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GORENSTEIN DIFFERENTIAL GRADED ALGEBRAS

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ABSTRACT

We propose a definition of Gorenstein Differential Graded Algebra. In order to give examples, we introduce the technical notion of Gorenstein morphism. This enables us to prove the following:

THEOREM: *Let A be a noetherian local commutative ring, let L be a bounded complex of finitely generated projective A-modules which is not homotopy equivalent to zero, and let $\mathcal{E} = \mathrm{Hom}_A(L, L)$ be the endomorphism Differential Graded Algebra of L. Then $\mathcal{E}$ is a Gorenstein Differential Graded Algebra if and only if A is a Gorenstein ring.*

THEOREM: *Let A be a noetherian local commutative ring with a sequence of elements $\mathbf{a} = (a_1, \ldots, a_n)$ in the maximal ideal, and let $\mathrm{K}(\mathbf{a})$ be the Koszul complex of $\mathbf{a}$. Then $\mathrm{K}(\mathbf{a})$ is a Gorenstein Differential Graded Algebra if and only if A is a Gorenstein ring.*

THEOREM: *Let A be a noetherian local commutative ring containing a field k, and let X be a simply connected topological space with $\dim_k \mathrm{H}_*(X;k) < \infty$, which has Poincaré duality over k. Let $\mathrm{C}^*(X;A)$ be the singular cochain Differential Graded Algebra of X with coefficients in A. Then $\mathrm{C}^*(X;A)$ is a Gorenstein Differential Graded Algebra if and only if A is a Gorenstein ring.*

The second of these theorems is a generalization of a result by Avramov and Golod from [4].

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0. Introduction

Some parts of the homological theory of Differential Graded Algebras can be viewed as a generalization of the homological theory of rings. One of the central notions of this last theory is that of Gorenstein rings. Hence it is natural to seek to define Gorenstein Differential Graded Algebras.

We propose such a definition, and give criteria for when some naturally occurring Differential Graded Algebras (abbreviated DGAs henceforth) are Gorenstein in our sense.

(0.1) BACKGROUND. The ring theoretical idea lying behind our definition of Gorenstein DGAs is the following: If A is a noetherian local commutative ring, then A is a Gorenstein ring precisely if the functor $\mathrm{RHom}_A(-,A)$ gives a duality, that is, a pair of quasi-inverse contravariant equivalences of categories,

$$\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A) \underset{\mathrm{RHom}_A(-,A)}{\overset{\mathrm{RHom}_A(-,A)}{\rightleftarrows}} \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A),$$

where $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$ is the derived category of bounded complexes of finitely generated A-modules, see [6, thm. (2.3.14)]. For this to happen is equivalent to the following two conditions:

- There is a natural isomorphism

$$M \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(M,A),A)$$

  for M in $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$.
- $\mathrm{RHom}_A(-,A)$ sends $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$ to $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$.

(0.2) GORENSTEIN DGAS. In section 2 these conditions are what we shall use as direct inspiration for our definition of Gorenstein DGAs, given as definition (2.1) below. The definition has two parts, [G1] and [G2], which generalize the two above conditions directly, using derived categories of Differential Graded modules (abbreviated DG-modules henceforth).

To show right away that our definition of Gorenstein DGAs is reasonable, section 2 continues by considering some ordinary rings as DGAs concentrated in degree zero, showing that they are Gorenstein DGAs precisely when they are Gorenstein rings in the approprate classical sense (propositions (2.5) and (2.6)).

(0.3) GORENSTEIN MORPHISMS OF DGAS. In section 3 we introduce in definition (3.4) the key technical tool of Gorenstein morphisms of DGAs. These morphisms are modeled on Gorenstein homomorphisms from ring theory (see [2], [3], and [13]).

The purpose of considering Gorenstein morphisms is to get a practical tool which will enable us to determine in section 4 when some DGAs occurring in nature are Gorenstein DGAs.

The two main results on Gorenstein morphisms are:

- Theorem (3.6) (Ascent): *Let $R \longrightarrow S$ be a Gorenstein morphism of DGAs. If R is a Gorenstein DGA, then S is also a Gorenstein DGA.*
- Proposition (3.10) (Partial Descent): *Let Q be a local commutative DGA with residue class field k, and let $Q \longrightarrow T$ be a (nice) Gorenstein morphism of DGAs. If T is a Gorenstein DGA, then Q satisfies the Gorenstein condition* $\dim_k \operatorname{Ext}_Q(k, Q) = 1$ *from* [2, sec. 3].

We conjecture in (3.7) that "Descent" holds in full generality, that is, if $R \longrightarrow S$ is a Gorenstein morphism of DGAs and S is a Gorenstein DGA, then R is a Gorenstein DGA, but are unable to prove this.

(0.4) Examples. In section 4 we determine when three naturally occurring types of DGAs, namely endomorphism DGAs of perfect complexes of modules, Koszul complexes, and singular cochain DGAs of topological spaces with Poincaré duality, are Gorenstein.

Let A be a noetherian local commutative ring, let L be a bounded complex of finitely generated projective modules (i.e., L is a so-called perfect complex) which is not homotopy equivalent to zero, and let $\mathcal{E} = \operatorname{Hom}_A(L, L)$ be the endomorphism DGA of L; see setup (4.1). We show the following "Ascent-Descent theorem",

- Theorem (4.5): *A is a Gorenstein ring $\Leftrightarrow$ $\mathcal{E}$ is a Gorenstein DGA.*

Also, let $\mathbf{a} = (a_1, \ldots, a_n)$ be a sequence of elements in the maximal ideal of A, and consider the corresponding Koszul complex $\mathrm{K}(\mathbf{a})$ which is a DGA; see setup (4.6). We show

- Theorem (4.9): *A is a Gorenstein ring $\Leftrightarrow$ $\mathrm{K}(\mathbf{a})$ is a Gorenstein DGA.*

Finally, suppose that A contains a field k. Let X be a simply connected topological space with $\dim_k \mathrm{H}_*(X; k) < \infty$, which has Poincaré duality over k, meaning that there is an isomorphism of graded $\mathrm{H}^*(X; k)$-modules

$$\mathrm{H}^*(X; k)' \cong \Sigma^d \mathrm{H}^*(X; k)$$

for some d, where the prime denotes dualization with respect to k; see setup (4.11). Consider $\mathrm{C}^*(X; A)$, the singular cochain DGA of X with coefficients in A; see paragraph (4.13). We show

- Theorem (4.16): *A is a Gorenstein ring $\Leftrightarrow$ $\mathrm{C}^*(X; A)$ is a Gorenstein DGA.*

Theorem (4.9) is a generalization of a result by Avramov and Golod from [4], which is confined to the case where $\mathbf{a}$ is a minimal set of generators for the maximal ideal of A.

(0.5) PERSPECTIVES. In the literature, there are several papers which consider Gorenstein conditions for *augmented* DGAs. In [9], Félix, Halperin, and Thomas consider augmented cochain DGAs; in [2], Avramov and Foxby consider augmented chain DGAs; and in the recent [8], Dwyer, Greenlees, and Iyengar consider more general augmented DGAs.

In a subsequent paper with Iyengar [11], we will show for most of the DGAs in question that the Gorenstein conditions from [2] and [9] coincide with our notion of Gorenstein DGA.

Note, however, that our setup differs from that of [2], [8], and [9], in that we do not use augmentations or other auxiliary data to define Gorenstein DGAs.

Also, we make it a point not to work only with chain or cochain DGAs, but rather to give a definition of Gorenstein DGA which is left/right symmetric.

Indeed, we shall see in section 4 that our theory can be applied to endomorphism DGAs which in general have no canonical augmentation, nor satisfy being either chain or cochain DGAs.

(0.6) ACKNOWLEDGEMENT. This paper owes a great debt to [21] which was the first paper to introduce dualizing complexes in a non-commutative situation, and hence the first paper that had to deal with such ensuing complications as left-, right-, and bi-structures of modules and functors.

Another paper we should mention is [12] in which duality over DGAs is employed to prove an existence result for dualizing complexes over rings.

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The diagrams were typeset with Paul Taylor's `diagrams.tex`.

1. Notation and terminology

The purpose of this section is to fix the notation and terminology we shall use. For more details, see [14] or [16, part III].

(1.1) DGAs. A Differential Graded Algebra (DGA) R over the commutative ground ring $\Bbbk$ is a graded algebra $\{R_i\}_{i\in\mathbb{Z}}$ over $\Bbbk$ which is equipped with a differential, that is, a $\Bbbk$-linear map $\partial^R\colon R_* \longrightarrow R_{*-1}$ with square zero, satisfying the Leibniz rule

$$\partial^R(rs) = \partial^R(r)s + (-1)^{|r|}r\partial^R(s)$$

when r is a graded element of degree $|r|$. Note that we almost exclusively employ homological notation, that is, lower indices and differentials of degree -1, and that we observe the Koszul sign convention of introducing a sign $(-1)^{mn}$ when graded objects of degrees m and n are interchanged.

The opposite DGA of R is denoted R^{opp} and is the same as R except that the product is changed to

$$r \overset{\mathrm{opp}}{\cdot} s = (-1)^{|r||s|} sr.$$

A morphism of DGAs over $\Bbbk$ is a morphism of graded algebras over $\Bbbk$ which is compatible with the differentials.

(1.2) NOTATION. In the rest of this section, R and S denote DGAs over the commutative ground ring $\Bbbk$.

Both in this section and in the rest of the paper, we will often suppress the ground ring $\Bbbk$ from the formulation of the results. If no canonical ground ring is present, then one can simply use $\Bbbk = \mathbb{Z}$.

(1.3) DG-MODULES. A Differential Graded R-left-module (DG-R-left-module) M is a graded left-module $\{M_i\}_{i\in\mathbb{Z}}$ over R (viewed as a graded algebra), which is equipped with a differential, that is, a $\Bbbk$-linear map $\partial^M\colon M_* \longrightarrow M_{*-1}$ with square zero, satisfying the Leibnitz rule

$$\partial^M(rm) = \partial^R(r)m + (-1)^{|r|} r\partial^M(m)$$

when r is a graded element of R of degree $|r|$.

DG-R-right-modules are defined similarly. Often we identify DG-R-right-modules with DG-R^{opp}-left-modules.

Note that we can also consider DG-modules having more than one DG-module structure, for instance DG-R-left-R-right-modules which would typically be denoted by ${}_RM_R$, or DG-R-left-S-right-modules which would typically be denoted by ${}_RN_S$. In such cases, all the different structures are required to be compatible; for a DG-R-left-R-right-module, compatibility means that the rule $(r_1m)r_2 = r_1(mr_2)$ holds. An example of a DG-R-left-R-right-module is R itself.

For a DG-module M we define the i'th suspension by

$$(\Sigma^i M)_j = M_{j-i}, \quad \partial_j^{\Sigma^i M} = (-1)^i \partial_{j-i}^M.$$

For each type of DG-modules (for instance, DG-R-left-modules or DG-R-left-R-right-modules), there is a notion of morphism. A morphism is a homomorphism of graded modules which is compatible with the differentials. Accordingly, each type of DG-modules forms an abelian category.

(1.4) HOMOLOGY. A DG-module M is in particular a complex, so has homology which we denote $\mathrm{H}(M)$ or $\mathrm{H}M$.

The product in R induces a product in $\mathrm{H}R$ which becomes a graded algebra, and the action of R on a DG-module M induces an action of $\mathrm{H}R$ on $\mathrm{H}M$ which becomes a graded $\mathrm{H}R$-module.

(1.5) QUASI-ISOMORPHISMS AND DERIVED CATEGORIES. If a morphism of DG-modules $M \longrightarrow N$ induces an isomorphism in homology $\mathrm{H}M \xrightarrow{\cong} \mathrm{H}N$, then the morphism is called a quasi-isomorphism, and is denoted $M \xrightarrow{\simeq} N$.

If we take one of the abelian categories of DG-modules introduced above and (formally) invert the quasi-isomorphisms, then we get the corresponding derived category of DG-modules which is a triangulated category. The derived category of DG-R-left-modules is denoted $\mathsf{D}(R)$.

Observe that when we identify DG-R-right-modules with DG-R^{opp}-left-modules, then we also identify the derived category of DG-R-right-modules with $\mathsf{D}(R^{\mathrm{opp}})$.

If $R \longrightarrow S$ is a morphism of DGAs over $\Bbbk$ which is moreover a quasi-isomorphism, then the derived categories $\mathsf{D}(R)$ and $\mathsf{D}(S)$ are equivalent as triangulated categories; see [16, III.4.2]. This obviously extends: If R and S are connected by a sequence of morphisms of DGAs over $\Bbbk$ all of which are quasi-isomorphisms,

$$R \xrightarrow{\simeq} T_1 \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} T_n \xleftarrow{\simeq} S,$$

then $\mathsf{D}(R)$ and $\mathsf{D}(S)$ are equivalent as triangulated categories. In this situation, R and S are called equivalent by a series of quasi-isomorphisms, or just equivalent, and are indistinguishable for homological purposes. In particular, R is a Gorenstein DGA in the sense of this paper if and only if S is a Gorenstein DGA, cf. paragraph (2.2).

(1.6) HOM AND TENSOR. On the abelian categories of DG-modules, we can define the functors Hom and $\otimes$:

If M and N are DG-R-left-modules, then $\mathrm{Hom}_R(M,N)$ is defined in a classical way, as the total complex of a certain double complex. The totaling is done taking products along diagonals.

Similarly, if A is a DG-R-right-module and B is a DG-R-left-module, then $A \otimes_R B$ is defined as the total complex of a certain double complex. The totaling is done taking coproducts along diagonals.

Note that extra structures on M, N, A, and B are inherited by Hom_R and $\otimes_R$. For instance, if ${}_RM$ is a DG-R-left-module and ${}_RN_S$ is a DG-R-left-S-right-module, then $\mathrm{Hom}_R({}_RM, {}_RN_S)$ is a DG-S-right-module.

(1.7) DERIVED HOM AND TENSOR. On the derived categories of DG-modules, we can define the functors right-derived Hom, denoted RHom, and left-derived $\otimes$, denoted $\overset{\mathrm{L}}{\otimes}$. The way to do this is to use appropriate resolutions:

Let P, I, and F be DG-R-left-modules. Then P is called K-projective, I is called K-injective, and F is called K-flat if the functors $\mathrm{Hom}_R(P,-)$, $\mathrm{Hom}_R(-,I)$, and $-\otimes_R F$ send quasi-isomorphisms to quasi-isomorphisms. By adjointness, a K-projective DG-module is also K-flat.

Now let M, N, and B be DG-R-left-modules. Then K-projective, K-injective, and K-flat resolutions of M, N, and B are quasi-isomorphisms of DG-R-left-modules $P \xrightarrow{\simeq} M$, $N \xrightarrow{\simeq} I$, and $F \xrightarrow{\simeq} B$ so that P is K-projective, I is K-injective, and F is K-flat. Such resolutions always exist; see [14] or [16, part III]. The original construction of such "unbounded" resolutions is due to [5] and [19].

With the resolutions, we can define $\mathrm{RHom}_R(M,N)$ as $\mathrm{Hom}_R(P,N)$ or $\mathrm{Hom}_R(M,I)$, and when A is a DG-R-right-module, we can define $A \overset{\mathrm{L}}{\otimes}_R B$ as $A \otimes_R F$. We could also define $A \overset{\mathrm{L}}{\otimes}_R B$ as $G \otimes_R B$, where $G \xrightarrow{\simeq} A$ is a K-flat resolution of A.

These definitions turn out to give well-defined functors on derived categories of DG-R-left- and DG-R-right-modules.

Extra structures on M, N, A, and B are inherited by RHom_R and $\overset{\mathrm{L}}{\otimes}_R$, but complications may arise: For instance, while it is always true that $\mathrm{RHom}_R({}_RM, {}_RN_S)$ is in the derived category of DG-S-right-modules, if we want to compute it as $\mathrm{Hom}_R(M,I)$ then we need a quasi-isomorphism ${}_RN_S \xrightarrow{\simeq} {}_RI_S$ of DG-R-left-S-right-modules so that ${}_RI$ is a K-injective DG-R-left-module. The existence of a resolution such as I is not guaranteed by [14] and [16, part III] (but see the next paragraph).

(1.8) EXISTENCE OF RESOLUTIONS. As we said in the previous paragraph, in case of DG-modules with two or more structures, existence of resolutions is a potential problem. We will comment on one important instance: Existence of a resolution of ${}_RR_R$ which is K-injective from the left and from the right, for the purpose of defining the functors $\mathrm{RHom}_R(-, {}_RR_R)$ and $\mathrm{RHom}_{R^{\mathrm{opp}}}(-, {}_RR_R)$ which play a large role in this paper.

Now, we can always define the derived functors $\mathrm{RHom}_R(-, {}_RR_R)$ and $\mathrm{RHom}_{R^{\mathrm{opp}}}(-, {}_RR_R)$, for we can simply use K-projective resolutions of the DG-R-left- and DG-R-right-modules in the first variables. However, it is valuable for computations (e.g., with biduality morphisms) also to be able to use a resolution in the second variable. To be precise, what we want is a quasi-isomorphism ${}_R R_R \xrightarrow{\simeq} {}_R I_R$ where ${}_R I$ and I_R are K-injective. This will give $\mathrm{RHom}_R(-, {}_R R_R) \simeq \mathrm{Hom}_R(-, {}_R I_R)$ and $\mathrm{RHom}_{R^{\mathrm{opp}}}(-, {}_R R_R) \simeq \mathrm{Hom}_{R^{\mathrm{opp}}}(-, {}_R I_R)$.

It is not clear how to get such an I, except in one case: If R itself is K-flat over the ground ring $\Bbbk$. In this case, we take the DG-R-left-R-right-module ${}_R R_R$ and view it as a DG-left-module over $R \otimes_{\Bbbk} R^{\mathrm{opp}}$, the "enveloping" DGA. It then has a K-injective resolution ${}_{R\otimes_{\Bbbk} R^{\mathrm{opp}}} R \xrightarrow{\simeq} {}_{R\otimes_{\Bbbk} R^{\mathrm{opp}}} I$. We can view this as a quasi-isomorphism of DG-R-left-R-right-modules ${}_R R_R \xrightarrow{\simeq} {}_R I_R$, and here ${}_R I$ and I_R turn out to be K-injective. For ${}_R I$, this follows from the computation

$$\mathrm{Hom}_R(-, {}_R I) \simeq \mathrm{Hom}_{R\otimes_{\Bbbk} R^{\mathrm{opp}}}(- \otimes_{\Bbbk} R, {}_{R\otimes_{\Bbbk} R^{\mathrm{opp}}} I),$$

which shows that $\mathrm{Hom}_R(-, {}_R I)$ is the composition of the functors $- \otimes_{\Bbbk} R$ and $\mathrm{Hom}_{R\otimes_{\Bbbk} R^{\mathrm{opp}}}(-, {}_{R\otimes_{\Bbbk} R^{\mathrm{opp}}} I)$, both of which send quasi-isomorphisms to quasi-isomorphisms, the first because R is K-flat over $\Bbbk$, the second because ${}_{R\otimes_{\Bbbk} R^{\mathrm{opp}}} I$ is K-injective over $R \otimes_{\Bbbk} R^{\mathrm{opp}}$.

In general, R is not K-flat over $\Bbbk$. However, by [15, lem. 3.2(a)] there always exists a morphism $\widetilde{R} \longrightarrow R$ of DGAs over $\Bbbk$ which is a quasi-isomorphism, so that $\widetilde{R}$ *is* K-flat over $\Bbbk$.

In other words, if we are willing to replace our DGA with a quasi-isomorphic DGA, then we can always assume that there is a resolution ${}_R R_R \xrightarrow{\simeq} {}_R I_R$ so that ${}_R I$ and I_R are K-injective. Let us remind the reader from paragraph (1.5) that quasi-isomorphic DGAs are indistinguishable for homological purposes.

(1.9) Definition (The category fin). Suppose that $\mathrm{H}_0 R$ is a noetherian ring. Then by $\mathsf{fin}(R)$ we denote the full subcategory of the derived category $\mathsf{D}(R)$ which consists of DG-modules M so that $\mathrm{H}M$ is bounded, and so that each $\mathrm{H}_i M$ is finitely generated as an $\mathrm{H}_0 R$-module.

(1.10) The centre. A graded element c in a graded algebra H (which could be a DGA) is called central if it satisfies $cd = (-1)^{|c||d|} dc$ for all graded elements d. An arbitrary element in H is called central if all its graded components are central. The centre of H is the set of all central elements, and H is called commutative if all its elements are central.

(1.11) DG-modules in the ring case. Note that an ordinary ring A can be viewed as a DGA concentrated in degree zero. A DG-module over A (when A is viewed as a DGA) is then the same thing as a complex of modules over A (when A is viewed as a ring); the various derived categories of DG-A-modules are the same as the various ordinary derived categories over A; and the derived functors of Hom and $\otimes$ of DG-A-modules are the ordinary RHom_A and $\overset{\mathrm{L}}{\otimes}_A$. When A is noetherian, the category $\mathsf{fin}(A)$ equals $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$, the derived category of complexes with bounded, finitely generated homology.

2. Gorenstein DGAs

This section defines our notion of Gorenstein DGA, and shows that it behaves sensibly when specialized to some important types of ordinary rings.

(2.1) DEFINITION (GORENSTEIN DGAS). Let R be a DGA for which H_0R is a noetherian ring. We call R a **Gorenstein DGA** if it satisfies:

[G1] For M in $\mathsf{fin}(R)$ and N in $\mathsf{fin}(R^{\mathrm{opp}})$ the following biduality morphisms are isomorphisms,

$$M \longrightarrow \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M, {}_RR_R), {}_RR_R),$$
$$N \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_{R^{\mathrm{opp}}}(N, {}_RR_R), {}_RR_R).$$

[G2] The functor $\mathrm{RHom}_R(-, {}_RR_R)$ maps $\mathsf{fin}(R)$ to $\mathsf{fin}(R^{\mathrm{opp}})$, and the functor $\mathrm{RHom}_{R^{\mathrm{opp}}}(-, {}_RR_R)$ maps $\mathsf{fin}(R^{\mathrm{opp}})$ to $\mathsf{fin}(R)$.

(2.2) INVARIANCE UNDER QUASI-ISOMORPHISM. Note that conditions [G1] and [G2] only concern functors on derived categories. So if two DGAs are equivalent, then they are Gorenstein simultaneously.

(2.3) REALIZING THE BIDUALITY MORPHISMS. From paragraph (1.8) we know that after replacing R by an equivalent DGA, we can assume that there exists a resolution ${}_RR_R \xrightarrow{\simeq} {}_RI_R$ so that ${}_RI$ and I_R are K-injective. Hence the biduality morphisms from condition [G1] can be realized as concrete biduality morphisms

$$M \longrightarrow \mathrm{Hom}_{R^{\mathrm{opp}}}(\mathrm{Hom}_R(M, {}_RI_R), {}_RI_R),$$
$$N \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_{R^{\mathrm{opp}}}(N, {}_RI_R), {}_RI_R).$$

(2.4) DUALITY. It is clear that if R is a Gorenstein DGA, then there is a duality, that is, a pair of quasi-inverse contravariant equivalences of categories,

$$\mathsf{fin}(R) \underset{\mathrm{RHom}_{R^{\mathrm{opp}}}(-,R)}{\overset{\mathrm{RHom}_R(-,R)}{\rightleftarrows}} \mathsf{fin}(R^{\mathrm{opp}}).$$

(2.5) PROPOSITION (COMMUTATIVE RINGS). *Let A be a noetherian commutative ring of finite Krull dimension. Then the following conditions are equivalent:*

(1) *When A is viewed as a DGA concentrated in degree zero, it is a Gorenstein DGA.*

(2) *The injective dimension* $\mathrm{id}_A(A)$ *is finite.*

(3) *For each prime ideal* $\mathfrak{p}$ *in* A, *the localization* $A_{\mathfrak{p}}$ *is a noetherian local commutative Gorenstein ring.*

Proof: It is well-known that (2) and (3) are equivalent.

(1) $\Rightarrow$ (3). We show that condition [G2] implies (3): Let $\mathfrak{p}$ be a prime ideal in A. It is clear that $A/\mathfrak{p}$ is in $\mathsf{fin}(A)$, so $\mathrm{RHom}_A(A/\mathfrak{p}, A)$ is in $\mathsf{fin}(A^{\mathrm{opp}}) = \mathsf{fin}(A)$ by condition [G2]. Localizing in $\mathfrak{p}$, we have that $\mathrm{RHom}_A(A/\mathfrak{p}, A)_{\mathfrak{p}}$ is in $\mathsf{fin}(A_{\mathfrak{p}})$. However, [6, lem. (A.4.5)] gives the first $\cong$ in

$$\begin{aligned}\mathrm{RHom}_A(A/\mathfrak{p}, A)_{\mathfrak{p}} &\cong \mathrm{RHom}_{A_{\mathfrak{p}}}((A/\mathfrak{p})_{\mathfrak{p}}, A_{\mathfrak{p}})\\ &\cong \mathrm{RHom}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}, A_{\mathfrak{p}}),\end{aligned}$$

so we have

$$\mathrm{RHom}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}, A_{\mathfrak{p}}) \in \mathsf{fin}(A_{\mathfrak{p}}).$$

In particular, $\mathrm{RHom}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}, A_{\mathfrak{p}})$ has bounded homology. As $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ is the residue class field of the noetherian local commutative ring $A_{\mathfrak{p}}$, this proves that $A_{\mathfrak{p}}$ is Gorenstein by [17, thm. 18.2].

(2) $\Rightarrow$ (1). We must see that conditions [G1] and [G2] hold. Note that since A is commutative, the two halves of condition [G1] are equivalent, and the two halves of condition [G2] are equivalent.

[G1]. Since we have $\mathrm{id}_A(A) < \infty$, there exists an injective resolution $A \xrightarrow{\simeq} I$ so that I is bounded. Also, any M in $\mathsf{fin}(A)$ has homology which is bounded to the right and consists of finitely generated A-modules. This shows that the biduality morphisms in condition [G1] are isomorphisms by [6, (A.4.24)].

[G2]. Given M in $\mathsf{fin}(A)$ we have $\mathrm{RHom}_A(M, A) \cong \mathrm{Hom}_A(P, A) \cong \mathrm{Hom}_A(M, I)$, where I is the resolution from above and $P \xrightarrow{\simeq} M$ is a projective resolution which can be chosen to consist of finitely generated projective A-modules. Now $\mathrm{Hom}_A(M, I)$ has bounded homology (because M has bounded homology while I is bounded), and $\mathrm{Hom}_A(P, A)$ has finitely generated homology modules (because P consists of finitely generated projective modules). Hence we have $\mathrm{RHom}_A(M, A) \in \mathsf{fin}(A)$, so [G2] holds. ∎

(2.6) PROPOSITION (NON-COMMUTATIVE LOCAL RINGS). *Let* k *be a field, and* A *a* k*-algebra which is noetherian semilocal PI (see* [20]*). Then the following conditions are equivalent:*

1. *When* A *is viewed as a DGA concentrated in degree zero, it is a Gorenstein DGA.*
2. *The injective dimensions* $\mathrm{id}_A(A)$ *and* $\mathrm{id}_{A^{\mathrm{opp}}}(A)$ *are finite.*

Proof: (1) $\Rightarrow$ (2). We show that condition [G2] implies (2): Let $\mathrm{J}(A)$ be the Jacobson radical of A, and let A_0 be $A/\mathrm{J}(A)$. Clearly, A_0 can be viewed either as a DG-A-left-module and as such it is in $\mathsf{fin}(A)$, or as a DG-A-right-module and as such it is in $\mathsf{fin}(A^{\mathrm{opp}})$. So condition [G2] implies that $\mathrm{RHom}_A(A_0, A)$ and $\mathrm{RHom}_{A^{\mathrm{opp}}}(A_0, A)$ have bounded homology. By [20, prop. 5.7(1)] this says $\mathrm{id}_A(A) < \infty$ and $\mathrm{id}_{A^{\mathrm{opp}}}(A) < \infty$.

(2) $\Rightarrow$ (1). This is completely analogous to the proof of (2) $\Rightarrow$ (1) in proposition (2.5). ∎

We suspect that proposition (2.6) is far from optimal. In fact, we propose the following conjecture:

(2.7) Conjecture (Non-commutative rings). The conditions in proposition (2.6) are equivalent for all noetherian rings of finite left- and right-Krull dimension.

Proposition (2.5) makes the conjecture seem reasonable. See [18, chap. 6] for the definition of Krull dimension over non-commutative rings.

3. Gorenstein morphisms of DGAs

This section defines what we call Gorenstein morphisms of DGAs, in order to make us able to give examples of Gorenstein DGAs. We show that these morphisms are capable of transporting Gorenstein properties back and forth between source and target, in a way analogous to ring theory.

The following is a generalization of finite ring homomorphisms of finite flat dimension:

(3.1) Definition (Finite morphisms). Let R and S be DGAs for which H_0R and H_0S are noetherian rings, and let $R \xrightarrow{\rho} S$ be a morphism of DGAs. We call ρ a **finite morphism** if it satisfies:

- The functor ${}_SS_R \overset{\mathrm{L}}{\otimes}_R -: \mathsf{D}(R) \longrightarrow \mathsf{D}(S)$ sends $\mathsf{fin}(R)$ to $\mathsf{fin}(S)$.
- The functor $- \overset{\mathrm{L}}{\otimes}_R {}_RS_S: \mathsf{D}(R^{\mathrm{opp}}) \longrightarrow \mathsf{D}(S^{\mathrm{opp}})$ sends $\mathsf{fin}(R^{\mathrm{opp}})$ to $\mathsf{fin}(S^{\mathrm{opp}})$.
- The functor $\rho^*: \mathsf{D}(S) \longrightarrow \mathsf{D}(R)$, restricting scalars from S to R, satisfies

$$M \in \mathsf{fin}(S) \Leftrightarrow \rho^* M \in \mathsf{fin}(R).$$

- The functor $\rho^*: \mathsf{D}(S^{\mathrm{opp}}) \longrightarrow \mathsf{D}(R^{\mathrm{opp}})$, restricting scalars from S to R, satisfies

$$M \in \mathsf{fin}(S^{\mathrm{opp}}) \Leftrightarrow \rho^* M \in \mathsf{fin}(R^{\mathrm{opp}}).$$

(Note that ρ^* is used to denote the functor which restricts scalars from S to R both on DG-S-left-modules and on DG-S-right-modules.)

(3.2) FINITE MORPHISMS IN THE RING CASE. If $A \xrightarrow{\varphi} B$ is a homomorphism of noetherian local commutative rings, then we can view φ as a morphism of DGAs. As such it is finite precisely if B, viewed as an A-module, is finitely generated and of finite flat dimension.

Let us next generalize finite Gorenstein homomorphisms of rings:

Reading [3, lem. (6.5), (7.7.1), and thm. (7.8)] one can see that if A and B are noetherian local commutative rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, and $A \xrightarrow{\varphi} B$ is a local homomorphism so that B viewed as an A-module is finitely generated and of finite flat dimension, then φ is "Gorenstein at $\mathfrak{n}$" in the sense of [3] precisely if $\mathrm{RHom}_A(B, A)$ is isomorphic to $\Sigma^{-n}B$ for some n. We shall attempt in definition (3.4) to generalize this to the world of non-commutative DGAs.

Already the non-commutativity makes a refinement necessary as also observed in [13], since in a non-commutative situation the complexes $\mathrm{RHom}_A(B, A)$ and $\mathrm{RHom}_{A^{\mathrm{opp}}}(B, A)$ have different structures (the first has B-left-A-right-structure, the second has B-right-A-left-structure). Hence the technical nature of the following two paragraphs.

(3.3) INDUCED MORPHISMS. Let $R \xrightarrow{\rho} S$ be a morphism of DGAs. Given a morphism ${}_SS_R \xrightarrow{\alpha} \mathrm{RHom}_R({}_RS_S, \Sigma^n({}_RR_R))$ and a DG-S-left-module M we can consider

$$\begin{array}{rcl}
\rho^* \mathrm{RHom}_S(M, {}_SS_S) & = & \mathrm{RHom}_S(M, {}_SS_R) \\
& \xrightarrow{\mathrm{RHom}_S(M,\alpha)} & \mathrm{RHom}_S(M, \mathrm{RHom}_R({}_RS_S, \Sigma^n({}_RR_R))) \\
& \xrightarrow{\text{adjointness}} & \mathrm{RHom}_R({}_RS_S \overset{\mathrm{L}}{\otimes}_S M, \Sigma^n({}_RR_R)) \\
& = & \mathrm{RHom}_R(\rho^* M, \Sigma^n({}_RR_R)).
\end{array}$$

In short, this gives an induced morphism

$$\rho^* \mathrm{RHom}_S(M, {}_SS_S) \longrightarrow \mathrm{RHom}_R(\rho^* M, \Sigma^n({}_RR_R)), \tag{1}$$

which is an isomorphism if α is an isomorphism.

Similarly, given a morphism ${}_RS_S \xrightarrow{\beta} \mathrm{RHom}_{R^{\mathrm{opp}}}({}_SS_R, \Sigma^n({}_RR_R))$ and a DG-S-right-module N there is an induced morphism

$$\rho^* \mathrm{RHom}_{S^{\mathrm{opp}}}(N, {}_SS_S) \longrightarrow \mathrm{RHom}_{R^{\mathrm{opp}}}(\rho^* N, \Sigma^n({}_RR_R)), \tag{2}$$

which is an isomorphism if β is an isomorphism.

(3.4) Definition (Gorenstein morphisms). Let R and S be DGAs for which $\mathrm{H}_0 R$ and $\mathrm{H}_0 S$ are noetherian, and let $R \xrightarrow{\rho} S$ be a finite morphism of DGAs. We call ρ a **Gorenstein morphism** if it satisfies:

(1) There are isomorphisms
 (a) ${}_S S_R \xrightarrow{\alpha} \mathrm{RHom}_R({}_R S_S, \Sigma^n({}_R R_R))$.
 (b) ${}_R S_S \xrightarrow{\beta} \mathrm{RHom}_{R^{\mathrm{opp}}}({}_S S_R, \Sigma^n({}_R R_R))$.

(2) The isomorphisms α and β are compatible in the following sense:
 (a) For each DG-S-left-module M the following square is commutative,

$$\begin{array}{ccc} \rho^* M & \xrightarrow{\rho^* t} & \rho^* \mathrm{RHom}_{S^{\mathrm{opp}}}(\mathrm{RHom}_S(M, {}_S S_S), {}_S S_S) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle b}\,\cong \\ \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(\rho^* M, \Sigma^n({}_R R_R)), \Sigma^n({}_R R_R)) & \underset{a}{\xrightarrow{\cong}} & \mathrm{RHom}_{R^{\mathrm{opp}}}(\rho^* \mathrm{RHom}_S(M, {}_S S_S), \Sigma^n({}_R R_R)), \end{array}$$

where s and t are biduality morphisms as in condition [G1], and where a and b are induced by α and β as explained in paragraph (3.3).

 (b) For each DG-S-right-module N there is a commutative square constructed like the one above.

(3.5) Gorenstein morphisms in the ring case. Note from the observations before paragraph (3.3) that if A and B are noetherian local commutative rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, and $A \xrightarrow{\varphi} B$ is a local ring homomorphism so that B viewed as an A-module is finitely generated and of finite flat dimension, then φ is a Gorenstein morphism of DGAs in the sense of definition (3.4) precisely if φ is "Gorenstein at $\mathfrak{n}$" in the sense of [3].

(3.6) Theorem (Ascent). *Let R and S be DGAs for which $\mathrm{H}_0 R$ and $\mathrm{H}_0 S$ are noetherian, and let $R \xrightarrow{\rho} S$ be a finite Gorenstein morphism of DGAs. Then*

$$R \text{ is a Gorenstein DGA } \Rightarrow S \text{ is a Gorenstein DGA.}$$

Proof: We prove the theorem by showing that condition [G1] for R implies condition [G1] for S, and that condition [G2] for R implies condition [G2] for S.

[G1]. Let us assume condition [G1] for R. In the first half of condition [G1] for S we are given M in $\mathsf{fin}(S)$ and must show that the biduality morphism

$$M \xrightarrow{t} \mathrm{RHom}_{S^{\mathrm{opp}}}(\mathrm{RHom}_S(M, {}_S S_S), {}_S S_S)$$

is an isomorphism. This is equivalent to showing that $\rho^* t$ is an isomorphism, because both things amount to seeing that t becomes bijective when the homology functor H is applied to it.

But $\rho^* t$ is one of the morphisms in the diagram in definition (3.4), part (2)(a), and if we can prove that the other arrows in the diagram are isomorphisms, then it follows that $\rho^* t$ is too. The only one of the diagram's other arrows which is not a priori an isomorphism is s. And in the situation at hand, ρ is a finite morphism so $\rho^* M$ is in $\mathsf{fin}(R)$, and condition [G1] for R then says that s is an isomorphism.

The second half of condition [G1] for S is proved in a symmetrical way.

[G2]. Let us assume condition [G2] for R. In the first half of condition [G2] for S we must show $\mathrm{RHom}_S(M, {}_S S_S) \in \mathsf{fin}(S^{\mathrm{opp}})$ for M in $\mathsf{fin}(S)$. Since ρ is a finite morphism, this is equivalent to showing $\rho^* \mathrm{RHom}_S(M, {}_S S_S) \in \mathsf{fin}(R^{\mathrm{opp}})$, and since ρ is a Gorenstein morphism, paragraph (3.3) implies that this is the same as showing

$$\mathrm{RHom}_R(\rho^* M, \Sigma^n({}_R R_R)) \in \mathsf{fin}(R^{\mathrm{opp}}). \tag{3}$$

But as ρ is a finite morphism, $\rho^* M$ is in $\mathsf{fin}(R)$, and condition [G2] for R then says that (3) holds.

The second half of condition [G2] for S is proved in a symmetrical way. ∎

Theorem (3.6) says that Gorenstein morphisms transfer the Gorenstein property of DGAs in the direction of the morphism. This is analogous to the situation in commutative ring theory, see [3, (7.7.2)], and non-commutative ring theory, see [13, thm. 4.7].

Moreover, in view of [3, (7.7.2)] and [13, thm. 4.7], we venture the following conjecture that Gorenstein morphisms also transfer the Gorenstein property of DGAs in the direction opposite to the morphism:

(3.7) Conjecture (Descent). In the situation of theorem (3.6), we have

$$S \text{ is a Gorenstein DGA } \Rightarrow R \text{ is a Gorenstein DGA.}$$

Unfortunately, we are unable to prove conjecture (3.7). As consolation, we aim for proposition (3.10) below.

(3.8) Local commutative DGAs. Let Q be a DGA. Then Q is called local commutative if it satisfies:

- $Q_i = 0$ for $i < 0$.
- Q is commutative and the commutative ring $\mathrm{H}_0 Q$ is noetherian and local.
- Viewed as a DG-Q-module, Q is in $\mathsf{fin}(Q)$.

In this case, the residue class field k of $\mathrm{H}_0 Q$ is also called the residue class field of Q. It is easy to see that one can get a DG-Q-module by placing k in degree zero, and zero in all other degrees. This DG-module is again denoted k.

Note that an ordinary noetherian local commutative ring placed in degree zero is a local commutative DGA.

(3.9) MORPHISMS WITH IMAGE IN THE CENTRE. Let Q and T be DGAs with Q commutative, and let $Q \xrightarrow{\varphi} T$ be a morphism of DGAs with image inside the centre of T (see paragraph (1.10)). Now φ makes it possible to view T as a DG-Q-left-Q-right-module in a way which is compatible with the structure of T as DG-T-left-T-right-module. In other words, T can be viewed as a DG-module with structure ${}_{Q,T}T_{Q,T}$.

Note that the Q-left- and Q-right-structures of ${}_{Q,T}T_{Q,T}$ are equivalent in the sense that $qt = (-1)^{|q||t|}tq$ holds for graded elements q and t in Q and T. In fact, Q behaves almost like a commutative ring of scalars, so we will frequently omit the subscripts indicating Q-structures.

(3.10) PROPOSITION (PARTIAL DESCENT). *Suppose we are given the following data:*

(1) *Q is a local commutative DGA with residue class field k.*
(2) *T is a DGA with H_0T noetherian.*
(3) *$Q \xrightarrow{\varphi} T$ is a finite Gorenstein morphism of DGAs which has image inside the centre of T.*
(4) *There is a K-projective resolution of T viewed as a DG-Q-module, $P \xrightarrow{\simeq} T$, so that P is minimal over Q, i.e., $P \otimes_Q k$ has zero differential.*
(5) *We have $T \overset{\mathrm{L}}{\otimes}_Q k \not\cong 0$.*

Then

$$T \text{ satisfies condition [G2]} \Rightarrow \dim_k \operatorname{Ext}_Q(k,Q) = 1.$$

Remark: The right hand side in the implication is the Gorenstein condition from [2, sec. 3].

Proof: The DG-Q-module k is clearly in $\mathsf{fin}(Q)$. So ${}_TT \overset{\mathrm{L}}{\otimes}_Q k$ is in $\mathsf{fin}(T)$ since φ is a finite morphism, so condition [G2] on T implies that $\mathrm{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT_T)$ is in $\mathsf{fin}(T^{\mathrm{opp}})$. Applying φ^*, the functor which restricts scalars from T to Q, gives that $\varphi^* \mathrm{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT_T)$ is in $\mathsf{fin}(Q)$, again since φ is a finite morphism. Writing this in a simpler way, we get

$$\mathrm{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT) \in \mathsf{fin}(Q). \tag{4}$$

However, φ is a Gorenstein morphism, so by paragraph (3.3) there is an isomorphism

$$\varphi^* \mathrm{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT_T) \xrightarrow{\cong} \mathrm{RHom}_Q(\varphi^*({}_TT \overset{\mathrm{L}}{\otimes}_Q k), \Sigma^n Q),$$

and writing this in a simpler way gives

$$\text{(5)} \qquad \operatorname{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT) \xrightarrow{\cong} \operatorname{RHom}_Q(T \overset{\mathrm{L}}{\otimes}_Q k, \Sigma^n Q).$$

Now, we have

$$T \overset{\mathrm{L}}{\otimes}_Q k \cong P \otimes_Q k \cong \coprod_{i \in I} \Sigma^{\beta_i} k$$

where the second $\cong$ is because P is minimal over Q. Here the β_i are integers and I is a non-empty index set since $T \overset{\mathrm{L}}{\otimes}_Q k \not\cong 0$. Substituting this into the right hand side of (5) gives

$$\begin{aligned} \operatorname{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT) &\xrightarrow{\cong} \operatorname{RHom}_Q(\coprod_{i \in I} \Sigma^{\beta_i} k, \Sigma^n Q) \\ &\cong \prod_{i \in I} \Sigma^{n-\beta_i} \operatorname{RHom}_Q(k, Q). \end{aligned}$$

By equation (4) the left hand side is in $\mathsf{fin}(Q)$, so the same must hold for the right hand side.

But then $\operatorname{RHom}_Q(k, Q)$ itself is certainly in $\mathsf{fin}(Q)$, so in particular $\operatorname{RHom}_Q(k, Q)$ has bounded homology, and by [2, thm. (3.1)] this implies $\dim_k \operatorname{Ext}_Q(k, Q) = 1$. ∎

There are examples of Gorenstein morphisms occurring in nature. Indeed, in the next section they are our chief tool to show that some DGAs occurring in nature are Gorenstein. The following lemma gives a way to obtain Gorenstein morphisms.

(3.11) LEMMA (A WAY TO OBTAIN GORENSTEIN MORPHISMS). *Suppose we are given the following data:*

1. *Q is a commutative DGA with $\mathrm{H}_0 Q$ noetherian.*
2. *T is a DGA with $\mathrm{H}_0 T$ noetherian.*
3. *$Q \xrightarrow{\varphi} T$ is a finite morphism of DGAs which has image inside the centre of T.*
4. *There is an isomorphism in the derived category of DG-T-left-T-right-modules,*

$$ {}_TT_T \xrightarrow{\gamma} \operatorname{RHom}_Q({}_TT_T, \Sigma^n Q).$$

Then φ is a Gorenstein morphism of DGAs.

Proof: Restricting the right-structure from T to Q, the morphism γ restricts to an isomorphism

$$ {}_TT_Q \xrightarrow{\alpha} \operatorname{RHom}_Q({}_QT_T, \Sigma^n({}_QQ_Q))$$

as in definition (3.4)(1)(a), and restricting the left-structure from T to Q, the morphism γ restricts to an isomorphism

$$ {}_Q T_T \xrightarrow{\beta} \mathrm{RHom}_{Q^{\mathrm{opp}}}({}_T T_Q, \Sigma^n({}_Q Q_Q)) $$

as in (3.4)(1)(b).

To prove the lemma, we must see that the α and β so obtained are compatible in the sense of (3.4)(2). So we must see that the diagrams from (3.4)(2), with Q and T in place of R and S, are commutative. This is easy, but tedious: Take, for instance, the diagram from (3.4)(2)(a). To see that it is commutative, we need to replace the various modules with suitable resolutions so that the derived Hom's become ordinary Hom's, the derived tensors become ordinary tensors, and the morphisms can be computed explicitly. A good choice is to start by picking a K-injective resolution $Q \xrightarrow{\simeq} I$ and go on by using ${}_T J_T = \mathrm{Hom}_Q({}_T T_T, \Sigma^n I)$ as a resolution of ${}_T T_T$. The DG-module ${}_T J_T$ has the virtue of being isomorphic to $\mathrm{RHom}_Q({}_T T_T, \Sigma^n Q)$, while being K-injective from the left and K-injective from the right. Having introduced resolutions, the computation to check commutativity is a matter of patience. ∎

4. Examples: Endomorphism DGAs, Koszul complexes, and singular cochain DGAs of topological spaces

This section considers three types of DGAs: Endomorphism DGAs of perfect complexes of modules, Koszul complexes, and singular cochain DGAs of topological spaces with Poincaré duality, all over noetherian local commutative rings. Using the theory of section 3, we give complete criteria for when these are Gorenstein DGAs: They are so if and only if the base ring is Gorenstein. The proofs work by showing that a suitable morphism from the base ring to the DGA in question is a Gorenstein morphism.

ENDOMORPHISM DGAS. The following paragraph recapitulates the definition of endomorphism DGAs of perfect complexes of modules; see [7] for more details.

(4.1) SETUP. In paragraphs (4.1) to (4.5) we consider the following situation: A is a noetherian local commutative ring, L is a bounded complex of finitely generated projective A-modules which is not homotopy equivalent to zero, and we look at $\mathcal{E} = \mathrm{Hom}_A(L, L)$.

A priori, $\mathcal{E}$ is just a complex of A-modules. However, there is a multiplication on $\mathcal{E}$ given by composition: An element ϵ in $\mathcal{E}_i$ is an A-linear map $L \xrightarrow{\epsilon} \Sigma^{-i} L$. If we also have an element ϵ' in $\mathcal{E}_j$, then we define the product $\epsilon\epsilon'$ as the composition

$\Sigma^{-j}(\epsilon) \circ \epsilon'$ which is an A-linear map $L \xrightarrow{\epsilon\epsilon'} \Sigma^{-(i+j)}L$, that is, an element in $\mathcal{E}_{i+j}$. It is not hard to check that with this multiplication, $\mathcal{E}$ is a DGA.

The complex L becomes a DG-$\mathcal{E}$-left-module with scalar multiplication $\epsilon\ell = \epsilon(\ell)$ for ϵ in $\mathcal{E}$ and ℓ in L. The $\mathcal{E}$-structure on L is compatible with the A-structure, so L is a DG-A-left-$\mathcal{E}$-left-module, ${}_{A,\mathcal{E}}L$. Moreover, the identification map is an isomorphism of DG-$\mathcal{E}$-left-$\mathcal{E}$-right-modules,

$$ {}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \xrightarrow{\cong} \mathrm{Hom}_A({}_{A,\mathcal{E}}L, {}_{A,\mathcal{E}}L). $$

(4.2) Remark. Note that $\mathcal{E}$ and its homology $\mathrm{H}\mathcal{E}$ are usually far from commutative. For instance, if L is the projective resolution of a finitely generated A-module of finite projective dimension, M, then we have $\mathrm{H}_0\mathcal{E} \cong \mathrm{End}_A(M)$. Also, $\mathcal{E}$ usually has non-zero homology both in positive and negative degrees.

(4.3) The morphism $\varphi_{\mathcal{E}}$. Since A is commutative, each element a in A gives a chain map $L \xrightarrow{a\cdot} L$ which is just multiplication by a. This chain map is an element in the degree 0 component of $\mathrm{Hom}_A(L, L)$. In other words, it is an element in $\mathrm{Hom}_A(L, L)_0 = \mathcal{E}_0$. One checks easily that this gives a morphism of DGAs,

$$ A \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}, \quad a \longmapsto (L \xrightarrow{a\cdot} L). $$

Here are three useful observations:

- The ring $\mathrm{H}_0\mathcal{E}$ is noetherian since it is a finitely generated A-module.
- The morphism $\varphi_{\mathcal{E}}$ has image inside the centre of $\mathcal{E}$, because an element ϵ in $\mathcal{E}_i$ is an A-linear map $L \xrightarrow{\epsilon} \Sigma^{-i}L$ whence

$$ \begin{aligned} (\varphi_{\mathcal{E}}(a)\epsilon)(\ell) &= (\Sigma^{-i}(\varphi_{\mathcal{E}}(a)) \circ \epsilon)(\ell) = a\epsilon(\ell) \\ &= \epsilon(a\ell) = (\epsilon \circ \varphi_{\mathcal{E}}(a))(\ell) = (\epsilon\varphi_{\mathcal{E}}(a))(\ell) \end{aligned} $$

  so $\varphi_{\mathcal{E}}(a)\epsilon = \epsilon\varphi_{\mathcal{E}}(a)$.
- The morphism $\varphi_{\mathcal{E}}$ is a finite morphism of DGAs since L and hence $\mathcal{E}$ are bounded complexes of finitely generated projective A-modules.

(4.4) Lemma ($\varphi_{\mathcal{E}}$ is Gorenstein). *The morphism $A \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}$ is a Gorenstein morphism.*

Proof: We shall use lemma (3.11) with the morphism $Q \xrightarrow{\varphi} T$ equal to $A \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}$. The observations in paragraph (4.3) show that the lemma's conditions (1) to (3) hold. If we can show that the lemma's condition (4) also holds, then the lemma gives our desired conclusion, that $\varphi_{\mathcal{E}}$ is Gorenstein.

So we must find an isomorphism ${}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \xrightarrow{\gamma} \mathrm{RHom}_A({}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}}, \Sigma^n A)$ in the derived category of DG-$\mathcal{E}$-left-$\mathcal{E}$-right-modules. Since $\mathcal{E}$ is K-projective over A, we have

$\mathrm{RHom}_A({}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}}, \Sigma^n A) \cong \mathrm{Hom}_A({}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}}, \Sigma^n A)$, so it is enough to find a quasi-isomorphism in the abelian category of DG-$\mathcal{E}$-left-$\mathcal{E}$-right-modules,

$${}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \xrightarrow{\simeq} \mathrm{Hom}_A({}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}}, \Sigma^n A).$$

However, ${}_{A,\mathcal{E}}L$ is a bounded complex of finitely generated projective A-modules, so by [1, sec. 1, thms. 1 and 2] the two so-called evaluation morphisms appearing as the last two arrows in the following diagram are isomorphisms,

$$\begin{array}{c} \mathrm{Hom}_A({}_{A,\mathcal{E}}L, {}_{A,\mathcal{E}}L) \\ \downarrow \cong \\ \mathrm{Hom}_A({}_{A,\mathcal{E}}L, A \odot_A {}_{A,\mathcal{E}}L) \\ \uparrow \cong \\ \mathrm{Hom}_A({}_{A,\mathcal{E}}L, A) \odot_A {}_{A,\mathcal{E}}L \xrightarrow{\cong} \mathrm{Hom}_A(\mathrm{Hom}_A({}_{A,\mathcal{E}}L, {}_{A,\mathcal{E}}L), A). \end{array}$$

Substituting ${}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \cong \mathrm{Hom}_A({}_{A,\mathcal{E}}L, {}_{A,\mathcal{E}}L)$ twice, this gives an isomorphism

$${}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \xrightarrow{\cong} \mathrm{Hom}_A({}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}}, A),$$

which is in particular a quasi-isomorphism, as desired. ∎

(4.5) THEOREM (ASCENT-DESCENT FOR ENDOMORPHISM DGAS). *In the situation of setup (4.1), we have*

A is a Gorenstein ring $\Leftrightarrow$ $\mathcal{E}$ is a Gorenstein DGA.

Proof: $\Rightarrow$: We will use theorem (3.6) with $R \xrightarrow{\rho} S$ equal to $A \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}$ to see this.

Paragraph (4.3) and lemma (4.4) say that the hypotheses of theorem (3.6) hold, so the theorem applies.

Now, A is noetherian commutative and has finite Krull dimension. So if A is a Gorenstein ring then proposition (2.5) says that A viewed as a DGA is a Gorenstein DGA. And theorem (3.6) then implies that $\mathcal{E}$ is a Gorenstein DGA.

$\Leftarrow$: We will use proposition (3.10) with $Q \xrightarrow{\varphi} T$ equal to $A \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}$ to see this.

Indeed, proposition (3.10) applies: The proposition's condition (1) clearly holds, and conditions (2) and (3) hold by paragraph (4.3) and lemma (4.4). Condition (4) holds since $\mathcal{E}$ is a bounded complex of finitely generated projective A-modules, hence has homology which is bounded and finitely generated over A, hence has a minimal projective resolution over A. Condition (5) holds since L is not homotopy equivalent to zero whence the homology of $\mathcal{E}$ cannot be zero, so $\mathcal{E} \overset{\mathrm{L}}{\otimes}_A k \not\cong 0$.

So if $\mathcal{E}$ is a Gorenstein DGA, then proposition (3.10) gives

$$\dim_k \operatorname{Ext}_A(k, A) = 1$$

which implies that A is a Gorenstein ring by [17, thm. 18.1]. ∎

KOSZUL COMPLEXES. The following paragraph recapitulates the definition of Koszul complexes.

(4.6) SETUP. In paragraphs (4.6) to (4.10) we consider the following situation: A is a noetherian local commutative ring, and $\mathbf{a} = (a_1, \ldots, a_n)$ is a sequence of elements in the maximal ideal of A.

We can construct the so-called Koszul complex $\mathrm{K}(\mathbf{a})$ of $\mathbf{a}$ which is a DGA: As a graded algebra, $\mathrm{K}(\mathbf{a})$ is simply the exterior algebra $\bigwedge F$ on the free module $F = Ae_1 \oplus \cdots \oplus Ae_n$. To get a DGA, we introduce the differential

$$\partial_j^{\mathrm{K}(\mathbf{a})}(e_{s_1} \wedge \cdots \wedge e_{s_j}) = \sum_i (-1)^{i+1} a_{s_i} e_{s_1} \wedge \cdots \wedge \widehat{e_{s_i}} \wedge \cdots \wedge e_{s_j},$$

where the hat indicates that e_{s_i} is left out of the wedge product.

(4.7) THE MORPHISM $\varphi_{\mathrm{K}(\mathbf{a})}$. There is a morphism of DGAs

$$A \xrightarrow{\varphi_{\mathrm{K}(\mathbf{a})}} \mathrm{K}(\mathbf{a})$$

given by noting that the degree zero component of $\mathrm{K}(\mathbf{a})$ is A itself. As in paragraph (4.3), here are three useful observations:

- The ring $\mathrm{H}_0\mathrm{K}(\mathbf{a})$ is noetherian since it is a finitely generated A-module. Also, $\mathrm{K}(\mathbf{a})$ is a commutative DGA.
- The morphism $\varphi_{\mathrm{K}(\mathbf{a})}$ has image inside the centre of $\mathrm{K}(\mathbf{a})$, since the centre is all of $\mathrm{K}(\mathbf{a})$.
- The morphism $\varphi_{\mathrm{K}(\mathbf{a})}$ is a finite morphism of DGAs since $\mathrm{K}(\mathbf{a})$ is a bounded complex of finitely generated projective A-modules.

(4.8) LEMMA ($\varphi_{\mathrm{K}(\mathbf{a})}$ IS GORENSTEIN). *The morphism $A \xrightarrow{\varphi_{\mathrm{K}(\mathbf{a})}} \mathrm{K}(\mathbf{a})$ is a Gorenstein morphism.*

Proof: Like the proof of lemma (4.4), this is based on lemma (3.11), and again, what we need is to show that the lemma's condition (4) holds. So we need to find an isomorphism $\mathrm{K}(\mathbf{a}) \xrightarrow{\gamma} \mathrm{RHom}_A(\mathrm{K}(\mathbf{a}), \Sigma^n A)$ (by commutativity of $\mathrm{K}(\mathbf{a})$ we need not worry about left- and right-structures here). Since $\mathrm{K}(\mathbf{a})$ is K-projective over A we have $\mathrm{RHom}_A(\mathrm{K}(\mathbf{a}), \Sigma^n A) \cong \mathrm{Hom}_A(\mathrm{K}(\mathbf{a}), \Sigma^n A)$, so it is enough to find a quasi-isomorphism

$$\mathrm{K}(\mathbf{a}) \xrightarrow{\simeq} \mathrm{Hom}_A(\mathrm{K}(\mathbf{a}), \Sigma^n A).$$

However, the degree n component of $\mathrm{K}(\mathbf{a})$ is $\bigwedge^n F$ which is A itself, so the projection of $\mathrm{K}(\mathbf{a})$ onto its degree n component has the form $\mathrm{K}(\mathbf{a}) \xrightarrow{\pi} \Sigma^n A$. It is now easy to check that there is an isomorphism

$$\mathrm{K}(\mathbf{a}) \xrightarrow{\cong} \mathrm{Hom}_A(\mathrm{K}(\mathbf{a}), \Sigma^n A), \quad k \longmapsto (\ell \longmapsto \pi(k \wedge \ell)),$$

which is in particular a quasi-isomorphism, as desired. ∎

(4.9) THEOREM (ASCENT-DESCENT FOR KOSZUL COMPLEXES). *In the situation of setup (4.6), we have*

$$A \text{ is a Gorenstein ring} \Leftrightarrow \mathrm{K}(\mathbf{a}) \text{ is a Gorenstein DGA}.$$

Proof: This is almost verbatim to the proof of theorem (4.5), in that we apply theorem (3.6) and proposition (3.10) to the finite Gorenstein morphism $A \xrightarrow{\varphi_{\mathrm{K}(\mathbf{a})}} \mathrm{K}(\mathbf{a})$. This gives the implications $\Rightarrow$ and $\Leftarrow$. ∎

(4.10) RELATION TO A RESULT BY AVRAMOV-GOLOD. Avramov and Golod proved in [4] a result which can be stated in the language of Avramov and Foxby [2] as follows: Let A be a noetherian local commutative ring, let $\mathbf{a} = (a_1, \ldots, a_n)$ be a minimal system of generators of the maximal ideal of A, and let $\mathrm{K}(\mathbf{a})$ be the Koszul complex of $\mathbf{a}$. Then

(6) $$A \text{ is a Gorenstein ring} \Leftrightarrow \dim_k \mathrm{Ext}_{\mathrm{K}(\mathbf{a})}(k, \mathrm{K}(\mathbf{a})) = 1,$$

where k is residue class field of $\mathrm{K}(\mathbf{a})$.

From theorem (4.9) follows the more general statement that (6) holds for the Koszul complex $\mathrm{K}(\mathbf{a})$ on *any* sequence $\mathbf{a}$ of elements in the maximal ideal of A. This is because the right hand sides of the bi-implications in theorem (4.9) respectively equation (6) are equivalent, as proved in [11, thm. I].

Equation (6) could also be proved for the Koszul complex $\mathrm{K}(\mathbf{a})$ on any sequence $\mathbf{a}$ in the maximal ideal of A by using [2, thm. (3.1)].

SINGULAR COCHAIN DGAS OF TOPOLOGICAL SPACES. In this subsection we break the habit of the rest of the paper and switch to cohomological notation, that is, upper indices on graded objects and differentials of degree +1.

The following three paragraphs give some facts on singular cochain DGAs of topological spaces; see [9] and [10] for more details.

(4.11) SETUP. In paragraphs (4.11) to (4.16) we consider the following situation: A is a noetherian local commutative ring which contains a field k, and X is a simply connected topological space with $\dim_k \mathrm{H}_*(X;k) < \infty$, which has Poincaré duality over k, in the sense that there is an isomorphism of graded $\mathrm{H}^*(X;k)$-modules

$$\mathrm{H}^*(X;k)' \cong \Sigma^d \mathrm{H}^*(X;k) \tag{7}$$

for some d, where the prime denotes dualization with respect to k.

Note that the isomorphism (7) implies

$$\mathrm{H}^d(X;k) \cong k \quad \text{and} \quad \mathrm{H}^i(X;k) = 0 \text{ for } i > d. \tag{8}$$

An important object is $\mathrm{C}^*(X;k)$, the singular cochain DGA of X with coefficients in k, which can be defined as

$$\mathrm{C}^*(X;k) = \mathrm{Hom}_k(\mathrm{C}_*(X;k), k),$$

where $\mathrm{C}_*(X;k)$ is the singular chain complex of X with coefficients in k. The multiplication which turns $\mathrm{C}^*(X;k)$ into a DGA is cup product, which is defined using the Alexander–Whitney map on $\mathrm{C}_*(X;k)$.

The singular cohomology $\mathrm{H}^*(X;k)$ is defined as the cohomology algebra of $\mathrm{C}^*(X;k)$. See, e.g., [10, chap. 5] for details on $\mathrm{C}^*(X;k)$ and $\mathrm{H}^*(X;k)$.

(4.12) INTRODUCING S. By the "free model" construction employed in [9, proof of thm. 3.6], we have that $\mathrm{C}^*(X;k)$ is equivalent to some R which is a DGA over k with $R^0 = k$ and $R^1 = 0$, and with each R^i finite dimensional over k.

Next, by the method employed in [10, ex. 6, p. 146], there exists a DG-ideal I in R so that the canonical surjection $R \longrightarrow R/I$ is a quasi-isomorphism, and so that the right-most non-vanishing component of $S = R/I$ has the same degree as R's right-most non-vanishing cohomology; namely, degree d (see equation (8)). So S looks like

$$\cdots \to 0 \to k \to 0 \to S^2 \to \cdots \to S^d \to 0 \to \cdots,$$

with each S^i finite dimensional over k. Note that S^0 is central in S because R^0 is central in R.

To sum up, $\mathrm{C}^*(X;k)$ is equivalent to R which is again equivalent to S.

(4.13) REMARKS ON $\mathrm{C}^*(X;A)$. Our main object of interest in this part of the paper is $\mathrm{C}^*(X;A)$, the singular cochain DGA of X with coefficients in A, which can be defined as

$$\mathrm{C}^*(X;A) = \mathrm{Hom}_k(\mathrm{C}_*(X;k), A).$$

Again, the multiplication is cup product, defined using the Alexander–Whitney map on $\mathrm{C}_*(X;k)$.

The purpose of the following paragraphs is to show that $\mathrm{C}^*(X;A)$ is a Gorenstein DGA if and only if A is a Gorenstein ring.

However, it will be an advantage not to work with $\mathrm{C}^*(X;A)$ itself but rather with an equivalent DGA which is more tractable: First, there is an evaluation morphism

$$\mathrm{Hom}_k(\mathrm{C}_*(X;k),k)\otimes_k A \longrightarrow \mathrm{Hom}_k(\mathrm{C}_*(X;k),k\otimes_k A)$$

which is a quasi-isomorphism because the homology

$$\mathrm{H}(\mathrm{C}_*(X;k)) = \mathrm{H}_*(X;k)$$

is finite dimensional over k, see [1, sec. 1, thm. 2]. This can also be read

$$\mathrm{C}^*(X;k)\otimes_k A \xrightarrow{\simeq} \mathrm{C}^*(X;A),$$

and it is not hard to check that this is a morphism of DGAs.

Secondly, by paragraph (4.12), we have that $\mathrm{C}^*(X;k)$ is equivalent to the DGA called S.

To sum up, we have that $\mathrm{C}^*(X;A)$ is equivalent to $\mathrm{C}^*(X;k)\otimes_k A$, and as A is flat over k, this is again equivalent to $S\otimes_k A$.

(4.14) THE MORPHISM φ_S. There is a morphism of DGAs

$$A \xrightarrow{\varphi_S} S\otimes_k A, \quad a \longmapsto 1_S\otimes a.$$

As in paragraphs (4.3) and (4.7), here are three useful observations:

- The ring $\mathrm{H}^0(S\otimes_k A)$ is noetherian because we have

  $$\mathrm{H}^0(S\otimes_k A)\cong \mathrm{H}^0(\mathrm{C}^*(X;A))\cong A,$$

  where the second $\cong$ holds because X is connected.
- The morphism φ_S has image inside $S^0\otimes_k A$, and as $S^0 = k$ is central in S and A is commutative, φ_S has image inside the centre of $S\otimes_k A$.
- The morphism φ_S is a finite morphism of DGAs since S is finite dimensional over k, whence $S\otimes_k A$ is a bounded complex of finitely generated projective A-modules.

(4.15) LEMMA (φ_S IS GORENSTEIN). *The morphism* $A \xrightarrow{\varphi_S} S\otimes_k A$ *is a Gorenstein morphism.*

Proof: Let us start with some computations. Since we have $\mathrm{H}(S) \cong \mathrm{H}(\mathrm{C}^*(X;k)) \cong \mathrm{H}^*(X;k)$, the Poincaré duality isomorphism (7) gives an isomorphism of graded $\mathrm{H}(S)$-modules

$$\Sigma^{-d}(\mathrm{H}(S)') \cong \mathrm{H}(S).$$

This means that $\Sigma^{-d}(\mathrm{H}(S)')$ is free; in other words, there is an element ξ in $(\Sigma^{-d}(\mathrm{H}(S)'))^0$ so that

$$\mathrm{H}(S) \ni c \longmapsto c\xi \in \Sigma^{-d}(\mathrm{H}(S)') \tag{9}$$

is an isomorphism.

Actually, ξ is a linear form $\mathrm{H}^d(S) \xrightarrow{\xi} k$. As S^d is the right-most non-vanishing component of S, we have a surjection $S^d \longrightarrow \mathrm{H}^d(S)$, and we can define a linear form $S^d \xrightarrow{\Xi} k$ as the composition

$$S^d \longrightarrow \mathrm{H}^d(S) \xrightarrow{\xi} k.$$

This can again be viewed as an element Ξ in $(\Sigma^{-d}(S'))^0$.

Now, Ξ is an element in the DG-S-left-S-right-module $\Sigma^{-d}(S')$ which satisfies $\Xi s = (-1)^{|\Xi||s|} s\Xi$ for any graded element s in S, as one proves easily using that Ξ is induced by ξ which is defined on the commutative graded algebra $\mathrm{H}(S) \cong \mathrm{H}^*(X;k)$. It is also mapped to zero by the differential of $\Sigma^{-d}(S')$. Hence we can define a morphism of DG-S-left-S-right-modules δ by

$$S \ni s \xrightarrow{\delta} s\Xi \in \Sigma^{-d}(S').$$

The cohomology of δ is easily seen to be the isomorphism (9), so we have that δ is a quasi-isomorphism, hence an isomorphism in the derived category of DG-S-left-S-right-modules.

Denoting source and target differently, δ reads

$$ {}_S S_S \xrightarrow{\delta} \mathrm{RHom}_k({}_S S_S, \Sigma^{-d} k). \tag{10}$$

We shall use this isomorphism below. (Note that by lemma (3.11), the existence of δ actually shows that the canonical morphism $k \longrightarrow S$ is Gorenstein.)

Now for the proof proper: To show that

$$A \xrightarrow{\varphi_S} S \odot_k A$$

is Gorenstein, we shall use lemma (3.11) with $Q \xrightarrow{\varphi} T$ equal to $A \xrightarrow{\varphi_S} S \odot_k A$. The observations in paragraph (4.14) show that the lemma's conditions (1) to (3) hold, so we must show that the lemma's condition (4) also holds.

This condition requires a certain isomorphism in the derived category of DG-$(S \odot_k A)$-left-$(S \odot_k A)$-right-modules, which we obtain as follows using the isomorphism δ from equation (10):

$$\begin{aligned} S \otimes_k A &\overset{\delta \otimes A}{\cong} \mathrm{RHom}_k(S, \Sigma^{-d}k) \otimes_k A \\ &\overset{(a)}{\cong} \mathrm{RHom}_k(S, \Sigma^{-d}k \odot_k A) \\ &\cong \mathrm{RHom}_k(S, \Sigma^{-d}A) \\ &\cong \mathrm{RHom}_k(S, \mathrm{RHom}_A(A, \Sigma^{-d}A)) \\ &\overset{(b)}{\cong} \mathrm{RHom}_A(S \overset{\mathrm{L}}{\odot}_k A, \Sigma^{-d}A) \\ &\cong \mathrm{RHom}_A(S \odot_k A, \Sigma^{-d}A), \end{aligned}$$

where (a) is an evaluation morphism which is a quasi-isomorphism and hence an isomorphism in the derived category, because the homology $\mathrm{H}(S) \cong \mathrm{H}(\mathrm{C}^*(X;k)) = \mathrm{H}^*(X;k)$ is finite dimensional over k, see [1, sec. 1, thm. 2], and where (b) is an adjunction isomorphism. ∎

(4.16) THEOREM (ASCENT-DESCENT FOR SINGULAR COCHAIN DGAS). *In the situation of setup (4.11), we have*

$$A \text{ is a Gorenstein ring } \Leftrightarrow \mathrm{C}^*(X;A) \text{ is a Gorenstein DGA.}$$

Proof: This is almost verbatim to the proof of theorem (4.5): Observe that as $\mathrm{C}^*(X;A)$ is equivalent to $S \odot_k A$ by paragraph (4.13), paragraph (2.2) implies that it is enough to show

$$A \text{ is a Gorenstein ring } \Leftrightarrow S \otimes_k A \text{ is a Gorenstein DGA.}$$

For this, we apply theorem (3.6) and proposition (3.10) to the finite Gorenstein morphism $A \xrightarrow{\varphi_S} S \odot_k A$. This gives the implications $\Rightarrow$ and $\Leftarrow$. ∎

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