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ANALYTIC AND POLYHEDRAL APPROXIMATION OF CONVEX BODIES IN SEPARABLE POLYHEDRAL BANACH SPACES

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ABSTRACT

A closed, convex and bounded set P in a Banach space E is called a polytope if every finite-dimensional section of P is a polytope. A Banach space E is called polyhedral if E has an equivalent norm such that its unit ball is a polytope. We prove here :

(1) Let W be an arbitrary closed, convex and bounded body in a separable polyhedral Banach space E and let $\varepsilon > 0$. Then there exists a tangential ε-approximating polytope P for the body W.

(2) Let P be a polytope in a separable Banach space E. Then, for every $\varepsilon > 0$, P can be ε-approximated by an analytic, closed, convex and bounded body V.

We deduce from these two results that in a polyhedral Banach space (for instance in $c_0(\mathbb{N})$ or in $C(K)$ for K countable compact), every equivalent norm can be approximated by norms which are analytic on $E\setminus\{0\}$.

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Introduction

Two kinds of approximation of convex bodies in finite-dimensional spaces are the mostly used in various areas of mathematics:

(1) Approximation by polytopes.

(2) Approximation by smooth (analytic) convex bodies.

By approximation we mean the following. Let W be a closed convex bounded body in a Banach space. For simplicity, let us suppose that $0 \in \operatorname{int}(W)$. We shall say that the closed, convex and bounded body V ε-approximates the body W if $V \subset W \subset (1+\varepsilon)V$. We say that a closed, convex and bounded body V such that $0 \in \operatorname{int}(V)$ is analytic if the Minkowski functional f_V of this body is analytic except at the origin.

In a finite-dimensional space E, the simplest way to approximate a closed, convex and bounded body W by an analytic closed, convex and bounded body V is apparently to proceed in two steps. First, we approximate W by a polytope

$$P = \{x \in E\colon \max f_i(x) \leq 1, f_i \in E^*, i = 1, \dots, n\}$$

and then we approximate the polytope P by a closed, convex and bounded body

$$V = \{x \in E\colon \sum_{i=1}^{n} f_i(x)^p \leq 1\}$$

with p even integer big enough. We shall see that this program also works in infinite dimensions under suitable assumptions on the space E.

Let P be a closed convex body subset of $\mathbb{R}^n$. Then P is called a (finitely dimensional) polytope if there exists a finite set $\{p_k\}_{k=1}^{l}$ such that $P = \operatorname{co}\{p_k\}_{k=1}^{l}$. A closed, convex and bounded set P in a normed space E is called a polytope (K-polytope in [10]) if every finite-dimensional section of P is a finite-dimensional polytope (for other definitions of infinite-dimensional polytope see [4] and [17]). A Banach space E is called (isomorphically) polyhedral if E admits an equivalent norm such that its unit ball is a polytope [16]. As it is proved in [8, 9], a separable Banach space is polyhedral if and only if it admits an equivalent norm with a countable boundary (a subset $B \subset S(E^*)$ of the unit sphere $S(E^*)$ of a dual Banach space E^* is called a boundary ([9, 12, 13]) if for each $x \in E$ there exists $f \in B$ such that $f(x) = \|x\|$). For example, each isometric predual of ℓ_1 (in particular $C(K)$ if K is countable compact) is polyhedral. On the other hand, polyhedral Banach spaces are saturated with $c_0(\mathbb{N})$, i.e. if E is a polyhedral Banach space, then every infinite-dimensional subspace of E contains a subspace isomorphic to $c_0(\mathbb{N})$ [9].

We shall show in this paper that the above approximation procedure works in polyhedral spaces. On the other hand, it is not difficult to see that if E is a Banach space and if there exist in E a closed, convex and bounded body with non empty interior and which is a polytope, then E is an isomorphically polyhedral Banach space. So the first step of the above program only works in isomorphically polyhedral spaces. The approximation of a closed, convex and bounded body W by an analytic closed, convex and bounded body V in Banach spaces which are not polyhedral (for instance in Hilbert spaces), is also possible, but new concepts need to be introduced. This will be treated elsewhere.

Let us mention that a polytope with empty interior (in the subspace that it generates) has also an empty algebraic interior [10]. We do not know if such examples do exist. Anyway all the polytopes under consideration through this paper are assumed to be closed, convex and bounded bodies with 0 in their interior.

Let us now describe the content of this paper.

In section 1, we give the characterization of separable polytopes in terms of convex bodies with countable boundary. Then we prove that each closed, convex and bounded body in a separable isomorphically polyhedral Banach space can be approximated by polytopes (even by tangential polytopes). Construction of approximating polytope uses some ideas from [1] and [18].

In section 2, we show that each closed, convex and bounded body in a vector normed space with countable algebraic basis can be approximated by polytopes.

In section 3, we prove that each closed, convex and bounded body with countable boundary can be approximated by analytic closed, convex and bounded bodies. Here we use some ideas from [6]. Together with the results of section 1 this yields that each closed, convex and bounded body in a separable polyhedral Banach space can be approximated by analytic closed convex and bounded bodies.

Since the approximation of a given closed, convex and bounded body W such that $0 \in \mathrm{int}(W)$, by some closed, convex and bounded body V such that $0 \in \mathrm{int}(V)$ is equivalent to the approximation of the Minkowski functional f_W by f_V (uniformly on bounded sets), we have shown that each equivalent norm on a separable polyhedral Banach space can be approximated by norms which are analytic on all the space except at the origin. This result answers the following question from [2]: find a Banach space such that each equivalent norm on it can be approximated by norms possessing smoothness of order higher than one.

1. Polyhedral approximation in Banach spaces

Recall that all the closed, convex and bounded bodies W under consideration possess the property $0 \in \operatorname{int}(W)$. We say that a polytope P is a tangential polytope for the body W if $P \supset W$ and each maximal face of P is tangent to the body W. Recall that a polytope P is ε-approximating for the body W if $W \subset P \subset (1+\varepsilon)W$.

The main purpose of this section is to prove the following:

THEOREM 1.1: *Let W be an arbitrary closed, convex and bounded body in a separable polyhedral Banach space E and $\varepsilon > 0$. Then there exists a tangential ε-approximating polytope P for body W.*

Before proving Theorem 1.1, we give some preliminary results. We say that a subset $B \subset W^o$ of the polar W^o of W is a boundary (for the set W) if for each $x \in \partial W$ there exists a functional $f \in B$ such that $f(x) = 1 = \max f(W)$. Of course $w^*\text{-cl}\,\operatorname{co}(B) = W^o$. We say that the space E has a countable boundary if its unit ball admits a countable boundary. The following theorem (see [9, 10, 11]) summarizes some properties of polyhedral Banach spaces that we will use. In its statement, $S(E^*)$ denotes the unit sphere of the space E^* and (ε_i) a decreasing sequence of real numbers converging to 0.

THEOREM A: *Let E be a Banach space of weight α such that its unit ball is a polytope. Then there exists a boundary $B \subset S(E^*)$ of cardinality α such that for every $f \in B$, the face $\{x \in S(E)\colon f(x) = 1\}$ has non-empty interior in the hyperplane $\{x \in E\colon f(x) = 1\}$.*

Conversely, if a separable Banach space E has a countable boundary $\{f_i\}$, then the unit ball U of the equivalent norm $|||x||| = \sup\{(1+\varepsilon_i)f(x_i)\colon i = 1,2,\dots\}$ is a polytope. U also admits a countable boundary $B = \{h_i\}$ (actually $h_i = (1+\varepsilon_i)f_i$) with the following property: every w^*-limit point f of the set B such that $|||f||| = 1$ does not attain its norm. Each functional $f \in E^*$ attaining its supremum on U belongs to the set $\operatorname{lin}\{h_i\}$, *(*$\operatorname{lin}\{h_i\}$ *is the linear span of the set $\{h_i\}$).*

The proof of the following proposition is based on Theorem A and uses some ideas from [7, 8, 11]. Its first part shows that the structure of the topological boundary of an infinite-dimensional separable polytope P (with non-empty interior) is similar to the structure of boundary of symetric one (see Theorem A), i.e. the boundary ∂P of P consists of countably many maximal faces that are solid parts of the corresponding hyperplanes.

PROPOSITION 1.2: *Let P be a polytope in a separable Banach space E. Then P admits a countable boundary $B \subset P^o$.*

Conversely, let A be a closed, convex and bounded body with countable boundary and $\varepsilon > 0$. Then there exists a polytope P with the following properties:

(1) $A \subset P \subset (1+\varepsilon)A$.

(2) *There exists a countable boundary $\{h_i\} \subset P^o$ such that each w^*-limit point h of the set $\{h_i\}$ with the property $h \in \partial P^o$ does not attain its supremum on P.*

(3) *For every sequence $\{\gamma_i\}$ of positive numbers, there exists a sequence $\{t_i\}$ of linear functionals such that:*

  (a) $\|h_i - t_i\| < \gamma_i$.

  (b) *Every sequence $\{l_i\}$ such that $\|l_i - t_i\| < \gamma_i/4$ satisfies w^*-cl co$\{l_i\} \supset P^o$.*

  (c) *The set $P_1 = \{x \in E\colon l_i(x) \leq 1\}$ is a polytope.*

Proof: Let $y \in$ int P. We define an affine mapping $A_y\colon P \to E$ by the formula

$$A_y(x) = 2y - x \qquad (x \in P).$$

Write $P_y = A_y(P)$ and $V_y = P \cap P_y$. Of course V_y is a symetric polytope, and, by Theorem A, there exists a countable boundary $B_y = \{f_y^j\}$ for V_y. It is obvious that $\partial V_y \subset \partial P \cup \partial P_y$. Let $x \in \partial V_y \cap \partial P$ be such a point that there exists $\delta > 0$ with the property $(x + \delta U(E)) \cap \partial P \subset \partial V_y$. If $f_y^j \in B_y$ is a supporting functional at the point x, i.e. $f_y^j(x) = 1 = \sup f_y^j(V_y)$, then it is easily verified that $f_y^j(x) = 1 = \sup f_y^j(P)$. Let $\{y_i\}$ be a dense subset in $inta(P)$. Simple consideration shows that for each $x \in \partial P$ there exist y_i and $\delta > 0$ such that $(x + \delta U(E)) \cap \partial P \subset \partial V_{y_i}$. Thus the countable set $B = \bigcup_{i=1}^{\infty}\{f_{y_i}^j; j = 1 \cdots \infty\}$ is a boundary for the polytope P.

Now let $A \subset E$ be a closed, convex and bounded body with a countable boundary $\{f_i\}$ and $\varepsilon > 0$. Without loss of generality we can assume that $0 \in$ int(A). Let $\{\varepsilon_i\}$ be a decreasing sequence of positive numbers tending to zero, $\varepsilon_1 < \varepsilon$. Write

$$h_i = \frac{1+\varepsilon_i}{1+\varepsilon} f_i, \qquad K = w^*\text{-cl co}\{h_i\}$$

and

$$P = \{x \in E\colon h_i(x) \leq 1, i = 1, 2, \ldots\}.$$

It is clear that

$$A \subset P \subset (1+\varepsilon)A, \qquad P^o = K, \qquad A^o \supset K \supset (1+\varepsilon)^{-1}A^o.$$

Let $h_0 \in \partial K$ be a w^*-limit point of the sequence $\{h_i\}$ ($h_0 = w^*\text{-}\lim h_{i_k}$) and let $x_0 \in \partial P$ be such that $h_0(x_0) = \max h_0(P) = 1$. Since $\varepsilon_i \to 0$ we have $h_0 = w^* \lim f_i/(1+\varepsilon)$ and therefore $h_0 \in (1+\varepsilon)^{-1}A^o$. Since $x_0 \in (1+\varepsilon)A$ we have $\sup x_0((1+\varepsilon)^{-1}A^o) \leq 1$ and so $h_0(x_0) = 1 = \max x_0((1+\varepsilon)^{-1}A^o)$. Since $\{(1+\varepsilon)^{-1}f_i\}$ is a boundary for $(1+\varepsilon)^{-1}A$ there exists a functional $(1+\varepsilon)^{-1}f_j$ such that $(1+\varepsilon)^{-1}f_j(x_0) = 1$. Thus $(1+\varepsilon_j)(1+\varepsilon)^{-1}f_j(x_0) = 1+\varepsilon_j > 1 = \max x_0(K)$. This contradiction shows that each w^*-limit point h_0 of the set $\{h_j\}$ such that $h_0 \in \partial K$ does not attain its supremum on the set P. Let $L \subset E$ be a finite-dimensional subspace. Then from the compactness of the set $L \cap P$, there exist a positive number α and an integer m such that for every $j > m$, $\sup\{h_j(x) : x \in L \cap P\} < 1 - \alpha$. This proves both that P is a polytope and that $\{h_i\}$ is a boundary. To prove (3), it is enough to put $t_j = (1+\gamma_i)h_i, i = 1, 2, \ldots$ and to observe that the property

$$w^*\text{-cl co}\{l_i\} \supset w^*\text{-cl co}\{h_i\} \tag{1}$$

is equivalent to the following one: for each $x \in E$

$$\max x(w^*\text{-cl co}\{l_i\}) \geq \max x(w^*\text{-cl co}\{h_i\}).$$

Since $\gamma_i \to 0$, it follows that each w^*-limit point of the set $\{t_i\}$ belongs to the set $w^*\text{-cl}\{h_i\}$. Thus P_1 is a polytope by the same reason that P is a polytope. The proof is completed. ∎

The following lemma is similar to lemma 4.1 [18] and to the first part of Theorem 1 [1]. We give a proof that is very similar to the consideration in [1].

LEMMA 1.3: *Let E be a Banach space with separable dual E^* and $\{x_i\} \subset S(E)$ be an M-basis of E such that the linear span of the biorthogonal system $\{x_i^*\}$ is dense in E^*. Let $W \subset E$ be a closed, convex and bounded body such that $0 \in \text{int}(W)$ and $0 < \varepsilon < 1/2$. Then there exists a w^*-compact subset $F \subset W^o$ such that:*

1. $(1+4\varepsilon)^{-1}W^o \subset w^*\text{-cl co}F \subset (1+\varepsilon)^{-1}W^o$.
2. *For each integer i, the set $x_i(F)$ is finite.*

Proof: Write $d = \inf\{\|g\| : g \in \partial W^o\}$ and $T_i = \{f(x_i) : f \in W^o\}$, $i = 1, 2, \ldots$. Each set T_i is bounded and thus there exists an $(\varepsilon/(2^{i+2}\|x_i^*\|)$-net C_i in T_i. Put

$$A = \left\{ \sum_{i=1}^{n} a_i x_i^* \in 1/(1+\varepsilon)W^o : n \in N, a_i \in C_i \right\}, \quad F = w^*\text{-cl}(A).$$

It is obvious that $x_i(F) = x_i(A) \subset C_i, i = 1, 2, \ldots$. Thus condition (2) is satisfied. Of course w^*-cl co$(F) \subset (1+\varepsilon)^{-1}W^o$. To check that $(1+4\varepsilon)^{-1}W^o \subset w^*$-cl co$(F)$, take $f \in (1+2\varepsilon)^{-1}W^o$, and, using the density of the set lin$\{x_i^*\}$ in E^*, find

$$g = \sum_{i=1}^{n} b_i x_i^* \in (1+2\varepsilon)^{-1}W^o$$

such that $\|f - g\| < \varepsilon d/6$. It is clear that $b_i \in T_i$, so there exists $a_i \in C_i$ such that $|b_i - a_i| < \varepsilon d/(62^i\|x_i^*\|), i = 1, 2, \ldots n$. Hence

$$\|g - \sum_{i=1}^{n} a_i x_i^*\| < \varepsilon d/6,$$

and now a straight verification shows that $h = \sum_1^n a_i x_i^* \in (1+\varepsilon)^{-1}W^o$. Thus, by definition, $h \in F$ and obviously $\|f - g\| < \varepsilon d/3$. From the last inequality, taking into account $0 < \varepsilon < 1/2$, one can deduce that $(1+4\varepsilon)^{-1}W^o \subset w^*$-cl co$(F)$. The proof is completed. ∎

The following Lemma 1.4 is also similar to some results from [18]. But because of our purpose, our formulation is slightly different and our proof is simpler. In Lemma 1.4, we use the notations of Lemma 1.3. In addition, we put $M_n = [x_i]_1^{n\perp}$, $n = 1, 2, \ldots$

Lemma 1.4: *For arbitrary $\varepsilon > 0$, there exist a sequence of points $\{g_k\}$ in the set F, a sequence of integers $\{n_k\}, n_k \to \infty$, and a decreasing sequence $\{F_\alpha\}$ of w^*-closed subsets of F such that:*

(1) $\bigcup_1^\infty \big((g_k + M_{n_k}) \cap F_k\big) = F$,
(2) $\operatorname{diam}\big((g_k + M_{n_k}) \cap F_k\big) < \varepsilon$.

Proof: Since F is w^*-compact in the separable dual space E^*, we have, according to the Baire category theorem: for every $\varepsilon > 0$, there exist a point $g \in F$ and a w^*-neighborhood G of g such that $G \cap F \neq \emptyset$ and diam $(G \cap F) < \varepsilon$.

In view of the structure of the set F, the sets $(h + M_n) \cap F$, for $h \in F$ and $n \in \mathbb{N}$, form a base of w^*-topology on F and each such set is both a closed and open subset of (F, w^*). Moreover, the family $\Im = \{h + M_n\colon h \in F, n \in \mathbb{N}\}$ contains countably many (different) sets and obviously each w^*-compact subset of F has the same structure as F.

By transfinite induction, we define, for each ordinal α, the sets F_α and $(g_\alpha + M_{n(\alpha)})$ as follows: $F_0 = F$, $F_{\alpha+1} = F_\alpha \setminus (g_\alpha + M_{n(\alpha)})$, where $(g_\alpha + M_{n(\alpha)})$ is a member of the family $\Im$ chosen so that $(g_\alpha + M_{n(\alpha)}) \cap F_\alpha \neq \emptyset$ and

$\operatorname{diam}((h_\alpha + M_{n(\alpha)}) \cap F_\alpha) < \varepsilon$; if α is a limit ordinal, then put $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$. Since the family $\Im$ is countable and each set F_α is w^*-compact, there exists a countable ordinal η such that $F_\eta \neq \emptyset$ and $F_{\eta+1} = \emptyset$. It is clear that

$$\bigcup_{\alpha\le\eta} \big((g_\alpha + M_{n(\alpha)}) \cap F_\alpha\big) = F.$$

Let us enumerate the countable family $\{g_\alpha + M_{n(\alpha)}\}_{\alpha\le\eta}$ in the following way: $\{g_k + M_{n_k}\}_{k=1}^{\infty}$. Since for each integer q there exist only finitely many members $g + M_n$ of the family $\Im$ such that $n \le q$, we have $n_k \to \infty$ as k tends to ∞. The lemma is proved. ∎

The following lemma was proved in [7] (see Theorem 2.2.3) and in [5] when the boundary B is equal to the set of extreme points of the dual unit ball of E. For the general case, see [14].

LEMMA 1.5: *Let X be a Banach space that admits a boundary that can be covered by a countable union of $\|.\|$-compacts. Then for every $\varepsilon > 0$, there exists an ε-isometric norm on X with a countable boundary.*

LEMMA 1.6: *Let E be a separable polyhedral Banach space with a norm possessing the properties of the second part of Theorem A. Let $L \subset E$ be a finite-dimensional subspace of E, $M = L^\perp$, and let $x \in E, h \in S(M)$ be such that $h(x) = \max x(U(M))$. Then there exists $h_0 \in \operatorname{lin}\{h_i\} \cap S(M)$ ($\{h_i\}$ come from Theorem A) such that $h_0(x) = \max x(U(M))$.*

Proof: If $\max x(U(M)) = 0$, then let h_0 be an arbitrary functional from $(\operatorname{lin}\{h_i\}) \cap S(M)$. In the case $h(x) \neq 0$, we can assume (without loss of generality) that $h_0(x_0) = 1$. Let $q\colon E \to E/L$ be the quotient map. Of course $\|q(x_0)\| = 1$, and, since L is a finite-dimensional subspace, there exists an element $x_1 \in S(E) \cap (x_0 + L)$. It is clear that $h(x_1) = h(x_0) = 1$. Thus the functional h attains its norm, and, using Theorem A, $h \in \operatorname{lin}\{h_i\}$. To complete the proof it is enough to put $h_0 = h$. ∎

LEMMA 1.7: *Let W be a closed, convex and bounded body in a Banach space E such that $0 \in \operatorname{int}(W)$, $\varepsilon > 0$ and A be a polytope possessing the property*

$$W \subset A \subset (1+\varepsilon)W. \tag{2}$$

Then for every $\varepsilon_1 > \varepsilon$, there exists a tangential polytope P_1 for the body W such that $W \subset P_1 \subset (1+\varepsilon_1)W$.

Proof: Let $\varepsilon_2 > 0$ be such that $(1+\varepsilon)(1+\varepsilon_2) < (1+\varepsilon_1)$. By Proposition 1, there exists a polytope P possessing the properties (1), (2), (3) with $\varepsilon = \varepsilon_2$.

From property (1) of Proposition 1 and assumption (2), we have $W \subset P \subset (1+\varepsilon_2)(1+\varepsilon)W$. Thus

$$W^o \supset P^o \supset \left((1+\varepsilon_2)(1+\varepsilon)\right)^{-1} W^o$$

and hence

$$W^o \supset (1+\varepsilon)(1+\varepsilon_2)(1+\varepsilon_1)^{-1} P^o \supset (1+\varepsilon_1)^{-1} W^o.$$

Write $\lambda = (1+\varepsilon)(1+\varepsilon_2)(1+\varepsilon_1)^{-1} < 1$. Since $\lambda P^o \subset \lambda W^o$, using the notation of Proposition 1, we can assert that:

(2^o) $\{\lambda h_i\}$ is a boundary for $\lambda^{-1}P$ such that each w^*-limit point h of the set $\{\lambda h_i\}$ belonging to $\partial(\lambda P^o)$ does not attain its supremum on the set $\lambda^{-1}P$.

(3^o) For every $\alpha > 0$, there exists a sequence of linear functionals $\{t_i\}$ possessing the properties:

(a) $\|\lambda h_i - t_i\| < \alpha/2^i$.

(b) For every sequence $\{l_i\}$ such that $\|l_i - t_i\| < \alpha/2^{i+2}$, the following holds: w^*-cl $\mathrm{co}\{l_i\} \supset \lambda P^o$.

(c) The set $P_1 = \{x \in E\colon l_i(x) \le 1, i = 1, 2, \dots\}$ is a polytope.

We fix $\alpha > 0$ small enough in order to have every sequence $\{l_i\}$ (from property (b)) inside $\mathrm{int}(W^o)$. Let $\{f_i\} \subset S(E^*)$ be an arbitrary sequence w^*-tending to zero. Denote by T_i the straight line containing the functionals l_i and $l_i + f_i$. Let u_i^1 and u_i^2 be the points of intersection of the line T_i with the boundary ∂W^o (recall that $l_i \in \mathrm{int}(W^o)$). Using the Bishop–Phelps Theorem about the density of the set of functionals that attain their supremum on the set W, it is not difficult to establish the existence of the functionals l_i ($\|l_i - t_i\| < \alpha/2^{i+2}$) and g_i ($\|f_i - g_i\| < 2^{-i}$) such that both functionals u_i^1 and u_i^2 attain their supremum on the set W. It is clear that

$$w^*\text{-}\lim(u_i^1 - \lambda h_i) = w^*\text{-}\lim(u_i^2 - \lambda h_i) = 0. \tag{3}$$

Since the point l_i is in the segment $[u_i^1, u_i^2]$, $i = 1, 2, \dots$, we get

$$w^*\text{-cl co}\{u_i^1, u_i^2\}_1^\infty \supset w^*\text{-cl co}\{l_i\} \supset \lambda P^o.$$

Denoting $P_1 = \{x \in E\colon u_i^1 \le 1, u_i^2 \le 1, i = 1, 2, \dots\}$, we have $P_1 \subset \lambda^{-1}P$. Let h be a w^*-limit point of the set $\{u_i^1, u_i^2\}_{i=1}^\infty$ such that $h \in \partial P_1^o$. From equality (3) and from the fact that $P_1^o \supset \lambda P^o$, it follows that $h \in \partial(\lambda P^o)$. Thus by ($2^o$) the functional h does not attain its supremum on the set $\lambda^{-1}P$. But

$\sup h(P_1) = \sup h(\lambda^{-1}P) = 1$ and $P_1 \subset \lambda^{-1}P$, hence the functional h does not attain its supremum on the set P_1. Thus P_1 is a polytope with boundary $\{u_i^1, u_i^2\}_{i=1}^{\infty}$. Since $P_1^o = w^*\text{-cl co}\{u_i^1, u_i^2\}_1^{\infty} \subset W^o$ and $P_1^o \supset \lambda P^o \supset (1+\varepsilon_1)^{-1}W^o$, we have

$$W \subset P_1 \subset (1+\varepsilon_1)W.$$

It is clear from the construction that P_1 is a tangential polytope for W. The proof is completed. ∎

We are now ready to prove the following:

THEOREM 1: *Let W be an arbitrary closed, convex and bounded body in a separable polyhedral Banach space E. Then, for every $\delta > 0$, there exists a tangential δ-approximating polytope P for the body W.*

Proof: We assume that the norm on the space E possesses all the properties of Theorem A. In view of Lemma 1.7, it is sufficient to prove the existence of a δ-approximating polytope. Let $\varepsilon > 0$ and let the w^*-compact $F \subset W^o$, the sequence of functionals $\{g_k\}$ and the subspaces $M_{n_k}, k = 1, 2, \ldots.$ be from Lemmas 1.3 and 1.4. It is clear that

$$\bigcup_1^{\infty}(g_k + \varepsilon U(M_{n_k})) \supset F. \tag{4}$$

By Lemmas 1.5 and 1.6, there exists, on the quotient-space $X_k = E/[x_i]_1^{n_k}$, an ε-isometric norm $|||.|||$ possessing a countable boundary. Let $V(M_{n_k})$ be the unit ball of the space $(M_{n_k}, |||.|||)$ and $\{v_i^k\}_{i=1}^{\infty} \subset V(M_{n_k})$ be a countable boundary. Of course we can assume that

$$(1-\varepsilon)V(M_{n_k}) \subset U(M_{n_k}) \subset V(M_{n_k}), \qquad k = 1, 2, \ldots.$$

From (4), we have

$$C = \bigcup_1^{\infty}(g_k + \varepsilon V(M_{n_k})) \supset F.$$

Write

$$B = \bigcup_{k=1}^{\infty}\{g_k + v_i^k, i = 1, \ldots, \infty\} \quad \text{and} \quad Q = \{x \in E: f(x) \le 1, f \in B\}.$$

We claim that B is a boundary for the body Q. First, we show that C is w^*-closed. Let $\{h_m\} \subset C$ be a sequence tending to h_0 in the w^*-topology. If infinitely many of h_m are in one of the sets $g_k + \varepsilon V(M_{n_k})$, then of course $h_0 \in C$. So assume that $h_m \in g_{k_m} + \varepsilon V(M_{n_{k_m}})$ for $m = 1, 2, \ldots$, with $k_m \to \infty$ as $m \to \infty$. Write $h_m = g_{k_m} + \varepsilon u_m$, with $u_m \in V(M_{n_{k_m}})$ for $m = 1, 2, \ldots$. Since $k_m \to \infty$, we have $n_{k_m} \to \infty$ (see Lemma 1.4) and therefore $w^*\text{-}\lim u_m = 0$. Thus $h_0 = w^*\text{-}\lim g_{k_m} \in F \subset C$. This proves that the set C is w^*-closed. Certainly

$$Q^o = w^*\text{-cl co}(C) = w^*\text{-cl co}(B)$$

and hence C is a boundary for Q. Let $x \in \partial Q$ and $h \in C$ such that $h(x) = 1 = \max x(Q^o)$. Since $h \in C$, there exists an integer k such that $h \in g_k + \varepsilon V(M_{n_k})$. If $g_k(x) = 1$, then $x(V(M_{n_k})) = 0$ and, for each i,

$$(g_k + v_i^k)(x) = 1.$$

If $g_k(x) < 1$, then

$$\sup x(V(M_{n_k})) = (1 - g_k(x))/\varepsilon \neq 0$$

and, using the fact that the set $\{v_i^k\}$ is a boundary, there exists a functional v_i^k such that $v_i^k(x) = \sup x(V(M_{n_k}))$. Thus we have proved that Q has a countable boundary and we can apply Proposition 1. It is clear that for ε small enough, we will be able to make δ-approximation. Theorem 1 is proved. ∎

2. Polyhedral approximation in normed spaces with countable algebraic basis

In this section, we prove that in normed spaces with countable algebraic basis, every closed convex body set can be approximated by polytopes, and that in such spaces, polytopes have countable boundary.

The following are some elementary properties of finite-dimensional polytopes:

(*) Every finite-dimensional polytope containing $\vec{0}$ as an interior point has a finite boundary (namely the set of the extremal points of the dual polytope).

(**) Suppose P is a convex body in $\mathbb{R}^n$, containing $\vec{0}$ as an interior point, $\varepsilon > 0$. Then there exists a polytope Q such that $P \subset Q \subset (1+\varepsilon)P$.

(***) Suppose P, Q are polytopes in $\mathbb{R}^n$, $\varepsilon > 0$ such that $P \cap \operatorname{span} Q \subset Q$, $\vec{0} \in \operatorname{int} Q$. Then $\operatorname{ext}((1+\varepsilon)Q) \subset \operatorname{ext}(\operatorname{co}(P \cup (1+\varepsilon)Q))$.

PROPOSITION 2.1: *In a normed space with countable algebraic basis, each polytope admits a countable boundary.*

Proof: A polytope K on X has a countable boundary. Indeed, let (X_n) be a sequence of finite-dimensional subspaces of X such that $X = \bigcup_{n \in \mathbb{N}} X_n$.

According to (*), each $K \cap X_n$ has a finite boundary $(f_1^n, f_2^n, \dots, f_{p_n}^n)$. Let $b_k^n \in X$ be a Hahn–Banach extension of f_k^n (i.e. $\sup_{K\cap X_n} f_k^n = \sup_K b_k^n$). The set $B := \{b_k^n; 1 \le k \le p_n, n \in \mathbb{N}\}$ is a countable boundary of K. ∎

THEOREM 2.2: *Let $(X, \|\cdot\|)$ be a normed linear space with countable algebraic basis . Then every closed convex body set B, $0 \in \operatorname{int}(B)$, can be approximated by polytopes.*

Proof of Theorem 2.2: Fix $\varepsilon > 0$ and choose a sequence $\varepsilon_k \searrow 0$, $\varepsilon_1 < 1/10$ so that $\prod_{k=1}^{\infty} (1+\varepsilon_k) < 1+\varepsilon$. Suppose $\{x_k\}_{k\in\mathbb{N}}$ is the algebraic basis of X. Denote by $X_n = \operatorname{span}\{x_k\}_{k=1}^n$, $B_n = X_n \cap B$. (We have $B_{n+1} \cap X_n = B_n$.) We construct by induction a sequence $\{K_n\}_{n\in\mathbb{N}}$ of polytopes in X_n satisfying

$$B_n \subset K_n \subset \prod_{k=1}^{n} (1+\varepsilon_k) B_n \quad \text{and} \quad K_{n+1} \cap X_n = (1+\varepsilon_{n+1}) K_n$$

as follows: first define $K_1 = (1+\varepsilon_1)B_1$.

INDUCTIVE STEP: If K_n has been constructed, denote by D_{n+1} a polytope, given by (**), such that

$$B_{n+1} \subset D_{n+1} \subset \left(1 + \frac{\varepsilon_{n+1}}{4}\right) B_{n+1}.$$

So we have

$$D_{n+1} \cap X_n \subset \left(1 + \frac{\varepsilon_{n+1}}{4}\right) B_n \subset \left(1 + \frac{\varepsilon_{n+1}}{4}\right) K_n.$$

Consequently

$$\left(1 + \frac{\varepsilon_{n+1}}{4}\right) (D_{n+1} \cap X_n) \subset \left(1 + \frac{3\varepsilon_{n+1}}{4}\right) K_n.$$

Put

$$K_{n+1} = \operatorname{co}(D_{n+1} \cup (1+\varepsilon_{n+1})K_n).$$

By (***), we have $K_{n+1} \cap X_n = (1+\varepsilon_{n+1})K_n$. Define $\tilde{K}_n = \prod_{k=n}^{\infty}(1+\varepsilon_k)K_n$. Then $B_n \subset \tilde{K}_n \subset (1+\varepsilon)B_n$ and $\tilde{K}_{n+1} \cap X_n = \tilde{K}_n$.

This allows us to define a new polytope $\tilde{K}$ in X such that $\tilde{K} \cap X_n = \tilde{K}_n$. The approximation condition obviously holds. ∎

3. Analytic approximation

Let $G \subset E$ be an open subset of a Banach space E. A real or complex function φ defined on a set U is said to be analytic if, for every $x \in G$, there exists a neighborhood V of x such that the Taylor expansion of φ at x converges uniformly to φ on V.

THEOREM 3.1: *Let E be a Banach space. Then every equivalent norm on X with countable boundary is a uniform limit (on bounded sets) of a sequence of analytic norms.*

Proof: Let us denote $\|\cdot\|$ an equivalent norm on E with a countable boundary B. Since B is countable, we can write $B = \{b_n; n \in \mathbb{N}\}$. We recall that B is a boundary of E means that B is a subset of the unit sphere of E^* such that, for each $x \in E$, there exists $n_0 = n_0(x)$ such that $b_{n_0}(x) = \|x\|$. Let $\varepsilon > 0$ and let (δ_n) be a sequence of positive real numbers such that $\delta_0 = \varepsilon$, $\delta_{n+1} < \delta_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \delta_n = 0$. We denote $h_n := (1+\delta_n)b_n$ and we define a new norm N on E by

$$N(x) = \sup\{|h_n(x)|; n \in \mathbb{N}\}.$$

It is clear that N is a norm on E that satisfies $\|x\| \leq N(x) \leq (1+\varepsilon)\|x\|$. Of course, N is not analytic, but satisfies the following property: for each $x \in E$ with $N(x) \leq 1$, there exists $n_0 = n_0(x) \in \mathbb{N}$ and $\alpha(x) > 0$ such that

$$|h_n(x)| < 1 - \alpha(x)$$

whenever $n > n_0$. Indeed, B is a boundary, so for each $x \in E$, there exists n_0 such that for every $n \in \mathbb{N}$, $b_n(x) \leq b_{n_0}(x)$. For $n > n_0$

$$(1+\delta_n)|b_n(x)| \leq (1+\delta_{n_0+1})|b_{n_0}(x)| = (1-\alpha)(1+\delta_{n_0})b_{n_0}(x) \leq 1 - \alpha(x)$$

where

$$\alpha(x) := \frac{\delta_{n_0} - \delta_{n_0+1}}{1+\delta_{n_0}}.$$

We now set

$$\varphi(x) = \sum_{n=1}^{+\infty} h_n(x)^{2n+p},$$

where p is an even integer to be fixed later. Finally, define

$$C := \{x \in E; \varphi(x) \leq 1\}.$$

The Minkowski functional $|||.|||$ of C is clearly the unit ball of an equivalent norm on E. Let us show that $|||.|||$ is analytic in a neighborhood of each point

$x \in E$ such that $|||x||| = 1$. The Minkowski functional $|||.|||$ of C is given by the equation $\varphi(x/|||x|||) = 1$, so $\varphi(x) = 1$. Let us denote by E^c the complexified space of E, $\tilde{N}$ the extension of N to E^c and define the extension $\tilde{\varphi}$ of φ to E^c by $\tilde{\varphi}(z) = \sum_{n=1}^{+\infty} h_n^c(z)^{2n+p}$. Since $\varphi(x) = 1$, there exists n such that $h_n(x) \neq 0$. So, we have $\varphi'(x).x = \sum_{n=1}^{+\infty}(2n+p)h_n(x)^{2n+p} > 0$ and $\tilde{\varphi}'(x) \neq 0$. We claim that $\tilde{\varphi}$ is well defined and holomorphic in a neighbourhood of x in E^c. Indeed, since $|||x||| \leq 1$, we also have $N(x) \leq 1$ (see the proof below). If $y \in E^c$ satisfies $\tilde{N}(y-x) < \alpha(x)/2$ and $n > n_0(x)$, then

$$|h_n^c(y)| \leq |h_n^c(y-x)| + |h_n(x)| \leq \tilde{N}(y-x) + 1 - \alpha(x) < 1 - \alpha(x)/2.$$

So the series defining $\tilde{\varphi}$ converges uniformly on the set

$$\{y \in E^c; \tilde{N}(x-y) < 1 - \alpha(x)/2\}.$$

According to the Implicit Function Theorem ([3]), the function A: $E^c \to \mathbb{C}$ defined by $\tilde{\varphi}\big((z/A(z)\big) = 1$ is holomorphic. Hence $|||.|||$, which is the restriction of A to E, is analytic.

It remains to prove that for p large enough one has

$$(1-\varepsilon)|||x||| \leq N(x) \leq |||x|||.$$

Indeed, assume first that $|||x||| \leq 1$. Since $\sum_{n=1}^{\infty} h_n(x/|||x|||)^{2n+p} = 1$, we have $\sum_{n=1}^{\infty} h_n(x)^{2n+p} \leq 1$. Hence for each n, $h_n(x) \leq 1$ and $N(x) \leq 1$. So $N(x) \leq |||x|||$.

On the other hand, let us assume that $N(x) \leq 1-\varepsilon$. Then $h_n(x) \leq 1-\varepsilon$ for all n. Hence $\sum_{n=1}^{\infty} h_n(x)^{2n+p} \leq \sum_{n=1}^{\infty}(1-\varepsilon)^{2n+p} \leq 1$ if p is large enough. Consequently, $|||x||| \leq 1$. This proves that $(1-\varepsilon)|||x||| \leq N(x)$ and completes the proof of the Theorem. ∎

The following Corollary is a consequence of Theorems 1.1 and 3.1.

COROLLARY 3.2: *Each equivalent norm on a separable polyhedral Banach space can be approximated by analytic norms.*

The following Corollary improves a result of R. Haydon [15] who showed the existence of an equivalent C^∞-smooth norm on the space $C(K)$ where K is a countable compact.

COROLLARY 3.3: *Let E be a Banach space predual to ℓ_1. Then each equivalent norm on E can be approximated by analytic norms.*

Remark 3.4: It is not clear from the above results that in vector normed spaces with countable algebraic basis, every equivalent norm can be approximated by analytic norms. Indeed, the application of the Implicit Function Theorem in the proof of Theorem 3.1 requires that the space is complete.

We conclude this section with a variant of Theorem 3.1 in the non-symmetric case.

THEOREM 3.1 (bis): *Let P be a polytope in a separable Banach space E. Then for every $\varepsilon > 0$, there exists an ε-approximating, analytic, closed, convex and bounded body V.*

Proof: In view of Proposition 1.2, we can assume without loss of generality that the polytope P satisfies property (2) of Proposition 1.2. It is easily verified that the boundary $\{h_i\}$ (coming from property (2)) has the following property: for each $x \in \partial P$, there exists an integer n such that

$$\sup\{h_i(x)\colon i > n\} < 1. \tag{5}$$

Of course, inequality (5) holds in some neighborhood of the point x and so the series

$$\varphi(x,\lambda) = \sum_{i=1}^{\infty} \exp\Big(2(i+p)\big(h_i(x/\lambda) - 1\big)\Big)$$

converges uniformly in some neighborhood $G(x)$ of x for each $\lambda \geq 1$ and for each integer p. Therefore $\varphi(x,\lambda)$ is analytic for x belonging to some neighbourhood of ∂P and for $\lambda \geq 1$. As in the proof of Theorem 3.1, we deduce from the Implicit Function Theorem that the function $\lambda = \lambda(x)$ defined by the equation $\varphi(x,\lambda) = 1$ is analytic. The function λ is the Minkowski functional of the set

$$V = \{x \in E\colon \lambda(x) \leq 1\}.$$

Moreover, one can prove as in Theorem 3.1 that if p large enough, we have

$$(1-\varepsilon)\lambda(x) \leq f_P(x) \leq \lambda(x),$$

where f_P is Minkowski functional of the set P. ∎

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