Remarks on a Possible Relation Between Gravitational Instantons and the Spin Thermodynamics of a Kerr Black Hole.

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Recently the theory of instantons has been applied to some solutions of Einstein equations. It was realized that it provides results agreeing with the thermodynamical theory of black holes (1). In this framework, the class of asymptotically flat instantons corresponds to thermal-equilibrium states of the gravitational field at some finite temperature. The Euclidean and quasi-Euclidean sections of Schwarzschild, Reissner-Nørdstrom, Kerr and Kerr-Newman solutions belong to this class and they provide black-hole temperatures which coincide with those obtained in other ways (2).

The idea of a gravitational instanton comes out from the path integral approach to quantum gravity. This approach consists in considering the action of the gravitational field

$$I(g) = (16\pi)^{-1}\int R[-g]^{\frac{1}{2}}\,\mathrm{d}^4x\,, \tag{1}$$

where g is the metric and R is its scalar curvature, and integrating it (with an appropriate « measure ») on the whole set of paths in the relevant configuration space.

The action is evaluated on a nonsingular section of a complexified space-time and can be related to the partition function for a canonical ensemble. This operation is performed by using the classical formula

$$Z = \operatorname{Tr}\exp[-\beta H] = \int \mathrm{d}[\phi]\exp[I(\phi)]\,, \tag{2}$$

where Z is the partition function for the canonical system of the field ϕ at temperature $T = \beta^{-1}$, H is the Hamiltonian and the path integral on the right-hand side is taken over all fields which are periodic in imaginary time.

This last property comes from the fact that on the Euclidean section where the action I is evaluated, the complexified time $\tau = it$ is periodic and assumes the meaning of angular co-ordinate around an axis.

(1) G. W. GIBBONS and S. W. HAWKING: *Phys. Rev. D*, **15**, 2752 (1977).
(2) S. W. HAWKING: *Commun. Math. Phys.*, **25**, 152 (1972).

In the Schwarzschild, Kerr, Kerr-Newman and Reissner-Nørdstrom solutions this axis corresponds to the subset of M where co-ordinate singularities are removed. Therefore it corresponds to the Killing horizons of the solutions [3].

In a recent paper [4] MAGNON ASHTEKAR proposed the following definition for a gravitation instanton (M, g):

a) g is a C^∞ metric with elliptic signature on a differentiable manifold M,

b) the geodesics of g are complete.

According to these requirements it follows that in the case of the Schwarzschild solution the Killing field $\partial/\partial\tau$ in the extended (Kruskal) metric plays the role of a Killing rotation centred in $r = 2m$, with a period $\beta_S = 8\pi m$ (where m is the mass of the black hole). Therefore, it turns out that the temperature T_S is associated to the period β_S of $\partial/\partial\tau$ on the event horizon which is a Killing horizon. When we have a solution possessing a Killing vector field with two (or more) horizons (for example, the Kerr solution) one expects that in principle two temperatures should exist.

However, in [4] the Kerr solution was excluded because the region between the two horizons is not timelike, as far as the Killing vector field is concerned. Therefore, it is impossible to obtain a Euclidean section from the complexification of such an extended solution. Consequently, MAGNON ASHTEKAR consider the case of the C-metric which is a solution endowed with two horizons with timelike Killing vector field between the horizons themselves. It is shown in [4] that such a solution can be suitably considered as a transition between two temperatures, *i.e.* between two instanton states.

We shall make here the following remarks. In our opinion, the definition of gravitational instanton given in [4] is too restrictive, since it excludes solutions like the Kerr and the Kerr-Newman ones. In fact we claim that it is possible to use the approach of Hawking and Gibbons also for the inner Killing horizon of the Kerr solution. In this way it is possible to define a temperature by considering the asymptotic region exterior to this horizon, the associated instanton and the Killing rotation taking place on the inner horizon itself.

To see this, in the Kerr solution describing a black hole having mass m and angular momentum $L_0 = ma$ it is necessary to distinguish between a « static » Killing vector (defining the stationarity with respect to the infinity) and a stationary Killing « bivector » defining the local stationarity [5]. The first one changes its timelike nature on the limit surface of stationarity $r = r_0 = (m + \sqrt{m^2 - a^2 \cos^2\theta})$, namely at the boundary of the ergosphere. The second one changes its sign on the Killing horizons, namely on the event horizon $r = r_+ = m + \sqrt{m^2 - a^2}$ and on the inner horizon $r = r_- = m - \sqrt{m^2 - a^2}$.

In Boyer-Lindquist co-ordinates r, θ, φ, t the magnitude of the static Killing vector $\partial/\partial t$ corresponds to the g_{tt} component of the metric tensor. Therefore, in the Kerr case the asymptotically flat complexified section exterior to the event horizon is not Euclidean.

HAWKING and GIBBONS, in fact, in [1] recognized the nonexistence of a Euclidean section and they evaluated the action of the gravitational field on a suitable section that they called « quasi-Euclidean ».

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[3] S. W. HAWKING: in *Les astres occlus* (New York, N. Y., 1973).

[4] A. MAGNON ASHTEKAR: *Contribution à l'étude des instantons gravitationnels*, in *Conference donnée aux Journées Relativistes, 1980*, Université de Caen.

[5] S. W. HAWKING and G. R. F. ELLIS: *The Large Scale Structure of Space-Time* (Cambridge, 1973), p. 167.

Actually, the path integral approach can be applied also on such a kind of space-time as described in (1).

In this way when evaluating the action, a rotation (in imaginary time) appears again. This rotation is recognized *a posteriori* to be closely related to the presence of a Killing horizon.

In the extended Kerr geometry, we have two Killing horizons (6). Therefore, even though the region between such two horizons is spacelike, the Hawking-Gibbons treatment works up to the inner horizon $r = r_-$. In fact the asymptotically flat Euclidean section can be extended by means of a Kruskal extension up to the inner horizon. For this horizon the regularity of the metric requires that the point (t, r, θ, φ) is identified with the point $(t+i2\pi\varkappa_-^{-1}, r, \theta, \varphi+i2\pi\Omega_-\varkappa_-^{-1})$, where $\varkappa_-$ is the surface gravity on the inner horizon and Ω_- is its angular velocity.

When such identification is performed using formula (2) we can obtain the expression of the partition function for a canonical ensemble at temperature β^{-1} (where β is the period of the field).

Therefore the path integral approach provides us with a temperature T_- on the inner horizon given by

$$T_- = \beta^{-1} = \frac{\varkappa_-}{2\pi} = \frac{1}{2\pi}\,\frac{A_- - A_+}{A_-}\,, \tag{3}$$

where A_- is the area of the inner horizon and A_+ is the area of the event horizon.

This temperature was already introduced in a preceding paper (7) as « spin temperature » for a rotating black hole. In fact it reveals an effective distribution of orbital angular momenta which may be interpreted as an inverted population; in (8,9) the temperature T_- was argued to be connected with emission phenomena depending on the spin of the hole (like superradiance and spontaneous spin-down).

In the spin temperature framework the black holes was assimilated to a spin system endowed with entropy $A_-/4$. In order to verify that the instanton formalism may be also applied to these new state variables, it is necessary to establish that even for a spin system there exists a partition function Z such that

$$\ln Z = -WT^{-1}, \tag{4}$$

where W is the free energy. This fact follows from Boltzmann's definition of entropy (10):

$$\sigma = -\sum_l P_l \log P_l\,, \tag{5}$$

where P_l is the Boltzmann factor:

$$P_l = \frac{\exp[-\varepsilon_l/T]}{Z}\,, \qquad \varepsilon_l = \text{energy of the state } l. \tag{6}$$

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(6) B. CARTER: in *Les astres occlus* (New York, N. Y., 1973).
(7) A. CURIR: *Nuovo Cimento B*, **51**, 262 (1979).
(8) A. CURIR and M. FRANCAVIGLIA: *Abstracts of contributed paper*, in *IX International Conference on General Relativity and Gravitation* (Jena, 1980).
(9) A. CURIR: *J. Gen. Rel. Grav.*, in press.
(10) C. KITTEL: *Termal Physics* (New York, N. Y., 1969).

The possibility of defining a Boltzmann factor for a spin system has been suggested by ABRAGHAM and PROCTOR in (11). RAMSEY developed a complete spin thermodynamics (12) in which a partition function was proposed in the form (4).

Therefore, (4) is applicable also to a Kerr black hole considered as a spin system. As a consequence, the instanton formalism shows the existence of a temperature even on the inner horizon. Such a temperature is the spin temperature associated to the afore-mentioned partition function.

In conclusion, on the ground of the path integral approach, both the C-metric and the Kerr solution may be considered as solutions with two temperatures. They both correspond to a suitable definition of gravitational instanton. However, owing to the fact that in the case of Kerr metric the « inner » and « outer » temperatures have different thermodynamical interpretations, the Kerr metric cannot be considered as a transition state.

Nevertheless, the characteristics of the Kerr solution could suggest a two-phase thermodynamics of a system, namely an effective presence of two temperatures into the same object (two phases far from equilibrium between them).

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(11) A. ABRAGAM and W. G. PROCTOR: *Phys. Rev.*, **109**, 1441 (1958).
(12) N. F. RAMSEY: *Phys. Rev.*, **103**, 20 (1956).