Classical Solutions for Fermionic Models.

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Summary. — Classical solutions for conformal invariant fermion-fermion interaction in two dimensions are given and their invariance properties are discussed. A possible generalization to four dimensions is given and corresponding instanton- and meron-like solutions are found. It turns out that these are a natural generalization of two-dimensional ones.

1. – Introduction.

Much attention has been paid recently to the properties of classical solutions for field systems, in particular when these contain no mass or dimensional parameter (1). The interest of this kind of theories lies in their larger symmetry properties as the conformal group or the local gauge group. These in-

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(**) Supported by an Italian Government Fellowship.
(1) Any list of references will presumably result not complete. For excellent reviews see, for instance, R. JACKIW: *Rev. Mod. Phys.*, **49**, 681 (1977); A. CHAKRABARTI: *Introduction to classical solutions of Yang-Mills field equations*, in *Centre de Physique Theorique de l'Ecole Politechnique Plateau de Palaiseau, France, 1968.*

vestigations have included scalar fields ([2]), Yang-Mills fields ([3]) and gravitation ([4]).

An interpretation of classical solutions can be given in the form of « vacuum » expectation values of quantum fields ([2]), as the manifestation of spontaneous breaking of some symmetry. Furthermore, the classical solution for a boson field is of the form $O(1/g)$ (where g is some coupling constant) as a consequence of the nonlinear character of the equation of motion; it then represents a large portion of the field in the small coupling limit, while the corrections are of the order $O(g^0)$ and represent a small portion of the field. When a fermion field $\psi(x)$ is present, as is the case for instance in a linear σ-model with Yukawa coupling, it is customary to consider it as $O(g^0)$: this amounts to taking equal to zero the « vacuum » expectation value of the field $\psi(x)$ (which is obvious) and of its bilinear combinations, such as $\bar{\psi}\psi$ ([5]).

A different point of view, however, can be taken where both the classical fermion and boson fields are of order $O(1/g)$ ([6]). It is easily recognized that this leads to problems at the quantum level, since canonical commutation relations would contain $O(1/g)$ terms, also leading to apparent violation of fermion number conservation. A recipe was proposed ([7]) which overcomes these difficulties by redefining anticommutators, but fixes, in the case of the linear scalar-fermion σ-model, the ratio of the coupling constants g_F/g_B. The aim of this paper is the investigation at the classical level of simple field-theoretical models involving only fermions, as a preliminary study of future quantum developments.

The study of self-coupled fermion systems in two dimensions has often been considered to provide an interesting group for research, since the Thirring model ([8]) up to the more recent versions of Gross and Neveu ([9]). In this work we discuss the simplest fermionic two-dimensional model with a $(\bar{\psi}\psi)^2$ coupling as far as classical solutions are concerned. Both instanton- and meron-like solutions are found and their invariance properties sketched.

The generalization of these results to the four-dimensional case is possible, provided one accepts to work with a Lagrangian and equations of motion of

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([2]) S. FUBINI: *Nuovo Cimento*, **34** A, 521 (1976).

([3]) A. A. BELAVIN, A. M. POLYAKOV, A. S. SCHWARTZ and YU. S. TYUPKIN: *Phys. Lett.*, **59** B, 85 (1975).

([4]) P. G. O. FREUND: *Phys. Rev. Lett.*, **37**, 1251 (1976).

([5]) V. DE ALFARO and G. FURLAN: *Nuovo Cimento*, **34** A, 555 (1976).

([6]) H. INAGAKI: *Phys. Lett.*, **9** B, 448 (1977); V. JA. FAINBERG and O. K. PASHAEV: *Phys. Lett.*, **77** B, 208 (1978).

([7]) H. INAGAKI: ICTP, Trieste, preprint IC/77/32.

([8]) W. E. THIRRING: *Ann. of Phys.*, **3**, 91 (1958).

([9]) D. J. GROSS and A. NEVEU: *Phys. Rev. D*, **10**, 3235 (1974).

third order in the derivatives of the field $\psi(x)$ (*). Determination of instanton- and meron-like classical configurations is then straightforward, confirming the analogous situation which occurs for the purely bosonic nonlinear σ-models [11].

2. – The two-dimensional model.

Let us consider the two-dimensional fermion-fermion interaction described by the conformal invariant Lagrangian (**)

$$\mathscr{L} = \frac{i}{2}\,\bar{\psi}\gamma\overset{\leftrightarrow}{\partial}\psi + g(\bar{\psi}\psi)^2\,, \tag{1}$$

where the fermion field $\psi(x)$ has scale dimension $\frac{1}{2}$ and the coupling constant g is dimensionless and positive ($g > 0$) [9]. No internal degrees of freedom will be ascribed to $\psi(x)$ in this paper.

The equation of motion which follows from this Lagrangian is

$$i\gamma\partial\psi = -\,2g(\bar{\psi}\psi)\,\psi \tag{2}$$

and we want to discuss some of its classical solutions.

According to well-known arguments, classical solutions can be related to spontaneous symmetry breaking of the full conformal group. They are then characterized by their being invariant under the transformations of a subgroup, which in turn reflects the final symmetry properties of the ground state $|\Omega\rangle$ of the system. This means that in our case we must concentrate, at this stage of the discussion, on the properties of the vacuum expectation value of a quantity like $\bar{\psi}(x)\,\psi(x)$ (rather than of simply $\psi(x)$, since $\langle\Omega|\psi(x)|\Omega\rangle = 0$).

Let us define

$$S(x) = \langle\Omega|\bar{\psi}(x)\,\psi(x)|\Omega\rangle\,, \tag{3}$$

$S(x)$ will be a function of x in order to have spontaneous symmetry breaking of the conformal group. Following the proposal of ref. [1], we introduce the

(*) An interesting alternative possibility has recently been proposed in the framework of generally covariant field theories [10]. It is shown there that a bosonic nonlinear σ-model coupled to gravitation leads to equations of motion with the normal number (two) of derivatives. A similar problem for fermionic fields is at present under investigation. (Private communication of G. FURLAN.)

(10) V. DE ALFARO, S. FUBINI and G. FURLAN: CERN preprint TH. 2584.

(11) V. DE ALFARO, S. FUBINI and G. FURLAN: ICTP, Trieste, preprint IC/78/69.

(**) For simplicity we work in Euclidean space, and continuation to Minkowski metrics will be mentioned at the appropriate moment.

operator

$$R_\mu = \frac{1}{2}\left(aP_\mu + \frac{1}{a}K_\mu\right), \tag{4}$$

where a is a parameter with the dimensions of a length, and require that

$$R_\mu S(x) \equiv \frac{i}{a}\left[\frac{a^2 - x^2}{2}\,\partial_\mu + (x\cdot\partial + 2d)\,x_\mu\right] S(x) = 0\,. \tag{5}$$

The relation of this condition to the invariance properties of the ground state can be intuitively understood by noticing that these are expressed as

$$R_\mu|\Omega\rangle = 0\,. \tag{6}$$

Then an admittedly heuristic use of the commutation relation

$$[\psi(x), R_\mu] = \frac{i}{a}\left[\frac{a^2 - x^2}{2}\,\partial_\mu + (x\cdot\partial + d)x_\mu + \Sigma_{\mu\nu}x_\nu\right]\psi(x) \tag{7}$$

gives eq. (5):

$$M_{\mu\nu}S(x) \equiv i(x_\mu\,\partial_\nu - x_\nu\,\partial_\mu)\,S(x) = 0\,. \tag{8}$$

In the above formulae $d = \frac{1}{2}$ is the scale dimension of the field ψ, $2d = 1$ the scale dimension of the $\bar{\psi}(x)\,\psi(x)$ bilinear as expected from a naive dimensional counting. Furthermore, since $M_{\mu\nu} \equiv M_{xt}$, the two-dimensional rotation generator, leaves $S(x)$ invariant too, we find that the solution of the constraints (5) and (8) is

$$S(x) = \frac{\text{const}}{(a^2 + x^2)^{2d}}\,. \tag{9}$$

The symmetry group of the solution (actually of the bilinear $S(x)$) is recognized to be the de Sitter group O_3 of generators $(R_\mu, M_{\mu\nu})$.

Thus starting from the conformally invariant Lagrangian (1) we look now for a classical solution of eq. (2) consistent with the residual symmetry of the system being the de Sitter one, which implies the form (9) for $S(x)$. We shall denote such solutions as of the instanton type.

We introduce the ansatz

$$\psi(x) = \frac{C_1 + \gamma\cdot x C_2}{a^2 + x^2}\,, \tag{10}$$

where C_1, and C_2 are constant spinors (*). Inserting this in (2) we find that

$$\psi(x) = \frac{a \pm i\gamma x}{a^2 + x^2}\, C\,, \qquad \bar{C}C = \pm\frac{a}{g}\,, \tag{11}$$

which indeed gives

$$S(x) = \pm\,\frac{a}{g(a^2 + x^2)}\,. \tag{9'}$$

A similar argument works for the solution, which we may call of the meron type, whose symmetry group is taken to be the $O_2 \times O_2$-subgroup of generators $M_{\mu\nu}$ again and D, the dilatation operator. Then the conditions

$$M_{\mu\nu} S(x) = 0\,, \qquad DS(x) \equiv i(x\cdot\partial + 2d)\, S(x) = 0 \tag{12}$$

lead with $2d = 1$ to

$$S(x) = \frac{\text{const}}{(x^2)^{\frac{1}{2}}}\,. \tag{13}$$

The form (13) then suggests a simple power ansatz for $\psi(x)$, *i.e.*

$$\psi(x) = \frac{C_1}{(x^2)^{\frac{1}{4}}} + \frac{\gamma x}{(x^2)^{\frac{3}{4}}}\, C_2\,. \tag{14}$$

Solving eq. (2) we find

$$\psi(x) = \frac{1}{(x^2)^{\frac{1}{4}}}\left[1 \pm \frac{i\gamma\cdot x}{(x^2)^{\frac{1}{2}}}\right] C\,, \qquad \bar{C}C = \pm\frac{1}{8q} \tag{15}$$

and

$$S(x) = \pm\,\frac{1}{4g}\,\frac{1}{(x^2)^{\frac{1}{2}}}\,. \tag{13'}$$

Having checked the symmetry properties of the above classical solutions, we proceed now to a further characterization by looking at the energy-momentum tensor. The Belifante energy-momentum tensor corresponding to our

(*) This ansatz must be compared with the possibility of considering C as a classical anticommuting object. Such an alternative is of relevance when scalar fields are present, as is the case for the supersymmetric nonlinear σ-model ([12]). We hope to return to these problems in future work.

([12]) See, *e.g.*, E. WITTEN: *Phys. Rev. D*, **16**, 2991 (1977).

specific Lagrangian (1) has the form

$$\theta^{\mathrm{B}}_{\mu\nu} = \frac{i}{4}(\psi\gamma_\mu \overleftrightarrow{\partial}_\nu \psi + \psi\gamma_\nu \overleftrightarrow{\partial}_\mu \psi) - \delta_{\mu\nu}\mathscr{L}. \tag{16}$$

In general it is preferable to work with the so-called improved energy-momentum tensor ([13]), which is connected in a simple way with dilatation and conformal currents and hence with the conformal invariance properties of the theory.

The improved energy-momentum tensor is obtained by adding a super-potential to the Belifante tensor

$$\theta_{\mu\nu} = \theta^{\mathrm{B}}_{\mu\nu} + \tfrac{1}{2}\partial_\lambda \partial_\varrho X_{\lambda\varrho\mu\nu}, \tag{17}$$

where the superpotential part is defined with

$$X_{\lambda\varrho\mu\nu} = \delta_{\lambda\varrho}\sigma^+_{\mu\nu} - \delta_{\lambda\mu}\sigma^+_{\nu\varrho} - \delta_{\lambda\nu}\sigma^+_{\mu\varrho} + \delta_{\mu\nu}\sigma^+_{\lambda\varrho} - \tfrac{1}{3}\delta_{\lambda\varrho}\delta_{\mu\nu}\sigma^+_{\alpha\alpha} + \tfrac{1}{3}\delta_{\lambda\mu}\delta_{\varrho\nu}\sigma^+_{\alpha\alpha}, \tag{18}$$

$\sigma^+_{\mu\nu}$ is the symmetric part of the tensor $\sigma_{\mu\nu}$ defined as

$$V_\mu = \partial_\nu \sigma_{\nu\mu} \tag{19}$$

and V_μ is the field virial

$$V_\mu = \pi_\nu[\mathrm{d}\delta_{\mu\nu} - \Sigma_{\mu\nu}]\psi. \tag{20}$$

Since, in our case, the field virial vanishes (as can easily be checked), the improved energy-momentum tensor is equal to the Belifante. This is divergenceless and traceless (as can be seen from (17)) as required by the dilatation and conformal invariance of the starting Lagrangian. Inserting the classical solution (11) into eq. (16) one can check that the energy-momentum tensor is vanishing,

$$\theta_{\mu\nu} = 0, \tag{21}$$

and that the action in Euclidean space is finite:

$$S \equiv \int \mathrm{d}^2x\, \mathscr{L} = -\frac{\pi}{g}. \tag{22}$$

([13]) C. G. CALLAN, S. COLEMAN and R. JACKIW: *Ann. of Phys.*, **59**, 42 (1970).

Repeating the argument for the singular solution (14) we find

$$\theta_{\mu\nu} = \frac{-1}{16g(x^2)^2}(x^2\delta_{\mu\nu} - 2x_\mu x_\nu)\,, \tag{23}$$

while the action is divergent in Euclidean space since $\mathscr{L} = (-1/16gx^2)$.

We can thus conclude that in the case of the two-dimensional conformal invariant Lagrangian (1) exploiting the same symmetry arguments as for gauge and scalar fields, we have found two types of classical solution in Euclidean space. One of them (11) leads to the vanishing of improved energy-momentum tensor and finite action and is called « instanton solution » as proposed earlier (1). The fact that it has vanishing energy-momentum tensor gives it a chance to be a candidate for a vacuum in the quantum world, since we interpret classical solutions as vacuum expectation values of the field $\bar{\psi}\psi$.

Solution (14) is singular in Euclidean space and leads to the divergent action and such a solution is called « meron solution » (14). One can improve it by shifting its singularities from 0, ∞ to arbitrary points. Performing a suitable conformal transformation (translation, inversion, translation), we obtain (*)

$$\psi^{\text{tit}}(x) = \frac{[(2u)^2]^{\frac{1}{4}} i\gamma\cdot(x+u)}{[(x-u)^2]^{\frac{1}{4}}[(x+u)^2]^{\frac{3}{4}}}\left\{1 + \frac{i\gamma\cdot[(2u)^2(x+u) - 2u(x+u)^2]}{[(2u)^2(x-u)^2(x+u)^2]^{\frac{1}{2}}}\right\}C\,. \tag{24}$$

Then the bilinear $S(x)$ becomes

$$S(x) = \frac{4\bar{C}C}{[(x-u)^2(x+u)^2]^{\frac{1}{2}}}\,. \tag{25}$$

This solution is continued to Minkowski space taking $x_4 = ix_0$ and $u_\mu = (1, \mathbf{0})$. We have

$$S(x) = \frac{4\bar{C}C}{[(1+t_+^2)(1+t_-^2)]^{\frac{1}{2}}}\,, \tag{26}$$

where $t_\pm = x_0 \pm |\boldsymbol{x}|$. This improved solution leads to the finite action and energy in Minkowski space

$$S \equiv \int \mathrm{d}x\,\mathrm{d}x_0\,\mathscr{L} = \frac{-\pi^2}{8g}\,, \tag{27}$$

$$E \equiv \int \theta_{00}\,\mathrm{d}x = \frac{\pi}{8g}\,. \tag{28}$$

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(14) V. DE ALFARO, S. FUBINI and G. FURLAN: *Phys. Lett.*, **5** B, 163 (1976).

(*) Transformation of a fermion field under the combined transformations mentioned above is given by

$$\psi^{\text{tit}}(x) = \frac{i\gamma\cdot(x+u)}{(x+u)^2}\,\psi(\text{tit}\,x)\,.$$

To conclude this section, we consider it useful to recall the way the above results influence the possible subsequent quantum developments. The idea is to separate the field into a classical part, which is big in the weak-coupling limit, and a small quantum fluctuation

$$\psi = \psi_{cl} + \psi' . \tag{29}$$

Since we ask that, consistent with our interpretation,

$$\bar{\psi}\psi = S(x) + (\bar{\psi}\psi)' , \tag{30}$$

one must assume that combinations of the form $\bar{\psi}_{cl}\psi'$, $\bar{\psi}'\psi_{cl}$ can be dropped. With this recipe it is easy to obtain from eq. (2) the equations of motion for the quantum field ψ', which read

$$i\gamma\partial\psi' = \frac{2a}{a^2 + x^2}\psi' \tag{31}$$

and

$$i\gamma\partial\psi' = \frac{1}{2(x^2)^{\frac{1}{2}}}\psi' \tag{32}$$

for the instanton and meron configurations, respectively.

We thus see that the effect of the classical background solutions amounts to producing the analogue of an external potential which determines the quantum aspects. One further recognizes that eqs. (31) and (32) are identical to the case in which the instanton or meron external field is due to an independent scalar particle. Some of these solutions are already known and we devote the appendix to a short summary of some properties. Internal degrees of freedom for a field ψ_α could be introduced trivially through a constant spinor C, *i.e.* $\bar{C}C = \sum \bar{C}_\alpha C_\alpha$, but space dependence and internal symmetry would still be separated. A nontrivial way would be mixing internal and space degrees, which is not discussed here.

3. – The four-dimensional model.

A conventional generalization of the Lagrangian (1) to four dimensions would be based on fermion fields of scale dimension $\frac{3}{2}$. This fact, together with a $(\bar{\psi}\psi)^2$ coupling, introduces a dimensional constant and hence violates conformal invariance. We rather propose to keep the scale dimension of the spinor field fixed to the value $\frac{1}{2}$. Fermion coupling constants are then dimensionless, but one must introduce higher derivatives in the Lagrangian density:

the analogue of eq. (1) will now be

$$\mathscr{L} = \bar{\psi}(i\gamma\partial)^3\psi + g(\bar{\psi}\psi)^4 . \tag{33}$$

Such kind of theories has often been considered as a specimen of models where ultraviolet divergences might be cured by the improved asymptotic behaviour of propagators [15]. On the other hand, it is well known that these theories are affected by the problem of ghost states. In spite of this fact, these theories have again become interesting in connection with the confinement problem: the strong infra-red singularities can give rise to an interaction increasing with distance [16]. Also conformal supergravity leads to higher-derivative Lagrangians [17].

Thus we consider it instructive to consider the theory (33) from the point of view of classical solutions. The equation of motion is

$$(i\gamma\partial)^3\psi = -4g(\bar{\psi}\psi)^3\psi . \tag{34}$$

Since the field has canonical dimension $\frac{1}{2}$ as before, symmetry requirements remain the same and we can use the same ansatz to solve eq. (34). The instanton solution is

$$\psi(x) = \frac{a \pm i\gamma\cdot x}{a^2 + x^2}\, C , \qquad (\bar{C}C)^3 = \pm \frac{12}{g}\, a^3 , \tag{35}$$

while the meron solution is

$$\psi(x) = \frac{1}{(x^2)^{\frac{1}{4}}}\left[1 \pm \frac{i\gamma\cdot x}{(x^2)^{\frac{1}{2}}}\right] C , \qquad (2\bar{C}C)^3 = \pm \frac{15}{32g} . \tag{36}$$

We see that the classical solutions are the same in form as the two-dimensional ones.

We must next investigate the properties of the energy-momentum tensor and of the action. In order to handle a higher-order derivative Lagrangian, the idea is to express it as an equivalent first-order derivative Lagrangian by introducing auxiliary fields. In our case, the Lagrangian which leads to the same equation of motion is

$$\mathscr{L} = \tfrac{1}{2}(\bar{\psi}_1 \overleftrightarrow{D}\psi_1 + \bar{\psi}\overleftrightarrow{D}\psi_2 + \bar{\psi}_2\overleftrightarrow{D}\psi) - (\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1) + g(\bar{\psi}\psi)^4 , \tag{37}$$

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[15] H. P. DURR and C. C. CHIANG: preprint MPI-PAE/PTh 2, Max-Planck-Institut für Physik und Astrophysik, München, B.D.R. (1975).

[16] H. NARNHOFER and W. THIRRING: *The taming of the dipole ghost*, Institut für Theoretische Physik, Wien, preprint (1978).

[17] S. FERRARA and B. ZUMINO: *Structure of conformal supergravity*, CERN preprint TH. 2418 (1977).

where $D = i\gamma\cdot\partial$ and ψ_1, ψ_2 are auxiliary fields satisfying the equations of motion

$$\left\{\begin{aligned} D\psi &= \psi_1 , \\ D\psi_1 &= \psi_2 , \\ D\psi_2 &= -4g(\bar{\psi}\psi)^3\psi . \end{aligned}\right. \tag{38}$$

These coupled equations of motion are obviously equivalent to (34), but they are of the first order, thus the problem is reduced to the standard one.

Consider now the Belifante energy-momentum tensor. For the Lagrangian (34) we obtain

$$\theta^{\text{B}}_{\mu\nu} = \frac{i}{4}\,[\bar{\psi}_1\gamma_\mu \overleftrightarrow{\partial}_\nu \psi_1 + \bar{\psi}_2\gamma_\mu \overleftrightarrow{\partial}_\nu \psi + \bar{\psi}\gamma_\mu \overleftrightarrow{\partial}_\nu \psi_2 + (\mu \leftrightarrow \nu)] - \delta_{\mu\nu}\,\mathscr{L} . \tag{39}$$

Since the dimensions of the fields ψ_1, $\frac{1}{2}$, ψ_2, $\frac{5}{2}$ are not the canonical ones of the four-dimensional fermion fields, the field virial (20) does not vanish now and is given by

$$V_\mu = i(\bar{\psi}\gamma_\mu\psi_2 - \bar{\psi}_2\gamma_\mu\psi) \tag{40}$$

and also by

$$\sigma_{\mu\nu} = i[(\partial_\mu\bar{\psi})\gamma_\nu\psi - \bar{\psi}\gamma_\nu\partial_\mu\psi] . \tag{41}$$

Using the solution (35) we get

$$\sigma^{+}_{\mu\nu} = 2a\bar{C}C\,\frac{\delta_{\mu\nu}}{(a^2 + x^2)^2} , \tag{42}$$

so that the improved energy-momentum tensor (17) vanishes,

$$\theta_{\mu\nu} = 0 , \tag{43}$$

as can easily be checked. The action is correspondingly

$$S \equiv \int \mathrm{d}^4x\;\mathscr{L} = -\,6\pi^2\left(\frac{12}{g}\right)^{\frac{1}{3}} . \tag{44}$$

Again, as in the two-dimensional case, solution (35) can really be considered as the analogue of the instanton solution.

For the solution (35) we obtain

$$\theta_{\mu\nu} = \frac{15}{16}\,\frac{\bar{C}C}{(x^2)^3}\,(4x_\mu x_\nu - x^2\,\delta_{\mu\nu}) , \tag{45}$$

while the action is divergent in Euclidean space. Thus we can again think of it as of an analogue of the meron configuration.

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APPENDIX

The aim of this appendix is to formulate in a compact way the quantum corrections for the models presented. We shall discuss only the two-dimensional case since the four-dimensional has the problem of ghost states. As discussed in the text, the idea is to divide the field into a classical part $O(1/g)$, which is big in a small coupling limit, and a small quantum correction

$$\bar{\psi}\psi = (\bar{\psi}\psi)_{\text{cl}} + (\bar{\psi}\psi)' + O(g^2) . \tag{A.1}$$

The next step is to find an equation of motion for the quantum correction ψ', which leads to eqs. (31) and (32). In order to carry this out in an intrinsic way, we project our Lagrangian (1) onto the surface of a three-dimensional sphere, as proposed in ref. [5]; the new co-ordinates ξ are defined by

$$\xi_\mu = \frac{2ax_\mu}{a^2 + x^2}, \qquad \xi_3 = \frac{a^2 - x^2}{a^2 + x^2}, \qquad \xi^2 = 1 . \tag{A.2}$$

Introduce the «rationalized» field (dimensionless)

$$\psi(x) = \frac{a^{\frac{1}{2}}}{a^2 + x^2}(a + i\gamma \cdot x)\,\chi(\xi) , \tag{A.3}$$

where ξ is the co-ordinate on the sphere. The O_3-invariant Lagrangian takes the form

$$\mathscr{L}_\Omega = -4\bar{\chi}(l \cdot S + \tfrac{1}{2})\,\chi + g(\bar{\chi}\chi)^2 , \tag{A.4}$$

where $l \cdot S$ is the operator in three-dimensional space defined by

$$l \cdot S = \tfrac{1}{2}[\xi\partial_\xi - (\Gamma\xi)(\Gamma\partial_\xi)] \tag{A.5}$$

and Γ_a are generalizations of γ_μ in the three-dimensional space defined as

(A.6) $$\Gamma_\mu = i\gamma_5\gamma_\mu\,, \qquad \Gamma_3 = \gamma_5\,.$$

In a small coupling limit $g \to 0$ we get the equation of motion

(A.7) $$(lS + \tfrac{1}{2})\,\chi = 0\,.$$

Using the definition of the J^2-operator in O_3 given by

(A.8) $$J^2 = \tfrac{1}{2} J_{ab} J_{ab}\,, \qquad J_{ab} = l_{ab} + S_{ab} \qquad (a, b = 1, 2, 3)\,,$$

where

(A.9) $$l_{ab} = i(\xi_a\,\partial_b - \xi_b\,\partial_a)\,, \qquad S_{ab} = \frac{i}{4}\,[\Gamma_a, \Gamma_b]\,,$$

we transform (A.7) into the eigenvalue problem of J^2

(A.10) $$(J^2 + \tfrac{1}{4})\,\chi = 0\,.$$

To find the quantum corrections for the field $\bar{\chi}\chi$, we use expansion (A.1) and get

(A.11) $$\bar{\chi}\chi = (\bar{\chi}\chi)_{cl} + (\bar{\chi}\chi)'\,, \qquad (\bar{\chi}\chi)_{cl} = \frac{1}{g}\,.$$

After the shift $\chi = \chi_{cl} + \chi'$ we have

(A.12) $$lS\chi' = 0\,.$$

This can again be written as the eigenvalue problem in O_3 as

(A.13) $$(J^2 - \tfrac{3}{4})\,\chi' = 0\,.$$

If we compare expressions for L_Ω and $\mathscr{L}(x)$, we see that we can go from one space to another just by changing $\chi \to \psi$ and $-4(lS + \frac{1}{2}) \to i\gamma\cdot\partial$ to get eqs. (31) and (32). Equations (31) and (32) can also be written as the eigenvalue problem in $O_{1,2}$. Using the definitions of the operator J^2 for the de Sitter $(M_{\mu\nu}, R_\mu)$ group and $O_2 \times O_2$ $(M_{\mu\nu}, D)$ group given by

(A.14) $$\begin{cases} J_D^2 = M^2 + D^2 = -x^2\Box - (\gamma\cdot x)(\gamma\cdot\partial)\,, \\[2ex] J_R^2 = M^2 + R^2 = -\dfrac{1}{4} - \left(\dfrac{a^2 + x^2}{2a}\right)^2 \Box - \dfrac{a^2 + x^2}{2a^2}\,(\gamma\cdot x)(\gamma\cdot\partial)\,, \end{cases}$$

we have

(A.15) $$(J_D^2 - \tfrac{1}{4})\,\psi' = 0\,, \qquad (J_R^2 - \tfrac{3}{4})\,\psi' = 0\,.$$

We conclude that the problem of solving eq. (31) reduces simply to the eigenvalue problem of the operator J^2 in three-dimensional space. Shifting the field $\psi(x)$ in O_3 corresponds to shifting the eigenvalue of J^2 from $-\frac{1}{4}$ to $\frac{3}{4}$. Similarly for J_D^2 eigenvalue 0 goes to eigenvalue $\frac{1}{4}$. We should note that the g-dependence in (A.15) disappears naturally, while the same is achieved in ref. ([5]) by fixing the ratio of fermion and boson coupling constants.

● RIASSUNTO (*)

Si danno soluzioni classiche per l'interazione fermione-fermione conforme invariante in due dimensioni e si discutono le loro proprietà d'invarianza. Si dà una possibile generalizzazione a quattro dimensioni e si trovano corrispondenti soluzioni del tipo istantone e del tipo merone. Ne risulta che queste sono una generalizzazione naturale di quelle a due dimensioni.

(*) *Traduzione a cura della Redazione.*

Классические решения для фермионных моделей.

Резюме (*). — Приводятся классические решения для конформного инвариантного фермион-фермионного взаимодействия в двух измерениях. Обсуждаются свойства инвариантности этих решений. Обсуждается возможное обобщение на случай четырех измерений. Получены соответствующие решения инстантонного и меронного типов. Оказывается, что полученные решения представляют естественное обобщение двумерных решений.

(*) *Переведено редакцией.*