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EXTREMAL SYMPLECTIC LATTICES

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ABSTRACT

We study the Jacobians of the genus 3 Picard and Fermat curves with respect to the problem of maximizing the minimum non-zero norm. We use criteria for symplectic lattices related to the criteria of perfect and eutactic for classical lattices. We show that the Picard curve is a local maximum, but the Fermat curve is not.

1. Introduction

The study of extremal symplectic lattices is motivated by the study of extremal Riemann surfaces. The goal is to generalize classic theorems about extremal lattices to Riemann surfaces. In the classic theory of lattices the problem is to maximize the minimum non-zero norm over the space of all n-dimensional lattices. For Riemann surfaces the problem is to maximize for a given genus either the minimum non-zero length in the length spectrum or the minimum non-zero norm in the Jacobian. These minimal elements are referred to as systoles. The systole problems for the length and Jacobi spectra appear to be closely related in genus ≤ 3 (see Quine [Q1][Q2]).

For an overview and results for large genus, we cite the papers of Gromov [G] and Buser and Sarnak [BS]. Extensive studies of the systole problem for the length spectrum can be found in papers of Schmutz [S1] [S2] [S3]. Some new perspectives on classical results about extreme lattices can be found in Conway and Sloane [CS2].

In the classical theory, a basic theorem is the theorem of Voronoi [V] that a lattice is extreme if and only if it is perfect and eutactic ((P) and (E), see section 3.2). The paper of Schmutz [S1] shows how to generalize these ideas to a study of

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the length spectrum. To apply these ideas to the Jacobian, a first step is to study the analogues of the properties (P) and (E) for symplectic lattice. This has been done recently by Bergé and Martinet [BM1], who have adapted the concepts of perfect and eutactic so they can be used to study symplectic and isodual lattices and in general any family of lattices invariant under a subgroup of the linear group. (They have also done a study of classical eutactic lattices [BM2].)

The purpose of this paper is to use the criteria of Bergé and Martinet to study Jacobians of certain genus 3 Riemann surfaces. The main example is the Jacobian of the Picard curve studied in [Q2] and in [CS3]. In this paper we show it is a local max on the space of 6-dimensional symplectic lattices for $\mu(L)$, the minimum non-zero norm. In [Q2] we showed how to compute this Jacobian and gave a cyclotomic construction for it. This lattice, $M(E_6)$, was discovered by Conway and Sloane but was not known to be symplectic or a Jacobian until [Q2]. Before that, the Klein curve seemed the most likely candidate for the extremal Jacobian in genus 3. So now the most likely genus 3 curve to maximize the length of the systole for both the length spectrum and the Jacobi spectrum seems to be the Picard curve (see Schmutz [S1] for the evidence on the length spectrum).

We will outline the contents of the paper. In section 2 we explain the concept of symplectic lattice with a particular emphasis on viewing it as embedded in $\mathbf{C}^g$ with the usual Hermitian inner product. We give three examples of Jacobians of genus 3 surfaces, the Klein curve, the Fermat curve and the Picard curve. The theory of the Jacobian of the Klein curve, $A_6^{(2)}$, is well known since it is one of only six extreme lattices of dimension 6. This classification was completed by Barnes [B] (see also Conway and Sloane [CS2]). We study it here as an illustration of the techniques which we set up to study these lattices in terms of their cyclotomic construction. In a cyclotomic construction the lattice is viewed as a ring of cyclotomic integers with a certain trace norm. The construction can also be given very explicitly as the set of integer combinations of a set of basis vectors in $\mathbf{C}^g$.

In section 3 we review the concepts of perfect and eutactic and the ideas of Bergé and Martinet as they apply to symplectic lattices. The basic theorem is analogous to the theorem of Voronoi. A symplectic lattice is strictly extremal for symplectic lattices of the same dimension if and only if it is perfect and eutactic for symplectic lattices. The form in which we state these conditions, (P(symp)) and (E(symp)) in section 3.4, is suited to studying these lattice in their complex form with a cyclotomic construction.

In section 4 we illustrate the results of section 3 by applying them to $A_6^{(2)}$, showing again that it is eutactic. Next we study the Jacobian of the Fermat curve. We show that it is eutactic for symplectic lattices, but not perfect for symplectic lattices. We show it is not a local max of $\mu(L)$ for symplectic lattices and that it is eutactic in the classical sense. Next we look at the Jacobian of the Picard curve. We show that it satisfies both (P(symp)) and (E(symp)), and is therefore a strict local max of $\mu(L)$ for symplectic lattices. We show that it is not eutactic in the classical sense. In most of these computations we can make use of the cyclotomic constructions. The sums involved can usually be reduced to $\sum_{k=0}^{n-1} \zeta^k = 0$ where ζ is a primitive nth root of unity.

We remark that for genus 3 surfaces, since the dimension of the Teichmüller space is the same as that of the Siegel upper half plane = the space of symplectic lattices, the study of extremal Jacobians and the study of extremal symplectic lattices is the same. In higher genus it will be necessary to develop a theory of the Jacobian as a function on the Teichmüller space.

2. Symplectic lattices

2.1. Definitions. Let L be a $2g$-dimensional lattice in $\mathbf{R}^{2g}$ considered as the set of integer combinations of a set of basis vectors together with the standard inner product inherited from $\mathbf{R}^{2g}$. We will think of vectors in $\mathbf{R}^{2g}$ as column vectors of the form $\begin{pmatrix} x \\ y \end{pmatrix}$ where x and y are $g \times 1$ dimensional column matrices. It is convenient to identify $\mathbf{R}^{2g}$ with $\mathbf{C}^g$ by identifying $\begin{pmatrix} x \\ y \end{pmatrix}$ with $z = x+iy$. The standard inner product on $\mathbf{R}^{2g}$ of $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ is $\begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} \begin{pmatrix} u \\ v \end{pmatrix}$. If $z = x + iy$ and $w = u + iv$, then

$$\Re(z^*w) = \begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad \Im(z^*w) = \begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} J \begin{pmatrix} u \\ v \end{pmatrix}$$

where

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Suppose the complex vectors z_j , $j = 1, \dots, 2g$ form a basis for L. The matrices

$$\Lambda_{\mathrm{r}} = \begin{pmatrix} x_1 & \dots & x_{2g} \\ y_1 & \dots & y_{2g} \end{pmatrix} \quad \text{and} \quad \Lambda_{\mathrm{c}} = (z_1, \dots, z_{2g})$$

are the real and complex (resp.) generator matrices of the lattice with respect to the given basis. Corresponding to these are the real and complex Gram matrices

$\Lambda_r^t\Lambda_r$ and $\Lambda_c^*\Lambda_c$. We have

$$\Re\Lambda_c^*\Lambda_c = \Lambda_r^t\Lambda_r \quad \text{and} \quad \Im\Lambda_c^*\Lambda_c = \Lambda_r^t J\Lambda_r.$$

We see that $\det\Re\Lambda_c^*\Lambda_c = \det\Im\Lambda_c^*\Lambda_c$. This value is independent of the basis and is by definition the determinant of the lattice.

The lattice L is said to be symplectic is there is a basis such that $\Im\Lambda_c^*\Lambda_c = J$, or equivalently, $\Lambda_r^t J\Lambda_r = J$. Such a basis is called a symplectic basis. It follows from the definition that if L is symplectic then its determinant is 1. To verify that a lattice is symplectic it is not necessary to find a symplectic basis. If the determinant is 1 and the imaginary part of the Gram matrix is integral, then by the theory of skew symmetric forms it is known that there is a symplectic basis.

A real matrix M is said to be symplectic if $M^t JM = J$. The set of such matrices forms a group, $\mathrm{Sp}_g(\mathbf{R})$, which is closed under transpose. The real generator matrix of a symplectic lattice with respect to a symplectic basis is symplectic. The corresponding Gram matrix is a symmetric symplectic matrix.

For a symplectic lattice multiplication by i sends L onto its dual L^*.

We now give some examples of symplectic lattices which can be obtained as Jacobians of genus three curves. In the following it is helpful to keep in mind the following remarks:

(1) If L' is a sublattice of a lattice L, then the number of elements of L/L' is $(\det L'/\det L)^{1/2}$.

(2) For a n-dimensional lattice L and a scalar a, $\det aL = |a|^{2n}\det L$.

2.2. JACOBIAN OF THE KLEIN CURVE. The Jacobian of the Klein curve $x^3y + y^3z + z^3x = 0$ is the lattice $7^{-1/4}A_6^{(2)}$ (see Quine [Q1], Mazur [M] for more details). The cyclotomic construction of $A_6^{(2)}$ is as follows. Let $\zeta = \exp(2\pi i/7)$. Identify an element $p(\zeta)$ of $\mathbf{Z}[\zeta]$ with the vector

$$v(p) = (p(\zeta)(1-\zeta)^2, p(\zeta^2)(1-\zeta^2)^2, p(\zeta^4)(1-\zeta^4)^2)^t$$

in $\mathbf{C}^3$ with the usual Hermitian inner product, making $\mathbf{Z}[\zeta]$ into a lattice, L. The elements $\zeta^k, k = 0, \ldots, 5$ form a basis. Using the Vandermonde determinant, the determinant of the real part of the Gram matrix with respect to this basis can be computed to be $7^9/2^6$. The imaginary part of the inner product is in $(7^{3/2}/2)\mathbf{Z}$. Thus $2^{1/2}7^{-3/4}L$ is a symplectic lattice and is the Jacobian of the Klein curve.

The lattice L can be identified as $7^{-1/4}A_6^{(2)}$ as follows. The number $p(\zeta)(1-\zeta)^2$, $p \in \mathbf{Z}[\zeta]$ can be written uniquely in the form $\sum_{n=0}^{6} a_n\zeta^n$ with $\sum_{n=0}^{6} a_n = 0$, and using this equation we can verify that

$$\tfrac{2}{7}v(p)^*v(p) = \sum a_k^2.$$

Since the derivative with respect to ζ of $p(\zeta)(1-\zeta)^2$ is zero at $\zeta = 1$, we have also $\sum_{n=1}^{6} na_n = 0 \pmod 7$. The two equations

$$\sum_{n=0}^{6} a_n = 0, \quad \sum_{n=1}^{6} na_n = 0 \pmod 7$$

give a description of $A_6^{(2)}$ as a sublattice of $\mathbf{Z}^7$.

2.3. Jacobian of the Fermat Curve. The following six-dimensional lattice can be constructed as the Jacobian of the Fermat curve $x^4 + y^4 = z^4$. See the book of Lang [L] and the appendix of Rohrlich to the paper of Gross [R] [G] for details on computing Jacobians of Fermat curves. Also Tretkoff [T] has details of this computation and explicit matrics for the genus three curve. The construction given here uses the Gaussian integers and is based on the relationship between codes and lattices. It has the advantage of giving the metric structure for the lattice.

Let $\mathcal{G}$ be the Gaussian integers $\mathbf{Z}[i]$, and let $\mathcal{G}_2 = \mathcal{G}/2\mathcal{G}$. Then $\mathcal{G}_2$ is a group of order 4 isomorphic to $(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/2\mathbf{Z})$. Let $\mathcal{C}$ be the code over $\mathcal{G}_2$ which is the subgroup of order 8 of $(\mathcal{G}_2)^3$ consisting of the elements $(0,0,0)^{\mathrm{t}}, (1,1,1)^{\mathrm{t}}, (i,i,1)^{\mathrm{t}}$, $(1+i, 1+i, 0)^{\mathrm{t}}$ and all vectors obtained from the latter two by permutation of coordinates. Let L be the sublattice of $(\mathcal{G})^3$ defined by

$$L = \bigcup_{\gamma \in \mathcal{C}} (\gamma + 2\mathcal{G}^3).$$

Since $(\mathcal{G})^3$ as a subset of $\mathbf{C}^3$ is isomorphic as a lattice to $\mathbf{Z}^6$, we have $\det(\mathcal{G})^3 = 1$. By the remarks at the end of section 2.1, $\det 2\mathcal{G}^3 = 2^{12}$ and $\det L = 2^6$.

We check that for γ and σ in $\mathcal{C}$, $\langle \gamma, \sigma \rangle \in 2\mathbf{Z}$. So $\det(1/\sqrt{2})L = 1$ and the imaginary part of the Hermitian inner product restricted to $(1/\sqrt{2})L$ is in $\mathbf{Z}$. Thus $(1/\sqrt{2})L$ is symplectic. It is the Jacobian of the Fermat curve.

2.4. Jacobian of the Picard Curve. The Jacobian of the Picard curve, $y^3 z = x^4 - z^4$ is the lattice $M(E_6)$ (see [Q2] and Conway–Sloane [CS3]). For a cyclotomic construction of this lattice, let $\zeta = \exp(\pi i/6)$, and let

$$\alpha = \sqrt{(3+\sqrt{3})/6}, \qquad \tilde{\alpha} = \sqrt{(3-\sqrt{3})/6}, \qquad \beta = \sqrt{1/\sqrt{3}}.$$

Identify $p(x)$ in $\mathcal{A} = \mathbf{Z}[x]/((x^2 - x + 1)(x^4 - x^2 + 1))$ with the vector

$$v(p) = (\alpha p(\zeta), \tilde{\alpha} p(\zeta^5), \beta p(\zeta^2))^{\mathrm{t}}$$

in $\mathbf{C}^3$. This makes $\mathcal{A}$ into a six-dimensional lattice L generated by $v(x^k)$, $k = 0, \ldots, 5$. The vectors $v(x^k) + v(x^{k+3})$ generate a copy of the lattice D_4 and the vectors $v(x^k) + v(x^{k+6})$ generate a copy of $(4/3)^{1/4}A_2$ orthogonal to the copy of D_4. The quotient $L/(D_4 \oplus (4/3)^{1/4}A_2)$ is of order 4 and representative elements are $v(x^k)$, $k = 0, \ldots, 4$. Since $\det(D_4 \oplus (4/3)^{1/4}A_2) = 16$ we see that $\det L = 1$. The imaginary part of $v(x^j)^* v(x^k)$ is integral for $j, k = 0, \ldots, 3$, and so it can be seen from the construction that L is symplectic.

3. Extremal lattices

3.1. Definitions. The study of extremal lattices is concerned with the minimum non-zero norm of the normalized lattice (normalized so $\det L = 1$),

$$\mu(L) = \min \left\{ z^* z \;\middle|\; z \in L, z \neq 0 \right\}.$$

A lattice L is extreme for $2g$-dimensional lattices if $\mu(L)$ is a local maximum. One may define an extreme symplectic lattice as one for which $\mu(L)$ is a local maximum when restricted to the set of symplectic lattices. The set of symplectic lattices of dimension $2g$ is a real $g(g+1)$-dimensional subspace of the real $g(2g+1)$-dimensional space of lattices of dimension $2g$. The Gram matrix P of a symplectic lattice with respect to a symplectic basis can be written as

$$\begin{aligned} P &= \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} \begin{pmatrix} Y^{-1} & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \\ &= \Re((I, Z)^* Y^{-1} (I, Z)) \end{aligned}$$

where $Z = X + iY$ is a $g \times g$ symmetric matrix with $Y > 0$. The set of such Z is the Siegel upper half plane of complex dimension $g(g+1)/2$.

3.2. Voronoi conditions. Let $M = M(L)$ be the set of minimum vectors (vectors of minimum non-zero norm) of a normalized lattice L,

$$M = \left\{ z \in L \;\middle|\; z^* z = \mu(L) \right\}.$$

The lattice L is said to be perfect if it satisfies the condition

(P) $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} \;\middle|\; z \in M \right\}$ spans the space of $2g \times 2g$ symmetric matrices.

It is said to be eutactic if there is a sequence c_z, $z \in M$ such that $c_z > 0$ and

$$\text{(E)} \qquad \sum_{z = x + iy \in M} c_z \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} = I.$$

(We use the notation $c_z > 0$ to mean $c_z > 0$ for all $z \in M$.) The numbers c_z are called eutactic coefficients. Voronoi's theorem states that a lattice is extreme if and only if it is perfect and eutactic.

Since the above conditions are quadratic in the minimum vectors, we adopt the convention that a minimum vector z is identified with its negative $-z$ when counting the minimum vectors and computing these sums.

To state the conditions (P) and (E) in complex form we introduce the following real vector spaces:

$$\begin{aligned} \mathcal{S}_{\mathbf{C}} &= \text{ the symmetric complex } g \times g \text{ matrices,} \\ \mathcal{S}_{\mathbf{R}} &= \text{ the symmetric real } 2g \times 2g \text{ matrices,} \\ \mathcal{H} &= \text{ the Hermitian symmetric } (A^* = A) \text{ complex } g \times g \text{ matrices.} \end{aligned}$$

There is a vector space isomorphism ϕ: $\mathcal{S}_{\mathbf{C}} \oplus \mathcal{H} \to \mathcal{S}_{\mathbf{R}}$ given by

$$\phi((p + iq) \oplus (r + is)) = \begin{pmatrix} p & q \\ q & -p \end{pmatrix} + \begin{pmatrix} r & s \\ -s & r \end{pmatrix}.$$

We note that

$$\phi\big(zz^{\mathrm{t}} \oplus zz^*\big) = 2\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}}.$$

Using the complex form of the lattice and the above isomorphism ϕ, the conditions above can be conveniently written as

(P) $$\{zz^{\mathrm{t}} \oplus zz^* | z \in M\} \text{ spans } \mathcal{S}_C \oplus \mathcal{H}.$$

There is a sequence c_z such that

(E) $$c_z > 0 \sum_{z \in M} c_z zz^{\mathrm{t}} = 0 \quad \text{and} \quad \sum_{z \in M} c_z zz^* = I.$$

3.3. Theorems for max-min problems. Condition (P) and (E) above are related to theorems about max-min problems from the theory of linear programming. The connection was pointed out by Barnes [B] and used also by Bergé and Martinet in [BM1]. In what follows $\min v$ will denote the minimum of the coordinates of a real vector v. We use the notation $v \geq 0$ to mean that all coordinates of v are ≥ 0 and similarly for $v > 0$. We observe that for a linear function A on $\mathbf{R}^n$ the condition $Ax \geq 0 \Rightarrow Ax = 0$ is equivalent to the statement that $\min Ax$ has a local max at $x = 0$. Likewise the condition $Ax \geq 0 \Rightarrow x = 0$ is equivalent to the statement that $\min Ax$ has a strict local max at $x = 0$.

The extremal properties of a smooth function and of its differential are related by the

MAX-MIN THEOREM: *Let $f\colon \mathbf{R}^p \to \mathbf{R}^s$ be differentiable with $f(0) = 0$ and the differential at 0 given by the $s \times p$ matrix A. If $\min Ax$ has a strict local max at $x = 0$ then $\min f(x)$ has a strict local max at $x = 0$. If $f(x) - Ax \geq 0$ for all x then the converse also holds.*

Proof of Max-min Theorem: Suppose $\min Ax$ has a strict local max at $x = 0$. If $x \neq 0$ then $\min Ax < 0$. Let $\epsilon > 0$ be defined by $-\epsilon = \max\min Ax$ where the max is taken over $|x| = 1$. For $x \neq 0$, we have $f(x) = |x|(A(x/|x|) + h(x))$ where $h(x) = \mathrm{O}(|x|)$. Now choose δ such that $|x| < \delta$ implies $h(x) < \epsilon$. Then $\min f(x) < 0$ for $|x| < \delta$ and $x \neq 0$ and consequently $\min f$ has a strict local max at 0.

Conversely if $f(x) - Ax \geq 0$ for all x then if $\min f$ has a strict local max at 0, the same is true for $\min A$.

The property that $\min Ax$ has a local max at $x = 0$ has an equivalent formulation based on the version of the fundamental theorem of linear programming called the

STIEMKE THEOREM: *Let A is a real matrix, x a suitable column matrix and c a suitable row matrix. The following are equivalent:*

(a) $Ax \geq 0 \Rightarrow Ax = 0$.
(b) *There is a $c > 0$ such that $cA = 0$.*

3.4. PERFECT AND EUTACTIC FOR SYMPLECTIC LATTICES. The above ideas have been used by Bergé and Martinet [BM] to generalize the Voronoi theorem to symplectic and isodual lattices. We will explain the idea briefly and give these conditions in complex form.

Consider the $g(g+1)$-dimensional subspace $\mathcal{T}$ of the Lie algebra $\mathrm{sp}_g(\mathbf{R})$ consisting of all matrices of the form

$$A = \begin{pmatrix} p & q \\ q & -p \end{pmatrix}$$

where p and q are symmetric matrices. This is the tangent space at the identity to the symmetric symplectic matrices. It can be identified with the vector space of symmetric complex matrices $p + iq$, and may also be characterized as the set of real symmetric matrices A satisfying $AJ = -JA$. Now the neighborhood of a given symplectic lattice with generator matrix Λ_{r} can be given by $\exp(A/2)\Lambda_{\mathrm{r}}$ for A in a neighborhood of the origin in $\mathcal{T}$.

To apply the Max-min Theorem, we let M be the set of minimum vectors of the lattice L with generator matrix Λ_{r}. Consider the function f from $\mathcal{T}$ ($=\mathbf{R}^{g(g+1)}$)

to $\mathbf{R}^s$, where s is the number of minimum vectors, where the value at A is given by components

$$f_z(A) = \begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} \exp A \begin{pmatrix} x \\ y \end{pmatrix} - \mu(L), \quad z = x + iy \in M.$$

From the expansion of $\exp A$ near $A = 0$ we see that $\exp A - I - A \geq 0$ for A near 0, so that the Max-min Theorem applies. For A small the minimum vectors of the lattice with generator matrix $\exp(A/2)\Lambda_{\mathrm{r}}$ are among the vectors $\exp(A/2) \begin{pmatrix} x \\ y \end{pmatrix}$, $z = x + iy \in M$. Note that the differential of f is the map sending A to the vector with components $\begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} A \begin{pmatrix} x \\ y \end{pmatrix}$, $z \in M$. Applying the Max-min Theorem and the Stiemke Theorem shows that the lattice L is an absolute local maximum for symplectic lattices if and only if the following conditions apply:

(P(symp)) $$\begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} A \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \text{for all } A \in \mathcal{T} \implies A = 0,$$

and there is a sequence c_z such that

(E(symp)) $$c_z > 0 \quad \text{and} \quad \sum_{z \in M} c_z \begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} A \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \text{for all } A \in \mathcal{T}.$$

We will consider a complex form of the above conditions. Suppose $A \in \mathcal{T}$ is written as above in terms of symmetric matrices p and q. Then

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} A \begin{pmatrix} x \\ y \end{pmatrix} = \Re(z^{\mathrm{t}}(p - iq)z) = \Re(zz^{\mathrm{t}}(p - iq))$$

where tr denotes the trace of the matrix. If α and β are in the vector space of symmetric complex matrices, $\mathcal{S}_{\mathbf{C}}$, then $\Re\,\mathrm{tr}(\alpha\beta^*)$ forms a non-degenerate inner product on $\mathcal{S}_{\mathbf{C}}$. Using these facts, we can see that the above conditions can be stated in complex form as

(P(symp)) $$\{zz^{\mathrm{t}} | z \in M\} \text{ spans } \mathcal{S}_C$$

and there is a sequence c_z such that

(E(symp)) $$c_z > 0 \quad \text{and} \quad \sum_{z \in M} c_z zz^{\mathrm{t}} = 0.$$

These should be compared with conditions (P) and (E) above. We note that if L is symplectic (P) implies (P(symp)) and (E) implies (E(symp)). If a symplectic lattice satisfies conditions (P(symp)) and E(symp) resp. we will say that it is resp. perfect and eutactic for symplectic lattices. These are analogues of the classical definitions of Voronoi and it follows from the Max-min theorem and the Stiemke theorem that

THEOREM: *A symplectic lattice is a strict local max of $\mu(L)$ for symplectic lattices of the same dimension iff it is perfect and eutactic for symplectic lattices.*

We are also interested in conditions under which we can show that $\mu(L)$ is not a local max. If L does not satisfy (P(symp)) then there are symmetric real matrices p_0 and q_0 such that for all $z = x + iy \in M$

$$\Re\left(zz^{\mathrm{t}}(p_0 - iq_0)\right)^{\mathrm{t}} = \begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} \begin{pmatrix} p_0 & q_0 \\ q_0 & -p_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

Now let A_0 be the above $2g \times 2g$ real matrix above. Suppose that A_0 satisfies the condition that $A_0 \begin{pmatrix} x \\ y \end{pmatrix} \neq 0$, or equivalently $(p_0 - iq_0)z \neq 0$, for all $z = x+iy \in M$. In this case the first term in the expansion of $\begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} \exp tA_0 \begin{pmatrix} x \\ y \end{pmatrix} - \mu(L)$ at $t = 0$ is $t^2 \begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} A_0^2 \begin{pmatrix} x \\ y \end{pmatrix}$, and this term is strictly positive for all $z \in M$, $t \neq 0$. So $\mu(L)$ will not be a local max.

3.5. SYMPLECTIC AUTOMORPHISMS. For a lattice with automorphisms, it is known (see Conway and Sloane [CS2]) that if the automorphism group acts as a transitive group of permutations on the set of minimum vectors, then we can find eutactic coefficients that are equal. We will prove a similar fact for symplectic lattices if we consider the group of symplectic automorphisms.

Recall that an element of the automorphism group corresponds to matrices T in $\mathrm{O}(2g)$ and $\tilde{T}$ in $\mathrm{GL}_{2g}(\mathbf{Z})$ such that $T\Lambda_{\mathrm{r}} = \Lambda_{\mathrm{r}}\tilde{T}$. If the lattice is symplectic then the automorphism is said to be symplectic if T (equivalently $\tilde{T}$) is symplectic. Now T induces a permutation $\pi = \pi(T)$ on the set of minimum vectors M. Also since T is symplectic the map $A \to T^{\mathrm{t}}AT$ is a vector space isomorphism of $\mathcal{T}$. So

$$\sum_{z \in M} a_z \begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} A \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \text{for all } A \in \mathcal{T}$$

if and only if

$$\sum_{z \in M} a_z \begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{t}} T^{\mathrm{t}}AT \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \text{for all } A \in \mathcal{T}.$$

It follows that

$$\sum_{z \in M} c_z \pi z (\pi z)^{\mathrm{t}} = \sum_{z \in M} c_{\pi^{-1} z} z z^{\mathrm{t}} = 0.$$

Summing over all symplectic automorphisms shows that if the group of permutations corresponding to symplectic automorphisms is transitive, we can take as eutactic coefficients $\sum c_{\pi(T)z}$, where the sum is over all symplectic automorphisms. By dividing through, we can replace them all by 1.

In the examples given below, it can be seen (see Quine [Q1] [Q2]) that the group of symplectic automorphisms is transitive. This makes it simpler to find eutactic coefficients since we can assume they are all equal to 1. The transitivity follows from the fact that the automorphisms of the surface are transitive on the corresponding cycles in the homology. The automorphisms of the Riemann surface induce symplectic automorphisms on the Jacobian.

4. Applications to Jacobians

4.1. THE KLEIN CURVE. It is known [B] that $A_6^{(2)}$ satisfies (P) and (E) and hence also (P(symp)) and (E(symp)). We will show how to use the cyclotomic construction of $A_6^{(2)}$ and the complex form of (E) that this lattice is eutactic in the classical sense.

From the description of $A_6^{(2)}$ as a sublattice of $\mathbf{Z}^7$, we can see that the vectors of minimum norm are (1,-1,-1,1,0,0,0), (1,-1,0,-1,1,0,0), and (1,0,-1,-1,0,1,0) and vectors obtained from these by permutation of coordinates. These correspond to

$$p(\zeta) = \zeta^k(1+\zeta), \quad p(\zeta) = \zeta^k(1+\zeta+\zeta^2) \quad \text{and} \quad p(\zeta) = \zeta^k(1+\zeta)(1+\zeta+\zeta^2),$$

$k = 0, \ldots, 6$. Denote the vectors $v(p)$ for p in each set of 7, by M_1, M_2, and M_3 respectively. The set of minimum vectors is $M = M_1 \cup M_2 \cup M_3$. The entries of $v(\zeta^k(1+\zeta))v(\zeta^k(1+\zeta))^{\mathrm{t}}$ are of the form $\zeta^{k(i+j)}(1+\zeta^i)(1+\zeta^j)(1-\zeta^i)^2(1-\zeta^j)^2$ with $i,j = 1,2,4$. Thus for $v \in M_1$ the entries of vv^{t} are of the form $\zeta^{k(i+j)}c(i,j)$ where $c(i,j)$ is independent of k. Summing over k we see that $\sum_{v\in M_1} vv^{\mathrm{t}} = 0$ since $i+j \neq 0 \pmod 7$. The same argument shows that the sum is zero over M_2 and M_3, so $\sum_{v\in M} vv^{\mathrm{t}} = 0$, and $A_6^{(2)}$ is eutactic for symplectic lattices. A similar argument shows $\sum_{v\in M} vv^* = 21I$, so $A_6^{(2)}$ is eutactic in the classical sense.

4.2. THE FERMAT CURVE. We will show from the cyclotomic construction that the Jacobian of the Fermat curve is eutactic in the classical sense and therefore also for symplectic lattices, but that it is not perfect for symplectic lattices and, in fact, is not a local max for $\mu(L)$.

From the description in section 2.3 we see that the minimum vectors are in four groups: $M_1 = \{(\epsilon_1,\epsilon_2,\epsilon_3)^{\mathrm{t}}\}$, $M_2 = \{(\epsilon_1 i,\epsilon_2 i,\epsilon_3)^{\mathrm{t}}\}$, $M_3 = \{(\epsilon_1 i,\epsilon_2,\epsilon_3 i)^{\mathrm{t}}\}$, $M_4 = \{(\epsilon_1,\epsilon_2 i,\epsilon_3 i)^{\mathrm{t}}\}$ where ϵ_j, $j=1,2,3$ are ±1. Each group accounts for four minimum vectors (recall that z and $-z$ are considered the same) for a total of 16. Now

$$\sum \epsilon_j \epsilon_k = \begin{cases} 0 & \text{if } j \neq k \\ 8 & \text{if } j = k \end{cases}$$

where the sum is over all 8 possibilities for $(\epsilon_1, \epsilon_2, \epsilon_3)$. So

$$\sum_{z \in M_1} zz^{\mathrm{t}} = 4I.$$

A similar argument shows

$$\sum_{z \in M_2} zz^{\mathrm{t}} = 4 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sum_{z \in M_3} zz^{\mathrm{t}} = 4 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\text{and} \quad \sum_{z \in M_4} zz^{\mathrm{t}} = 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

so that finally

$$\sum_{z \in M} zz^{\mathrm{t}} = 0,$$

and the lattice is eutactic for symplectic lattices (E(symp)). In fact, a similar argument shows that

$$\sum_{z \in M} z\bar{z}^{\mathrm{t}} = 16I$$

so that the lattice is eutactic in the classical sense (E).

To show that the lattice does not satisfy (P(symp)), consider the three symmetric complex matrices

$$\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Considering again the non-degenerate inner product $\Re \alpha\beta^*$ on symmetric complex matrices we see that the three matrices are perpendicular to zz^{t} for every $z \in M$ since the diagonals of zz^{t} contain real entries. Thus the matrices $\{zz^{\mathrm{t}} \mid z \in M\}$ do not span the symmetric complex matrices.

If α is any one of the above three matrices, $\alpha z \neq 0$ for all $z \in M$ since no coordinate of z vanishes. Thus by considerations at the end of section 3.4, $\mu(L)$ is not a local max.

4.3. The Picard Curve. We will show that the Jacobian of the Picard curve is perfect and eutactic for symplectic lattices and is therefore a strict local max for $\mu(L)$. Again, the cyclotomic construction aids in the computations required to show (P(symp)) and (E(symp)) in the complex form.

We will show the Jacobian lattice is eutactic for symplectic lattices by showing that

$$\sum_{z\in M} zz^{\mathrm{t}} = 0.$$

Following 2.3, let $v_k = v(x^k)$. Recall that the minimum vectors are v_k and $v_{k+1} - v_k$, $k = 0, \ldots, 11$. Thus we must show

$$\sum_{k=0}^{11} v_k v_k^{\mathrm{t}} = \sum_{k=0}^{11} v_k v_{k+1}^{\mathrm{t}} = 0.$$

We compute that the entries of $v_k v_k^{\mathrm{t}}$ are

$$\alpha^2\zeta^{2k} \quad \zeta^{6k} \quad \alpha\beta\zeta^{3k} \quad \alpha^{-2}\zeta^{10k} \quad \alpha^{-1}\beta\zeta^{7k} \quad \beta^2\zeta^{4k}$$

and the entries of $v_k v_{k+1}^{\mathrm{t}}$ are

$$\alpha^2\zeta^{2k+1} \quad \zeta^{6k+5} \quad \alpha\beta\zeta^{3k+2} \quad \alpha^{-2}\zeta^{10k+5} \quad \alpha^{-1}\beta\zeta^{7k+2} \quad \beta^2\zeta^{4k+2}.$$

Each of these sums to 0 over $k = 0, \ldots, 11$. Hence the lattice satisfies (E(symp)).

A similar argument shows that

$$\sum_{z\in M} z\bar{z}^{\mathrm{t}} = 12 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3}\sqrt{3} \end{pmatrix}.$$

Thus the lattice does not satisfy (E) and is not eutactic is the classical sense. The identity matrix in the upper left corner above reflects the fact that D_4 is eutactic.

To show this lattice satisfies (P(symp)), we must show that the vectors

$$P_k = (\zeta^{2k}, \zeta^{6k}, \zeta^{3k}, \zeta^{10k}, \zeta^{7k}, \zeta^{4k})$$

together with the vectors

$$Q_k = (\zeta^{2k+1}, \zeta^{6k+5}, \zeta^{3k+2}, \zeta^{10k+5}, \zeta^{7k+2}, \zeta^{4k+2}),$$

$k = 0, \ldots, 11$ span $\mathbf{C}^6$. Let P be the matrix with rows P_k and Q the matrix with rows Q_k, $k = 0, \ldots, 5$. By direct computation it can be shown that

$$\det \begin{pmatrix} P & \bar{P} \\ Q & \bar{Q} \end{pmatrix} \neq 0,$$

thus the vectors do span $\mathbf{C}^6$ and the Jacobian of the Picard curve is perfect for symplectic lattices.

From the classification of perfect six-dimensional lattices by Barnes [B], we know the lattice does not satisfy (P). The only perfect (P) symplectic lattice is the Jacobian of the Klein curve.

Note: Paul Schmutz Schaller has recently informed me that the above examples have been studied also by Christophe Bavard [Bav], using similar methods. He has additional results, including that extremal symplectic lattices have Gram matrices with algebraic entries.

References

[B] E. S. Barnes, *The perfect and extreme senary forms*, Canadian Journal of Mathematics **9** (1957), 235–242.

[Bav] C. Bavard, *Systole et invariant d'Hermite*, Journal für die reine und angewandte Mathematik **482** (1997), 93–120.

[BM1] A–M. Bergé and J. Martinet, *Densité dans des familles réseaux, application aux réseaux isoduaux*, L'Enseignement Mathématique **41** (1995), 335–365.

[BM2] A–M. Bergé and J. Martinet, *Sur la classification des réseaux eutactiques*, Journal of the London Mathematical Society (2) **53** (1996), 417–432.

[BS] P. Buser and P. Sarnak, *On the period matrix of a Riemann surface of large genus (with an appendix by J. H. Conway and N. J. A. Sloane)*, Inventiones Mathematicae **117** (1994), 27–56.

[CS1] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, Berlin, 1988.

[CS2] J. H. Conway and N. J. A. Sloane, *Low dimensional lattices III, perfect forms*, Proceedings of the Royal Society of London, Series A **418** (1988), 43–80.

[CS3] J. H. Conway and N. J. A. Sloane, *On lattices equivalent to their duals*, Journal of Number Theory **48** (1994), 373–382.

[G] M. Gromov, *Systoles and intersystolic inequalities*, Actes de la Table Ronde de Géométrie Différentièlle (Luminy, 1992), Sémin. Congr., 1, Soc. Math. France, Paris, 1996, pp. 291–362.

[G2] B. Gross, *On the periods of Abelian integrals and a formula of Chowla and Selberg*, Inventiones Mathematicae **45** (1978), 193–211.

[G3] B. Gross, *Group representations and lattices*, Journal of the American Mathematical Society **3** (1990), 929–960.

[L] S. Lang, *Introduction to Algebraic and Abelian Functions*, Springer-Verlag, Berlin, 1982.

[M] B. Mazur, *Arithmetic on Curves*, Bulletin of the American Mathematical Society **14** (1986), 207–259.

[Q1] J. R. Quine, *Systoles of two extremal Riemann surfaces*, The Journal of Geometric Analysis **6** (1996), 461–474.

[Q2] J. R. Quine, *Jacobian of the Picard curve*, Contemporary Mathematics, Vol. 201, *Extremal Riemann Surfaces*, American Mathematical Society, 1997, pp. 33–41.

[R] D. Rohrlich, *The periods of the Fermat Curve*, Appendix to B. Gross, Inventiones Mathematicae **45** (1978), 208–211.

[S1] P. Schmutz, *Riemann surfaces with shortest geodesic of maximal length*, Geometric and Functional Analysis **3** (1993), 564–631.

[S2] P. Schmutz, *Systoles on Riemann surfaces*, Manuscripta Mathematica **85** (1994), 429–447.

[S3] P. Schmutz, *Congruence subgroups and maximal Riemann surfaces*, The Journal of Geometric Analysis **4** (1994), 207–217.

[T] C. L. Tretkoff and M. D. Tretkoff, *Combinatorial group theory, Riemann surfaces and differential equations*, Contemporary Mathematics, Vol. 33, American Mathematical Society, 1984, pp. 467–519.

[V] G. F. Voronoi, *Nouvelles applications des paramètres continus à la théorie des formes quadratique: 1 Sur quelques propriétés des formes quadratique positive parfaites*, Journal für die reine und angewandte Mathematik **133** (1908), 97–178.