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θ-Vacua in Two-Dimensional Quantum Chromodynamics (*).

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Summary. — It is shown that θ-vacua exist in two-dimensional non-Abelian gauge theories, as well as in the Abelian theories.

1. – Introduction.

In the last few years, it has been realized that U_1 gauge theories in two dimensions possess not just a single vacuum, but a one-parameter family of vacua, the « θ-vacua » (1). This is true regardless of whether the U_1 symmetry is spontaneously broken. When the U_1 symmetry is not spontaneously broken, the θ-vacua correspond to the possible presence of a background electric field (2). When the U_1 symmetry is spontaneously broken, the existence of θ-vacua is connected with the existence of instantons (3).

In this paper it will be shown that an analogous phenomenon, multiple vacuum states, occurs in non-Abelian gauge theories in two dimensions. However, instead of a one-parameter family of vacua, there exists in this case only a finite, discrete set of possible vacuum states.

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(1) J. LOWENSTEIN and J. SWIECA: *Ann. of Phys.*, **68**, 172 (1971).

(2) S. COLEMAN: *Ann. of Phys.*, **101**, 239 (1976).

(3) A. M. POLYAKOV: *Nucl. Phys.*, **121** B, 429 (1977); G. 'T HOOFT: *Nucl. Phys.*, **72** B, 461 (1974); C. CALLAN, R. DASHEN and D. J. GROSS: *Phys. Lett.*, **63** B, 334 (1976).

When the gauge symmetry is not spontaneously broken, then, as in the U_1 case, the θ-vacua are associated with a possible background electric field. However, as we will see, in the non-Abelian case, a background electric field must be treated as a q-number, not a c-number.

When the gauge symmetry is spontaneously broken, the θ-vacua are related instead, as in the Abelian case, to instantons. Moreover, as in the Abelian case, the pattern of the θ-vacua (how many there are, and certain of their properties to be discussed) is the same whether the gauge symmetry is spontaneously broken or not.

2. – Theories with unbroken gauge symmetry.

In this section we will consider, for definiteness, a SU_N gauge group. For definiteness, we will choose all charged fields to be fermions. As for choosing representations of the gauge group for the fermions ψ, we will consider several possibilities, such as the fundamental representation ψ^i, the antisymmetric tensor ψ^{ij} and the adjoint representation ψ^i_j. The Lagrangian is thus

$$\mathscr{L} = \operatorname{Tr}\left(\bar{\psi}i(\gamma\cdot D)\psi - M\bar{\psi}\psi - \tfrac{1}{4}G^{\mu\nu}G_{\mu\nu}\right), \tag{1}$$

where the fermion representation has not been specified. The conclusions that follow would be the same if the fermions were scalar fields.

Perhaps the easiest way to see that θ-vacua are possible in (1) is to use the path integral representation

$$\begin{aligned} Z &= \int \mathrm{d}\psi\,\mathrm{d}\bar{\psi}\,\mathrm{d}A\,\exp\left[i\int \mathrm{d}^2x\,\mathscr{L}\right] = \\ &= \int \mathrm{d}\psi\,\mathrm{d}\bar{\psi}\,\mathrm{d}A\,\exp\left[i\left(\int \mathrm{d}^2x\,\operatorname{Tr}\left(\bar{\psi}i(\gamma\cdot D)\psi - M\bar{\psi}\psi - \tfrac{1}{4}G^{\mu\nu}G_{\mu\nu}\right)\right)\right] \end{aligned} \tag{2}$$

In an Abelian gauge theory, one well-known way to see the existence of θ-vacua is the following. We modify the path integral by inserting into the integrand an extra factor $\exp\left[(i\theta/2\pi)\oint \mathrm{d}x^\mu A_\mu\right]$, where θ is an arbitrary angle and the contour of integration runs at the boundary of our two-dimensional space-time world (fig. 1). Thus we consider

$$Z(\theta) = \int \mathrm{d}\psi\,\mathrm{d}\bar{\psi}\,\mathrm{d}A\left(\exp\left[i\int \mathrm{d}^2x\,\mathscr{L}\right]\right)\left(\exp\left[i\frac{\theta}{2\pi}\oint \mathrm{d}x^\mu A_\mu\right]\right). \tag{3}$$

This modified path integral obviously is still gauge invariant. It is also obviously Lorentz invariant, since a contour at space-time infinity is mapped by a Lorentz transformation into a contour at space-time infinity. It is also true, and will be made obvious below by introduction of a Hamiltonian formulation, that the modification of the path integral preserves properties such as positivity of the metric and cluster decomposition. Thus the new path integral defines a quantum field theory.

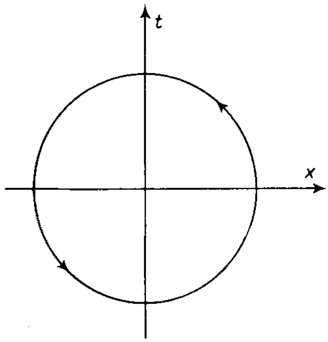

Fig. 1. – A contour at the boundary of space-time.

This quantum field theory satisfies the same equations of motion as before, since we have added a surface term that does not change the equations of motion. Thus (3) describes a new solution of the equations of motion of the theory, or, differently put, a new vacuum state.

In this form, it is fairly obvious how to construct non-Abelian θ-vacua. Instead of $\exp\left[(i\theta/2\pi)\oint \mathrm{d}x^\mu A_\mu\right]$, we choose an arbitrary representation U of the Lie algebra of the gauge group and write $\mathrm{Tr}_U P \exp\left[i\oint \mathrm{d}x^\mu A_\mu\right]$, where P stands for path ordering along a contour at infinity, and Tr_U stands for taking the trace in the representation U of the algebra. Thus we write

$$Z(U) = \int \mathrm{d}\psi\,\mathrm{d}\bar{\psi}\,\mathrm{d}A_\mu \exp\left[i\int \mathrm{d}^2x\mathscr{L}\right]\mathrm{Tr}_U P\exp\left[i\oint \mathrm{d}x^\mu A_\mu\right]. \tag{4}$$

The non-Abelian « θ-vacua » are defined by (4). Instead of the continuous parameter θ we have available only the discrete choice of the representation U.

In the Abelian theory, a nonzero θ corresponds, as COLEMAN has shown ([2]), to a world with a fractional charge of strength $e\theta/2\pi$ at the right-hand end of the world, $x = +\infty$, and one of strength $-e\theta/2\pi$ at the left-hand end, $x = -\infty$. By the same token, in the non-Abelian theory, a nontrivial choice of U corresponds to a world with a charge in the representation U of the gauge group at $x = +\infty$ and one in the complex conjugate representation $\bar{U}$ at $x = -\infty$. In fact, our way of modifying the basic path integral (2) has been to introduce a path-ordered exponential with a contour at the ends of the world. Such path-ordered exponentials are often used ([4]) (with contours that are ordinarily

([4]) K. WILSON: *Phys. Rev. D*, **10**, 2445 (1974).

at finite distances) to represent the effects of externally introduced, static charges, and in particular we are dealing here with static charges at $x = \pm\infty$.

In a non-Abelian gauge theory, even a static (that is, nonmoving) charge cannot be regarded as a c-number, because it has color degrees of freedom and it can exchange color with the other charges in the system. In fact, for nontrivial choices of U, it is necessary to work not with the usual Hilbert space V of the theory in the ordinary vacuum, but with an enlarged space. Let V_U and $V_{\bar{U}}$ be finite-dimensional vector spaces of dimension equal to the dimension of U and $\bar{U}$. These spaces will represent the color degrees of freedom of charges at the right- and left-hand ends of the world. Then the Hilbert space of this theory in a vacuum with nontrivial U will turn out to be $V \otimes V_U \otimes V_{\bar{U}}$.

To explain those facts, and also to understand the physical properties of the nontrivial vacua, it is useful to introduce a Hamiltonian formulation. This can be done directly from (4), by following the standard procedure for going from a path integral to a Hamiltonian, via the transfer matrix [5]. However, we will follow a simpler procedure, *i.e.* the canonical quantization in the gauge $A_1 = 0$.

In this gauge A_0 is a dependent variable. Before quantization A_0 should be eliminated by means of Gauss's law [6]

$$\frac{\mathrm{d}}{\mathrm{d}x} E^a = qJ_0^a\,, \tag{5}$$

where E^a is the «electric field», in this gauge $E^a = (\partial/\partial x)A_0^a$, and where J_0^a is the charge density, which is, of course, a quantum field operator, $J_0^a = \bar{\psi}\gamma_0\lambda^a\psi$. The general solution of (5) is

$$E^a(x) = \tfrac{1}{2}g\int \mathrm{d}y\,\varepsilon(x-y)\,J_0^a(y) + gC^a\,, \tag{6}$$

where $\varepsilon(x-y) = \pm 1$ for x greater or smaller than y, and where gC^a is a constant of integration, to be discussed later. (The factor of g is included for convenience.) Finally, we write the Hamiltonian, including the energy of the electric field:

$$H = \int \mathrm{d}x \left(\mathrm{Tr}\left(\bar{\psi}i\gamma^1 \frac{\partial}{\partial x}\psi + M\bar{\psi}\psi\right) + \sum_a (E^a)^2\right). \tag{7}$$

The usual vacuum corresponds to $C^a = 0$ in (6); now we must discuss what nonzero choices of C^a are allowed.

([5]) K. G. Wilson and J. Kogut: *Phys. Rep.*, **12** C (1974), Chapter 10.

([6]) For a related discussion, see I. Bars and M. B. Green: *Phys. Rev. D*, **17**, 537 (1978).

First, we should realize that C^a is, in general, an operator, not a c-number. In fact, all quantities in (6) are operators: J_0^a is the quantum field operator for the charge density and E^a is the electric-field operator. Thus we must expect also C^a to be an operator.

Second, we must realize that, for an arbitrary choice of C^a, we will not obtain a Lorentz-invariant theory. We are working in a non–Lorentz-invariant gauge $A_1 = 0$, and choices for C^a will lead to Lorentz-invariant theories only if they can be derived from some Lorentz-invariant and gauge-invariant starting point.

Taking the path integral as our Lorentz-invariant, gauge-invariant starting point, we must choose a nonzero C^a that corresponds to an invariant modification of the basic path integral (2). The only such modification is (4), thus we must determine which C^a corresponds to (4).

As has already been noted, (4) describes a world with an external charge in the representation U at $x = +\infty$ and one in the representation $\overline{U}$ at $x = -\infty$. The appropriate choice of C^a is simply that C^a must represent the contribution to the charge density coming from these boundary charges. The appropriate formula is

$$E^a(x) = \tfrac{1}{2} g \int \mathrm{d}y\, \varepsilon(x-y)\, J^a_{0\,\mathrm{tot}}(y)\,, \tag{8}$$

where $J^a_{0\,\mathrm{tot}} = J^a_0 + J^a_{0\,\mathrm{bdry}}$; $J^a_{0\,\mathrm{tot}}$ includes the contribution J^a_0 to the charge density coming from the quantum fields and also the contribution $J^a_{0\,\mathrm{bdry}}$ from the boundary charges.

In general, the charge density operator at a point is the operator that generates gauge transformations at that point. In particular, the contribution of the boundary charges to the charge density is $J^a_{0\,\mathrm{bdry}}(x) = \delta(x = +\infty) T^a_U + \delta(x = -\infty) T^a_{\overline{U}}$, where T^a_U and $T^a_{\overline{U}}$ are the generators of the gauge group acting on the boundary charges in the U and $\overline{U}$ representations. Inserting this information in (8), we learn

$$E^a(x) = \tfrac{1}{2} g \int \mathrm{d}y\, \varepsilon(x-y)\, J^a_0(y) + \tfrac{1}{2} g (T^a_U - T^a_{\overline{U}})\,. \tag{9}$$

The Hamiltonian is found by inserting (9) in the general formula (7).

In summary, the passage from the usual vacuum to a nontrivial « θ-vacuum » is obtained by a substitution $V \to V \otimes V_U \otimes V_{\overline{U}}$ on the Hilbert space, to represent the degrees of freedom of boundary charges, and a substitution $E^a \to E^a + \frac{1}{2} g (T^a_U - T^a_{\overline{U}})$ in the electric-field operator, to represent the physical effects of such charges.

Now we must discuss more tangible properties of these new vacua.

First, let us calculate, in the weak-coupling regime, the energy of the θ-vacua. We should consider only states that are overall color singlet states—states in which all of the charges present, including the charges at infinity, are coupled to total color zero. In rough terms, the reason is that this is a confining theory.

Formally, in writing down (9) and (5), we have dropped from the Hamiltonian a term which is zero for overall color singlet states and infinite for other states. This occurs as follows. The world with charges at infinity must be regarded as the limit $x_0 \to \infty$ of a world with charges at $x = \pm x_0$. For any finite (but large) x_0 one must consider in the Hamiltonian a potentially infinite term, the electric-field energy in the infinite region $x > x_0$ or $x < -x_0$. The electric field for $x > x_0$ or $x < -x_0$ is $E^a = \pm\frac{1}{2}gQ^a$, Q^a being the total color of the world. If the total Q^a is not zero, the electric field is nonzero for $x > x_0$ and for $x < -x_0$ and the total field energy in this infinite region is infinite. Therefore, we must now use (9) and (5) only in color singlet states.

Now, to calculate the energy difference between a θ-vacuum and the ordinary vacuum in the leading weak-coupling approximation, we ignore vacuum polarization, and thus ignore the J_0^a-term in (9). Thus we take $E^a = \frac{1}{2}g(T^a_{\bar{U}} - T^a_U)$ and the vacuum energy is $W = (E^a)^2 = (g^2/4)(T^a_{\bar{U}} - T^a_U)^2$. This is to be evaluated in a state with U and $\bar{U}$ coupled to total color zero, which means $T^a_{\bar{U}} + T^a_U$ annihilates the vacuum. For such states,

$$W = \frac{g^2}{2}\left((T^a_{\bar{U}})^2 + (T^a_U)^2\right) - \frac{g^2}{4}(T^a_{\bar{U}} + T^a_U)^2 = \frac{g^2}{2}\left((T^a_{\bar{U}})^2 + (T^a_U)^2\right).$$

But $(T^a_U)^2$ and $(T^a_{\bar{U}})^2$ are a c-number, a Casimir operator of the representation U and in terms of this Casimir, which we will call $C(U)$, the vacuum energy, relative to the energy of the ordinary vacuum, is $W = g^2 C(U)$.

Now we can see that not all of the θ-vacua are stable. The reasoning proceeds by analogy with Coleman's argument that in the Abelian case the physics is a periodic function of θ. For, whenever it is energetically favorable, the vacuum will emit a charged particle-antiparticle pair, which will travel to $x = \pm\infty$, in order to screen the boundary charges. This will occur whenever, by coupling in an appropriate way to some of the physical charged particles, it is possible to reduce $C(U)$, the total Casimir operator of the charges at $x = +\infty$. As a result of this mechanism, there is always only a finite number of stable θ-vacua. How many there are depends on the group representations of the charged matter fields. For a SU_N gauge group, the number of stable vacua is never larger than N.

For example, suppose that there are charged fields in the fundamental (N-dimensional) representation φ^i. Then the only stable vacuum is the ordinary vacuum, because any representation of SU_N can be formed by combining the fundamental representation with itself enough times, so that any boundary charge can be screened.

On the other hand, suppose that all charged particles are in the adjoint representation of the group. In this case it turns out that there are in all N stable vacua. Apart from the ordinary vacuum, these correspond to choosing for U the fundamental representation v^i, the second-rank antisymmetric tensor

v^{ij}, the third-rank antisymmetric tensor v^{ijk}, and so on up to the $(N-1)$-rank antisymmetric tensor $v^{ijk...p}$.

These are all stable because they transform differently under the center of the group, while the adjoint representation is invariant under the center. Under a transformation by an element $\exp[2\pi i/N]$ of the center of SU_N, any representation of SU_N transforms by a factor $\exp[2\pi ik/N]$, where k is an integer modulo N that is sometimes called the N-ality. Since the adjoint representation has N-ality zero, the N-ality cannot be changed by coupling to particles in the adjoint representation, and, therefore, there are at least N stable vacua, one for each N-ality. The k-th rank antisymmetric tensor has N-ality k, and these are the stable vacua.

To see that other possible vacua are unstable, consider for instance a boundary charge in the symmetric-tensor representation s^{ij}. A symmetric tensor can combine with a particle in the adjoint representation φ^i_j to make an antisymmetric tensor $s^{ik}\varphi^j_k - s^{jk}\varphi^i_k$. This lowers the energy, because the Casimir operator $C(U)$ is less for the antisymmetric tensor than for the symmetric tensor and, therefore, a world with a symmetric-tensor charge at infinity will decay, via creation of a pair of particles in the adjoint representation, into a world in which the total charge at infinity is in the antisymmetric-tensor representation.

As another example of a possible choice of the gauge group representation of the charged fields, let us suppose that all charged fields are second-rank antisymmetric tensors φ^{ij}. In this case the physics depends very much on whether N is even or odd.

If N is odd, the only stable vacuum is the ordinary vacuum, because for odd N every representation can be made by combining antisymmetric tensors. For instance, the fundamental representation, from which everything else can be built, can be constructed for $N=3$ as $\varphi^{ij}\varphi^{kl}\varepsilon_{ijk}$, and for $N=5$ as $\varphi^{ij}\varphi^{kl}\varphi^{mn}\varepsilon_{ijklm}$.

But for even N, there are two stable vacua—the ordinary vacuum and also a world with a charge v^i in the fundamental representation at infinity. The latter possibility exists because for even N the fundamental representation does not arise in combining second-rank tensors.

We are finally ready to discuss the physical properties of the new vacua. Here we will find some surprising results, somewhat analogous to some results of Coleman in the Abelian problem.

Let us consider first the case in which the physical charged fields are in the adjoint representation ψ^i_j.

In the ordinary vacuum, in theories of the type considered here with unbroken gauge symmetry, these ψ particles are confined. However, in the nontrivial θ-vacua, the ψ particles are not confined—but color is still confined, the physical ψ particles being color singlets.

Consider, for instance, the first nontrivial θ-vacuum with a charge v^i in the N representation at $x=+\infty$ and a charge $\bar{v}_i$ in the $\bar{N}$ representation at

$x = -\infty$. If we ignore vacuum polarization (which is negligible in the weak-coupling regime), this θ-vacuum can be described by simply coupling v^i and $\bar{v}_j$ to total color zero, $v^i\bar{v}_i$. There is in this world, as we have discussed, a background non-Abelian electric field and an energy density greater by $\frac{1}{2}g^2 C(U)$ than the energy of the ordinary vacuum, but the field cannot decay and the vacuum is stable. The θ-vacuum is indicated in line one of fig. 2.

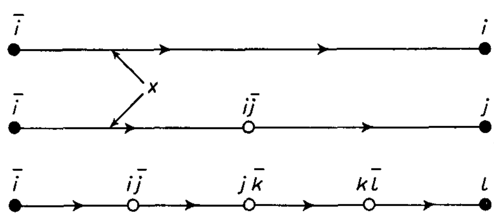

Fig. 2. – θ-vacua in a theory with quarks in the adjoint representation.

In the ordinary vacuum, the ψ^i_j particles are confined, and introducing a single ψ^i_j particle would increase the total energy by an infinite amount. The surprise we now encounter is that in the θ-vacuum the ψ^i_j particles are not confined.

In fact, it is possible in the θ-vacuum to introduce a single ψ^i_j particle and obtain a state of total color zero by coupling to the charges at infinity in the form $v^j\psi^i_j\bar{v}_i$. This is indicated in line two of fig. 2. This state differs in energy from the θ-vacuum by a finite amount, which in weak coupling is just the bare mass of the ψ^i_j particle.

To see that there is only a finite energy difference between the states shown in lines one and two of fig. 2, note that the potentially infinite term is the electric-field energy to the left or right of the ψ particle. But this field energy has the same value in line two as in line one. Consider, for instance, the electric field at the point x in the diagram, which is to the left of the ψ particle. The electric-field energy at this point depends only on the representation of SU_N to which the total charge to the left of x (or to the right of x) is coupled. To the left of x there is in each case a single charge in the $\bar{N}$ representation, and to the right of x there is a single charge in the N representation (line one) or a pair of charges coupled to the N representation (line two), which leads to the same energy.

In a similar fashion, there exist finite energy excitations of the θ-vacuum with an arbitrary number of ψ particles. For instance, in the last line of fig. 2 there is sketched a state with three particles, coupled with the charges at infinity to total color zero, in the pattern $v^l\psi^k_{1l}\psi^j_{2k}\psi^i_{3j}\bar{v}_i$. The energy of this state remains finite even as one separates the three ψ-particles from each other.

Thus, in this world ψ particles are not confined. If, for instance, there is a conserved quantum number—baryon number or quark number—and the

ψ field has fractional baryon number, then the state shown in the second line of fig. 1 is a finite-energy physical state of fractional baryon number.

Although ψ particles are not confined, color is still confined, and anyone living in this world would regard the physical ψ particles as color singlet states. This can easily be seen from the lack of degeneracy of the states shown in fig. 2. There is only one finite-energy excitation of the θ-vacuum with a single ψ particle present, because there is only one way to couple ψ with the charges at infinity to get total color zero. Because there is only one state with a single ψ particle, the physical ψ particle, as opposed to the ψ field, has no internal degrees of freedom—it is a singlet.

These results are rather reminiscent of some results of Coleman ([2]) concerning the Abelian problem at $\theta = \pi$. Even closer analogies with Coleman's results appear if we choose the ψ particles to be, not in the adjoint representation ψ^i_j, but in the antisymmetric-tensor representation ψ^{ij}. In this case, as has been mentioned, if the gauge group is SU_N for even N, there is a single stable θ-vacuum. As in the previous problem, this vacuum can be considered to have a charge v^i at $x = +\infty$ and a charge $\bar{v}_j$ at $x = -\infty$.

Unlike the previous example, there do not now exist finite-energy states with a single ψ^{ij} particle, because it is impossible to couple ψ^{ij} with the surface charges v^i and $\bar{v}_j$ to make a color singlet.

On the other hand, there certainly exist finite-energy states with one ψ^{ij} particle and one $\bar{\psi}_{ij}$ antiparticle. In fact, ψ, $\bar{\psi}$ and the boundary charges can be coupled to total color zero in more than one way.

One particularly interesting coupling is $v^k\bar{\psi}_{jk}\psi^{ij}\bar{v}_i$ (lines two and three of fig. 3). States with a particle and an antiparticle coupled in this way certainly have finite energy, at least as long as we do not try to separate the particle and the antiparticle. What happens, however, when we try to separate them?

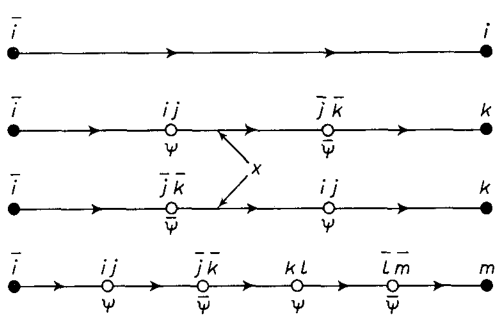

Fig. 3. – θ-vacua in a theory with quarks in the second-rank tensor representation.

It turns out that the energy remains finite if we take the particle to the left and the antiparticle to the right (line two of fig. 3), but it diverges if we take the particle to the right and the antiparticle to the left (line three). The difference between the two cases stems from the electric field in the region between ψ and $\bar{\psi}$, indicated as x in the diagram. The electric-field energy at x depends, of course, only on the total SU_N representation of all the charges to the left (or to the right) of x. In line two there are two charges, ψ and $\bar{v}$, to the left of x, coupled as $\psi^{ij}\bar{v}_j$ to the N representation. This is the smallest nontrivial representation of SU_N and corresponds to an electric-field energy density equal to the energy density of the θ-vacuum. Therefore, even as ψ and $\bar{\psi}$ are pulled apart, the configuration in line two is a finite-energy excitation of the θ-vacuum.

On the other hand, in line three the charges to the left of x are $\bar{v}$ and $\bar{\psi}$, which cannot combine to the N or $\bar{N}$ representation of SU_N, but combine to form larger (third-rank tensor) representations. In this case, the electric-field energy density is larger than in the θ-vacuum, and the energy, relative to the energy of the θ-vacuum, would diverge as one tried to separate ψ and $\bar{\psi}$.

Thus ψ and $\bar{\psi}$ can be separated, but only if one removes ψ to the left and $\bar{\psi}$ to the right. They are half-asymptotic particles, like the states found by COLEMAN in the Abelian case at $\theta = \pi$.

As in the previous example, because the multiplicity of states in line two is one, the physical half-asymptotic states are color singlets.

It is possible (line four of fig. 3) to have an arbitrary number of ψ and $\bar{\psi}$ particles, which can be separated arbitrarily far from each other at only a finite cost in energy, but only if the ψ and $\bar{\psi}$ particles alternate and the leftmost particle is a ψ particle.

3. – Non-Abelian Higgs theories in two dimensions.

In this section we will change orientation and consider Higgs theories in which the gauge symmetry is spontaneously broken. For simplicity, we will consider only theories in which there are no unbroken gauge symmetries.

Our purpose is to show that θ-vacua still exist in this case, that they are associated with instantons and that certain qualitative properties of the θ-vacua depend only on the representations of the gauge group in which one places the charged particles, and not on whether or not the gauge symmetries are spontaneously broken.

We will thus consider Lagrangians of the general type

$$\mathscr{L} = \mathrm{Tr}\left(D_\mu \varphi^* D_\mu \varphi - V(\varphi^*, \varphi) - \tfrac{1}{4} G_{\mu\nu}^2\right), \tag{10}$$

where φ and φ^* are scalar fields and $V(\varphi^*, \varphi)$ is a potential chosen so that the gauge symmetries are completely broken. The scalar multiplets may be several in number, if this is needed in order to completely break the gauge symmetries.

The gauge group we will still take to be SU_N. Moreover, we will assume that there are no accidental degeneracies in the vacuum state—any two possible vacua can be obtained from each other by a gauge transformation.

What does it mean to say that the gauge symmetries are completely broken? Letting φ_0 be a typical minimum of the Higgs potential, one would usually say that this means that the only element g of SU_N which leaves φ_0 invariant is $g = 1$. Actually, however, a refinement in the definition is necessary.

Suppose that all of the φ-fields are in the adjoint representation of the group. Then any transformation $g = \exp[2\pi ik/N]$ in the center of the group leaves the charged fields invariant—for arbitrary values of φ, not just $\varphi = \varphi_0$. Such a « transformation » cannot be regarded as a symmetry of Lagrangian (10), since it is not really a transformation at all—it is the identity operation on all the fields.

As another example, suppose that all charged fields are in the tensor representation φ^{ij}, and that the N of SU_N is even. Then, in addition to $g = 1$, also $g = -1$ leaves the φ^{ij} invariant. But if there are fields in the fundamental representation φ^i, only $g = 1$ leaves them invariant.

In saying that the gauge symmetries are completely broken, what one really means is that the only elements g of SU_N that leave a typical vacuum φ_0 invariant are elements of the center of SU_N that have a trivial action on all of the charged fields.

Now, let us ask whether our theory has instantons. (For a related discussion, see ref. (7).)

Any finite-action field configuration must, as $x \to \infty$, approach some gauge transform of the typical vacuum $\varphi = \varphi_0$, $A_\mu = 0$. Thus at large x we have $\varphi(x) = U(x)\varphi_0$, $A_\mu = (\partial_\mu U)\,U^{-1}$. $U(x)$ is defined essentially on a large circle at the boundary of the world (fig. 1). Because this contour is a circle, we are dealing with maps from a circle into the gauge group SU_N. Because these maps are all topologically trivial $(\pi^1(SU_N) = 0)$ it seems at first sight that arbitrary boundary conditions can be deformed into the standard conditions $\varphi = \varphi_0$, $A_\mu = 0$, so that there would be no instantons.

If U were completely well defined by the formulae $\varphi(x) = U(x)\varphi_0$, $A_\mu = (\partial_\mu U)\,U^{-1}$, this conclusion would be correct—there would be no instantons. However, it may happen that, even when $\varphi(x)$ and A_μ are given, there is more than one solution for $U(x)$—more than one branch in the definition of U. In this case, it might happen that on going around a large circle at infinity one would, on returning to the starting point, arrive at a different branch in the definition of U. This change in branch of U on travelling around a large circle would then be a topological invariant classifying the instanton.

(7) For analogous comments on classification of instantons and vortices, see G. 'T HOOFT: *Nucl. Phys.*, **138** B, 1 (1978).

An ambiguity in the formula $\varphi(x) = U(x)\varphi_0$, once φ is given, corresponds to $U \to Ug$, where g leaves φ_0 invariant. In view of the comments above about what it means to have the gauge symmetries completely broken, g must be an element of the center of the group that leaves all of the charged fields invariant. Therefore, $g = \exp[2\pi ik/N]$ for some k, the allowed values of k depending on what charged fields are present. It is this k that will label the different topological classes.

If there are matter fields in the fundamental representation of the gauge group, they are not invariant under any element of the center of the group except $g = 1$. Therefore, there are no nontrivial topological classes, no instantons and no nontrivial θ-vacua. This is reminiscent of the conclusion in the previous section that theories with unbroken gauge symmetry, and with some of the matter fields in the fundamental representation, have no nontrivial θ-vacua.

On the other hand, suppose all charged fields are in the adjoint representation of the gauge group. Since this representation is invariant under the center of the group, it is possible for $U(x)$, on going around a large circle, to change by an arbitrary factor $\exp[2\pi ik/N]$, for an arbitrary integer k, which is defined only modulo N.

In this case, there are N topological classes, corresponding to N possible values of k. We may regard k as the « instanton number ». But, because k is only defined modulo N, the instanton number is well defined (is a topological invariant) only modulo N. We may now define θ-vacua by modifying the usual Feynman path integral prescription to weight all fields with an extra factor $\exp[i\theta k]$. Since k is defined only modulo N, the permitted values of θ, to make $\exp[i\theta k]$ well defined, are

$$\theta = 0\,,\; \frac{2\pi}{N}\,,\; \frac{4\pi}{N}\,,\; \ldots\,,\; \frac{2\pi(n-1)}{N}\,.$$

Thus there are N possible θ-vacua.

This may be reminiscent of the conclusion in sect. **2** that, when the gauge symmetry is not spontaneously broken, and the charged particles are all in the adjoint representation of the gauge group, there are N θ-vacua.

As a final example, let us consider theories in which all charged matter fields are in the second-rank tensor representation φ^{ij}. In this case, if N is even, the change in U in going around a large circle must be a multiplicative factor ± 1, since 1 and -1 are the only elements of SU_N that leave the second-rank tensors invariant. Thus, there are two topological classes of fields. And there are also two vacua: apart from the usual vacuum, there is an additional vacuum in which fields in the nontrivial topological class are weighted with a factor -1.

On the other hand, if N is odd, only the identity element of SU_N leaves the representation φ^{ij} invariant. Therefore, for odd N there are no nontrivial topological classes and no nontrivial vacua.

These results mirror the claim in the last section that, in theories with unbroken gauge symmetry and with the charged fields in the φ^{ij} representation, a single nontrivial θ-vacuum exists if and only if N is even.

These examples show that the number of θ-vacua depends only on the representations of the charged fields, and not on whether the gauge symmetries are spontaneously broken. However, the interpretation of the vacua—in terms of a background field, or instantons—is different in the two cases.

Additional similarities between the θ-vacua in theories with unbroken gauge symmetry and those in Higgs theories can be found by considering the expectation value of $\mathrm{Tr}_V P \exp\left[i \oint A_\mu \, \mathrm{d}x^\mu\right]$ for large but finite contours and various choices of the SU_N representation V. Whether one finds area law decay depends on the choice of V and the θ-vacuum. As in the Abelian case, instantons give area decay for the same choices of V (and the same θ-vacua) for which one finds area decay in theories in which the θ-vacua are interpreted in terms of a background electric field.

In theories with unbroken gauge symmetry, we defined the θ-vacua by including a factor $\mathrm{Tr}_U P \exp\left[i \oint A_\mu \, \mathrm{d}x^\mu\right]$, with a contour at the boundary of the world, in the definition of the path integral. For various choices of U one obtains the various vacuum states. Actually, the same formal procedure yields the θ-vacua in Higgs theories. Indeed, $\mathrm{Tr}_U P \exp\left[i \oint A_\mu \, \mathrm{d}x^\mu\right]$ will, in gauge fields that are pure gauges at infinity, simply measure the topological class, and assign a phase factor to each topological class. If k is the topological class, as defined above, and n is the N-ality (discussed in sect. **2**) of the representation U, this phase factor is $\exp\left[2\pi i n k/N\right]$. To obtain the n-th θ-vacuum, one picks U to have N-ality n. Thus, despite their different physical interpretations, the θ-vacua can formally be defined the same way, whether or not the gauge symmetry is spontaneously broken.

4. – Conclusions.

Several aspects of this subject seem to be worthy of some note.

First, the correct treatment of the θ-vacua in theories with unbroken gauge symmetry requires that one takes seriously the fact that a background Yang-Mills field is an operator, not a c-number.

Second, it is interesting to see that many formal properties of the θ-vacua do not depend on whether the theory has symmetry breaking and instantons.

Finally, it is interesting to see, in some of the cases treated above, explicit examples in which color is confined, but quarks are not confined.

● RIASSUNTO (*)

Si mostra che esistono vuoti θ in teorie di gauge bidimensionali non abeliane, così come in teorie abeliane.

(*) *Traduzione a cura della Redazione.*

θ вакуумы в двумерной квантовой хромодинамике.

Резюме (*). — Показывается, что θ вакуумы существуют в двумерных неабелевых калибровочных теориях, а также в абелевых теориях.

(*) *Переведено редакцией.*