case when both R and R^d are right Goldie

(3) *$Q(R) = RT^{-1}$, where $Q(R)$ is the Goldie localization of R and RT^{-1} is a localization of R at the set T of regular elements of R^d.*

(4) *$Q(R)^{\overline{d}} = Q(R^d)$, where $\overline{d}$ is the unique extension of d to $Q(R)$.*

Proof: (2). In light of Theorem 2.2 the equivalence (2) holds if we will prove that R is right nonsingular provided R^d is semiprime Goldie. Suppose R^d is semiprime Goldie. Let $\operatorname{char} F = 0$. Then the semiprimeness of R^d implies $R^d = R_d$. Indeed, by Leibniz formula, $d^{n-1}(R_d)^2 = 0$ and $d^{n-1}(R_d)$ is a two–sided ideal of R^d. Let $a \in Z(R) \cap R^d$. Then $K = \text{r.ann}_R a$ is an essential right d–ideal of R. Thus, by Lemma 2.3(1) $K_d = K \cap R_d$ is essential in $R_d = R^d$, i.e. $a \in Z(R^d) = 0$. This shows that $Z(R) \cap R^d = 0$ and implies $Z(R) = 0$, as required.

Let $\operatorname{char} F = p > 0$ and let $\delta = d^{p^m}$ be such that $R_d = R_\delta = R^\delta$. Since R^d is semiprime Goldie, Proposition 1.1 and Lemma 1.6 yield that R^δ has finite Goldie rank and $Z(R^\delta) = 0$, i.e. R^δ is semiprime Goldie. Therefore, while proving $Z(R) = 0$, we may replace d by δ and assume that $R_d = R^d$. Now, as in the case $\operatorname{char} F = 0$, $Z(R) = 0$ follows.

(3) and (4). Now assume that both R and R^d are semiprime Goldie. We claim that T is an Ore set of R. Since the right annihilator of a constant element is d–invariant and every non–zero right d–ideal has a non–zero intersection with R^d, all elements of T are regular in R. Now let $s \in T$ and $a \in R$. Then $K = sR$ is an essential right d–ideal of R and $\widehat{K} = \bigcap_{j=0}^{n+k} \{x \in K \mid d^j(a)x \in K\}$ is an essential right d–ideal contained in K (see the proof of Lemma 2.3). By Lemmas 2.3 and 1.6, $\widehat{K}^d$ is essential in R^d, so $\widehat{K} \subseteq K$ contains a regular element $t \in T$. It means that $sR \cap aT \neq \emptyset$, as claimed.

Now we can consider the localization RT^{-1} and the extension $\overline{d}$ of d to RT^{-1} via $\overline{d}(at^{-1}) = d(a)t^{-1}$. Clearly d satisfies the same identity (3) as d and $(RT^{-1})^{\overline{d}} = R^dT^{-1}$ is a semisimple right Artinian algebra. Applying Theorem 2.5 to RT^{-1} we obtain that RT^{-1} is semisimple Artinian. Thus $Q(R) = RT^{-1}$ which gives the proof of (3) and (4).

The statement (1) is a direct consequence of Theorem 2.5 and (4). ∎

We conclude this paper with the following extension of J. Bergen's result [2, Theorem 3.4] to the action of arbitrary solvable finite dimensional Lie algebra acting by algebraic derivations.

THEOREM 2.7: *Let R be an algebra with no non–zero nilpotent elements and let L be a finite dimensional solvable Lie algebra acting on R by algebraic derivations. Then:*

(1) *R is right Artinian if and only if R^L is right Artinian,*

(2) *R is right Goldie if and only if R^L is right Goldie.*

Furthermore, in the case where both R and R^L are right Goldie

(3) *$Q(R) = RT^{-1}$, where T is the set of regular elements of R^L,*

(4) *$Q(R)^L = Q(R^L)$.*

Proof: The algebra R is without non-zero nilpotents, so any its non–zero subalgebra is semiprime. We will proceed by induction on $\dim L$. If $\dim L = 1$, then the theorem is a special case of Proposition 2.6. Assume $\dim L > 1$. Then L contains an ideal M of co–dimension one. Let $z \in L \setminus M$. Then $L = M \oplus Fz$ as linear spaces and $R^L = (R^M)^d$, where $d = \Phi(z)$. Applying the inductive hypothesis to the extension $R^L = (R^M)^d \subseteq R^M \subseteq R$ we easily get statements (1) and (2) of the theorem. From the same reasons, it is clear that:

(i) every element from T is regular in R,

(ii) T is an Ore set in R^M such that $Q(R^M) = R^MT^{-1}$,

(iii) $\overline{T}$ is an Ore set in R and $Q(R) = R\overline{T}^{-1}$, where $\overline{T}$ is the set of all regular elements of R^M.

Let $s \in T$ and $0 \neq a \in R$. Then, by (i) and (iii) $sR \cap a\overline{T} \neq \emptyset$. Take $0 \neq a\bar{t} \in sR \cap a\overline{T}$. By (ii) and (iii) $R^M\overline{T}^{-1} = R^MT^{-1}$. Hence there are $r \in R^M$, $t \in T$ such that $\bar{t}^{-1} = rt^{-1}$. It means that $\bar{t}r = t \in T$. Therefore $0 \neq at = a\bar{t}r \in sR \cap aT$, i.e., T is an Ore set in R.

Now extending the action of L to the action on RT^{-1} by $d(at^{-1}) = d(a)t^{-1}$, for all $d \in \Phi(L)$, we see that L acts on RT^{-1} by algebraic derivations and

$(RT^{-1})^L = R^L T^{-1} = Q(R^L)$. Making use of (1), we get immediately that RT^{-1} is semisimple Artinian, so $Q(R) = RT^{-1}$. This ends the proof of (3) and (4). ∎

Let us remark that the assumption imposed on R is essential in the above theorem.

Example 2: (Bergen, [2]). Let $R = M_2(S)$, where $S = F\{x,y\}$ is a non-commutative free algebra in two variables over a field F of arbitrary characteristic. Let d_i, $i = 1,2,3$ be inner derivations of R adjoint to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$, respectively. Then $L = Fd_1 \oplus Fd_2 \oplus Fd_3$ is a three dimensional solvable Lie algebra of algebraic derivations of R. One can check that $R = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in F\}$. Thus R^L is semisimple Artinian however R is neither Artinian nor Goldie. ∎

Notice, at the end, that if a finite dimensional solvable Lie algebra L acts on a semiprime algebra R by separable derivations (i.e., $R^d = R_d$ for any $d \in \Phi(L)$) then the analogue of Theorem 2.7 holds, because R^d is semiprime for every $d \in \Phi(L)$ in this case (cf. [3]).

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