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Estimate of the Sticking Probability for Cells in Uniform Shear Flow with Adhesion Caused by Specific Bonds

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Abstract

A theory is developed for the aggregation rate of cells in uniform shear flow when the cell–cell adhesion is mediated by bonds between specific molecules on the cell surfaces such as antigen and antibody or lectin and carbohydrate. The theory is based on estimates of the frequency and duration of cell–cell collisions and of the number of bonds formed and required to hold the cells together. For high shear rates, the sticking probability is a function of a single dimensionless parameter, A, that is proportional to G^{-2}, with G the shear rate. For low shear rates, the sticking probability is a function of a second dimensionless parameter, $A' \sim G^{-1}$. In either case the rate of cell–cell sticking is a maximum when A (or A') $\simeq 1.0$. For small values of A (or A') the cells collide frequently, but do not stick, whereas for large values of A (or A') the cells collide infrequently, but stick with larger probability. Studies in a Couette viscometer or other flow having approximately uniform shear can test these models.

Index Entries: Sticking probability, of specifically bonded cells; probability, sticking, of specifically bonded cells; cell sticking probability, of specifically bonded cells; bonds between cells, and sticking probability; flow, uniform shear, cell sticking probability in; shear flow, uniform, cell sticking probability in; Couette viscometer, in cell sticking probability studies; viscometer, in cell sticking probability studies; adhesion, cell–cell; aggregation, of cells.

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Introduction

For many reasons, biologists are interested in the sticking together of cells, mediated either by interactions between specific complementary cell surface molecules or by soluble multivalent ligands that can form intercellular bridges. For example, in immunology, the agglutination of red cells is often used as a sensitive assay for the presence of some agglutinating antibody and aggregates between lymphocytes and red cells are often used to indicate the presence of certain kinds of lymphocytes in a rosette assay. Agglutination of cells by lectins has also been much studied, partly because of the finding that normal and transformed cells differ in their agglutinability and partly because of a general feeling that transformed cells may have altered adhesive properties compared to normal cells.

In all of these cases one deals with the interaction between cells mediated by specific molecules rather than by nonspecific electrical forces. I have recently developed a theoretical framework (*1, 2*) for treating such interactions when the surface molecules on at least one of the cells can move about, by diffusion, on the membrane in such a way as to locate a reactive partner on the other cell.

In this paper, I apply these considerations to estimate the rate of cell aggregation in a particular experimental system, namely when the cells are in a solution subject to uniform shear. This situation may be closely approached in the Couette viscometer, a device in which a solution is placed between two closely spaced cylinders, one of which is rotating. Indeed, several investigators have already applied such techniques to the measurement of cell adhesion (*3, 4*). However, they were mainly concerned with interactions caused by nonspecific van der Waals forces rather than by specific molecular interactions.

The basic problem is to estimate the sticking probability for two cells colliding with relative velocity V_r and impact parameter b (see Fig. 1). I do this by estimating a collision duration and contact area, which together with the rate of bond formation discussed earlier (*1, 2*), gives the expected number of bonds formed during the collision. If this exceeds a critical number of bonds required to hold the cells together, sticking is achieved. If not, the cells separate after the collision.

Analysis

Estimate of the Sticking Probability for Cells in Uniform Shear Flow

Consider spherical cells of radius R in a fluid where the velocity exhibits uniform shear. In particular, let us choose a rectangular coordinate system (x, y, z) such that the fluid velocity is

$$\mathbf{V} = (V_0 + Gx)\mathbf{y} \qquad [1]$$

where the shear rate is G and **y** is a unit vector in the y direction. Assume that an individual cell moves with the velocity characteristic of the x coordinate of its center. Thus two cells can collide with velocity difference of magnitude $\leq 2GR$.

Consider a collision between cells of impact parameter b; that is, b would be the distance of closest approach of the cell centers if the cells were to move without interacting with each other. Let the difference in x coordinates of the cells be $\Delta x = b \cos \theta$ $(0 \leq \theta \leq 2\pi)$; Fig. 1. Then the relative velocity difference, V_r, is in magnitude,

$$V_r = Gb \mid \cos \theta \mid \qquad [2]$$

If the average number of cells per unit volume is n, then the expected number of collisions, dC, that a particular cell will make per unit time with other cells having impact parameters within db about b and angle within $d\theta$ about 0 is

$$\begin{aligned} dC &= V_r n\, b\, db\, d\theta \\ &= nG \mid \cos \theta \mid b^2 db\, d\theta \end{aligned} \qquad [3]$$

The collision rate, C, can now be obtained (*4, 5*) by integrating Eq. [3] over θ $(0 \leq \theta \leq 2\pi)$ and b $(0 \leq b \leq 2R)$. The result is

$$C = 32/3\, Gn\, R^3 \qquad [4]$$

as found by von Smoluchowski (*5*). C represents the collision rate of other cells with a give cell. To obtain the collision rate, c, per unit volume, we must multiply C by n and divide by 2 since each collision is counted twice, thus

$$c = 1/2\, C = 32/6\, Gn^2R^3 \qquad [5]$$

In order to estimate the probability that two cells colliding with b and θ stick together by the formation of specific intermolecular bonds, I assume that each

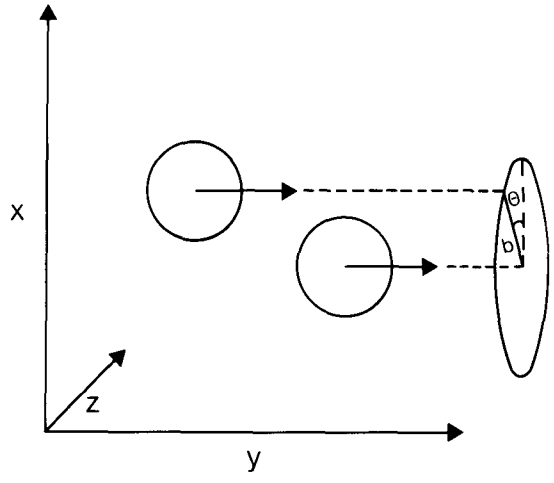

Fig. 1. Coordinate system showing two cells with impact parameter b. The cells move in the y direction and the difference in the x coordinates of their centers is $b \cos \theta$.

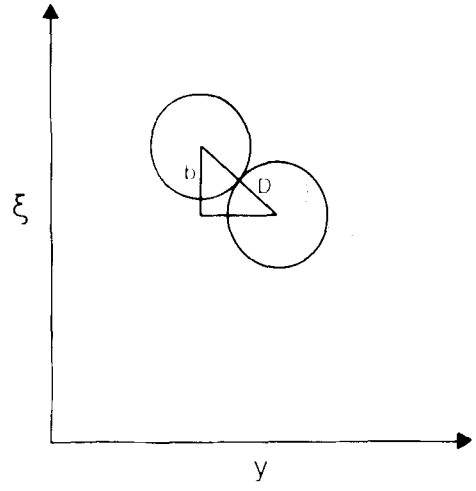

Fig. 2. Two colliding cells. ζ is a direction in the x,z plane such that $x\zeta = \cos\theta$. D is a cell diameter and b is the impact parameter. Evidently the difference in y coordinates of the cell centers is $(D^2 - b^2)^{1/2}$.

collision can be characterized by a number of quantities, namely, the duration of the collision, the area of intercellular contact, the rate of intercellular bond formation per unit area of interface, and the number of bonds that must form in order to hold the cells together.

(a) Duration of Collision Referring to Fig. 2, we see that when spherical cells make contact, the difference in y coordinates of their centers is $(D^2 - b^2)^{1/2}$ where $D = 2R$ is a cell diameter. I assume that the collision duration is twice this distance divided by the relative cell velocity

$$\text{Collision duration} \cong 2\sqrt{D^2 - b^2}/V_r \qquad [6]$$

(b) Contact Area The area on one cell that is in contact with the other will vary during the course of the collision, depending on how deformable the cells are, whether they have surface projections such as microvilli, and so on. For deformable cells, we might expect the maximum contact area during the collision to be proportional to $D^2 - b^2$ and I assume that the average contact area can be represented as a constant η times this quantity. Thus

$$\text{Contact area} \cong \eta\,(D^2 - b^2) \qquad [7]$$

(c) Rate of Bond Formation It is assumed that when two cells are in contact, specific intercellular bonds can form at a rate B per unit area. B can in turn be related to the number of complementary receptors per unit area on each cell, their diffusion constants for motion in the membranes and intrinsic reaction rates as described in refs. (*1*) and (*6*). If for example, the colliding cells have n_1 and

n_2 complementary receptors per unit area with diffusion constants D_1 and D_2 for motion in the plane of the membrane, we may write

$$\dot{B} \cong n_1 n_2 \, 2\pi \, \epsilon \, (D_1 + D_2) \qquad [8]$$

where ϵ is a constant by which the reaction rate falls short of its diffusion limit. If the intercellular bonds are mediated by soluble ligands, $\dot{B}$ will depend on the ligand concentration, ligand–cell binding constants, and other considerations discussed in ref. (*2*).

By multiplying the collision duration, contact area, and bond formation rate, we obtain an estimate of the number, N, of bonds formed in a collision,

$$N \cong \frac{2\eta\dot{B} \, (D^2 - b^2)^{3/2}}{V_r} \qquad [9]$$

(d) Number of Bonds Required to Hold Two Cells Together In ref. (*1*) I estimated the force that is required in order to break intermolecular bonds such as those between antigen and antibody. In particular, I estimated that the number of uniformly stressed bonds required in order to hold a cell in a fluid stream of velocity V against Stokes drag is $n_c V$ with n_c a constant ($n_c = 6\pi\eta R/f_c$, where η is here the fluid viscosity and f_c is the critical force per bond). For the particular case of lymphocytes held in water by antigen–antibody bonds, $R \simeq 4\mu$m, $\eta \simeq 10^{-2}$ g/cm-s, $f_c \simeq 4 \times 10^{-6}$ dynes/bond, we have $n_c \cong 19$ with V in cm/s. I assume that the colliding cells will stick together if the expected number of bonds formed exceeds $n_c V_r$ and not otherwise. Thus, the sticking probability, P is

$$\begin{aligned} P &= 1 \text{ if } N > n_c V_r \\ &= 0 \text{ if } N < n_c V_r \end{aligned} \qquad [10]$$

It is probably not possible to take numerical values of n_c directly from the considerations of ref. (*1*) since the complex dynamics of the collision are not such that the bonds between the cells will be uniformly stressed when the cells begin to separate during the latter half of the collision.

For sufficiently small relative velocities, $n_c V_r$ will be less than one. In this case it is a better approximation to let the required number of bonds be a constant such as one. A theory for this case is given in the following section. Of course, a single bond has only a finite lifetime, but it may be likely that a single bond will hold the cells together sufficiently long that further bonds form, creating a stable aggregate.

From the foregoing assumptions, the sticking probability may be related to experimental parameters and its average value computed. By introducing Eqs. [9] and [2] and in Eq. [10] we find

$$\begin{aligned} P &= 1 \text{ if } 2\eta\dot{B}(D^2 - b^2)^{3/2} > n_c G^2 b^2 \cos^2\theta \\ &= 0 \text{ otherwise} \end{aligned} \qquad [11]$$

For finite values of all parameters, P will be one for $\cos\theta$ sufficiently small, or θ sufficiently near to $\pi/2$ and $3\pi/2$. For very small values of b, which correspond to nearly head-on collisions, $P = 1$ for all values of θ.

Defining the constant A by

$$A = \frac{2\eta\dot{B}D}{n_c G^2} \qquad [12]$$

and the variable β by $b = \beta D$, we have

$$P = 1 \quad \text{if} \quad \cos^2\theta < A\frac{(1-\beta^2)^{3/2}}{\beta^2} \qquad [13]$$

Moreover, let β_m be the value of β in Eq. [13], corresponding to equality with $\cos\theta = 1$, i.e.,

$$\beta_m^2 = A(1 - \beta_m^2)^{3/2} \qquad [14]$$

This equation has a unique solution in the interval $0 < \beta_m < 1$ for any finite A. For $\beta > \beta_m$, let the value of θ, $0 < \theta < \pi/2$, which gives the equality in Eq. [13] be θ_m, i.e.,

$$\cos\theta = A\frac{(1 - \beta^2)^{3/2}}{\beta^2} \qquad \beta > \beta_m \qquad [15]$$

We are now in a position to compute the expected number of collisions, $\bar{P}C$ per unit time, that result in sticking. We proceed as before in computing the collision rate, i.e.,

$$\bar{P}C = 4Gn\int_0^b b^2 db \int_0^{\pi/2} P\cos\theta\, d\theta \qquad [16]$$

where we have considered only one quadrant in θ and multiplied by 4, since the other quadrants are identical.

For small values of b ($0 < b < \beta_m D$) all values of θ give $P = 1$, while for larger b only those values, $\theta > \theta_m$, have $P = 1$. Thus

$$\bar{P}\theta = 4Gn\int_0^{\beta_m D} b^2 db \int_0^{\pi/2} \cos\theta\, d\theta + \int_{\beta_m D}^{D} b^2 db \int_{\theta_m}^{\pi/2} \cos\theta\, d\theta \qquad [17]$$

and, on working out the integrals and using Eq. [4] for C we find

$$P = 1 - 3\int_{\beta_m}^1 \sin(\theta_m)\beta^2 d\beta =$$

$$1 - 3\int_{\beta_m}^1 \sqrt{1 - A\beta^{-2}(1 - \beta^2)^{3/2}}\,\beta^2 d\beta \qquad [18]$$

This integral has been evaluated numerically and results are shown in Fig. 3. An approximate value is derived in the appendix.

The sticking probability is a monotonically increasing function of A and hence decreases with increasing shear rate G. The collision rate given by Eq. [4] increases with increasing shear rate. Their product, $\bar{P}C$, which is the rate at

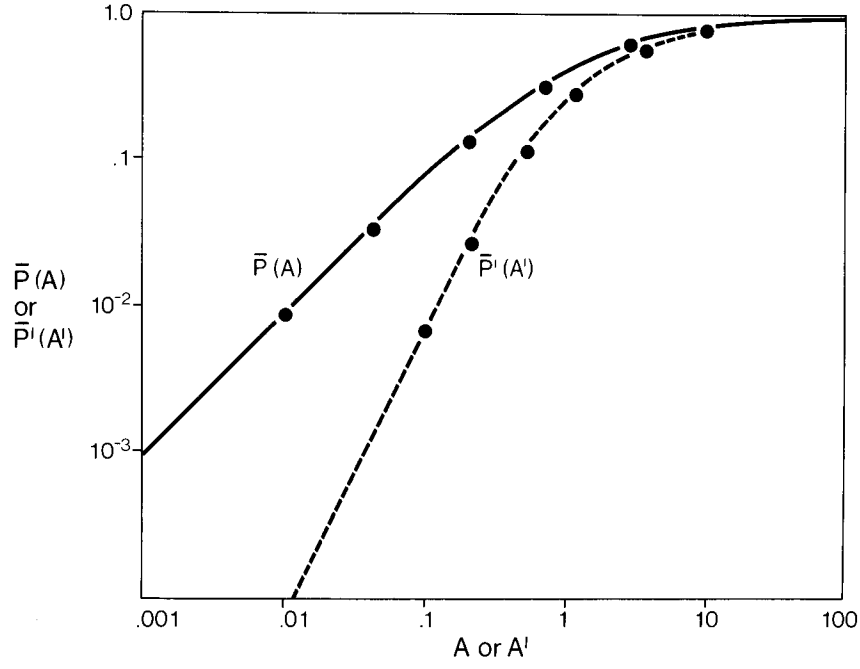

Fig. 3. Sticking probabilities $\bar{P}(A)$ as given by Eq. [18] for high shear rates and $\bar{P}'(A')$ as given by Eq. [24] for low shear rates. The curves represent the results of numerical integration, whereas approximate values given by Eqs. [A2] and [A5] are represented by the points. It is indicated in Appendix 2 that whichever is the smaller of the two quantities $\bar{P}(A)$ and $\bar{P}'(A')$ should be used for the sticking probability.

which a given cell sticks to other cells, has a unique maximum as a function of G, as shown in Fig. 4. To see this, note that for small G, (large A), P approaches one, so $\bar{P}C \sim G$. For large G (small A), $P \sim A \sim G^{-2}$ (Appendix 1), so $PC \sim G^{-1}$. From Fig. 4 we see that the maximum sticking rate occurs for A near unity.

Treatment for Small Shear Rates

For small collision velocities, which are always to be found for low shear rates, $n_c V_r$ may be small compared to one bond. In particular, if the maximum collision velocity, DG, is such that $DG\, n_c < 1$, the model predicts that even one, or only a few bonds will hold the cells together. In this case, a better approximation is to let the required number of bonds for sticking be some fixed number, n_c' of order unity. The criterion [11] is then replaced by

$$\begin{aligned} P &= 1 \quad \text{if } 2\eta\dot{B}(D^2 - b^2)^{3/2} > n_c' Gb \mid \cos\theta \mid \\ &= 0 \quad \text{otherwise} \end{aligned} \qquad [19]$$

As before we may define a new parameter A' by

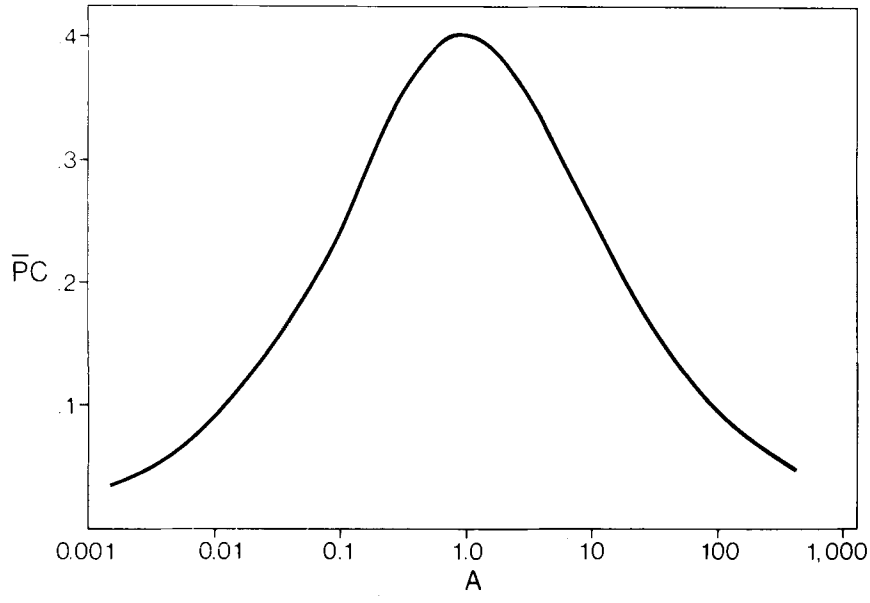

Fig. 4. Rate of cell sticking $\bar{P}C$, as a function of A. The shear rate is assumed to be the only parameter being varied. Units of $\bar{P}C$ are arbitrary and the actual quantity plotted is $\bar{P}A^{-1/2} \sim \bar{P}G$.

$$A' = \frac{2\eta\dot{B}D^2}{n_c' G} \qquad [20]$$

so that

$$P = 1 \qquad \text{if } | \cos\theta | < A' \frac{(1 - \beta^2)^{3/2}}{\beta} \qquad [21]$$

Let the maximum value of β for which Eq. [21] is satisfied for $\cos\theta = 1$, be β_m', i.e.,

$$\beta_m' = A'[1 - (\beta_m')^2]^{3/2} \qquad [22]$$

and for $\beta > \beta_m'$, let $(0 < \theta_m' < \pi/2)$

$$\cos\theta_m' = A' \frac{(1 - \beta^2)^{3/2}}{\beta} \qquad [23]$$

With these definitions of A', β_m' and θ_m', *and* θ_m' Eq. [17] holds with the primed quantities replacing unprimed, and

$$P' = 1 - 3\int_{\beta_m}^{1} \sin\theta'_m\, \beta^2 d\beta =$$

$$1 - 3\int_{\beta_m}^{1} \sqrt{1 - (A')^2\beta^{-2}(1-\beta^2)^3}\, d\beta \qquad [24]$$

This expression has been evaluated numerically and the results are shown in Figs. 3 and 5. An approximate expression is also derived in Appendix 1.

Refinements

Two further refinements may be considered. First of all, one might let the required number of bonds be the greater of n'_c and $n_c V_r$, to include both slow and fast collisions. Unfortunately, the integrals over β and θ then become sometimes partitioned into a number of pieces depending on both A and A'. The treatment is given in the Appendix 2, where it is shown that a good approximation is to use whichever of the quantities, $\overline{P}$ (A) or $\overline{P}'$ (A'), is the smaller.

The other feature is to note that even when the average rate of bond formation is too small to produce a single bond, so that we would have set $P = 0$, there is a finite probability that any number of bonds would form. If N is the expected number of bonds formed in a particular collision, the actual number will presumably have a Poisson distribution about N, such that the probability, p (n), of forming n bonds is

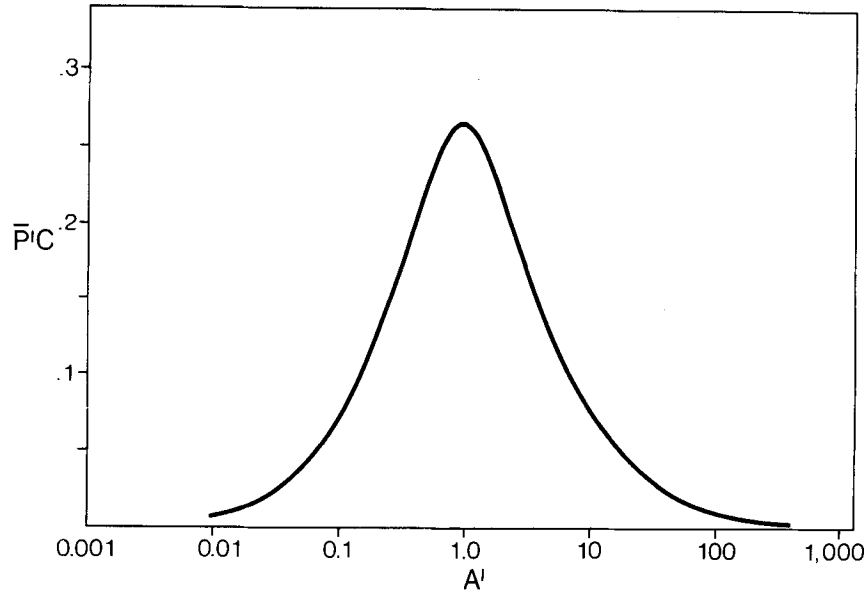

Fig. 5. Rate of cell sticking for the low shear rates, $\overline{P}'C$ as a function of A'. The shear rate is assumed to be the only parameter being varied. Units of $\overline{P}'C$ are arbitrary and the actual quantity plotted is $P'/A \sim \overline{P}'G$.

$$p(n) = \frac{N^n e^{-N}}{n!} \qquad [25]$$

Thus, the probability of forming at least one bond is

$$p(n \geq 1) = 1 - e^{-N} \qquad [26]$$

If Eq. [9] were used for N in this expression, it does not seem possible to do even the first quadrature explicity. In effect we have above approximated the quantity $1 - e^{-N}$ by a step function, namely 0 for $N < 1$ and 1 for $N > 1$.

Discussion

For high shear rates we have derived an estimate of the average sticking probability, $\overline{P}$, shown in Fig. 3, that depends only on the single dimensionless param eter, A. The parameter A, Eq. [12], is proportional to the rate of intercellular bond formation, $\dot{B}$, (*1, 2*) and to the cell diameter, D, and inversely proportional to the square of the shear rate, G, and to a critical number of bonds. Some of these parameters are under the experimenter's control so that the predicted dependence on A could be checked in many ways. The sticking rate as a function of shear rate, Fig. 4, is predicted to have a maximum near $A = 1$; for smaller values of G the collision rate is low and the sticking probability is near unity, while for larger values of G, the sticking probability is low.

For small shear rates, an alternative model is presented in which the required number of bonds is a fixed number, such as one, rather than being proportional to the relative cell velocity. In this case the sticking probability, $\overline{P}'$ in Fig. 3, depends on another dimensionless parameter, A', given by Eq. [20], and the sticking rate as a function of shear rate has a maximum near $A' = 1$ as shown in Fig. 5. It is argued in Appendix 2 that a reasonable approximation for all shear rates is to use the smaller of $\overline{P}$ (A) and $\overline{P}$ (A').

It is clear from the derivation of the equations that our model of the collision between two cells is highly simplified. We hope that by including the parameter η/n_c we have allowed sufficient flexibility in the model to permit it to be a useful way of treating real collisions, but only comparison with experiments can test this.

In addition, the treatment of bond formation by an average rate, $\dot{B}$ neglects several considerations. For one thing, the number of bonds formed in a prescribed collision is really a random variable with some distribution, probably Poisson, about its average value. This means that the sticking probability that we treated as a step function, for example, in Eq. [10] should be a smoothly varying function. However, since we average over a wide range of collisions, we may expect that the neglect of fluctuations in the number of bonds formed is not a serious approximation. If all collisions were the same, it would be important to consider fluctuations. If for example, we wanted to compute an aggregation rate

for cells due to Browian motion, it might be important to account for random fluctuations both in the number of bonds formed and in the collision duration.

We have also neglected the tendency of receptors to accumulate in an area of cell–cell contact during a collision (*7*). So long as there are many more receptors in the contact area than are required to form the critical number of bonds, this should not be a serious limitation. If the receptors were very sparse and the cells were coming together at very low shear rates, hence long collisions, receptor redistribution might become important.

As noted in the introduction, there are many biological contexts in which it is clear that cell–cell interactions and agglutination are mediated by bonds between specific known molecules. In other contexts, including the adhesion of blood cells—lymphocytes, granulocytes, and platelets—to endothelial cells, tumor cells to endothelial cells, or in many situations in developmental biology, the nature of the cell– cell–interactions is less clear, but their specificity suggests that adhesion is caused by the interaction between specific molecules. Some of these examples might be fruitfully studied using approaches suggested by this paper.

For experimental tests of this model, the exclusive use of red cells is undesirable because they have relatively immobile receptors and rather odd shapes. Rosette assays, in particular the sticking of red cells to lymphocytes might be good candidates because the lymphocytes have mobile receptors and one can vary the number of complementary surface molecules on the red cells in a controlled manner. Alternatively, one might consider the agglutination of lymphocytes by multivalent ligands such as lectins or antibodies. In this case some care might be required in order to prevent receptor redistribution (capping) by the multivalent ligand. Some preliminary experiments relating to these questions will be reported in a subsequent manuscript (*8*).

The rate of sticking of suspended cells such as leukocytes to a layer of cells such as endothelial cells could be studied in a viscometer having one surface composed of the layer of cells.

Appendix 1

Approximate Value for Sticking Probability

An approximate value for $\bar{P}$, Eq. [18] may be derived in the following way. Rewrite $\bar{P}$ as

$$\bar{P} = \beta_m^3 + 3\int_{\beta_m}^1 [1 - \sqrt{1 - A\beta^{-2}(1 - \beta^2)^{3/2}}]\, \beta^2 d\beta \qquad [A1]$$

Let us expand the square root $\sqrt{1 - f(\beta)} \cong 1 - f(\beta)/2$. Since $0 \leq f(\beta) \leq 1, 0 \leq 1 - \sqrt{1 - f(\beta)} \leq 1$, while $0 < f(\beta)/2 < \frac{1}{2}$, so that the approximate integrand and $\bar{P}$, will never be off by more than a factor two. Writing the approximate value for $\bar{P}$ as $\tilde{P}$,

$$\tilde{P} = \beta_m^3 + 3A/2 \int_{\beta_m}^1 (1 - \beta^2)^{3/2}\, d\beta$$
$$= \beta_m^3 + 3A/2 \{3\pi/16 - 1/4 [\beta_m (1 - \beta_m^2)^{3/2} + 3/2\, \beta_m (1 - \beta_m^2)^{1/2} + 3/2 \sin^{-1}\beta_m]\} \quad [A2]$$

For small values of β_m, $A \simeq \beta_m^2$ and

$$\tilde{P} \simeq 9\pi A/32 - 1/2\, A^{3/2} \quad [A3]$$

For values of β close to 1, the first term in Eq. (A2) is dominant and

$$\tilde{P} \simeq \beta_m^3 \simeq 1 - 3/2\, A^{-3/2} \quad [A4]$$

The comparison between these approximate values and the numerical solution to Eq. [18] is shown in Fig. 3.

For small shear rates, such that $n_c GD < 1$, Eq. [24] is a better approximation for $\tilde{P}$. In this case, approximating the square root gives

$$\tilde{P}' = \beta_m^3 + 3/2\, A'^2 \int_{\beta_m'}^1 (1 - \beta^2)^3 d\beta$$
$$= \beta_m'^3 + 3/2\, (A')^2\, (0.4571 - \beta_m' + \beta_m'^3 - 3/5\, \beta_m'^5 + 1/7\, \beta_m'^7) \quad [A5]$$

This approximate value of $\tilde{P}$ is compared with the numerical solution to Eq. [24] in Fig. 3, and seen to be in good agreement. In this case, for small β_m', $A' \simeq \beta_m'$ so that

$$\tilde{P}' \simeq 0.69(A')^2 \quad (A6)$$

Appendix 2

Uniform Treatment for High and Low Shear Rates

Let the required number of bonds be the greater of n_c' and $n_c Gb \mid \cos\theta \mid$. That is, we assume that the actual number of bonds must exceed the greater of these quantities before sticking will take place. This means that $P = 1$ if both Eqs. [13] and [21] are satisfied:

$$P = 1 \quad \text{if} \quad A\, \beta^{-2}(1 - \beta^2)^{3/2} > \cos^2\theta \quad [A6]$$
$$\text{and} \quad A'\beta^{-1}(1 - \beta^2)^{3/2} > \mid \cos\theta \mid \quad [A6']$$

For prescribed values of A and A', we must now consider for what values of (β,θ) $(0 \leq \beta < 1, 0 < \theta < \pi/2)$ one or the other of these inequalities is limiting.

Let

$$\gamma = A/A' = \frac{n_c'}{n_c GD} \quad [A7]$$

If $\gamma > 1$, we have already argued that the low shear rate (primed) approximation applies, and inequality [A6′] must be limiting. To see this, chose any pair of β,θ values satisfying [A6′]. To obtain the left side of [A6] we multiply that in [A6′] by γ/β (>1) and to get the right side of [A6] we multiply the right side of [A6′] by $\cos\theta$ (≤ 1). Hence if [A6′] is satisfied, so also is [A6], and [A6′] is limiting if $\gamma > 1$.

For $\gamma < 1$, consider β_m and β'_m as defined in Eqs. [14] and [22]

$$A(1 - \beta_m^2)^{3/2} = \beta_m^2$$

which using [A7] may be written as

$$A'(1 - \beta_m^2)^{3/2} = \beta_m^2/\gamma \qquad [A8]$$

and

$$A'(1 - \beta_m'^2)^{3/2} = \beta'_m \qquad [A8']$$

Note that if $\beta_m = \gamma$, the two equations are the same so that $\beta_m = \beta'_m$. If $\beta_m > \gamma$, then letting $\gamma/\beta_m = R$, Eq. [A8] may be written

$$RA'(1 - \beta_m^2)^{3/2} = \beta_m \qquad [A9]$$

which is the same as Eq. [A8′] except that $RA' < A'$ replaces A'. Since β'_m is a monotonically increasing function of A', it follows that $\beta'_m > \beta_m$. Conversely if $\beta_m < \gamma$ so that $R > 1$, $\beta_m > \beta'_m$. We thus have two cases to consider

I. $\beta'_m > \beta_m > \gamma$
II. $\beta'_m < \beta_m < \gamma$

as sketched in Fig. 6. If $\beta < \max(\beta_m, \beta'_m)$ the region where $P = 1$ is clearly determined by the smaller of β_m, β'_m. However, when $\beta > \max(\beta_m, \beta'_m)$ will

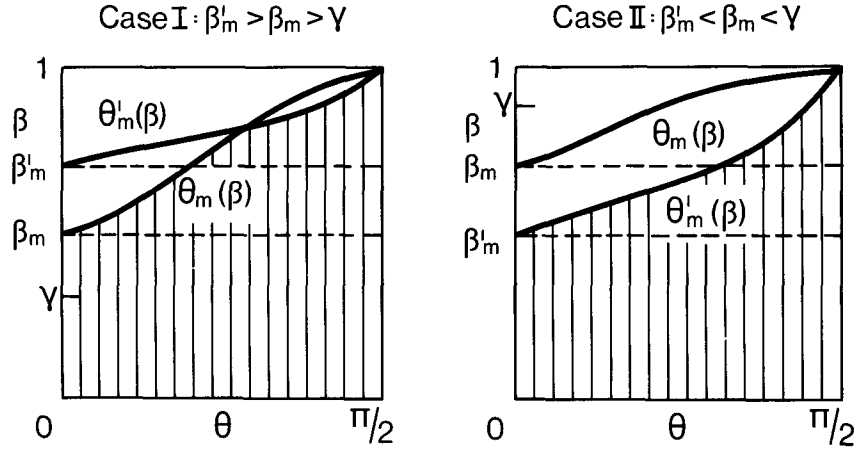

Fig. 6. Domains of (θ,β) where $P = 1$ are cross hatched. The curves $\theta_m(\beta)$ and $\theta'_m(\beta)$ are schematic and indicate that in case I there is a crossing of the curves while in case II there is not.

the two curves $\theta_m(\beta)$ and $\theta'_m(\beta)$ ever cross? θ_m and θ'_m are defined by Eqs. [15] and [23], here written as

$$\cos \theta'_m = A'(1 - \beta^2)^{3/2}\beta^{-1} \qquad [A10']$$

$$\cos \theta_m = \frac{\gamma A'(1 - \beta^2)^{3/2}\beta^{-2}}{\cos \theta_m} \qquad [A10]$$

The right-hand sides are equal and $\theta_m = \theta'_m$, if $\cos \theta_m = \gamma/\beta$. Let the corresponding value of β be denoted by β_c, which satisfies

$$\gamma = A'(1 - \beta_c^2)^{3/2} \qquad [A11]$$

This equation will have a solution ($0 < \beta_c < 1$) only if $A' > \gamma$. By comparing Eqs. [A11] and [A8'], it is seen that if $\beta'_m > \gamma$ (case I), then $\beta_c > \beta'_m$, while if $\beta'_m < \gamma$, then $\beta_c < \beta'_m$. Hence, β_c lies in the required range only for case I. We conclude that only for case I and for $A' > \gamma$ will there be solutions of Eqs. [A10] and [A10'] with $\theta_m = \theta'_m$ in the range $0 \leq \theta_m \leq \pi/2$, $\max(\beta_m\beta'_m) \leq \beta \leq 1$.

Note that since $\beta_m < 1$ and $\beta_m < A'$, the condition $\beta_m > \gamma$ includes both conditions $A' > \gamma$ and $1 > \gamma$. We have thus partitioned the positive A', A quadrant into two parts by the curve $\beta_m = \gamma$, as sketched in Fig. 7. For $\beta_m \leq \gamma$ the previous $\bar{P}'$ treatment is correct while for $\beta_m > \gamma$ we need a new treatment, in which unprimed quantities are used for $0 \leq \beta \leq \beta_c$ and primed quantities for $\beta_c < \beta \leq 1$.

Hence the new treatment gives, for $\beta_m > \gamma$

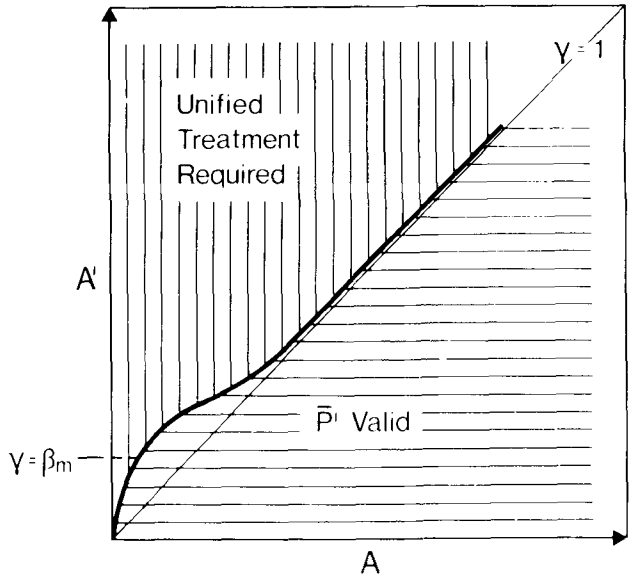

Fig. 7. Regions of A', A space where unified treatment, Eq. [A12],is required.

TABLE 1
Values of $\bar{P}''$ From Numerical Integration of Eq. [A12][a]

β_m	A	γ	β_c	$\bar{P}''$
0.99	349	0.5	0.996	0.978
0.98	122	0.5	0.992	0.956
0.95	29.6	0.5	0.979	0.892
		0.25	0.992	0.893
0.90	9.78	0.5	0.956	0.791
		0.25	0.983	0.793
0.80	2.96	0.5	0.899	0.608
		0.25	0.961	0.614
		0.10	0.989	0.615
0.70	1.34	0.50	0.821	0.451
		0.25	0.933	0.461
		0.10	0.981	0.462
0.60	0.703	0.50	0.706	0.316
		0.25	0.895	0.333
		0.10	0.970	0.334
0.50	0.385	0.25	0.838	0.227
		0.10	0.955	0.229
0.40	0.208	0.25	0.742	0.141
		0.10	0.931	0.145
0.30	0.104	0.25	0.535	0.0743
		0.10	0.889	0.0809
		0.05	0.957	0.0810
		0.025	0.983	0.0812
0.20	0.0425	0.10	0.787	0.0358
		0.05	0.921	0.0361
		0.025	0.970	0.0362
0.10	0.0102	0.10	0.100	6.89×10^{-3}
		0.05	0.779	8.3×10^{-3}
		0.025	0.919	8.71×10^{-3}
		0.010	0.977	8.84×10^{-3}
0.04	1.6×10^{-3}	0.025	0.683	1.37×10^{-3}
		0.010	0.918	1.41×10^{-3}
0.02	4.00×10^{-4}	0.01	0.777	3.548×10^{-4}
		0.005	0.918	3.53×10^{-4}
0.01	1.00×10^{-4}	0.01	0.100	6.85×10^{-5}
		0.005	0.777	8.70×10^{-5}
		0.0025	0.918	8.82×10^{-5}
		0.001	0.977	8.83×10^{-5}

[a]Calculations were done for the listed values of β_m and various choices of $\gamma < \beta_m$. The underlined values do not change, to three significant figures, as γ is further *de*creased. That is, they represent $\bar{P}$. Note that the deviations from P are slight.

$$\overline{P''} = 1 - 3\int_{\beta m}^{\beta_c} \sin\theta_m \beta^2 d\beta - 3\int_{\beta_c}^{1} \sin\theta'_m \beta^2 d\beta \qquad [A12]$$

where β_m, β_c, θ_m, and θ'_m are defined by Eqs. [A8], [A11], [A10], and [A10'], respectively.

Note that there is no region of parameter space for which $\overline{P}$ is valid. This is be cause there are always collisions at large β, but small relative velocities ($\cos\theta << 1$) for which there are less than n'_c bonds formed but for which the sticking criterion [10] is satisfied. For such collisions $n'_c > n'_c Gb \cos\theta$ and n'_c is the limiting number of bonds.

The sticking probability as computed from Eq. [A12] is shown in Table 1. The results show that except for a small region near $\beta_m = \gamma$, the new Eq. [A12] gives results very close to $\overline{P}$. From Fig. 6 it appears reasonable to always use the smaller of $\overline{P}$ (A) as $\overline{P}'$ (A') and the results in Table 1 show that this is a good approximation.

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