LETTERE AL NUOVO CIMENTO VOL. 25, N. 4 26 Maggio 1979

Infinite Diffusion Velocity and Causality in Special Relativity.

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(ricevuto il 23 Febbraio 1979)

In relativistic kinetic theory (see, *e.g.*, ref. (1-3) for general reviews) the various transport properties of the relativistic gas are generally dealt with a covariant extension of the well-known Chapman-Enskog method. However, one has to face the following problem: the resulting heat conduction equation is *parabolic* and hence heat propagates with an infinite velocity whereas one would expect at most a propagation with the speed of light. Incidentally, this is a manifestation of the fact that while « causality implies the Lorentz group », the converse statement (4) is not true. It should also be emphasized that this problem is not specific of relativity: the usual heat conduction equation (or diffuse equation) is also parabolic and this feature is known since a long time (the wayout to the difficulty of infinite diffusion velocity is now well understood (5)). What is however specific of relativity is the question of *causality* and one would like—at least in behalf of principles—to get a hyperbolic heat conduction equation *or* possibly reconcile its parabolic nature with causality.

Several attempts to solve this problem have been performed along various lines (3.6-8) and the most recent one consists in the addition of a more or less *ad hoc* term in the definition of the heat flux (7); this new term being next justified by a suitable modification (8) of the Chapman-Enskog method needed to solve the kinetic equation under consideration. On the other hand, one can argue (9) that the approximations performed (within relativistic kinetic theory) in order to get the heat conduction equation, are no longer valid much before a regime where velocities are close to the speed of light is reached: in this optic, it is not surprising that a pathology arises (*i.e.* the lack of causality

(1) J. EHLERS: *Rendiconti S.I.F.*, Course LXVII (New York, N. Y., 1972).
(2) W. ISRAEL: *The relativistic Boltzmann equation*, in *General Relativity: Papers in Honour of J. L. Synge*, edited by L. O'RAIFEARTAIGH (Oxford, 1972).
(3) J. M. STEWART: *Nonequilibrium Relativistic Kinematic Theory* (Berlin, 1971).
(4) E. C. G. ZEEMAN: *Journ. Math. Phys.*, **5**, 490 (1964).
(5) See, *e.g.*, R. M. MAZO: *Brownian motion*, in *Stochastic Processes in Nonequilibrium Statistical Mechanics (1978 Sutges International School on Statistical Mechanics*; Berlin), to appear.
(6) M. KRANYŠ: *Nuovo Cimento*, **42** B, 51 (1966).
(7) W. ISRAEL: *Ann. Phys.*, **100**, 310 (1976).
(8) W. ISRAEL and J. M. STEWART: *Phys. Lett.*, **58** A, 213 (1976).
(9) J. L. ANDERSON: private discussion.

in the results) when one is outside of the domain of validity of the theory. However, although correct this argument does not settle the problem of principle.

In this note we would like to present a very simple model that sheds a different light on this question. This model exhibits the interesting feature of being *causal* and nevertheless possessing an *infinite diffusion velocity*. This model is a covariant generalization of the usual random walk ([10]) and, at this stage, we would like to emphasize strongly that—unlike the conventional random walk problem, which may be considered as a model for Brownian motion—it is not intended to represent any actual problem whatsoever: its virtue lies merely in its heuristic value.

For the sake of simplicity a two-dimensional space-time is considered in the following (only one spatial dimension). From an origin 0 the future light cone is drawn and, in its interior, a curvilinear lattice is constructed (see fig. 1), determined from the curves

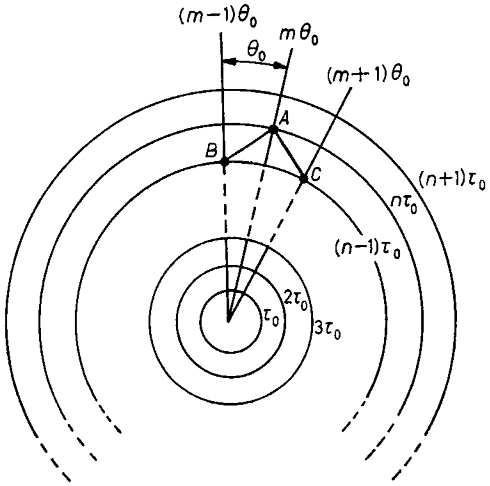

Fig. 1. – This Euclidean representation of the space-time grid emphasizes its invariant under discrete Lorentz transformations. Point A (co-ordinates n, m) can be reached only from point B $(n-1, m-1)$ or from point C $(n-1, m+1)$.

$$x^\mu \cdot x_\mu \equiv \tau^2 = n^2 \tau_0 \qquad (n = 1, 2, \ldots),$$

$$x^0 = m\theta_0 x^1 \qquad (m = 1, 2, \ldots)$$

(with $\mu = 0, 1$). τ_0 and θ_0 are arbitrarily small (but not zero) quantities. Rather than the Cartesian co-ordinates (x^0, x^1) we shall use the «spherical» ones (τ, θ) defined by

$$x_0 = \tau \operatorname{ch} \theta ,$$

$$x^1 = \tau \operatorname{sh} \theta ,$$

([10]) See, *e.g.*, N. WAX: *Selected Papers on Noise and Stochastic Processes* (New York, N. Y., 1954).

so that our grid is characterized by the set of integers (n, m), in such a way that $\tau = n\tau_0$, $\theta = m\theta_0$. Notice that this grid is Lorentz invariant in the sense that it is invariant under *discrete* Lorentz transformations $\theta \to n\theta$.

As usual we consider a particle that is at $\theta = 0$ at « time » τ_0; furthermore at each step (*i.e.* at each time interval τ_0) it must jump either to the right or to the left, of one unit, with equal *a priori* probabilities. It follows that the probability $P\{n\tau_0, m\theta_0\}$ for a particle to reach the point $m\theta_0$ at « time » $n\tau_0$ is given by

$$P\{n\tau_0, m\theta_0\} = \tfrac{1}{2}\{P\{(n-1)\tau_0, (m+1)\theta_0\} + P\{(n-1)\tau_0, (m-1)\theta_0\}\}, \tag{1}$$

with the initial condition $P\{\tau_0, 0\} = 1$. As usual [10], by substracting the quantity $P\{(n-1)\tau_0, m\theta_0\}$ to both sides of this last equation, dividing by τ_0 and going over to the limit $\tau_0 \to 0$, $\theta_0 \to 0$, one is led to the following diffusion equation:

$$(\partial/\partial\tau) f(\tau, \theta) = D(\partial^2/\partial\theta^2) f(\tau, \theta), \tag{2}$$

where we have set $\lim \theta_0^2/2\tau = D$. To eq. (2) for $f(\tau, \theta)$, the probability *density* for a particle to be found at point θ and at time τ, one must add the initial condition $f(0, 0) = \delta(\theta) \times \delta(\tau)$. $f(\tau, \theta)$ is normalized through

$$\int_{-\infty}^{+\infty} d\theta\, f(\tau, \theta) = 1. \tag{3}$$

The diffusion equation (2) is obviously Lorentz-invariant since i) τ is itself an invariant and ii) the operator $\partial/\partial\theta$ is invariant under the change $\theta \to \theta + \xi$, representing itself a pure Lorentz transformation.

The solution of eq. (2), obeying our initial condition, has the customary form

$$f(\tau, \theta) = 1/(2\pi D\tau)^{\frac{1}{2}} \exp[-\theta^2/2D\tau], \tag{4}$$

so that, at an arbitrarily small time τ, there is a nonvanishing probability for the random particle to be found at an arbitrarily large distance θ: the diffusion velocity is still infinite in this however covariant model.

Nevertheless, although the diffusion velocity is infinite, the random particle always lies *within* the light cone (this property is embodied in the condition $\tau \geqslant 0$) and hence causality is preserved. If we now come back to Cartesian co-ordinates and take an hyperplane $t = \text{const}$ (or any other spacelike three-surface; see fig. 2), this shows that the particle has travelled only on a distance less than ct.

Let us now draw a few conclusions from the above considerations.

First, one clearly sees that the adopted notion of instantaneity (*i.e.* the « slicing » of space-time by a family of spacelike three-surfaces) does play an important role in the problem: the model is causal as long as one uses a slicing by spacelike three-surfaces that cut the future null cone; it loses this property when considering *e.g.* hyperboloids inside this null cone.

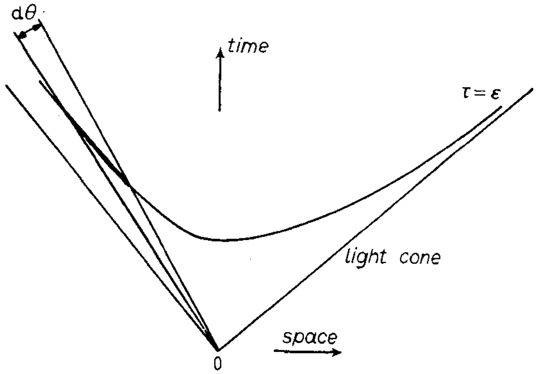

Fig. 2. – The random particle can be found at an arbitrarily large distance within a small interval $d\theta$ with a nonvanishing probability and at an arbitrarily small « time » $\tau = \varepsilon$ leading thereby to an infinite diffusion « velocity » while causality is preserved since the particle is still within the null cone: its actual velocity is less than that of light.

Next comes the most important point. If the jumps of the random particle are to be considered as due to collisions (with a subjacent medium of molecules, the collision time being τ_0 and the mean free path $c\theta_0\tau_0$), the limiting process $(\theta_0, \tau_0) \to 0$ amounts to a *change in the scale of length and time*: in a loose sense, time intervals (respectively distances) of the order of τ_0 (respectively of the order of $c\theta_0\tau_0$) are considered as being vanishingly small; one takes a long-time limit (respectively a long-distance limit). This feature is, of course, not new and appears not only in the usual random walk approach ([10]) to Brownian motion but also in all modern views of statistical mechanics where changes of scales are a part of the art. Although not new, this property has not been fully considered in relativistic kinetic theory and, in particular, in approximation methods.

Let us briefly specify this last point more precisely in the case of the relativistic Chapman-Enskog expansion. This latter represents an expansion in the small parameter

$$\varepsilon \equiv \frac{l}{L} = \frac{v_{\text{th}}\tau}{L}, \tag{5}$$

where L is a typical hydrodynamical distance; l is the mean free path; v_{th} is the thermal velocity (whatever its definition in special relativity ([11])) and τ the collision time. As emphasized in any text book on kinetic theory, this also amounts to using new time and space variables, say $(\boldsymbol{t}, \boldsymbol{x})$, connected to the short scale ones (t, x) through

$$t = \tau \boldsymbol{t}, \qquad x = v_{\text{th}}\tau \boldsymbol{x}.$$

Such a transformation *does not* preserve the light cone $c^2t^2 - x^2 = 0$. With these new variables, rather it becomes $\{c^2\boldsymbol{t}^2 - v_{\text{th}}^2\boldsymbol{x}^2\} = 0$, *i.e.*, *it gets flat*, allowing thereby motions with apparent speeds greater than the velocity of light. This is certainly the reason why, eventhough covariant, the Chapman-Enskog method used in relativistic kinetic theory is not causal.

([11]) R. Hakim and A. Mangeney: *Phys. Fluids*, **14**, 2751 (1971) (appendix).

A possible wayout to this difficulty can be found in the following remark. While for an ordinary Boltzmann gas there exists essentially one typical velocity, *i.e.* the thermal velocity, the relativistic gas involves one more velocity, *i.e.* the speed of light. It follows that now we have *two* expansion parameters (*i.e.*, ε and $\eta \equiv c\tau/L$) at our disposal and, only a correct interplay of the two parameters can guarantee that causality is maintained in the approximation scheme. It is also worth noticing that causality is preserved in two cases: i) when the light cone is preserved in the approximation method used and ii) when it closes as a timelike straightline, as is the case in the above model (this can be seen by making x and t tending to zero in the model given above; one then easily sees that $x/t \to 0$).

We shall elaborate further on these ideas in a forthcoming paper.

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We are indebted to Drs. E. Alvarez, D. Gerbal and A. Mangeney for discussions.