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SELECTING A PROPORTION OF CHARACTERS

BY

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ABSTRACT

We use majorizing measures to provide a simpler proof of the following unpublished result of J. Bourgain. For any set of characters on a compact group there exists a subset of proportional size such that, on the span of this subset, the ℓ^1 and ℓ^2 norm are equivalent up to a factor $(C \log n \log\log n)^{1/2}$.

1. Introduction

In a remarkable paper, J. Bourgain proves, for $p > 2$, the existence of $\Lambda(p)$ sets of "maximal density" [B1]. The proof of this result is probabilistic. The use of general tools has allowed this author to show that the heart of Bourgain's result can be seen as a result on restriction of operators. Before we state this result, we need to recall that a Banach space W is called **2-smooth** if, for some constant C, we have for each vector x, y of W that

$$\|y\| = 1 \Rightarrow \|y + x\| + \|y - x\| \leq 2 + C\|x\|^2. \tag{1.1}$$

The number C will be called the **smoothness constant** of W.

We denote by $(e_i)_{i \leq n}$ the canonical basis of ℓ_n^2.

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THEOREM A ([T2]): *Consider an operator U from ℓ^2_n to a 2-smooth Banach space W. Consider a number $0 \le \delta < 1$ and, for $i \le n$, consider independent random variables $(\delta_i)_{i \le n}$ with $\delta_i \in \{0,1\}$ and $E\delta_i = \delta$. Consider the random set $I = \{i \le n: \delta_i = 1\}$ and denote by $U|_I$ the restriction of U to the space generated by the vectors $(e_i)_{i \in I}$. Then*

$$E(\|U|_I\|) \le \frac{K(C)}{\sqrt{\log \frac{1}{\delta}}} \Big(\|U\| + \sup_{i \le n} \|U(e_i)\| \sqrt{\log n} \Big)$$

where $K(C)$ depends upon C only.

The problem with Theorem A is that it involves a hypothesis on the whole space, and that there are natural situations (such as the one considered in the abstract) where it is desirable that have a "local" version of Theorem A, i.e., where the geometric condition on the entire space is replaced by a geometrical condition, say, on the vectors $\big(U(e_i)\big)_{i \le n}$.

We will introduce such a condition, as follows.

Definition 1.1: Consider a number $C > 0$. We say that a family of vectors $(x_i)_{i \le n}$ of a Banach space W satisfies condition $H(C)$ provided for each vector y of W, with $\|y\| = 1$, and each sequence $(\alpha_i)_{i \le n}$ of numbers, we have

$$P\Big(\|y + \sum_{i \le n} \alpha_i g_i x_i\| \le 1 + C^2 \sum_{i \le n} \alpha_i^2 \Big) \ge \frac{1}{4} \tag{1.2}$$

where $(g_i)_{i \le n}$ is a sequence of independent standard normal random variables.

Certainly this condition looks unappealing at first sight. It turns out, however, that it is a weakening of 2-smoothness as is shown by the following.

PROPOSITION 1.2: *If the Banach space W satisfies* (1.1), *then any sequence of vectors $(x_i)_{i \le n}$ in W satisfies condition $H(5C \max_{i \le n} \|x_i\|)$.*

The appeal of condition $H(C)$ does not lie in its aesthetics, but in the fact that it is rather easy to check in the case motivated by the situation of the abstract. We recall that for a function f a probability space, its L_{Ψ_2} norm is given by

$$\|f\|_{\Psi_2} = \inf \Big\{ \lambda > 0; E\Big(\exp\Big(\frac{f^2}{\lambda^2}\Big)\Big) \le 2 \Big\}.$$

We denote by L_{Ψ_2} the Banach space of functions f for which $\|f\|_{\Psi_2} < \infty$. This space is **not** 2-smooth. However, we have the following.

PROPOSITION 1.3: *Consider a sequence $(x_i)_{i\leq n}$ of functions on a probability space. Assume that $\|x_i\|_\infty \leq 1$ for $i \leq n$. Then the sequence $(x_i)_{i\leq n}$ satisfies condition $H(C)$ in L_{Ψ_2}, where C is universal.*

The usefulness of condition $H(C)$ is demonstrated by the following.

THEOREM 1.4: *Consider an operator U from ℓ_n^2 to a Banach space W. Denote by $\|\cdot\|$ the norm on W, and consider another norm $\|\cdot\|_\sim$ on W such that $\|\cdot\| \leq \|\cdot\|_\sim$. Assume that the sequence $\big(U(e_i)\big)_{i\leq n}$ satisfies condition $H(C)$ in $(W, \|\cdot\|_\sim)$. Then, with the notation of Theorem 1, we have*

$$E\|U|_I\| \leq \frac{K}{\sqrt{\log 1/\delta}}\left[\|U\| + C\sqrt{\log n}\right]$$

where K is universal.

The usefulness of considering two different norms is related to rather technical reasons. So, in order to compare this statement with Theorem 1, we assume that $\|\cdot\|_\sim = \|\cdot\|$. In that case, the main difference is that the hypothesis of Theorem A that W is 2-smooth has been replaced in Theorem 1.4 by the hypothesis that the vectors $x_i = U(e_i)$ satisfy condition $H(C)$. Thus Proposition 1.2 shows that Theorem 1.4 is an extension of Theorem 1.1.

The main motivation at present behind Theorem 1.4 is the following.

THEOREM 1.5 (J. Bourgain [B2]): *Consider a sequence $(\varphi_i)_{i\leq n}$ of orthogonal functions on a probability space (Ω, P). Assume that $\|\varphi_i\|_\infty \leq 1$, and let $\alpha = \inf_{i\leq n}\|\varphi_i\|_2$. Then there is $\delta > 0$, depending on α only, such that for most subsets I of $\{1,\ldots,n\}$, with $\operatorname{card} I \leq n\delta$, we have for each number $(a_i)_{i\in I}$ that*

$$\Big\|\sum_{i\in I} a_i\varphi_i\Big\|_1 \geq \frac{\alpha^2}{K\sqrt{\log n \log\log n}}\Big(\sum_{i\in I} a_i^2\Big)^{1/2}. \tag{1.3}$$

It is known that the term $\log n$ is necessary. Whether the term $\log\log n$ is necessary is a rather interesting question. Such terms are usually parasitic. There is, however, no readily identifiable place in the arguments we use where an obvious weakness would create this term. It seems to me that removing this term if at all possible would require a deeper analysis rather than a simple modification of the present approach.

2. Exploring condition $H(C)$

We start with Proposition 1.2. The easy proof helps one to understand the nature of condition $H(C)$. A basic ingredient of the proof is the fact that 2-smoothness implies type 2. Since the proof of [Li-Tz] is somewhat indirect, we will in passing give a direct proof. We assume (1.1), that is

$$\forall y, x \in W, \quad \|y\| = 1 \Rightarrow \|y+x\| + \|y-x\| \le 2 + C\|x\|^2 \tag{2.1}$$

where $C \ge 1$.

LEMMA 2.1: *Consider $y \in W$, $y^* \in W^*$ with $\|y^*\| = \|y\| = 1 = y^*(y)$. Then*

$$\forall x \in W, \quad \|y+x\| \le 1 + y^*(x) + C\|x\|^2. \tag{2.2}$$

Proof: Set

$$A = \|y+x\| - y^*(y+x) = \|y+x\| - 1 - y^*(x),$$

so that $A \ge 0$. Similarly

$$B = \|y-x\| - 1 + y^*(x) \ge 0.$$

Now, by (2.1), we have $A + B \le C\|x\|^2$, so that $A \le C\|x\|^2$. ∎

LEMMA 2.2: *Consider $y, x \in W$, and a standard normal random variable g. Then*

$$E\|y+gx\|^2 \le \|y\|^2 + 6C\|x\|^2.$$

Proof: First, by the triangle inequality

$$\|y+gx\| \le \|y\| + |g|\|u\|$$

so that

$$\|y+gx\|^2 \le \|y\|^2 + 2\|y\|\|x\||g| + g^2\|x\|^2$$

and, taking expectation, and since $E|g| \le 1$, $Eg^2 = 1$,

$$E\|y+gx\|^2 \le \|y\|^2 + \|x\|^2\left[1 + 2\frac{\|y\|}{\|x\|}\right]. \tag{2.3}$$

Next, using (2.2) and homogeneity

$$\|y+gx\| \le \|y\| + Cg^2\frac{\|x\|^2}{\|y\|} + gy^*(x)$$

where $\|y^*\| = 1 = y^*\left(\frac{y}{\|y\|}\right)$.

Taking squares, taking expectation, and using the fact that $Eg = Eg^3 = 0$, we get

$$E\|y+gx\|^2 \le \|y\|^2 + |y^*(x)|^2 + 2C\|x\|^2 + C^2 Eg^4 \frac{\|x\|^4}{\|y\|^2}.$$

Now $|y^*(x)| \le \|x\|$, so that, since $C \ge 1$ and $Eg^4 = 3$, we get

$$E\|y+gx\|^2 \le \|y\|^2 + 3C\|x\|^2\left(1 + C\frac{\|x\|^2}{\|y\|^2}\right). \tag{2.4}$$

The result then follows from (2.3) if $\|y\| \le C\|x\|$ and from (2.4) if $\|y\| \ge C\|x\|$.

Proof of Proposition 1.2: By (2.2), we have

$$\Big\|y + \sum_{i\le n}\alpha_i g_i x_i\Big\| \le 1 + y^*\Big(\sum_{i\le n}\alpha_i g_i x_i\Big) + C\Big\|\sum_{i\le n}\alpha_i g_i x_i\Big\|^2. \tag{2.5}$$

By Lemma 2.2 and induction, we have

$$E\Big\|\sum_{i\le n}\alpha_i g_i x_i\Big\|^2 \le 6C\sum_{i\le n}\alpha_i^2\|x_i\|^2$$

so that

$$P\Big(\Big\|\sum_{i\le n}\alpha_i g_i x_i\Big\|^2 \le 24C\sum_{i\le n}\alpha_i^2\|x_i\|^2\Big) \ge \frac{3}{4}.$$

Now by symmetry,

$$P\Big(y^*\Big(\sum_{i\le n}\alpha_i g_i x_i\Big) \le 0\Big) \ge \frac{1}{2}$$

so that, with probability at least $\frac{1}{4}$, we have

$$y^*\Big(\sum_{i\le n}\alpha_i g_i x_i\Big) \le 0; \quad \Big\|\sum_{i\le n}\alpha_i g_i x_i\Big\|^2 \le 24C\sum_{i\le n}\alpha_i^2\|x_i\|^2,$$

and thus

$$\Big\|y + \sum_{i\le n}\alpha_i g_i x_i\Big\| \le 1 + 24C^2\Big(\sum_{i\le n}\alpha_i^2\Big)\sup_{i\le n}\|x_i\|^2. \qquad \blacksquare$$

We now turn to the proof of Proposition 1.3.

The functions in L_{Ψ_2} are defined on a probability space (Ω, P). For simplicity, we write $E(z)$ for $\int z dP$. On the other hand, we consider gaussian r.v. $(g_i)_{i\le n}$. Expectation and probability with respect to these variables are denoted respectively by E_g and P_g.

We start with some simple observations.

LEMMA 2.3: *For any function z, we have*

$$\|z\|_{\Psi_2}^2 \leq E\exp(z^2).$$

Proof: This follows from the fact that if $a \geq 1$ we have

$$\exp\frac{t^2}{a^2} - 1 \leq \frac{1}{a}(\exp t^2 - 1) \tag{2.6}$$

so that

$$\exp\frac{t^2}{a^2} \leq 1 + \frac{1}{a^2}\exp t^2.$$

Using this for $a^2 = E\exp(z^2)$ we get

$$E\exp\frac{z^2}{a^2} \leq 2. \qquad \blacksquare$$

LEMMA 2.4: *Consider two functions u, v, with $\|v\|_{\Psi_2} < 1$. Then*

$$E\exp(u + v^2) \leq 2^{\|v\|_{\Psi_2}^2} E\exp\frac{u}{1 - \|v\|_{\Psi_2}^2}.$$

Proof: We use Holder's inequality

$$E\exp(u + v^2) \leq \Big(E\exp\frac{u}{\alpha}\Big)^{\alpha}\Big(E\exp\frac{v^2}{\beta}\Big)^{\beta}$$

with $\beta = \|v\|_{\Psi_2}^2, \alpha = 1 - \beta$. $\blacksquare$

LEMMA 2.5: *Consider functions $(x_i)_{i\leq n}$ with $\|x_i\|_\infty \leq 1$ and independent standard normal random variables $(g_i)_{i\leq n}$. Then for all numbers $(\alpha_i)_{i\leq n}$, we have*

$$P_g\Big(\Big\|\sum_{i\leq n}\alpha_i g_i x_i\Big\|_{\Psi_2} \leq 4\Big(\sum_{i\leq n}\alpha_i^2\Big)^{1/2}\Big) \geq \frac{7}{8}.$$

Proof: We observe that $E\exp\beta g^2/4 \leq \sqrt{2}$ for $\beta \leq 1$, so that

$$E_g E\exp\frac{\Big(\sum_{i\leq n}\alpha_i g_i x_i\Big)^2}{4\sum_{i\leq n}\alpha_i^2} \leq \sqrt{2}$$

by inverting the expectations, and observing that $\sum_{i\le n}\alpha_i g_i x_i(\omega)$ is distributed like βg, for $\beta = \left(\sum_{i\le n}\alpha_i^2 x_i(\omega)^2\right)^{1/2} \le 1$. Combining with (2.6) for $a^2 = 4$, we get

$$E_g\left(E\exp\frac{(\sum_{i\le n}\alpha_i g_i x_i)^2}{16\sum_{i\le n}\alpha_i^2} - 1\right) \le \frac{\sqrt{2}-1}{4} \le \frac{1}{8}$$

and the result follows from Chebyshev's inequality. ∎

We now start the proof of Proposition 1.3. We consider y with $\|y\|_{\Psi_2} = 1$, functions $(x_i)_{i\le n}$ with $\|x_i\|_\infty \le 1$, and numbers $(\alpha_i)_{i\le n}$. For simplicity we set $r = \left(\sum_{i\le n}\alpha_i^2\right)^{1/2}$.

If we combine Lemma 2.5 with the triangle inequality,

$$\Big\|y + \sum_{i\le n}\alpha_i g_i x_i\Big\|_{\Psi_2} \le 1 + \|\sum \alpha_i g_i x_i\|_{\Psi_2},$$

we see that there is nothing to prove unless $r \le 1/8$. So we assume $r \le 1/8$.

We try to prove that there is a universal constant K such that, if we set $A = 1 + Kr^2$, with probability P_g at least $1/4$, we have

$$E\exp\frac{\left(y + \sum_{i\le n}\alpha_i g_i x_i\right)^2}{A^2} \le 2.$$

We appeal to Lemma 2.4 for

$$u = \frac{1}{A^2}\left(y^2 + 2y\sum_{i\le n}\alpha_i g_i x_i\right); \quad v = \frac{1}{A}\sum_{i\le n}\alpha_i g_i x_i.$$

By Lemma 2.5, we have $\|v\|_{\Psi_2} \le 4r$ with probability P_g at least $7/8$. Thus, it suffices to prove that with probability at least $3/8$, we have

$$E\exp\frac{u}{1-16r^2} \le 2^{1-16r^2}.$$

Setting $B = A^2(1-16r^2)$, consider the random variable

$$X = E\left(\exp\frac{y^2}{B}\left(\exp\frac{2yh}{B} - \frac{2yh}{B} - 1\right)\right),$$

where for simplicity we set $h = \sum_{i\le n}\alpha_i g_i x_i$.

We observe the crucial fact that X is non-negative (since $1+x\le e^x$) so that

$$P_g(X\le 8E_gX)\ge \tfrac{7}{8}.$$

To bound E_gX from above, we permute the expectations E and E_g, and we observe that $E_gh=0$ and

$$E_g\exp\frac{2yh}{B}\le \exp\frac{2y^2r^2}{B^2}.$$

Indeed, h is distributed like $(\sum \alpha_i^2x_i^2)^{1/2}g$, $E_g\exp\alpha g=\exp\alpha^2/2$ and $\sum\alpha_i^2x_i^2\le \sum\alpha_i^2\le r^2$.

Thus, when $B\ge 1$, with probability P_g at least $7/8$, we have

$$X\le 8E\left(\exp\frac{y^2}{B}\left(\exp\frac{2y^2r^2}{B}-1\right)\right).$$

Now, the random variable $E\left((2yh/B)\exp(y^2/B)\right)$ is symmetric, so it is smaller than or equal to zero with probability P_g at least $1/2$. Thus, with probability P_g at least $3/8$, we have

$$\begin{aligned}E\Big(\exp\frac{y^2}{B}\Big(\exp\frac{2yh}{B}-1\Big)\Big)&\le 8E\Big(\exp\frac{y^2}{B}\Big(\exp\frac{2y^2r^2}{B}-1\Big)\Big)\\&\le E\Big(\exp\frac{y^2}{B}\Big(\exp\frac{16y^2r^2}{B}-1\Big)\Big)\end{aligned}$$

using the inequality $8(e^x-1)\le e^{8x}-1$. Thus, with probability P_g at least $3/8$, we have

$$E\Big(\exp\frac{y^2+2yh}{B}\Big)\le E\exp y^2\frac{(1+16r^2)}{B}.$$

Thus, the proof is finished provided the last term is at most 2^{1-16r^2}. Now, for $B\ge 1+16r^2$, by Holder's inequality we have

$$E\exp\Big(y^2\frac{1+16r^2}{B}\Big)\le (E\exp y^2)^{(1+16r^2)/B}\le 2^{(1+16r^2)/B}.$$

Thus, it suffices that

$$B\ge\frac{1+16r^2}{1-16r^2}$$

or, equivalently,

$$A^2\ge\frac{1+16r^2}{(1-16r^2)^2}.$$

Since $(1-x)^{-1}\le 1+2x$ for $x\le 1/2, A=1+Kr^2$ will work for K large enough. ∎

3. Proportion of characters

In this section we prove Theorem 1.5. On the space of measurable functions, we consider the norm $\|\cdot\|_{\Psi_2}$ and its dual norm $\|\cdot\|^*_{\Psi_2}$. We consider the norm $\|\cdot\|^*$ given by

$$\|y\|^* = \|y\|_2 + \sqrt{\log n}\|y\|^*_{\Psi_2}. \tag{3.1}$$

We consider the dual norm $\|\cdot\|$ of $\|\cdot\|^*$, and the Banach space W with norm $\|\cdot\|$. We observe that

$$\begin{aligned} \|y\| &\leq \|y\|_2, \\ \|y\| &\leq \frac{1}{\sqrt{\log n}}\|y\|_{\Psi_2}. \end{aligned} \tag{3.2}$$

Consider the operator U from ℓ^2_n to W given $U(e_i) = \varphi_i$, where $(e_i)_{i\leq n}$ is the canonical basis of ℓ^2_n. Thus, by (3.2) and since the functions φ_i are orthogonal, we have $\|U\| \leq 1$.

Consider the norm

$$\|y\|_\sim = \frac{1}{\sqrt{\log n}}\|y\|_{\Psi_2}.$$

By Proposition 1.3, the functions $\varphi_i = U(e_i)$ have property $H(K/\sqrt{\log n})$ in $(W, \|\cdot\|_\sim)$, where K is universal.

It then follows from Theorem 1.4 that, with the notations of that theorem, we have

$$E\|U|_I\| \leq K/\sqrt{\log 1/\delta}$$

where K is universal. Thus, we can fix δ depending on α only such that $E\|U|_I\| \leq \alpha/8$. We now show that (1.3) holds under the condition $\|U|_I\| \leq \alpha/4$. This will finish the proof, since $P(\operatorname{card} I \geq \delta n)$ is about $1/2$.

Lemma 3.1: *Consider a function h, with $\|h\|_2 \leq 1$ and $a = P(|h| \neq 0)$. Then (for $a \leq 1/2$), we have*

$$\|h\|^*_{\Psi_2} \leq K\sqrt{a \log \frac{1}{a}}.$$

Proof: Consider f with $\|f\|_{\Psi_2} \leq 1$. Then, as is well known, for $p \geq 1$, we have $\|f\|_p \leq K\sqrt{p}$. Thus, by Holder's inequality

$$E(fh) \leq K\sqrt{p}\|h\|_q \leq K\sqrt{p}a^{1-q/2}$$

where $1/p + 1/q = 1$. Taking q such that $q = 1 + 1/\log(1/a)$ finishes the proof. ∎

To prove (1.3), we can assume $\sum_{i\in I} a_i^2 = 1$. Thus, if $f = \sum_{i\in I} a_i\varphi_i$, we have $\alpha \le \|f\|_2 \le 1$. Consider

$$h = \frac{f}{\|f\|_2} 1_A,$$

where $P(A) = a$, and where A will be determined later.

From (3.1) and Lemma 3.1, we see that

$$\|h\|^* \le 1 + K\Big(a \log \frac{1}{a} \log n\Big)^{1/2}.$$

Thus, $\|h\|^* \le 2$ whenever

$$K^2 a \log \frac{1}{a} \log n \le 1. \tag{3.3}$$

When this occurs, since $\|U|_I\| \le \alpha/4$, we have

$$E\left(\frac{f^2}{\|f\|_2} 1_A\right) = E(fh) \le \|U|_I\| \|h\|^* \le frac\alpha 2.$$

Thus,

$$E\left(\frac{f^2}{\|f\|_2} 1_{A^c}\right) = \|f\|_2 - E\left(\frac{f^2}{\|f\|_2} 1_A\right) \ge \frac{\alpha}{2}$$

and, if we now take $A = \{|f| \ge b\}$, we have

$$bE(|f|) \ge E(f^2 1_{A^c}) \ge \alpha^2/2$$

and thus $E(|f|) \ge \alpha^2/2b$. Since $P(|f| \ge b) \le 1/b^2$, this holds provided $a = 1/b^2$ satisfies (3.3). ∎

4. Majorizing measures

The aim of this section is to prove Theorem 1.4. We keep the notation and the hypothesis of this theorem. First, we observe that if a family $(x_i)_{i\le n}$ of vectors satisfies condition $H(C)$, for $\lambda > 0$ the family $(\lambda x_i)_{i\le n}$ satisfies $H(\lambda C)$. Thus, by homogeneity, there is no loss of generality to make the technical assumption that $C \ge 1$.

For $x^* \in W^*$, $i \le n$, we set $h_i(x^*) = \operatorname{sign}\big(x^*(x_i)\big)x^*(x_i)^2$, and $h(x^*) = \big(h_i(x^*)\big)_{i\le n}$. We consider the set

$$T = \{h(x^*); x^* \in W_1^*\}$$

where $W_1^* = \{x^* \in W_1^*; \|x^*\| \le 1\}$.

It is shown in [T2] that to prove Theorem 1.4, it suffices to prove that

$$\gamma_1(T, d_\infty) \le K(\|U\|^2 + C^2 \log n) \tag{4.1}$$

where d_∞ denotes the distance induced on $\mathbb{R}^n$ by the supremum norm and where, for a metric space (T, d),

$$\gamma_1(T, d) = \inf_\mu \sup_{t \in T} \int_0^\infty \log \frac{1}{\mu\big(B(t, \epsilon)\big)} d\epsilon$$

where the infimum is taken over all (atomic) probability measures on T.

Given $u \in T, k \ge 0$, we define

$$C_k(u) = \{x^* \in W_1^*; \|h(x^*) - u\|_\infty \le 2r^{-k}\}. \tag{4.2}$$

There, and in the rest of the section, $r = 16$. We observe that $C_k(u)$ is convex.

We now define, for u in T,

$$\begin{aligned}
\varphi_k^1(u) &= \sum_{i \le n} \left(|u_i| - \min(|u_i|, 2r^{-k})\right), \\
\varphi_k^2(u) &= \inf\{\|x^*\|_\sim; x^* \in C_k(u)\}, \\
\varphi_k^3(u) &= (1 - r^{-k}) \log n
\end{aligned}$$

and

$$\varphi_k(u) = \varphi_k^1(u) + C^2 \log n \varphi_k^2(u) + \varphi_k^3(u).$$

It is obvious that

$$\varphi_{k+1}(u) \ge \varphi_k(u), \tag{4.3}$$

$$\begin{aligned}
\varphi_k(u) &\le \|U\|^2 + C^2 \log n + \log n \\
&\le \|U\|^2 + 2C^2 \log n,
\end{aligned} \tag{4.4}$$

since $\|x^*\|_\sim \le \|x^*\| \le 1$ for $x^* \in W_1^*$, and since $C \ge 1$.

It is shown in [T2], Section 4 that to prove (4.1), it suffices to prove the following.

PROPOSITION 4.1: *Consider $k \in \mathbb{N}, u \in T$, points $u_1, \ldots, u_N$ of T such that*

$$\forall \ell \leq N, \quad \|u - u_\ell\|_\infty \leq r^{-k}, \tag{4.5}$$

$$\forall \ell, \ell', \quad 1 < \ell < \ell' \leq N, \quad \|u_\ell - u_{\ell'}\|_\infty \geq r^{-k-1}. \tag{4.6}$$

Then

$$\max_{\ell \leq N} \varphi_{k+2}(u_\ell) \geq \varphi_k(u) + \frac{r^{-k}}{M} \log N, \tag{4.7}$$

where M is a universal constant.

We now observe that we can assume without loss of generality that $M \geq 4$; using the contribution of φ_k^3, we see that (4.7) holds if $N \leq n^2$. We can hence assume in the rest of the proof that $N \geq n^2$.

LEMMA 4.2: *If $t \geq r^{-k-1}/4$, we have*

$$|s - t| \leq r^{-k} \Rightarrow s - \min(s, 2r^{-k}) + r^{-k-1}/8 \leq t - \min(t, 2r^{-k-2}). \tag{4.8}$$

Proof: The right-hand side is

$$t - 2r^{-k-2} \geq \frac{r^{-k-1}}{4} - 2r^{-k-2} \geq \frac{r^{-k-1}}{8}$$

since $r = 16$. Thus the claim is proved if $s \leq 2r^{-k}$. If $s \geq 2r^{-k}$, the claim reduces to

$$s - 2r^{-k} + \frac{r^{-k-1}}{8} \leq t - 2r^{-k-1}$$

which is true since $|s - t| \leq r^{-k}$, $r^{-k-1}/8 + 2r^{-k-2} \leq r^{-k}$. ∎

We now start the proof of (4.7). Since $|u_i - u_{\ell,i}| \leq r^{-k}$, Lemma 4.2 shows that

$$\varphi_{k+2}^1(u_\ell) \geq \varphi_k^1(u) + \frac{r^{-k-1}}{8} \operatorname{card}\{i \leq n; |u_{\ell,i}| \geq r^{-k-1}/4\}. \tag{4.9}$$

Consider a parameter K_1 to be determined later.

CASE 1: For some $\ell \leq N$, we have

$$\operatorname{card}\left\{i \leq n; |u_{\ell,i}| \geq \frac{r^{-k-1}}{4}\right\} \geq \frac{\log N}{K_1}.$$

In this case, by (4.9) we have

$$\varphi_{k+2}^1(u_\ell) \geq \varphi_k^1(u) + \frac{r^{-k}}{8K_1 r} \log N.$$

Also, since $C_{k+2}(u_\ell) \subset C_k(u)$, we have

$$\varphi^2_{k+2}(u_\ell) \geq \varphi^2_k(u)$$

so that

$$\varphi_{k+2}(u_\ell) \geq \varphi_k(u) + \frac{r^{-k}}{8K_1 r}\log N$$

and (4.7) holds provided $M \geq 8K_1 r$.

CASE 2: We have

$$\forall \ell \leq N, \operatorname{card}\left\{i \leq n, |u_{\ell,i}| \geq \frac{r^{-k-1}}{4}\right\} \leq \frac{\log N}{K_1}. \tag{4.10}$$

This is the main case; the purpose of the functional φ^1 was actually to create this condition.

Consider the interval $D = [-r^{-k-1}/4, r^{-k-1}/4]$. For a number $q \in \mathbb{Z}$, we set

$$D(q) = D \cup \left(D + q\frac{r^{-k-1}}{2}\right).$$

LEMMA 4.3 (Separation): *If $N \geq n^2$ and if K_1 is chosen appropriately, we can find a subset L of $\{1, \dots, N\}$ such that $\operatorname{card} L \geq \sqrt{N}$ and integers $(q_i)_{i\leq n}$ such that*

$$\forall i \leq n,\ \forall \ell \in L, \quad u_{\ell,i} \in D(q_i). \tag{4.11}$$

Proof: For each $i \leq n$, consider the set Q_i of integers q such that

$$\left(D + q\frac{r^{-k-1}}{2}\right) \cap [u_i - r^{-k}, u_i + r^{-k}] \neq \emptyset.$$

Thus $\operatorname{card} Q_i \leq 4r + 4$. Consider the set $Q = \Pi_{i\leq n} Q_i$, provided with the normalized counting measure P.

The key point is that, for any $\ell \leq N$, we have

$$P\left(\{(q_i)_{i\leq n} \in Q; \forall i \leq n, u_{\ell,i} \in D(q_i)\}\right) \geq 1/\sqrt{N}. \tag{4.12}$$

Indeed, once (4.12) is proved, (4.11) follows by the Fubini theorem.

To prove (4.12) we fix ℓ, and we observe that there is a sequence $(q_{\ell,i}) \in Q$ such that $u_{\ell,i} \in D(q_{\ell,i})$ whenever $i \leq n$. Consider the set

$$I = \{i \leq n; |u_{\ell,i}| > r^{-k-1}/4\}.$$

For $i \notin I$, we have $|u_{\ell,i}| \le r^{-k-1}/4$, so that $u_{\ell,i} \in D(q_i)$ whatever the value of q_i. Thus

$$\forall i \in I, \quad q_i = q_{\ell,i} \Rightarrow \forall i \le n, \quad u_{\ell,i} \in D(q_i).$$

Thereby the left-hand side of (4.12) is at least

$$\Big(\prod_{i \in I} \operatorname{card} Q_i\Big)^{-1} = (4r+4)^{\operatorname{card} I} \ge 1/\sqrt{N}$$

by (4.10) and provided $K_1 = 2\log(4+4r)$. ∎

In the rest of the proof, we fix $(q_i)_{i \le n}$ as provided by Lemma 4.3, and we set

$$J = \{i \le n; |q_i| \ge 2\}.$$

For $\ell \in L$, we consider the set

$$R(\ell) = \{i \in J; u_{\ell,i} \in D + \tfrac{1}{2} q_i r^{-k-1}\}.$$

Thus, $u_{\ell,i} \in D$ for $i \notin R(\ell)$.

By (4.6), if $\ell \ne \ell'$, there is $i \le n$ such that $|u_{\ell,i} - u_{\ell',i}| \ge r^{-k-1}$. Since $|u_{\ell,i} - u_{\ell',i}| \le r^{-k-2}/2$ when $i \notin J$, when $i \in R(\ell) \cap R(\ell')$, or when $i \in J \backslash \big(R(\ell) \cup R(\ell')\big)$, we must have $R(\ell) \ne R(\ell')$.

There exists at least $\operatorname{card} L$ different sets of the type $R(\ell)$. Since we assume $N \ge n^2$, we have $\operatorname{card} L \ge \sqrt{N} \ge n$. Consider then the largest integer p such that $\operatorname{card} L \ge \binom{n}{p}$. Then $p \ge 1$, and

$$\operatorname{card} L \le \binom{n}{p+1} \le n^{p+1}$$

so that

$$p \ge \frac{p+1}{2} \ge \frac{\log \operatorname{card} L}{2 \log n} \ge \frac{\log N}{4 \log n}. \tag{4.13}$$

We now appeal to the Sauer–Shelah lemma [Sa], [Sh] and we see that there is a subset I of J with $\operatorname{card} I \ge p$ that is shattered by the family of sets $R(\ell)$, $\ell \in L$. This means that for every subset B of I, we can find $\ell(B)$ in L such that

$$B = I \cap R\big(\ell(B)\big).$$

For each such subset B we consider a vector x_B^* in $C_{k+2}(u_{\ell(B)})$ such that

$$\|x_B^*\| = \varphi_{k+2}^2(u_{\ell(B)}). \tag{4.14}$$

We now consider standard normal r.v. $(g_i)_{i \in I}$.

LEMMA 4.4: *We have*

$$E \sup_{B \subset I} x_B^* \Big(\sum_{i \in I} g_i x_i \Big) \geq \frac{1}{K} r^{-k/2} \operatorname{card} I. \tag{4.15}$$

Proof: Consider the random variable $Y_B = \sum_{i \in I} g_i x_B^*(x_i)$.

STEP 1: We prove that

$$\big(E|Y_B - Y_{B'}|^2 \big)^{1/2} \geq \frac{r^{-k/2}}{K} \sqrt{\operatorname{card}(B \Delta B')}. \tag{4.16}$$

Indeed, the left-hand side is

$$\Big(\sum_{i \in I} |x_B^*(x_i) - x_{B'}^*(x_i)|^2 \Big)^{1/2}.$$

Observe now that we have

$$|h\big(x_B^*(x_i)\big) - u_{\ell(B),i}| \leq 2r^{-k-2}.$$

If $i \notin B$, we have $u_{\ell(B),i} \in D$, and thus

$$|x_B^*(x_i)|^2 \leq \frac{r^{-k-1}}{4} + 2r^{-k-2} \leq \frac{3r^{-k-1}}{8},$$

so that

$$|x_B^*(x_i)| \leq (\tfrac{3}{8})^{1/2} r^{-(k+1)/2}.$$

If $i \in B$, then $u_{\ell(B),i} \in D + q_i r^{-k-1}/2$, so that $|u_{\ell(B),i}| \geq 3r^{-k-1}/4$ since $|q_i| \geq 2$, and

$$|x_B^*(x_i)|^2 \geq \frac{3r^{-k-1}}{4} - 2r^{-k-2} \geq \frac{5}{8} r^{-k-1}$$

and

$$|x_B^*(x_i)| \geq (\tfrac{5}{8})^{1/2} r^{-(k+1)/2}.$$

The same holds for B' rather than B. Thus

$$i \in B \Delta B' \Rightarrow |x_B^*(x_i) - x_B^*(x_i)| \geq r^{-k/2}/K$$

and this implies (4.16).

STEP 2: A simple counting argument shows that there is a family $\mathcal{B}$ of subsets of I such that

$$\log \operatorname{card} \mathcal{B} \geq \frac{1}{K} \operatorname{card} I,$$
$$B, B' \in \mathcal{B}, B \neq B' \Rightarrow \operatorname{card}(B \Delta B') \geq \tfrac{1}{3} \operatorname{card} I.$$

Then, by Sudakov minoration (L-T, Th. 3.18) we have

$$E \sup_{B \in \mathcal{B}} Y_B \geq \frac{1}{K} \sqrt{\log \operatorname{card} \mathcal{B}} \inf\{(E|Y_B - Y_{B'}|^2)^{1/2}; B, B' \in \mathcal{B}, B \neq B'\}. \qquad \blacksquare$$

Consider $\epsilon > 0$. Since $C_k(u)$ is convex and weak* closed, we can find y in W with $\|y\|_\sim = 1$ and

$$\forall y^* \in C_k(u), \quad y^*(y) \geq \varphi_k^2(u) - \epsilon. \tag{4.17}$$

LEMMA 4.5: *If* $\operatorname{card} I \geq K$, *then for each* $\alpha > 0$ *we have*

$$P\Big(\sup_{B \subset I} x_B^*(y + \alpha \sum_{i \in I} g_i x_i) \geq \varphi_k^2(u) - \epsilon + \frac{\alpha r^{-k/2}}{K} \operatorname{card} I\Big) > \frac{3}{4}. \tag{4.18}$$

Proof: Using (4.17) we have

$$\sup_{B \subset I} x_B^*(y + \alpha \sum_{i \in I} g_i x_i) \geq \varphi_k^2(u) - \epsilon + \alpha Z$$

where $Z = \sup_{B \subset I} x_B^*(\sum_{i \in I} g_i x_i)$.

I claim that $|x_B^*(x_i)| \leq 2r^{-k/2}$ for $i \in I, B \subset I$. To see this, we first observe that if $\ell_0 = \ell(\emptyset)$, then $u_{\ell_0, i} \in D$ for $i \notin R(\ell_0)$; but $R(\ell_0) \cap I = \emptyset$, so $|u_{\ell_0,i}| \leq r^{-k}$. Now, for any B, we have $x_B^* \in C_k(u)$, so that $|h_i(x_B^*) - u_i| \leq r^{-k}$. Since $|u_{\ell_0,i} - u_i| \leq r^{-k}$, we have $|h_i(x_B^*)| \leq 3r^{-k}$, and $|x_B^*(x_i)| \leq 2r^{-k/2}$.

Consider

$$\sigma^2 = \sup_{B \subset I} \sum x_B^*(x_i)^2.$$

Thus $\sigma \leq 2r^{-k/2}\sqrt{\operatorname{card} I}$. Now, the concentration of the measure phenomenon for the gaussian measure (such as in [L-T], Section 1.1) shows that

$$P(Z \geq EZ - K\sigma) > \tfrac{3}{4}.$$

The result follows. $\blacksquare$

We observe that the condition $\operatorname{card} I \geq K$ is not restrictive; indeed by (4.13), $\operatorname{card} I \geq \log N/4\log n$, so that $\operatorname{card} I \geq K$ provided $\log N \geq M\log n$ and M is large enough. On the other hand, if $\log N \leq M\log n$, to prove (4.7) it suffices to see that for every ℓ, we have $\varphi_{k+2}(u_\ell) \geq \varphi_k(u)$. We have already observed that $\varphi^2_{k+2}(u_\ell) \geq \varphi^2_k(u)$. Also,

$$\varphi^1_{k+2}(u_\ell) \geq \varphi^1_k(u) \tag{4.19}$$

follows from the elementary fact that, for $s, t \geq 0$,

$$|s-t| \leq r^{-k} \Rightarrow s - \min(s, 2r^{-k}) \leq t - \min(t, 2r^{-k-2}).$$

We now prove (4.7). The first part of the proof is somewhat secondary. It is to establish that

$$r^{-k} \operatorname{card} I \leq KC^2. \tag{4.20}$$

To see this, we observe that, for any $\alpha \geq 0$,

$$\alpha\Big\|\sum_{i\in I} g_i x_i\Big\|_\sim \leq 1 + \Big\|y + \sum_{i\in I}\alpha g_i x_i\Big\|_\sim.$$

Thus, by condition $H(C)$,

$$P\Big(\Big\|\sum_{i\in I} g_i x_i\Big\|_\sim \leq \frac{1}{\alpha}(2 + C^2\alpha^2 \operatorname{card} I)\Big) \geq \frac{1}{4}$$

and thus, by optimization over α,

$$P\Big(\Big\|\sum_{i\in I} g_i x_i\Big\|_\sim \leq 2\sqrt{2}C\sqrt{\operatorname{card} I}\Big) \geq \frac{1}{4}$$

and

$$P\Big(\Big\|y + \alpha\sum_{i\in I} g_i x_i\Big\|_\sim \leq 1 + 2\sqrt{2}C\alpha\sqrt{\operatorname{card} I}\Big) \geq \frac{1}{4}.$$

Combining with (4.18), we get

$$\varphi^2_k(u) - \epsilon + \frac{\alpha}{K} r^{-k/2}\sqrt{\operatorname{card} I} \leq 1 + 2\sqrt{2}C\alpha\sqrt{\operatorname{card} I}.$$

This holds for each $\alpha > 0$, so that (4.20) holds.

The main argument starts now. Combining (4.18) with condition $H(C)$, we find that, given α, there exists a vector z $(= y + \sum_{i\in I} \alpha_i g_i)$ such that

$$\sup_{B\subset I} x_B^*(z) \geq \varphi_k^2(u) - \epsilon + \frac{\alpha r^{-k/2}}{K} \operatorname{card} I$$

while

$$\|z\|_\sim \leq 1 + C^2\alpha^2 \operatorname{card} I.$$

Thus, there exists $B \subset I$ such that

$$\begin{aligned}\|x_B^*\|_\sim &\geq \frac{\varphi_k^2(u) - \epsilon + \alpha r^{-k/2} \operatorname{card} I/K}{1 + C^2\alpha^2 \operatorname{card} I} \\ &= \varphi_k^2(u) - \epsilon + \frac{\alpha r^{-k/2} \operatorname{card} I/K - C^2\alpha^2 \operatorname{card} I}{1 + C^2\alpha^2 \operatorname{card} I}.\end{aligned}$$

We see that a good choice for α is $r^{-k/2}/KC^2$. Using (4.20), we get

$$\|x_B^*\|_\sim \geq \varphi_k^2(u) - \epsilon + \frac{r^{-k} \operatorname{card} I}{KC^2}.$$

Since ϵ is arbitrary, this implies that there exists $\ell \leq N$ for which

$$\varphi_{k+2}^2(u_\ell) \geq \varphi_k^2(u) + \frac{r^{-k} \operatorname{card} I}{KC^2}.$$

Thus, using (4.13) and (4.19),

$$\varphi_{k+2}(u_\ell) \geq \varphi_k(u) + \frac{r^{-k}}{K} \log N. \qquad \blacksquare$$

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