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LATTICE-EMBEDDING L^p INTO ORLICZ SPACES

BY

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ABSTRACT

Given $0 < \alpha \leq p \leq \beta < \infty$, we construct Orlicz function spaces $L^F[0,1]$ with Boyd indices α and β such that L^p is lattice isomorphic to a sublattice of $L^F[0,1]$. For $p > 2$ this shows the existence of (non-trivial) separable r.i. spaces on $[0,1]$ containing an isomorphic copy of L^p. The discrete case of Orlicz spaces $\ell^F(I)$ containing an isomorphic copy of $\ell^p(\Gamma)$ for uncountable sets $\Gamma \subset I$ is also considered.

Introduction

The symmetric structure of rearrangement invariant (r.i.) Banach function spaces has been studied in the memoirs of Johnson, Maurey, Schechmann and Tzafriri [J-M-S-T] and Kalton [K_2] (see also [L-T_3]). In ([K_1] Theorem 3.2) Kalton proved that if a r.i. Banach function space X on $[0,1]$, without isomorphic copies of c_0, has a sublattice isomorphic to $L^1[0,1]$, then $X[0,1]$ is precisely $L^1[0,1]$. This result is even valid replacing sublattice for subspace and was also obtained in ([J-M-S-T] Corollary 5.4) under a slightly stronger assumption.

One of the purposes of this paper is to analyze possible extensions of the above result of Kalton ([K_1]). We study, for $0 < p < \infty$, the existence of separable r.i. function spaces $X[0,1]$ different from $L^p = L^p[0,1]$ having a *sublattice* which is lattice isomorphic to L^p. Obviously, the interest of this question consists in considering separable r.i. function spaces for a probabilistic measure jointly with the requirement of L^p be lattice embeddable as a sublattice (so the usual

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isomorphic embedding of L^p via p-stable random variables, for $p < 2$, is not useful here). Moreover, in some cases the existence of an embedding of L^p into $X[0,1]$ as subspace is equivalent to the existence of an embedding as sublattice (f.i. for $p > 2$ and Orlicz spaces [J-M-S-T] p. 195; also for $1 < p < 2$ [K_2] Theorem 10.9).

It is known that the answer to the above question is negative for the class of Lorentz function spaces $L_{p,q}[0,1]$ and $L_{w,q}[0,1]$ for submultiplicative weights (see Carothers [C_1], [C_2]). Here we give a positive answer within the class of separable Orlicz function spaces $L^F[0,1]$. Thus the main result of Section II is the following:

THEOREM A: *Let $0 < \alpha < p \leq \beta < \infty$. There exists an α-convex Orlicz function space $L^F[0,1]$ with indices $\alpha_F^\infty = \alpha$ and $\beta_F^\infty = \beta$ such that L^p is lattice-isomorphic to a sublattice of $L^F[0,1]$.*

In general an α-convex Orlicz function space $L^F[0,1]$, different of L^α, cannot contain a subspace isomorphic to L^α for $1 < \alpha \neq 2$. This follows from ([J-M-S-T] Theorem 7.1). Thus the case $p = \alpha$ is solved in Section III removing the α-convexity:

THEOREM A': *Let $0 < \alpha = p \leq \beta < \infty$. There exists an Orlicz function space $L^F[0,1] \neq L^p$ with indices $\alpha_F^\infty = \alpha$ and $\beta_F^\infty = \beta$ such that L^p is lattice-isomorphic to a sublattice of $L^F[0,1]$.*

In particular the spaces obtained in Theorems A and A' for the case $p > 2$ are (as far as we know) the first examples of non-trivial separable r.i. Banach function spaces on $[0,1]$ containing a *subspace* isomorphic to L^p. Also, Theorems A and A' for the case $p = 1$ show that the above Kalton result [K_1] cannot be extended to the class of r.i. quasi-Banach function spaces. The proofs depend on some technical Lemmas and the built Orlicz spaces $L^F[0,1]$ are rather sophisticated (comparing with the spaces in the $(0,\infty)$ case [H-Ru]).

The second aim of this paper concerns with a similar question but now in the setting of Banach spaces X with an uncountable symmetric basis. Thus, we analyze when the $\ell^p(\Gamma)$ spaces, for uncountable sets Γ, can be isomorphically embedded into X.

In [T] Troyanski proved that if a Banach space X with a symmetric basis $(e_i)_{i\in I}$ contains an isomorphic copy of $\ell^1(\Gamma)$ for Γ uncountable then $X = \ell^1(I)$. Recently, in [H-T], it has been proved the impossibility of embedding $\ell^p(\Gamma)$-spaces for Γ uncountable into any Lorentz space $d(w,p,I)$ for any non trivial weight w

and $0 < p < \infty$. Here, we prove a positive result for the class of reflexive Orlicz spaces $\ell^F(I)$, extending a previous result given in ([H-T], Proposition 7) for the non-reflexive case. Thus, our main result in Section I is the following:

THEOREM B: *Let $0 < \alpha < p \leq \beta < \infty$. There exists an α-convex Orlicz space $\ell^F(I)$ with indices $\alpha_F = \alpha$ and $\beta_F = \beta$ containing a lattice isomorphic copy of $\ell^p(\Gamma)$ for any set $\Gamma \subset I$.*

Note that an α-convex Orlicz space $\ell^F(I)$, different from $\ell^\alpha(I)$, cannot contain an isomorphic copy of $\ell^\alpha(I)$. This follows from ([H-T], Proposition 5 and [R], Corollary 2.4). Thus the case $p = \alpha$ is solved in Section III without α-convexity:

THEOREM B': *Let $0 < \alpha = p \leq \beta < \infty$. There exists an Orlicz space $\ell^F(I) \neq \ell^p(I)$, with indices $\alpha_F = \alpha$ and $\beta_F = \beta$, containing a lattice isomorphic copy of $\ell^p(\Gamma)$ for any set $\Gamma \subset I$.*

1. Proof of Theorem B

Before to give the proof of Theorem B let us give some definitions and notations.

Given a set I and an Orlicz function F, we denote by $l^F(I)$ the Orlicz space consisting of all real-valued functions $x(i)$ defined on I for which

$$m_F\left(\frac{x}{\lambda}\right) = \sum_{i\in I} F\left(\frac{|x(i)|}{\lambda}\right) < \infty$$

for some $\lambda > 0$, endowed vith the F-norm

$$||x|| = \inf\left\{\lambda > 0 : \ m_F\left(\frac{x}{\lambda}\right) \leq \lambda\right\}.$$

This F-norm is equivalent to the Luxemburg norm when F is convex.

If F satisfies the Δ_2^0-condition (i.e. there exists $C > 0$ and $t_0 > 0$ such that $F(2t) \leq C \cdot F(t)$ for $0 \leq t \leq t_0$) then the unit vectors $(e_i)_{i\in I}$ are a symmetric basis in $\ell^F(I)$. We refer to [T], [D] and [H-T] for general properties of Banach spaces and F-spaces with an uncountable symmetric basis (f.i. all symmetric basis are equivalent).

In the countable case, the structure theory of Orlicz sequence spaces $\ell^F(N) = \ell^F$ has been extensively studied (cf. [L-T$_1$], [L-T$_2$]). For the uncountable case, the necessary and sufficient conditions for the isomorphic embedding of Orlicz spaces $\ell^G(\Gamma)$ into $\ell^F(I)$ for uncountable sets $\Gamma \subset I$ have been given recently in [R] (see also [H-T]).

Let us denote by $\Sigma_{F,1}$ the set of all the Orlicz functions G which are equivalent at 0 to a function

$$H(x) = \int_0^1 \frac{F(sx)}{F(s)} d\mu(s) \quad (0 < x < 1)$$

where μ is a probability measure on $(0,1]$. It holds that $\ell^F(I)$ contains a (lattice) isomorphic copy of $\ell^G(\Gamma)$ for uncountable sets $\Gamma \subset I$ if and only if $G \in \sum_{F,1}$.

The set $\sum_{F,1}$ is contained in $C_{F,1}$, up to equivalence, however in general is not compact. The following properties of the sets $\sum_{F,1}$ will be used: for every $q > 0$, $x^q \sum_{F,1} \cong \sum_{x^q F,1}$, and, if F_q denotes the q-convexification of the function F (i.e. $F_q(x) = F(x^q)$) then

$$\sum\nolimits_{F_q,1} = \Big(\sum\nolimits_{F,1}\Big)_q = \Big\{G_q :\ G \in \sum\nolimits_{F,1}\Big\}.$$

In the proof of Theorem B we need the following three Lemmas:

LEMMA 1.1: *There exists two sequences (α_n) and (δ_n) of 0's and 1's numbers such that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} \alpha_n \delta_{n+k} = 1, \quad \textit{and} \quad \sum_{n=k}^{\infty} \alpha_n \delta_{n-k} \le (k+2)^2$$

for every $k \in N = \{0,1,\ldots\}$.

Proof: This is a consequence of Lemma 8 in [H-T]: There exists two increasing sequences of natural numbers (m_i) and (k_i) such that $(m_{i+1} - m_i) \to \infty$ and the function

$$f(x) = \sum_{i=0}^{\infty} \chi_{[m_i, m_i+1)}(x)$$

satisfies

$$\sum_{i=0}^{\infty} f(x + k_i) = 1$$

for every $x \ge 0$. Moreover, it follows from (22) in [H-T] that

$$\sum_{i=0}^{\infty} f(k_i - k) \le (k+2)^2$$

for every $k = 1, 2, \ldots$. Then, if we take $\alpha_{k_i} = \delta_{m_i} = 1$ for $i = 0, 1, 2, \ldots$ and $\alpha_j = \delta_{j'} = 0$ in the other cases, we have

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \ \sum_{n=0}^{\infty} \alpha_n \delta_{n+k} = 1 \quad \text{and} \quad \sum_{n=k}^{\infty} \alpha_n \delta_{n-k} \le (k+2)^2.$$

for every $k \in N$. ∎

LEMMA 1.2: *Let* $\epsilon > 0$. *There exist two sequences* (α_n) *and* (ϵ_n) *of positive numbers such that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \epsilon_n \leq c\epsilon_{n+1}$$

for $n \in N$ *and* $c = 2^{\epsilon} > 1$, *verifying*

$$A \leq \sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \leq B$$

for every $k \in N$ *and where* A *and* B *are positive constants.*

Proof: We apply Lemma 1.1. Let $M = \{m_i : i = 0, 1, 2, \ldots\}$ $(m_0 = 1)$,

$$M_i = (M + i) \setminus \bigcup_{j=0}^{i-1} (M + j)$$

and

$$a_k = \sum_{n=k}^{\infty} \alpha_n \delta_{n-k} \leq (k+2)^2$$

for $k = 1, 2, \ldots$. Let us define the sequence $(\epsilon_n)_{n=0}^{\infty}$ by $\epsilon_0 = 0$, and

$$\epsilon_n = \begin{cases} 1 & \text{if } n \in M = M_0, \\ c^{-k} = \delta_{n-k} c^{-k} & \text{if } n \in M_k. \end{cases}$$

It is clear that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$ and $\epsilon_n \leq c\ \epsilon_{n+1}$ $(n \in N)$ as well as

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \geq 1$$

for every $k \in N$.

On the other hand,

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} = \sum_{i=0}^{\infty} \sum_{n+k \in M_i} \alpha_n \epsilon_{n+k}.$$

Now, as

$$\sum_{i=0}^{k} \sum_{n+k \in M_i} \alpha_n \epsilon_{n+k} \leq \sum_{i=0}^{k} \sum_{n=0}^{\infty} \frac{\alpha_n \delta_{n+k-i}}{2^{i\epsilon}} \leq \sum_{i=0}^{k} \frac{1}{2^{i\epsilon}} < \frac{1}{1 - 2^{-\epsilon}}$$

and

$$\sum_{i=k+1}^{\infty} \sum_{n+k\in M_i} \alpha_n \epsilon_{n+k} \leq \sum_{i=k+1}^{\infty} \sum_{n} \alpha_n \frac{\delta_{n+k-i}}{2^{i\epsilon}} = \sum_{i=k+1}^{\infty} \frac{a_{i-k}}{2^{i\epsilon}}$$
$$\leq \sum_{i=1}^{\infty} \frac{a_i}{2^{i\epsilon}} \leq \sum_{i=1}^{\infty} \frac{(i+2)^2}{2^{i\epsilon}} < \infty,$$

we deduce

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \leq \frac{1}{1-2^{-\epsilon}} + \sum_{i=1}^{\infty} \frac{(i+2)^2}{2^{i\epsilon}} = B < \infty$$

which concludes the proof. ∎

LEMMA 1.3: *Let $c_k = (k+1)^4$ for $k \in N$. There exists two sequences (α_n) and (ϵ_n) of positive numbers such that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \epsilon_n \leq c_k \epsilon_{n+k}$$

for $n, k \in N$, verifying

$$A \leq \sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \leq B$$

for every $k \in N$, and where A and B are positive constants.

Proof: We proceed as in Lemma 1.2, defining now the sequence $(\epsilon_n)_{n=0}^{\infty}$ by $\epsilon_0 = 0$, and

$$\epsilon_n = \begin{cases} 1 & \text{if } n \in M = M_0, \\ c_k^{-1} = \frac{\delta_{n-k}}{c_k} & \text{if } n \in M_k. \end{cases}$$

It is clear that $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\epsilon_n \leq c_k \epsilon_{n+k}$ $(n \in N)$ and

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \geq 1$$

for every $k \in N$. Now, the upper inequality follows from

$$\sum_{i=0}^{k} \sum_{n+k\in M_i} \alpha_n \epsilon_{n+k} \leq \sum_{i=0}^{k} \sum_{n=0}^{\infty} \frac{\alpha_n \delta_{n+k-i}}{c_i} \leq \sum_{i=0}^{\infty} \frac{1}{c_i} < \infty$$

and

$$\sum_{i=k+1}^{\infty} \sum_{n+k\in M_i} \alpha_n \epsilon_{n+k} \leq \sum_{i=k+1}^{\infty} \sum_{n} \frac{\alpha_n \delta_{n+k-i}}{c_i} = \sum_{i=k+1}^{\infty} \frac{a_{i-k}}{c_i}$$
$$\leq \sum_{i=1}^{\infty} \frac{a_i}{c_i} \leq \sum_{i=1}^{\infty} \frac{(i+2)^2}{(i+1)^4} < \infty. \quad ∎$$

Proof of Theorem B: We will consider three cases:

(i) First the case $\alpha = 1 < p < \beta < \infty$: We will find a convex Orlicz function F with indices $\alpha_F = 1$ and $\beta_F = \beta = (p+\epsilon)$ such that $\ell^F(I)$ contains an isomorphic copy of $\ell^p(I)$.

We will make use of Lemma 1.2: let f be the function defined by

$$f(x) = \sum_{n=0}^{\infty} \epsilon_n \chi_{(2^{-(n+1)},2^{-n}]}(x) \qquad (0 \le x \le 1)$$

and we consider the convex function $F(x)$ given by

$$F(x) = \int_0^x (x-t)t^{p-2} f(t)dt$$

for $0 \le x \le 1$. Using Lemma 1.2 we have

$$A \le \sum_{n=0}^{\infty} \alpha_n f\left(\frac{x}{2^n}\right) \le B$$

for $0 < x \le 1$. This implies by integration and the Beppo–Levi Theorem that

$$(*) \qquad \frac{Ax^p}{p(p-1)} \le \sum_{n=0}^{\infty} \alpha_n 2^{pn} F\left(\frac{x}{2^n}\right) \le B\frac{x^p}{p(p-1)}$$

for $0 \le x \le 1$.

Since

$$F\left(\frac{1}{2^n}\right) = \sum_{k=n}^{\infty} \epsilon_k \int_{2^{-k-1}}^{2^{-k}} (2^{-n}-t)t^{p-2}dt = \sum_{k=n}^{\infty} \epsilon_k 2^{-(p-1)k}(a2^{-n} - b2^{-k}),$$

with

$$a = \frac{1-2^{-(p-1)}}{p-1} \qquad \text{and} \qquad b = \frac{1-2^{-p}}{p},$$

we deduce

$$(**) \qquad 2^{pn}F(2^{-n}) = \sum_{k=0}^{\infty} (a - b2^{-k})2^{-(p-1)k}\epsilon_{n+k}.$$

Let us show that the lower index $\alpha_F = 1$. It is enough to check that

$$\sup_{m,n} \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} = \infty$$

for every $q > 1$. Indeed, for $m = m_i - n > m_{i-1}$ we have using $(**)$ that

$$2^{p(m+n)}F(2^{-m-n}) \geq (a-b)\epsilon_{m+n} = (a-b)$$

and

$$\begin{aligned} 2^{pm}F(2^{-m}) &\leq a\left(\frac{\epsilon_m}{1-2^{-(p-1)}} + \sum_{k=0}^{\infty} \epsilon_{m_i+k} 2^{-(p-1)(n+k)}\right) \\ &\leq \frac{a}{1-2^{-(p-1)}}(\epsilon_m + 2^{-(p-1)n}) \end{aligned}$$

with $\epsilon_m \to 0$ for $i \to \infty$ and n fixed. Then

$$\sup_m \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} \geq \frac{a-b}{a}(1-2^{-(p-1)})2^{(q-1)n}$$

and

$$\sup_{m,n} \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} = \infty.$$

Let us see now that $\beta_F = p + \epsilon$ (and hence F satisfies the Δ_2^0-condition). It follows from $(**)$ that

$$\frac{2^{-pn}F(2^{-m})}{F(2^{-m-n})} \leq \frac{a}{a-b} \frac{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon_{m+k}}{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon_{m+n+k}} \leq \frac{a}{a-b} 2^{\epsilon n},$$

so we deduce that $\beta_F \leq p + \epsilon$. In order to show the converse inequality, let us consider $m = m_i < m_{i+1} - n$. Then

$$(***) \qquad 2^{pm}F(2^{-m}) \geq (a-b)\epsilon_m = a - b$$

and

$$\begin{aligned} 2^{p(m+n)}F(2^{-m-n}) &\leq \frac{a}{1-2^{-(p-1)}}(\epsilon_{m+n} + 2^{-(p-1)m_{i+1}}) \\ &= \frac{a}{1-2^{-(p-1)}}(2^{-\epsilon n} + 2^{-(p-1)m_{i+1}}). \end{aligned}$$

Hence, making $i \to \infty$, we find

$$\sup_m \frac{2^{-pn}F(2^{-m})}{F(2^{-m-n})} \geq \frac{a-b}{a}(1-2^{-(p-1)})2^{\epsilon n},$$

which implies that $\beta_F \geq p + \epsilon$. Thus $\beta_F = p + \epsilon$.

Finally, it remains to show that $\ell^F(I)$ contains a subspace isomorphic to $\ell^p(\Gamma)$ for Γ uncountable. (The countable case is well-known: [L-T$_2$], Theorem 4 a 8.) Indeed, if μ denotes the discrete measure on $(0,1]$ defined by $\mu(2^{-n}) = \alpha_n 2^{pn} F(2^{-n})$, we consider the function

$$G(x) = \int_0^1 \frac{F(xt)}{F(t)} d\mu \qquad (0 \leq x \leq 1).$$

Then, by $(*)$, the function G is equivalent to x^p at 0, so $x^p \in \Sigma_{F,1}$ and, using Theorem B of [R] (or [H-T], Proposition 5), we conclude that $\ell^F(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$.

(ii) The case $\alpha = 1 < p = \beta < \infty$. We proceed as in the above case but now using the sequence (ϵ_n) of Lemma 1.3 in order to define the functions f and F: In the same way as above it is proved that $\alpha_F = 1$ and $x^p \in \sum_{F,1}$.

Now it holds that $\beta_F = p$. Indeed, using that $\epsilon_n \leq c_k \epsilon_{n+k}$, we have

$$\frac{2^{-pn}F(2^{-m})}{F(2^{-m-n})} \leq \frac{a}{a-b} \frac{\sum_{k=0}^{\infty} 2^{-(p-1)k} \epsilon_{m+k}}{\sum_{k=0}^{\infty} 2^{-(p-1)k} \epsilon_{m+n+k}} \leq \frac{a}{a-b}(n+1)^4,$$

which implies that $\beta_F \leq p$, hence $\beta_F = p$.

(iii) The general case $0 < \alpha < p \leq \beta < \infty$. It follows from the above cases (i) and (ii) that there exists a convex Orlicz function F with indices $\alpha_F = 1 < p/\alpha \leq \beta_F = \beta/\alpha$ such that $\ell^F(I)$ contains an isomorphic copy of $\ell^q(\Gamma)$ for $q = p/\alpha$. Now, if we consider the α-convex function $F_\alpha(x) = F(x^\alpha)$, we get easily, using the properties of the sets $\Sigma_{F_\alpha,1}$ and ([R], Theorem A, or [H-T], Proposition 5), that $\ell^{F_\alpha}(I)$ verifies $\alpha_{F_\alpha} = \alpha < \beta_{F_\alpha} = \beta$ and $\ell^{F_\alpha}(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$. ∎

Remark: Given an Orlicz space $\ell^F(I)$, we consider the **index** γ_F defined by

$$\gamma_F = \overline{\lim_{n\to\infty}} \frac{\log F(2^{-n})}{\log 2^{-n}}.$$

It holds that $\alpha_F \leq \gamma_F \leq \beta_F$. It follows from ([R], [H-T]) that if $\ell^q(\Gamma)$ is isomorphically embedded into $\ell^F(I)$ for $\Gamma \subset I$ uncountables, then $\alpha_F \leq q \leq \gamma_F$.

Note that the Orlicz spaces $\ell^F(I)$ constructed in Theorem B verify $\gamma_F = p$. This follows easily from inequality $(***)$.

Remark: In the non-locally bounded case the above Theorem is also true: given $\alpha = 0 < p < \beta < \infty$ there exists $\ell^F(I)$ with $\alpha_F = 0$ and $\beta_F = \beta$ such that $\ell^F(I)$ contains a subspace isomorphic to $\ell^p(\Gamma)$.

The proof is similar to Theorem B considering now the function

$$F(x) = \int_0^x t^{p-1} f(t)dt, \qquad 0 \le x \le 1,$$

where

$$f(t) = \sum_{n=0}^{\infty} \epsilon_n \chi_{(2^{-n-1}, 2^{-n}]}(t),$$

and using now ([H-T], Proposition 5 (ii)).

2. Proof of Theorem A

Our notation in this section is standard and we refer to [L-T3].

Given an Orlicz function F, let us denote by $\sum_{F,1}^{\infty}$ the set of all Orlicz functions G which are equivalent at ∞ to a function

$$H(x) = \int_0^{\infty} \frac{F(xs)}{F(s)} d\mu(s), \qquad \text{for } x \ge 1,$$

where μ is a probability measure on $(0, \infty)$ satisfying

$$\int_0^{\infty} \frac{d\mu(s)}{F(s)} \le 1.$$

The following criteria given in ([J-M-S-T], Theorem 7.7) for lattice-embeddings of function spaces into a convex Orlicz space $X = L^F[0,1]$ (as sublattices of type X_g) holds also in the quasi-Banach case (see [H-Ru]).

PROPOSITION 2.1: *Let $L^F[0,1]$ be an Orlicz space with $0 < \alpha_F^{\infty} \le \beta_F^{\infty} < \infty$. If $G \in \sum_{F,1}^{\infty}$ then $L^G[0,1]$ is lattice-isomorphic to a sublattice of $L^F[0,1]$.*

The set $\sum_{F,1}^{\infty}$ is contained in $C_{F,1}^{\infty}$, up to equivalence, but in general is not compact. For any $q > 0$ it holds that $\sum_{x^qF,1}^{\infty} = x^q \sum_{F,1}^{\infty}$ and $\sum_{F_q,1}^{\infty} = (\sum_{F,1}^{\infty})_q$ where F_q is the q-convexification of the function F at ∞.

In the proof of Theorem A we need the following:

LEMMA 2.2: *Let $\epsilon > 0$. There exists two sequences (α_n) and (ϵ_n) of positive numbers such that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \epsilon_{n+1} \le c\epsilon_n$$

for $n = 0, 1, 2, \ldots$ and $c = 2^{\epsilon} > 1$, verifying

$$A \le \sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \le B$$

for every $k = 0, 1, 2, \ldots$ and A and B positive constants. Furthermore, for every $p > 0$,

$$(+) \qquad \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{pn}} \sum_{k=0}^{n} 2^{pk} \epsilon_k < \infty.$$

Proof: Let us apply Lemma 1.1: Let $M = \{m_i\colon i = 0, 1, \ldots\}$ with $(m_{i+1} - m_i) \to \infty$ and

$$M_i = (M - i) \setminus \bigcup_{j=0}^{i-1} (M - j)$$

for $i = 1, 2, \ldots$. Let us define the sequence (ϵ_n) by

$$\epsilon_n = \begin{cases} 1 & \text{if } n \in M = M_0, \\ c^{-k} = \delta_{n+k} c^{-k} & \text{if } n \in M_k. \end{cases}$$

It is clear that if we take (α_n) as in Lemma 1.1 we have

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \epsilon_{n+1} \le c\epsilon_n \text{ for } n = 0, 1, \ldots \quad \text{and}$$

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \ge 1 = A \quad \text{for every } k = 0, 1, 2, \ldots.$$

On the other hand,

$$\begin{aligned} \sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} &= \sum_{i=0}^{\infty} \sum_{n+k \in M_i} \alpha_n \epsilon_{n+k} \\ &\le \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \alpha_n \delta_{n+k+i} c^{-i} = \sum_{i=0}^{\infty} c^{-i} = \frac{c}{c-1} = B < \infty \end{aligned}$$

for every $k = 0, 1, \ldots$.

We pass now to show (+). Let $\{k_i\colon i = 0,1,\ldots\} = \{n\colon \alpha_n = 1\}$. For fixed $k_i \in N$, let us consider

$$m_i = \max\{m_j\colon m_j \leq k_i\} \quad \text{and} \quad s_i = \min\{m_j\colon m_j > k_i\}.$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\alpha_n}{2^{pn}} \sum_{k=0}^{n} 2^{pk}\epsilon_k &\leq \sum_{i=0}^{\infty} \frac{1}{2^{pk_i}} \Big(\sum_{k=0}^{r_i} 2^{pk} + \sum_{r_i+1}^{k_i} 2^{pk}\epsilon_k \Big) \\ &\leq \frac{1}{1-2^{-p}} \sum_{i=0}^{\infty} \frac{1}{2^{p(k_i-r_i)}} + \frac{1}{1-2^{-p}} \sum_{i=0}^{\infty} \frac{1}{c^{s_i-k_i}} \\ &\leq \frac{1}{1-2^{-p}} \sum_{n=0}^{\infty} \frac{(n+2)^2}{2^{pn}} + \frac{1}{1-2^{-p}} \sum_{n=0}^{\infty} \frac{1}{c^n} < \infty,\end{aligned}$$

since from $\sum_{n=0}^{\infty} \alpha_n \delta_{n-k} \leq (k+2)^2$ (Lemma 1.1), it follows that there are at most $(n+2)^2$ values of i such that $k_i - r_i = k_i - m_j = n$, and, from $\sum_{n=0}^{\infty} \alpha_n \delta_{n+k} = 1$, there is at most one value of i such that $s_i - k_i = m_j - k_i = n$. This concludes the proof. ∎

LEMMA 2.3: *Let $c_k = (k+1)^2$ for $k \in N$. There exists two sequences (α_n) and (ϵ_n) of positive numbers such that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \epsilon_{n+k} \leq c_k \epsilon_n$$

for $n, k \in N$, verifying

$$A \leq \sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \leq B$$

for every $k \in N$, and A and B positive constants. Furthermore, for every $p > 0$,

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{2^{pn}} \sum_{k=0}^{n} 2^{pk}\epsilon_k < \infty.$$

Proof: It is similar to Lemma 2.2 considering now the sequence $(\epsilon_n)_{n=0}^{\infty}$ defined by

$$\epsilon_n = \begin{cases} 1 & \text{if } n \in M = M_0, \\ {c_k}^{-1} = \delta_{n+k}/c_k & \text{if } n \in M_k. \end{cases}$$ ∎

Proof of Theorem A: It is sufficient to consider the case $\alpha = 1 < p \leq \beta$ and F convex, since the general case can be deduced from this by considering the α-convex function $F_\alpha(x) = F(x^\alpha)$ and Proposition 2.1.

(i) Let $\alpha = 1 < p < \beta = p + \epsilon$. We define the function f on $[1, \infty)$, by

$$f(x) = \sum_{n=1}^{\infty} \epsilon_n \chi_{(2^{n-1}, 2^n]}(x),$$

where (ϵ_n) are as in Lemma 2.2, and the convex function $F(x)$ is defined by

$$F(x) = 1 + \int_1^x (x - t) t^{p-2} f(t) dt,$$

for $x \geq 1$. It follows from Lemma 2.2 that the function f satisfies

$$A \leq \sum_{n=0}^{\infty} \alpha_n f(2^n x) \leq B$$

for $x \geq 1$. Now, integrating and using the Beppo–Levi Theorem we get

$$A \frac{x^{p-1} - 1}{p - 1} \leq \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{(p-1)n}} (F'(2^n x) - F'(2^n)) \leq B \frac{x^{p-1} - 1}{p - 1}$$

for $x \geq 1$.

It holds that

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{2^{(p-1)n}} F'(2^n) < \infty.$$

Indeed, this follows from Lemma 2.2 and the equality

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{(p-1)n}} F'(2^n) &= \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{(p-1)n}} \sum_{k=1}^{n} \int_{2^{k-1}}^{2^k} t^{p-2} f(t) dt \\ &= \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{(p-1)n}} \sum_{k=1}^{n} \frac{\epsilon_k}{p-1} 2^{(p-1)k} (1 - 2^{-(p-1)}). \end{aligned}$$

Thus, for $x \geq 1$ we have

$$A \frac{x^{p-1}}{p-1} + a_0 \leq \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{(p-1)n}} F'(2^n x) \leq B \frac{x^{p-1}}{p-1} + b_0$$

and

$$\begin{aligned} A \frac{x^p}{p(p-1)} + a_0 x + a_1 &\leq \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{pn}} [F(2^n x) - F(2^n)] \\ &\leq B \frac{x^p}{p(p-1)} + b_0 x + b_1 \end{aligned}$$

where a_0, a_1, b_0 and b_1 are constants.

It holds also that

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{2^{pn}} F(2^n) < \infty.$$

Indeed,

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{2^{pn}} \int_1^{2^n} (2^n - t)t^{p-2} f(t)dt \leq \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{(p-1)n}} \int_1^{2^n} t^{p-2} f(t)dt$$
$$= \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{(p-1)n}} F'(2^n) < \infty.$$

Hence we deduce that there exists constants $A_0, B_0 > 0$ such that F satisfies

$$(-) \qquad A_0 x^p \leq \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{pn}} F(2^n x) \leq B_0 x^p$$

for $x \geq 1$.

Let us deduce now that $L^F[0,1]$ contains a sublattice lattice-isomorphic to L^p. Indeed, if μ is the discrete measure on $[1,\infty)$ defined by

$$\mu(2^k) = \frac{\alpha_k 2^{-pk} F(2^k)}{\sum_{n=0}^{\infty} \alpha_n 2^{-pn} F(2^n)},$$

we get from $(-)$ that the function

$$G(x) = \int_1^{\infty} \frac{F(xt)}{F(t)} d\mu(t), \qquad \text{for } x \geq 1,$$

is an Orlicz function equivalent to x^p at ∞. Now, as F satisfies the Δ_2^{∞}-condition (we prove it below), we can apply Proposition 2.1 to conclude that L^p is lattice embedded into $L^F[0,1]$.

We pass to compute the associated indices to F. Since

$$(*) \quad F(2^n) = 1 + \sum_{k=1}^{n} \int_{2^{k-1}}^{2^k} (2^n - t)t^{p-2} f(t)dt = 1 + \sum_{k=1}^{n} \epsilon_k 2^{(p-1)k}(a2^n - b2^k)$$

where $a = (1 - 2^{-(p-1)})/(p-1)$ and $b = (1 - 2^{-p})/p$, we have

$$\frac{F(2^{m+n})}{2^n F(2^m)} \leq \frac{\sum_{k=1}^{m+n} \epsilon_k 2^{(p-1)k}(a2^{m+n} - b2^k)}{2^n \sum_{k=1}^{m} \epsilon_k 2^{(p-1)k}(a2^m - b2^k)} \leq \frac{a}{a-b}\left(1 + \frac{\sum_{k=m+1}^{m+n} \epsilon_k 2^{(p-1)k}}{\sum_{k=1}^{m} \epsilon_k 2^{(p-1)k}}\right)$$

$$(**) \qquad \leq \frac{a}{a-b}\left(1 + \frac{\sum_{k=m+1}^{m+n} \epsilon_k 2^{(p-1)k}}{2^{(p-1)m}\epsilon_m}\right).$$

Now, let us prove that $\alpha_F^\infty = 1$. Indeed, taking $m = m_i$ and $n < m_{i+1} - m_i$, we have using $(\overline{**})$ that

$$\frac{F(2^{m+n})}{2^n F(2^m)} \le \frac{a}{a-b}\left(1 + \frac{\sum_{k=m+1}^{m+n} \epsilon_k 2^{(p-1)k}}{2^{(p-1)m}}\right) \le \frac{a}{a-b}\left(1 + \frac{2^{(p-1)n}}{1-2^{-(p-1)}}\epsilon_{m+n}\right).$$

This implies that

$$\inf_m \frac{F(2^{m+n})}{F(2^m)} \le \frac{2^n a}{a-b}$$

since $\epsilon_{m_i+n} \to 0$. Hence

$$\inf_{m,n} \frac{F(2^{m+n})}{2^{qn}F(2^m)} = 0$$

for every $q > 1$, which means that $\alpha_F^\infty = 1$.

Finally, let us prove that $\beta_F^\infty = p + \epsilon = \beta$, and hence F satisfies the Δ_2^∞-condition. First let us see that $\beta_F^\infty \le p + \epsilon$. Indeed, from $(\overline{**})$ we get for $c = 2^\epsilon$ that

$$\frac{F(2^{m+n})}{2^n F(2^m)} \le \frac{a}{a-b}\left(1 + \frac{2^{(p-1)n}c^n}{1-c^{-1}}\right)$$

and

$$\frac{F(2^{m+n})}{2^{pn} F(2^m)} \le \frac{a}{a-b}\left(\frac{1}{2^{(p-1)n}} + \frac{2^{\epsilon n}}{1-2^{-\epsilon}}\right).$$

And this implies that $\beta_F^\infty \le p + \epsilon$.

Let us prove now the converse $\beta_F^\infty \ge p + \epsilon$. By using $(\overline{*})$ with $m = m_{i+1} - n > m_i$ we have

$$(\overline{***}) \qquad F(2^{n+m}) - 1 \ge 2^{(p-1)(m+n)}(a-b)2^{m+n}$$

and

$$\begin{aligned} F(2^m) - 1 &\le \sum_{k=1}^{m} \epsilon_k 2^{(p-1)k} a 2^m \\ &\le a2^m\left(\sum_{k=1}^{m_i} 2^{(p-1)k} + \sum_{k=m_i+1}^{m} \epsilon_k 2^{(p-1)k}\right) \\ &\le a2^m(1-2^{-(p-1)})^{-1}(2^{(p-1)m_i} + \epsilon_m 2^{(p-1)m}). \end{aligned}$$

Hence

$$\frac{F(2^{m+n})-1}{F(2^m)-1} \ge \frac{a-b}{a}(1-2^{-(p-1)})\frac{2^{(p-1)(m+n)}2^n}{2^{-n\epsilon}2^{(p-1)m} + 2^{(p-1)m_i}}.$$

Now, making $i \to \infty$ we have

$$\sup_m \frac{F(2^{m+n})}{F(2^m)} \geq (\frac{a-b}{a})(1-2^{-(p-1)})2^{(p+\epsilon)n}.$$

And this implies that $\beta_F^\infty \geq p+\epsilon$, hence $\beta_F^\infty = p+\epsilon$.

(ii) The case $\alpha = 1 < p = \beta < \infty$. We proceed as in the above case using now the sequence (ϵ_n) of Lemma 2.3 in order to define the functions f and F. In the same way as above it is proved that $\alpha_F^\infty = 1$ and $x^p \in \Sigma_{F,1}^\infty$. It holds also that $\beta_F^\infty = p$. Indeed, using that $\epsilon_{n+k} \leq c_k \epsilon_n$, we have

$$\begin{aligned}\frac{F(2^{m+n})}{2^n F(2^m)} &\leq \frac{a}{a-b}\Big(1+\frac{\sum_{k=m+1}^{m+n}\epsilon_k 2^{(p-1)m}}{2^{(p-1)m}\epsilon_m}\Big)\\ &\leq \frac{a}{a-b}\Big(1+c_n\sum_{k=m+1}^{m+n}2^{(p-1)(k-m)}\Big)\\ &\leq \frac{a}{a-b}\Big(1+\frac{2^{(p-1)n}c_n}{1-2^{-p}}\Big),\end{aligned}$$

which implies that $\beta_F^\infty \leq p$, hence $\beta_F = p$. ∎

Remark: In the special case of $p = 1$, Theorem A proves that there exist r.i. quasi-Banach function spaces $X[0,1]$ containing a sublattice isomorphic to L^1. (Compare with the convex case [K$_1$], Theorem 3.2.)

3. Proof of Theorems A' and B'

Proof of Theorem B': We can only consider w.l.o.g. $1 < \alpha = p \leq \beta < \infty$.

(i) The case $1 < \alpha = p = \beta < \infty$. Let (α_n) and (ϵ_n) be as in Lemma 1.3. We consider

$$\epsilon_n' = \sum_{k=0}^{\infty}\frac{\epsilon_{n+k}}{c_k}, \qquad \text{where } c_k = (k+1)^4.$$

Then $\epsilon_n' \leq c_k \epsilon_{n+k}'$ and there exists positive constants A' and B' such that

$$A' \leq \sum_{n=0}^{\infty}\alpha_n \epsilon_{n+k}' \leq B'$$

for $k \in N$. Let

$$f(x) = \sum_{n=0}^{\infty}\epsilon_n' \chi_{(2^{-(n+1)},2^{-n}]}(x)$$

for $0 < x \leq 1$, and

$$F(x) = \int_0^x (x-t)t^{p-2}f(t)dt.$$

It holds that for $0 \leq x \leq 1$,

$$(\sharp) \qquad A'\frac{x^p}{p(p-1)} \leq \sum_{n=0}^{\infty} \alpha_n 2^{pn} F\left(\frac{x}{2^n}\right) \leq B'\frac{x^p}{p(p-1)},$$

which implies, by using ([R], Theorem B or [H-T], Proposition 5), that $\ell^F(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$ for Γ uncountable.

Let us show that $\alpha_F = p = \beta_F$. Like in Theorem B we have

$$(\sharp\sharp) \qquad 2^{pn}F(2^{-n}) = \sum_{k=0}^{\infty}(a - b2^{-k})2^{-(p-1)k}\epsilon'_{n+k}$$

and

$$\frac{2^{pn}F(2^{-m-n})}{F(2^{-m})} \leq \frac{a}{a-b}\frac{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon'_{m+n+k}}{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon'_{m+k}} \leq \frac{a}{a-b}c_n = \frac{a}{a-b}(n+1)^4,$$

since

$$\epsilon'_m \geq \sum_{k=0}^{\infty}\frac{\epsilon_{m+n+k}}{c_{n+k}} \geq \frac{1}{c_n}\sum_{k=0}^{\infty}\frac{\epsilon_{m+n+k}}{c_k} = \frac{1}{c_n}\epsilon'_{m+n}.$$

Hence $\alpha_F \geq p$, so $\alpha_F = p$. Also, from $(\sharp\sharp)$ it follows that

$$\frac{2^{-pn}F(2^{-m})}{F(2^{-m-n})} \leq \frac{a}{a-b}\frac{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon'_{m+k}}{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon'_{m+n+k}} \leq \frac{a}{a-b}(n+1)^4,$$

which implies $\beta_F \leq p$, hence $\beta_F = p$.

Finally, let us note that the constructed function F is not equivalent to x^p at 0. Indeed, it follows from $(\sharp)$ and $\sum \alpha_n = \infty$ that $\underline{\lim}_{n\to\infty} 2^{pn}F(2^{-n}) = 0$.

(ii) The case $1 < \alpha = p < \beta < \infty$. We proceed as in the above case but considering now (α_n) and (ϵ_n) as defined in Lemma 1.2.

Let

$$\epsilon'_n = \sum_{k=0}^{\infty}\frac{\epsilon_{n+k}}{c_k}, \qquad \text{where } c_k = (k+1)^2.$$

Then $\epsilon'_n \le 2^{k\epsilon}\epsilon'_{n+k}$ for $\epsilon = (\beta - p)$, and there exist positive constants A' and B' such that

$$A' \le \sum_{n=0}^{\infty} \alpha_n \epsilon'_{n+k} \le B'$$

for $k \in N$. Let

$$f(x) = \sum_{n=0}^{\infty} \epsilon'_n \chi_{(2^{-(n+1)}, 2^{-n}]}(x)$$

and

$$F(x) = \int_o^x (x-t)t^{p-2} f(t)dt$$

for $0 \le x \le 1$. Reasoning as in (i) we get that $\ell^F(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$ for uncountable Γ, and also that $\alpha_F = p$.

Finally, let us show that $\beta_F = (p+\epsilon)$. Since

$$2^{pn}F(2^{-n}) = \sum_{k=0}^{\infty} (a - b2^{-k})2^{-(p-1)k}\epsilon'_{n+k}$$

we have

$$\frac{2^{-pn}F(2^{-m})}{F(2^{-m-n})} \le \frac{a}{a-b} \frac{\sum_{k=0}^{\infty} 2^{-(p-1)}\epsilon'_{m+k}}{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon'_{m+n+k}} \le \frac{a}{a-b}2^{\epsilon n},$$

which implies that F is $(p+\epsilon)$-concave and $\beta_F \le (p+\epsilon)$. Now for $m = m_i < m_{i+1} - n$ we have

$$2^{pm}F(2^{-m}) \ge (a-b)\epsilon'_m \ge a - b$$

and

$$2^{p(m+n)}F(2^{-m-n}) \le a\sum_{k=0}^{\infty} \epsilon'_{m+n+k}2^{-(p-1)k}$$
$$\le a\epsilon'_{m+n}\sum_{k=0}^{\infty} c_k 2^{-(p-1)k} = a'\epsilon'_{m+n}.$$

Hence

$$\sup_m \frac{2^{-pn}F(2^{-m})}{F(2^{-m-n})} \ge \sup_{m=m_i} \frac{a-b}{a'}\frac{1}{\epsilon'_{m+n}} \ge a''2^{\epsilon n}$$

where a'' is a positive constant, since

$$\epsilon'_{m+n} \le 2^{-\epsilon n} \sum_{k=0}^{m_{i+1}-m-n-1} \frac{1}{c_k} + \sum_{k=m_{i+1}-m-n}^{\infty} \frac{1}{c_k} \longrightarrow 2^{-\epsilon n}\sum_{k=0}^{\infty}\frac{1}{c_k}.$$

And this implies that $\beta_F \geq (p+\epsilon)$, so $\beta_F = (p+\epsilon)$. ∎

Proof of Theorem A': Let $0 < \alpha \leq p \leq \beta < r < \infty$, we consider $\alpha_0 = r - \beta$, $p_0 = r - p$ and $\beta_0 = r - \alpha$. It follows from Theorem B' that there exists an Orlicz function F_0 with indices $\alpha_{F_0} = \alpha_0$, $\beta_{F_0} = \beta_0$ such that the function

$$G_0(x) = \int_0^1 \frac{F_0(xt)}{F_0(t)} d\mu_0(t),$$

for $0 \leq x \leq 1$, is equivalent to x^{p_0} at 0, where μ_0 is a probability measure on $(0,1]$. Since $r > \beta_0$, we can assume w.l.o.g. that $F_0(st) \geq s^r F_0(t)$ for $0 \leq s$, $t \leq 1$.

We consider now the non-decreasing function F defined by

$$F(x) = x^r F_0\Big(\frac{1}{x}\Big), \qquad \text{for } x \geq 1.$$

It holds that $\alpha_F^\infty = r - \beta_0 = \alpha$ and $\beta_F^\infty = r - \alpha_0 = \beta$.

Furthermore, the function $G(x) = x^r G_0(x^{-1})$ for $x \geq 1$ verifies

$$G(x) = \int_1^\infty \frac{F(xt)}{F(t)} d\mu(t) \qquad (x \geq 1),$$

where μ is the probability measure on $[1, +\infty)$ defined by $\mu(t) = \mu_0(\frac{1}{t})$, and G is equivalent to the function $x^{r-p_0} = x^p$ at ∞. Hence $x^p \in \sum_{F,1}^\infty$ and, by Proposition 2.1, we conclude that L^p is lattice-isomorphic to a sublattice of $L^F[0,1]$. ∎

Note that the above proof can be also used in proving partially Theorem A. Given an Orlicz space $L^F[0,1]$ we consider the **index** γ_F^∞ defined by

$$\gamma_F^\infty = \overline{\lim_{n\to\infty}} \frac{\log F(2^n)}{\log 2^n}.$$

It holds that $\alpha_F^\infty \leq \gamma_F^\infty \leq \beta_F^\infty$. Let us denote by Q_F^∞ the set of $q > 0$ such that L^q is isomorphically embedded into $L^F[0,1]$. It follows from ([J-M-S-T], Theorem 7.1) that if $\alpha_F^\infty > 1$, then

$$Q_F^\infty \subset [\gamma_F^\infty, \beta_F^\infty \text{v} 2] \cup \{2\}.$$

And in the case of $\gamma_F^\infty < 2$ we have, by using ([J-M-S-T], Proposition 8.9), that

$$(\gamma_F^\infty, 2] \subset Q_F^\infty.$$

Thus, as a direct consequence of Theorems A and A' and inequality $(\overline{***})$ we have the following:

COROLLARY 3.1: *Let* $1 < \alpha \le p \le \beta \le 2$. *There exists an Orlicz function space* $L^F[0,1]$ *with indices* $\alpha_F^\infty = \alpha$, $\gamma_F^\infty = p$ *and* $\beta_F^\infty = \beta$ *such that* $Q_F^\infty = [\gamma_F^\infty, 2]$.

Notice that in the above result the isomorphic embedding of L^p into $L^F[0,1]$ for $p = \gamma_F^\infty$ cannot be obtained using ([J-M-S-T], Proposition 8.9). Indeed, the function $t^{-1/p} \notin L^F[0,1]$ since

$$\int_1^\infty \frac{F(x)}{x^{p+1}} dx \ge \frac{1}{2^p} \sum_{n=1}^\infty \frac{F(2^n)}{2^{np}}$$

and this series is divergent because $\overline{\lim_{n\to\infty}} F(2^n)/2^{np} > 0$.

The constructed Orlicz spaces $L^F[0,1]$ having a sublattice isomorphic to L^p for $p > 0$ verify $x^p \in \sum_{F,1}^\infty$, so the inclusion map $L^p[0,1] \hookrightarrow L^F[0,1]$ holds. This suggests asking whether there exists any sublattice of $L^F[0,1]$ where the norms $\| \ \|_F$ and $\| \ \|_p$ are equivalent: in other words, whether the inclusion map $L^p[0,1] \hookrightarrow L^F[0,1]$ is or is not disjointly strictly-singular. Recall that an operator T from a Banach lattice E to a Banach space is **disjointly strictly-singular** ([H-R], p. 48) if there is not a disjoint sequence of non-null vectors (x_n) in E such that the restriction of T to the span $[x_n]$ is an isomorphism.

PROPOSITION 3.2: *Let* $L^F[0,1]$ *be a Orlicz space different from* L^p *with* $0 < \alpha_F^\infty \le \beta_F^\infty < \infty$. *If* $x^p \in \sum_{F,1}^\infty$ *then the inclusion map* $L^p[0,1] \hookrightarrow L^F[0,1]$ *is disjointly strictly-singular.*

Proof: From $x^p \in \sum_{F,1}^\infty$ and the Δ_2^∞-condition we get that there exist positive constants A and B and a positive sequence (α_n) such that

$$Ax^p \le \sum_{-\infty}^{+\infty} \alpha_n F(2^n x) \le Bx^p$$

for $x \ge 1$, and $\sum_{-\infty}^{+\infty} \alpha_n < \infty$.

We claim that $\sum_{n=0}^\infty \alpha_n 2^{pn} = \infty$. Indeed, assume that $\sum_{n=0}^\infty \alpha_n 2^{pn} < \infty$. Then, for $M > 0$ verifying $F(x) \le Mx^p$ at ∞, there exists $m \in M$ such that $\sum_{n=m+1}^\infty \alpha_n 2^{pn} < A/2M$, and hence

$$Ax^p \le \sum_{-\infty}^{m} \alpha_n F(2^n x) + M \sum_{m+1}^{\infty} \alpha_n 2^{pn} x^p \le \sum_{-\infty}^{m} \alpha_n F(2^m x) + \frac{A}{2} x^p$$

for $x \geq 1$. Hence

$$\frac{A}{2}x^p \leq \Big(\sum_{-\infty}^{m} \alpha_n\Big) F(2^m x)$$

for $x \geq 1$, which implies that $F \sim x^p$ at ∞, a contradiction.

Now, for any constant $C > 0$ there exists an integer m such that

$$C \sum_{n=0}^{\infty} \alpha_n F(2^n x) \leq CBx^p \leq \sum_{0}^{m} \alpha_n (2^n x)^p$$

for any $x \geq 1$. And, using Proposition 3.2 (b) in ([H-R]) which holds also for $p < 1$, we conclude that the inclusion $L^p[0,1] \hookrightarrow L^F(0,1]$ is disjointly strictly-singular. ∎

In particular if $\alpha_F^\infty > 2$ and $L^F[0,1]$ contains an isomorphic copy of L^p for $p \neq 2$, then the inclusion map $L^p[0,1] \hookrightarrow L^F[0,1]$ is disjointly strictly-singular. This follows from the above Proposition and Theorem 7.7 in [J-M-S-T].

Remark: For a fixed Orlicz space $L^F[0,1]$, let P_F^∞ be the set $\{p > 0\colon x^p \in \sum_{F,1}^{\infty}\}$. It would be interesting to know the structure of the sets P_F^∞.

References

[C₁] N. L. Carothers, *Rearrangement-invariant subspaces of Lorentz function spaces*, Israel Journal of Mathematics **40** (1981), 217–228.

[C₂] N. L. Carothers, *Rearrangement-invariant subspaces of Lorentz function spaces II*, Rocky Mountain Journal of Mathematics **17** (1987), 607–616.

[D] L. Drewnowski, *On symmetric bases in nonseparable Banach spaces*, Studia Mathematica **85** (1987), 157–161.

[H-R] F. L. Hernández and B. Rodríguez-Salinas, *On ℓ^p-complemented copies in Orlicz spaces II*, Israel Journal of Mathematics **68** (1989), 27–55.

[H-Ru] F. L. Hernández and C. Ruiz, *Universal classes of Orlicz function spaces*, Pacific Journal of Mathematics **155** (1992), 87–98.

[H-T] F. L. Hernández and S. L. Troyanski, *On the representation of the uncountable symmetric basic sets and its applications*, Studia Mathematica **107** (1993), 287–304.

[J-M-S-T] W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Memoirs of the American Mathematical Society **217** (1979), 1–298.

[K1] N. J. Kalton, *Embedding L^1 in a Banach lattice*, Israel Journal of Mathematics **32** (1979), 209–220.

[K2] N. J. Kalton, *Lattice structures on Banach spaces*, Memoirs of the American Mathematical Society **493** (1993), 1–92.

[L-T1] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces III*, Israel Journal of Mathematics **14** (1973), 368–389.

[L-T2] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence Spaces*, Springer-Verlag, Berlin, 1977.

[L-T3] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II, Function Spaces*, Springer-Verlag, Berlin, 1979.

[R] B. Rodríguez-Salinas, *Subspaces with symmetric basis in the Orlicz spaces* $\ell^F(I)$, preprint.

[T] S. L. Troyanski, *On non-separable Banach spaces with a symmetric basis*, Studia Mathematica **53** (1975), 253–263.