Analysis of the New Conservation Law in Electromagnetic Theory.

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Summary. — The new conserved tensor in electromagnetic theory introduced by Lipkin is shown to be a member of an infinite sequence of moments constructed from the wave vector. The physical interpretation is discussed.

1. – Introduction.

In a recent paper (1) on the classical electromagnetic field, Lipkin exhibits a symmetric traceless tensor in the form of the integral of a conserved density over a spacelike surface. The physical interpretation is left open, apart from an indication that the sign of the quantities depends on the sense of circular polarization (in quantal terms, on the helicity).

In the present paper, this tensor and the energy-momentum vector are displayed as integrals over invariant amplitudes for components of definite polarization and wave-vector, and shown to be members of an infinite sequence of wave-vector « moments »; it is conjectured that half of these « moments » are susceptible of display as integrals of conserved densities over spacelike surfaces in space-time.

2. – Formalism.

In this Section, the solution $\boldsymbol{E}$ and $\boldsymbol{H}$ of Maxwell's equations without sources are expressed in terms of invariant amplitudes $q^{\pm}(\boldsymbol{k})$ for right and left circu-

(1) D. M. LIPKIN: *Journ. Math. Phys.*, **5**, 696 (1964).

larly polarized components of wave-vector $\boldsymbol{k}$; vector and pseudovector potentials are also introduced.

Suppose that we have an electromagnetic disturbance of the vacuum of finite total energy

$$P_0 = \frac{1}{2}\int (\boldsymbol{E}^2 + \boldsymbol{H}^2)\,\mathrm{d}^3\boldsymbol{r} < \infty \,. \tag{1}$$

By Plancherel's theorem, we can introduce Fourier transforms

$$\boldsymbol{E}^*(\boldsymbol{k}, t) = (2\pi)^{-\frac{3}{2}}\int \boldsymbol{E}(\boldsymbol{r}, t)\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}]\mathrm{d}^3\boldsymbol{r} \tag{2a}$$

and

$$\boldsymbol{H}^*(\boldsymbol{k}, t) = (2\pi)^{-\frac{3}{2}}\int \boldsymbol{H}(\boldsymbol{r}, t)\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}]\,\mathrm{d}^3\boldsymbol{r}\,. \tag{2b}$$

From Maxwell's equations, these can be written

$$\boldsymbol{E}^*(\boldsymbol{k}, t) = k^{-1}[\boldsymbol{e}(\boldsymbol{k})\exp[-ikt] + \boldsymbol{e}^*(-\boldsymbol{k})\exp[ikt]]\,, \tag{3a}$$

and

$$\boldsymbol{H}^*(\boldsymbol{k}, t) = k^{-1}[\boldsymbol{h}(\boldsymbol{k})\exp[-ikt] + \boldsymbol{h}^*(-\boldsymbol{k})\exp[ikt]]\,, \tag{3b}$$

where $k = |\boldsymbol{k}|$ (with $c = 1$) and $\boldsymbol{e}$, $\boldsymbol{h}$ satisfy

$$\boldsymbol{k}\cdot\boldsymbol{e} = 0\,, \qquad \boldsymbol{k}\cdot\boldsymbol{h} = 0\,, \qquad \boldsymbol{k}\times\boldsymbol{e} = k\boldsymbol{h}\,, \qquad \boldsymbol{k}\times\boldsymbol{h} = -k\boldsymbol{e}\,. \tag{4}$$

It is convenient to introduce standard solutions $\boldsymbol{u}^{\pm}(\boldsymbol{k})$, which in the quantized theory would be the wave functions for right and left circularly polarized photons. They are defined apart from an arbitrary phase by

$$i\boldsymbol{k}\times\boldsymbol{u}^{\pm}(\boldsymbol{h}) = \pm\,\boldsymbol{u}^{\pm}(\boldsymbol{k})\,, \tag{5}$$

with the invariant normalization

$$[\boldsymbol{u}^{\pm}(\boldsymbol{k})]^*\cdot\boldsymbol{u}^{\pm}(\boldsymbol{k}) = k^2. \tag{6}$$

We can now expand

$$\boldsymbol{e}(\boldsymbol{k}) \pm i\boldsymbol{h}(\boldsymbol{k}) = q^{\pm}(\boldsymbol{k})\,\boldsymbol{u}^{\pm}(\boldsymbol{k})\,, \tag{7}$$

where the amplitudes $q^{\pm}(\boldsymbol{k})$ are subject to no restriction other than the trans-

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form of (1), namely,

$$P_0 = \frac{1}{2}\int [|q^+(\boldsymbol{k})|^2 + |q^-(\boldsymbol{k})|^2]\,\mathrm{d}^3\boldsymbol{k} < \infty \,. \tag{8}$$

The covariance of the formalism under Lorentz transformations and in particular the invariance of the amplitudes $q^{\pm}$ is most easily verified in van der Waerden's spinor notation; to avoid new definitions this will be omitted.

In the next Section we need the vector potential A_μ, defined as usual up to a gauge transformation $A_\mu \to A_\mu + \partial_\mu A$ by

$$-\dot{\boldsymbol{A}} - \nabla A_0 = \boldsymbol{E}\,, \tag{9a}$$

$$\nabla \times \boldsymbol{A} = \boldsymbol{H}\,, \tag{9b}$$

$$\nabla \cdot \boldsymbol{A} + \dot{A}_0 = 0\,. \tag{9c}$$

Similarly, we define a pseudovector potential C_μ up to a gauge transformation by

$$\dot{\boldsymbol{C}} + \nabla C_0 = \boldsymbol{H}\,, \tag{10a}$$

$$\nabla \times \boldsymbol{C} = \boldsymbol{E}\,, \tag{10b}$$

$$\nabla \cdot \boldsymbol{C} + \dot{C}_0 = 0\,. \tag{10c}$$

3. – Wave-vector moments and conserved tensors.

If the tensor $T^{\mu\nu\ldots\varrho}$ satisfies

$$\partial_\varrho T^{\mu\nu\ldots\varrho} = 0\,, \tag{11}$$

then its integral

$$U^{\mu\nu\ldots} = \int T^{\mu\nu\ldots\varrho}\,\mathrm{d}\sigma_\varrho \tag{12}$$

over a spacelike surface is independent of the choice of surface. Examples of such conserved tensors are the stress-energy density of rank two and Lipkin's « zilch » density of rank three.

In addition to these, we may discuss an example of a vector (first-rank tensor).

Let

$$S_\mu = A^\nu \partial_\mu C_\nu - C^\nu \partial_\mu A_\nu\,, \tag{13}$$

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then

$$\partial^\mu S_\mu = 0\,, \tag{14}$$

from Maxwell's equations and (9*c*), (10*c*). Also the conserved quantity

$$S = \int S^\mu \mathrm{d}\sigma_\mu \tag{15}$$

is invariant under independent gauge transformations of A_μ, C_μ, although its density S_μ is not. Introducing the invariant amplitudes $q^\pm(\boldsymbol{k})$ defined by (7), we obtain

$$S = \int [\,|q^+(\boldsymbol{k})|^2 - |q^-(\boldsymbol{k})|^2]\, \mathrm{d}^3\boldsymbol{k}/2k\,. \tag{16}$$

We may call S the «screw action»; in the quantized theory it is $\hbar$ times the number of right-handed photons less the number of left-handed photons. Similarly we have the energy-momentum vector

$$P_\mu \equiv \int T^\nu_\mu \mathrm{d}\sigma_\nu = \int [\,|q^+(\boldsymbol{k})|^2 + |q^-(\boldsymbol{k})|^2]\, k_\mu \mathrm{d}^3\boldsymbol{k}/2k\,. \tag{17}$$

(16) and (17) may be generalized to give the «moment» of any order in the wave-number vector,

$$U^\pm_{\mu\nu\ldots} = \int [\,|q^+(\boldsymbol{k})|^2 \pm |q^-(\boldsymbol{k})|^2]\, k_\mu k_\nu \ldots \mathrm{d}^3\boldsymbol{k}/2k\,. \tag{18}$$

It appears possible to write this tensor as the integral of a conserved density over a spacelike surface in space-time only for the combinations of plus sign with odd rank and of minus sign with even rank. This remark can be verified by attempting to construct the $U^\pm_{00\ldots}$ component as an integral over 3-space; the additional powers of k, as compared with the expression (16) for S, can only be supplied by additional time derivatives, each of which changes the relative sign of the $q^\pm$ contributions.

We may therefore expect the next homologue of S and P_μ to take the form

$$Z_{\mu\nu} = \int [\,|q^+(\boldsymbol{k})|^2 - |q^-(\boldsymbol{k})|^2]\, k_\mu k_\nu\, \mathrm{d}^3\boldsymbol{k}/2k\,. \tag{19}$$

It is readily proved that (apart from a factor 2) this coincides with Lipkin's «zilch» tensor.

The physical interpretation of «zilch» is contained in the expression (19).

In view of the existence of the whole sequence (18) of « moments », half of which I conjecture can be expressed as integrals of conserved densities, it seems unlikely that any special significance or application can be found for « zilch ».

ADDENDUM

After this paper was written, I received a preprint « Conservation Laws for Free Fields » (Imperial College, London, I.C.T.P. 64/55) by Dr. T. W. B. KIBBLE, in which generalizations of Lipkin's work are also discussed. We appear to disagree on the number of sets of conserved tensors which can be constructed as quadratic expressions in the electromagnetic field.

Introduce the usual notation (j, j') for the finite (nonunitary) irreducible representations of the homogeneous Lorentz group. Then the electromagnetic fields belonging to the representation $(1, 0) \oplus (0, 1)$, and the irreducible quadratic expressions belong to the reduction of $[(0, 1) \oplus (1, 0)]^* \otimes [(0, 1) \oplus (1, 0)]$ into irreducible components. We have

$$(0, 1)^* \otimes (0, 1) = (1, 0) \otimes (0, 1) = (1, 1)$$

and

$$(1, 0)^* \otimes (1, 0) = (0, 1) \otimes (1, 0) = (1, 1)\,.$$

Symmetric and antisymmetric combinations of these two (which are symmetric traceless tensors) lead to the energy-momentum set and the « zilch » set. These are Kibble's fourth and fifth sets, and our $U^+_{\mu\nu}\,...$ and $U^-_{\mu\nu}\,...$, respectively.

The other combinations are

$$(0, 1)^* \otimes (1, 0) = (1, 0) \otimes (1, 0) = (2, 0) \oplus (1, 0) \oplus (0, 0)$$

and

$$(1, 0)^* \otimes (0, 1) = (0, 1) \otimes (0, 1) = (0, 2) \oplus (0, 1) \oplus (0, 0)$$

which fall into four sets when expressed in tensor notation: two scalars, one antisymmetric second-rank tensor $(0, 1) \oplus (1, 0)$ and one highly-symmetric fourth-rank tensor $(0, 2) \oplus (2, 0)$. When integrated over a spacelike surface, tensors formed from these and powers of the wave vector give zero, as Kibble has shown explicitly for the first three sets. That this is necessarily so is easily seen in the Fourier transform, since these sets involve integrals over $(q^+)^* q^-$ and $(q^-)^* q^+$, which represent « helicity flip » in quantal terms.

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I am most grateful to Dr. KIBBLE for communicating his work before publication. I should also like to thank Dr. D. M. FRADKIN for his preprint on the same subject (« Conserved Quantities Associated with Symmetry Transformations of Relativistic Free Particle Equations of Motion », to appear in *Journal Math. Phys*, May 1965).

RIASSUNTO (*)

Si mostra che il nuovo tensore conservato della teoria elettromagnetica introdotto da Lipkin fa parte di una successione infinita di momenti costruiti a partire dal vettore d'onda. Se ne discute l'interpretazione fisica.

(*) *Traduzione a cura della Redazione.*