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ASYMPTOTICS FOR THE STANDARD AND THE CAPELLI IDENTITIES

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ABSTRACT

Let $\{c_n(St_k)\}$ and $\{c_n(C_k)\}$ be the sequences of codimensions of the T-ideals generated by the standard polynomial of degree k and by the k-th Capelli polynomial, respectively. We study the asymptotic behaviour of these two sequences over a field F of characteristic zero. For the standard polynomial, among other results, we show that the following asymptotic equalities hold:

$$c_n(St_{2k}) \simeq c_n(C_{k^2+1}) \simeq c_n(M_k(F)),$$
$$c_n(St_{2k+1}) \simeq c_n(M_{k\times 2k}(F) \oplus M_{2k\times k}(F)),$$

where $M_k(F)$ is the algebra of $k \times k$ matrices and $M_{k\times l}(F)$ is the algebra of $(k+l) \times (k+l)$ matrices having the last l rows and the last k columns equal to zero. The precise asymptotics of $c_n(M_k(F))$ are known and those of $M_{k\times 2k}(F)$ and $M_{2k\times k}(F)$ can be easily deduced. For Capelli polynomials we show that also upper block triangular matrix algebras come into play.

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1. Introduction

In this paper we study the asymptotic behaviour of the sequence of codimensions of the T-ideals generated by the standard polynomial and the Capelli polynomial over a field of characteristic zero. Let F be a field, $\operatorname{char} F = 0$. Recall that if $F\langle X\rangle$ is the free associative algebra on the countable set $X = \{x_1, x_2, \ldots, y_1, y_2, \ldots\}$, a T-ideal I of $F\langle X\rangle$ is an ideal invariant under all endomorphisms of $F\langle X\rangle$. Also, $I = Id(A)$ is the ideal of polynomial identities of some PI-algebra A. To each T-ideal I one associates a numerical sequence called the sequence of codimensions $\{c_n(I)\}_{n\geq 1}$ of I or A. We also write $c_n(Id(A)) = c_n(A)$. Each $c_n(I)$ measures the dimension of the multilinear part of $\frac{F\langle X\rangle}{I}$ in n fixed variables. Thus, if $P_n = \mathrm{Span}_F\{x_{\sigma(1)}\cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ denotes the space of multilinear polynomials in the first n variables $x_1, \ldots, x_n$, we have that $c_n(I) = \dim_F P_n/P_n \cap I$.

When $\operatorname{char} F = 0$, I is determined by its multilinear part and the sequence of codimensions is of special interest in this case.

It is well known ([14]) that for a proper ideal I, $c_n(I)$ is exponentially bounded. Moreover, the precise asymptotics of $c_n(I)$ were computed for some important classes of T-ideals ([1], [12], [15]), e.g., the ideal of polynomial identities of $n \times n$ matrices ([15]). Recently in [5] and [6] the exponential behaviour of $c_n(I)$, for $I = Id(A)$ a proper ideal, was studied and it was shown that $\exp(I) = \lim_{n\to\infty} \sqrt[n]{c_n(I)}$, the exponent of I, exists and is a non-negative integer. We also write $\exp(I) = \exp(A)$.

For a polynomial (or set of polynomials) V, let $\langle V\rangle_T$ be the T-ideal generated by V and write $c_n(V) = c_n(\langle V\rangle_T)$. In PI-theory a prominent role is played by the standard and the Capelli polynomials; here we shall study their T-ideals and the asymptotics of the corresponding codimensions.

Let S_m be the symmetric group on $\{1, \ldots, m\}$. Recall that

$$St_m(x_1, \ldots, x_m) = \sum_{\sigma\in S_m} (\operatorname{sgn}\sigma) x_{\sigma(1)} \cdots x_{\sigma(m)}$$

is the standard polynomial of degree m and that

$$c_{m+1}(x_1, \ldots, x_{m+1}; y_1, \ldots, y_m) = \sum_{\sigma\in S_{m+1}} (\operatorname{sgn}\sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_m x_{\sigma(m+1)}$$

is the $(m+1)$-th Capelli polynomial. Let C_{m+1} denote the set of 2^m polynomials obtained from c_{m+1} by deleting any subset of variables y_i (by evaluating the variables y_i to 1 in all possible ways).

In this paper we try to find a close relation among the asymptotics of $c_n(St_m), c_n(C_{m+1})$ and $c_n(M_k(F))$ where $M_k(F)$ is the algebra of $k \times k$ matrices over F. Two other algebras play a role in this description: $M_{k\times l}(F)$, the algebra of $(k+l) \times (k+l)$ matrices over F having the last l rows and the last k columns equal to zero, and $UT(d_1, d_2)$, the algebra of upper block triangular matrices of size d_1 and d_2 over F (see details in the next sections).

Suppose that F is algebraically closed. It is well known ([15]) that $\exp(M_k(F)) = k^2$. Also, from [10] it follows that $\exp(M_{k\times 2k}(F)) = \exp(M_{2k\times k}(F)) = k^2$ and from [7] we have that $\exp(UT(d_1, d_2)) = d_1^2 + d_2^2$.

Here we show that

$$\langle St_{2k}\rangle_T = Id(M_k(F) \oplus B),$$
$$\langle St_{2k+1}\rangle_T = Id(M_{k\times 2k}(F) \oplus M_{2k\times k}(F) \oplus D)$$

and

$$\langle C_{k^2+1}\rangle_T = Id(M_k(F) \oplus E),$$

where B, D, E are finite dimensional algebras whose exponent is strictly smaller than k^2. It follows that asymptotically

$$c_n(St_{2k}) \simeq c_n(C_{k^2+1}) \simeq c_n(M_k(F))$$

and

$$c_n(St_{2k+1}) \simeq c_n(M_{k\times 2k}(F) \oplus M_{2k\times k}(F)).$$

We remark that the precise asymptotics of $c_n(M_k(F))$ were computed in [15] and those of $M_{k\times 2k}(F)$ and $M_{2k\times k}(F)$ can be easily deduced from [10].

For the Capelli polynomials C_{m+1}, it was shown in [13] that $m-3 \leq \exp(C_{m+1}) \leq m$. Here we examine the two cases: $\exp(C_{m+1}) = m$ or $m-1$. The case $\exp(C_{m+1}) = m$ has been already described above since $\exp(C_{m+1}) = m$ if and only if m is a square. When m is not a square and $m-1$ is a square or the sum of two squares (i.e., $\exp(C_{m+1}) = m-1$), we prove that $Id(C_{m+1}) = Id(A_1 \oplus \cdots \oplus A_t \oplus B)$ where B is a finite dimensional algebra of exponent smaller than $m-1$ and $A_1, \ldots, A_t$ are algebras of the type $M_{k\times 2k}(F), M_{2k\times k}(F)$ or $UT(d_1, d_2)$ whose existence depends whether $m-1 = k^2$ or $m-1 = d_1^2 + d_2^2$. Asymptotic inequalities for $c_n(C_{m+1})$ are given in this case.

The main results of this paper were announced in [9].

2. Reduced algebras

An important ingredient of this paper is the exponent of a T-ideal or of a PI-algebra (or of a proper variety of algebras) and its existence was proved in [5] and [6]. If $\mathcal{V}$ is a variety of algebras, we denote by $Id(\mathcal{V})$ the T-ideal of $F\langle X\rangle$ of polynomial identities satisfied by all the algebras of $\mathcal{V}$. Also, if $\mathcal{V}$ is generated by the algebra A, we write $\mathcal{V} = \mathrm{var}(A)$, $c_n(\mathcal{V}) = c_n(A)$ and $\exp(\mathcal{V}) = \exp(A)$. First we reduce all computations to the case of algebraically closed fields.

Let $V \subseteq F\langle X\rangle$ be a subset of multilinear polynomials and let $I = \langle V\rangle_T$ be the T-ideal generated by V. Consider an extension field $\bar{F} \supseteq F$ and let $\bar{I}$ be the corresponding T-ideal generated by V in the free associative $\bar{F}$-algebra $\bar{F}\langle X\rangle$. It is not difficult to show that $\bar{I} = I \otimes_F \bar{F}$; hence the nth codimension $c_n(\bar{I})$ in $\bar{F}\langle X\rangle$ coincides with the nth codimension $c_n(I)$ in $F\langle X\rangle$. Since we are mostly interested in codimensions of multilinear identities, throughout we shall assume, as we may, that F is an algebraically closed field of characteristic zero.

Let G be the infinite dimensional Grassmann algebra over F and let $G = G^{(0)} \oplus G^{(1)}$ be its natural $\mathbb{Z}_2$-grading. If $A = A^{(0)} \oplus A^{(1)}$ is a superalgebra over F, then $G(A) = A^{(0)} \otimes G^{(0)} \oplus A^{(1)} \otimes G^{(1)}$ is called the Grassmann envelope of A. We recall that by a result of Kemer ([11, Theorem 2.3]), if $\mathcal{V}$ is a proper variety then there exists a finite dimensional superalgebra A such that $\mathcal{V} = \mathrm{var}(G(A))$.

The exponent of $\mathcal{V}$ is computed as follows: let $\mathcal{V} = \mathrm{var}(G(A))$ where $A = A^{(0)} \oplus A^{(1)}$ is a finite dimensional superalgebra over F. By the Wedderburn–Malcev theorem (see [4, Theorem 72.19]), A can be written as $A = B + J$ where B is a maximal semisimple subalgebra of A and $J = J(A)$ is the Jacobson radical of A. It is not difficult to see that J is a $\mathbb{Z}_2$-invariant subspace of A. It also follows from [17] that B can be chosen with induced $\mathbb{Z}_2$-grading and $B = A_1 \oplus \cdots \oplus A_t$ is the direct sum of simple superalgebras $A_1, \ldots, A_t$. Write all possible products of the form $A_{i_1} J A_{i_2} J \cdots J A_{i_k} \neq 0$ for distinct $A_{i_1}, \ldots, A_{i_k}$ and set $\dim_F(A_{i_1} \oplus \cdots \oplus A_{i_k}) = d_{i_1,\ldots,i_k}$. Then it was shown in [6, Proposition 1 and Proposition 2] that $\exp(\mathcal{V}) = \max_{i_1,\ldots,i_k} \{d_{i_1,\ldots,i_k}\}$.

Motivated by this construction we make the following definition

Definition 1: Let $A = A_1 \oplus \cdots \oplus A_r + J$ be a finite dimensional superalgebra where $A_1, \ldots, A_r$ are simple superalgebras and $J = J(A)$. We say that A is reduced if $A_1 J A_2 J \cdots J A_r \neq 0$.

We prove in the next theorem that the reduced algebras can be used as building blocks of any proper variety. We first prove a lemma (see [2, Lemma 3.2]) that will be used throughout the paper.

LEMMA 1: *If A and B are PI-algebras, then $c_n(A), c_n(B) \leq c_n(A \oplus B) \leq c_n(A) + c_n(B)$. Hence $\exp(A \oplus B) = \max\{\exp(A), \exp(B)\}$.*

Proof: Recall that if R is any algebra, then

$$c_n(R) = \dim_F \frac{P_n}{P_n \cap Id(R)}.$$

Hence, since $Id(A), Id(B) \supseteq Id(A \oplus B)$, we have that $c_n(A), c_n(B) \leq c_n(A \oplus B)$, for all $n \geq 1$.

Now, the map

$$\varphi\colon P_n \to \frac{P_n}{P_n \cap Id(A)} \oplus \frac{P_n}{P_n \cap Id(B)},$$

defined by $\varphi(a) = (a + P_n \cap Id(A), a + P_n \cap Id(B))$, has kernel $P_n \cap Id(A) \cap Id(B)$. Thus, since $Id(A \oplus B) = Id(A) \cap Id(B)$, we have that $\frac{P_n}{P_n \cap Id(A \oplus B)}$ embeds into $\frac{P_n}{P_n \cap Id(A)} \oplus \frac{P_n}{P_n \cap Id(B)}$ and $c_n(A \oplus B) \leq c_n(A) + c_n(B)$ follows. ∎

THEOREM 1: *Let $\mathcal{V}$ be a proper variety of algebras. Then there exists a finite number of reduced superalgebras $B_1, \ldots, B_t$ and a finite dimensional superalgebra D such that*

$$\mathcal{V} = \mathrm{var}(G(B_1) \oplus \cdots \oplus G(B_t) \oplus G(D))$$

where $\exp(\mathcal{V}) = \exp(G(B_1)) = \cdots = \exp(G(B_t))$ *and* $\exp(G(D)) < \exp(\mathcal{V})$.

Proof: Let A be a finite dimensional superalgebra such that $Id(\mathcal{V}) = Id(G(A))$. Write $A = A_1 \oplus \cdots \oplus A_s + J$ where $A_1, \ldots, A_s$ are simple superalgebras and $J = J(A)$. Suppose $\exp(\mathcal{V}) = d$. Then, as it was mentioned above, there exist distinct simple superalgebras $A_{i_1}, \ldots, A_{i_k}$ such that

$$A_{i_1} J \cdots J A_{i_k} \neq 0 \quad \text{and} \quad \dim_F(A_{i_1} \oplus \cdots \oplus A_{i_k}) = d.$$

Let $L_1, \ldots, L_t$ be all possible subsets of $\{1, \ldots, s\}$ with the following property: if, say, $L_j = \{i_1, \ldots, i_k\}$, then

$$\dim_F(A_{i_1} \oplus \cdots \oplus A_{i_k}) = d \quad \text{and} \quad A_{\sigma(i_1)} J A_{\sigma(i_2)} J \cdots J A_{\sigma(i_k)} \neq 0,$$

for some permutation $\sigma \in S_k$. For any such L_j, $j = 1, \ldots, t$, then define $B_j = A_{i_1} \oplus \cdots \oplus A_{i_k} + J$. By the characterization of the exponent, it follows that $\exp(G(B_1)) = \cdots = \exp(G(B_t)) = d = \exp(G(A))$.

Let $D_1, \ldots, D_p$ be all subalgebras of A of the type $A_{j_1} \oplus \cdots \oplus A_{j_q} + J$ where $1 \leq j_1 < \cdots < j_q \leq s$ and $\dim(A_{j_1} \oplus \cdots \oplus A_{j_q}) < d$. If we set $D = D_1 \oplus \cdots \oplus D_p$, then $\exp(G(D)) < \exp(G(A))$.

We are left to show that $\mathrm{var}(G(B_1)\oplus\cdots\oplus G(B_t)\oplus G(D)) = \mathrm{var}(G(A))$. Since for $i=1,\dots,t$, $G(B_i), G(D) \in \mathrm{var}(G(A))$, then

$$\mathrm{var}((G(B_1)\oplus\cdots\oplus G(B_t)\oplus G(D)) \subseteq \mathrm{var}(G(A)).$$

Let $f = f(x_1,\dots,x_n)$ be a multilinear polynomial and suppose that $f \notin Id(G(A))$. Then there exist $a_1,\dots,a_n \in A, g_1,\dots,g_n \in G$ such that

$$f(a_1\odot g_1,\dots,a_n\odot g_n) \neq 0.$$

Since f is multilinear, we may assume that

$$a_1\odot g_1,\dots,a_n\odot g_n \in A^{(0)}\odot G^{(0)} \oplus A^{(1)}\odot G^{(1)} = G(A)$$

are homogeneous in the $\mathbb{Z}_2$-grading. It follows that

$$f(a_1\odot g_1,\dots,a_n\odot g_n) = f^*(a_1,\dots,a_n)\odot g_1\cdots g_n$$

where $f^*(x_1,\dots,x_n)$ is a multilinear polynomial which differs from f only on the sign of some of its coefficients (see [11, Lemma 1.1]). Clearly $f^*(a_1,\dots,a_n)\neq 0$. We may also assume that $a_1,\dots,a_n \in A_1\cup\cdots\cup A_s\cup J$. Since for $i\neq j$, $A_iA_j=0$, by the property of d described above, we must have that

$$a_1,\dots,a_n \in A_{i_1}\oplus\cdots\oplus A_{i_k} + J$$

for some $A_{i_1},\dots,A_{i_k}$ such that $\dim(A_{i_1}\oplus\cdots\oplus A_{i_k}) \le d$. It follows that f is not an identity for one of the algebras $G(B_1),\dots,G(B_t)$, $G(D)$. Hence $\mathrm{var}(G(A)) \subseteq \mathrm{var}(G(B_1)\oplus\cdots\oplus G(B_t)\oplus G(D))$ and the proof is complete. ∎

In case of varieties generated by a finite dimensional algebra, the previous theorem has a simplified form as follows. Recall that we may regard an algebra as a superalgebra with trivial grading

Corollary 1: *Let A be a finite dimensional algebra. Then there exist a finite number of reduced algebras $B_1,\dots,B_t$ and a finite dimensional algebra D such that* $\mathrm{var}(A) = \mathrm{var}(B_1\oplus\cdots\oplus B_t\oplus D)$ *and* $\exp(A) = \exp(B_1) = \cdots = \exp(B_t)$, $\exp(D) < \exp(A)$.

Another application of Theorem 1 is given in terms of codimensions. Recall that if $f(n)$ and $g(n)$ are two functions of a natural argument, then we say that $f(n)$ and $g(n)$ are asymptotically equal and we write $f(n)\simeq g(n)$ if $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 1$. Then we have

COROLLARY 2: *For any proper variety $\mathcal{V}$, there exists a finite number of reduced superalgebras $B_1, \dots, B_t$ such that*

$$c_n(\mathcal{V}) \simeq c_n(G(B_1) \oplus \cdots \oplus G(B_t)).$$

Proof: Let $\mathcal{V} = \mathrm{var}(G(B_1) \oplus \cdots \oplus G(B_t) \oplus G(D))$ as in the theorem above. By Lemma 1 we have that

$$c_n(G(B_1) \oplus \cdots \oplus G(B_t)) \leq c_n(\mathcal{V}) \leq c_n(G(B_1) \oplus \cdots \oplus G(B_t)) + c_n(G(D)).$$

Recalling that $\exp(G(D)) < \exp(G(B_1)) = \exp(G(B_1) \oplus \cdots \oplus G(B_t))$, we have the asymptotic equality $c_n(\mathcal{V}) \simeq c_n(G(B_1) \oplus \cdots \oplus G(B_t))$. ∎

3. Evaluating polynomials

In this section we study the case of a finite dimensional reduced algebra of a special type. Throughout this section we assume that

$$R = A + J$$

where $A = M_k(F)$ is the algebra of $k \times k$ matrices over F and $J = J(R)$.

We shall determine the standard and the Capelli identities of minimal degree vanishing on R. We start with the following key lemma.

LEMMA 2: *The Jacobson radical J can be decomposed into the direct sum of four A-bimodules*

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where, for $p, q \in \{0,1\}$, J_{pq} is a left faithful module or a 0-left module according as $p = 1$ or $p = 0$, respectively. Similarly, J_{pq} is a right faithful module or a 0-right module according as $q = 1$ or $q = 0$, respectively. Moreover, for $p, q, i, l \in \{0,1\}$, $J_{pq}J_{ql} \subseteq J_{pl}$, $J_{pq}J_{il} = 0$ for $q \neq i$ and there exists a finite dimensional nilpotent algebra N such that $J_{11} \cong A \odot_F N$ (isomorphism of A-bimodules and of algebras).

Proof: Let e be the unit element of $A = M_k(F)$. Denote by $L_e, R_e\colon J \to J$ the linear transformations of J of left and right multiplication by e, respectively. Since $L_e^2 = L_e$ and $R_e^2 = R_e$, they both are diagonalizable linear transformations with eigenvalues 0 and 1. Moreover, $L_eR_e = R_eL_e$ and J decomposes into the sum of its eigenspaces $J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$, as desired. The inclusions $J_{pq}J_{ql} \subseteq J_{pl}$ and the equalities $J_{pq}J_{il} = 0$ for $i \neq q$ are clear.

Let now $J_{11} = V_1 \oplus \cdots \oplus V_m$ be the decomposition of J_{11} into irreducible A-bimodules. Each irreducible V_i is isomorphic to ${}_AA_A$; hence V_i contains a non-zero element d_i (unique up to a scalar) commuting with A and $V_i = Ad_i$. Moreover, for every $i,j \in \{1,\ldots,m\}$, d_id_j commutes with A, hence d_id_j is a linear combination of $d_1,\ldots,d_m$. It follows that $N = \mathrm{Span}\{d_1,\ldots,d_m\}$ is a subalgebra of R and $J_{11} = AN \cong A \otimes_F N$. ∎

LEMMA 3: *Suppose that $J_{01} \oplus J_{10} \neq 0$. Then $St_{2k} \notin Id(R)$ and $C_{k^2+1} \not\subseteq Id(R)$.*

Proof: Suppose $J_{10} \neq 0$ and let $d \in J_{10}$, $d \neq 0$. Then $dA = 0$ and $e_{ii}d \neq 0$ for some $i \in \{1,\ldots,k\}$.

If $k = 1$, the conclusion of the lemma follows; therefore assume that $k \geq 2$.

Since St_{2k-1} is not an identity for $A = M_k(F)$, there exist elements $a_1,\ldots,a_{2k-1} \in A$ such that $St_{2k-1}(a_1,\ldots,a_{2k-1}) = e_{ji}$ for some $j \neq i$ (for instance, the staircase $e_{i+1,i+1}, e_{i+1,i+2},\ldots,e_{nn},e_{n1},e_{11},\ldots,e_{ii}$ will do). But then

$$e_{ij}St_{2k}(a_1,\ldots,a_{2k-1},d) = e_{ij}St_{2k-1}(a_1,\ldots,a_{2k-1})d = e_{ij}e_{ji}d = e_{ii}d \neq 0$$

and $St_{2k} \not\equiv 0$ on A.

It is well known that $M_k(F)$ does not satisfy the k^2-th Capelli polynomial. Also, there exist $a_1,\ldots,a_{k^2},b_1,\ldots,b_{k^2-1} \in A$ such that

$$c_{k^2}(a_1,\ldots,a_{k^2};b_1,\ldots,b_{k^2-1}) = e_{kk}$$

(see, for instance, [16, Proposition 1.4.7]). We now compute

$$\begin{aligned} &c_{k^2+1}(a_1,\ldots,a_{k^2},d;b_1,\ldots,b_{k^2-1},e) \\ &= c_{k^2}(a_1,\ldots,a_{k^2};b_1,\ldots,b_{k^2-1})ed = e_{kk}ed = e_{kk}d \neq 0. \end{aligned}$$

Hence $c_{k^2+1} \not\equiv 0$ on A. A similar proof holds in case $J_{01} \neq 0$. ∎

LEMMA 4: *Write $J_{11} \cong A \otimes_F N$ as in Lemma 2. If N is not commutative, then $St_{2k+1} \notin Id(R)$ and $C_{k^2+2} \not\subseteq Id(R)$.*

Proof: Let $J_{11} = AN \cong A \otimes_F N$ be as in Lemma 2 and pick $d_1, d_2 \in N$ such that $d_1d_2 \neq d_2d_1$. We claim that for any choice of $x_1,\ldots,x_{2k-1} \in A$,

$$St_{2k+1}(x_1,\ldots,x_{2k-1},d_1,d_2) = \gamma St_{2k-1}(x_1,\ldots,x_{2k-1})[d_1,d_2],$$

for some non-zero constant γ.

In order to simplify the notation, let us write $2k-1 = n$. Denote $\mathcal{A} = \sum_{\sigma\in S_n}(\mathrm{sgn}\sigma)\sigma \in FS_n$ and recall the left action of S_n on the space of multilinear polynomials in $x_1,\ldots,x_n$.

Expand $St_{n+2}(x_1,\ldots,x_{2k-1},d_1,d_2)$ according to the last two variables in a monomial. There are the following three possibilities for these variables: (1) both are x's. (2) One x and one d. (3) Both are d's. Accordingly we have

$$\begin{gathered}St_{n+2}(x_1,\ldots,x_{2k-1},d_1,d_2)=\\ \mathcal{A}\Big(\sum_{1\le i<j\le n} x_1\cdots x_{i-1}d_1\cdots x_{j-1}d_2\cdots x_nx_ix_j - x_1\\ \cdots x_{i-1}d_2\cdots x_{j-1}d_1\cdots x_nx_ix_j\Big)\\ -\mathcal{A}\Big(\sum_{i=1}^{n} x_1\cdots x_{i-1}d_1\cdots x_nx_id_2 + x_1\cdots x_{i-1}d_2\cdots x_nd_1x_i\\ -x_1\cdots x_{i-1}d_2\cdots x_nx_id_1 - x_1\cdots x_{i-1}d_1\cdots x_nd_2x_i\Big)\\ +\mathcal{A}(x_1\ldots x_n(d_1d_2-d_2d_1)).\end{gathered}$$

Since d_1 and d_2 commute with $x_1,\ldots,x_n$, we obtain that

$$\begin{aligned}St_{n+2}(x_1,\ldots,x_n,d_1,d_2)=\sum_{1\le i<j\le n} & St_n(x_1,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,x_n,x_i,x_j)[d_1,d_2]\\ &+St_n(x_1,\ldots,x_n)[d_1,d_2]\end{aligned}$$

where $\widehat{x}$ means that the variable x is missing.

On the other hand,

$$St_n(x_1,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,x_n,x_i,x_j)=(-1)^{n-j+n-i-1}St_n(x_1,\ldots,x_n).$$

Since $(-1)^{n-j+n-i-1}=(-1)^{i+j-1}$ and

$$\sum_{i=1}^{j-1}(-1)^{i+j-1}=\begin{cases}1 & \text{if } j \text{ is even}\\ 0 & \text{if } j \text{ is odd}\end{cases}$$

we get that

$$St_{2k+1}(x_1,\ldots,x_n,d_1,d_2)=\gamma St_n(x_1,\ldots,x_n)[d_1,d_2]$$

where $\gamma=1+0+1+\cdots\neq 0$ as claimed.

Since $M_k(F)$ does not satisfy St_{2k-1}, it is clear that $St_{2k+1}\notin Id(R)$.

Let now $v_1,\ldots,v_{k^2}$ be an ordered basis of A consisting of all matrix units e_{ij} such that $v_1=e_{11}$ and let $a_0,a_1,\ldots,a_{k^2}\in A$ be such that

$$a_0v_1a_1\cdots a_{k^2-1}v_{k^2}a_{k^2}=e_{11}$$

and

$$a_0 v_{\sigma(1)} a_1 \cdots a_{k^2-1} v_{\sigma(k^2)} a_{k^2} = 0$$

for all $\sigma \in S_{k^2}$, $\sigma \neq 1$.

Recalling that d_1 and d_2 commute with A, we now take $d_1' = e_{11}d_1, d_2' = e_{11}d_2$ and compute

$$\begin{aligned} &c_{k^2+2}(d_1', v_1, \ldots, v_{k^2}, d_2'; a_0, a_1, \ldots, a_{k^2}) \\ &= d_1'e_{11}d_2' - d_2'e_{11}d_1' - d_1'd_2'e_{11} - e_{11}d_1'd_2' + e_{11}d_2'd_1' + d_2'd_1'e_{11} \\ &= e_{11}[d_2, d_1] \neq 0. \end{aligned}$$

Hence $c_{k^2+2} \notin Id(R)$ and the proof is complete. ∎

LEMMA 5: *If $J_{01}J_{10} + J_{10}J_{01} + J_{10}J_{00} + J_{00}J_{01} \neq 0$, then $St_{2k+1} \notin Id(R)$ and $C_{k^2+2} \not\subseteq Id(R)$.*

Proof: Suppose first that $k = 1$. If $J_{10}J_{01} \neq 0$, let $a \in J_{10}, b \in J_{01}$ be such that $ab \neq 0$.

Then, if $1 = 1_{M_k(F)}$, $1a = a, b1 = b$ and $a1 = 1b = 0$. It follows that $St_3(1, a, b) = 2ab + ba$. Since $ab \in J_{10}J_{01} \subseteq J_{11}$, $ba \in J_{01}J_{10} \subseteq J_{00}$, and $J_{11} \cap J_{00} = 0$, we obtain that $St_3 \notin Id(R)$ and so $C_3 \not\subseteq Id(R)$. The other cases $J_{01}J_{10} \neq 0, J_{10}J_{00} \neq 0, J_{00}J_{01} \neq 0$ are dealt with similarly.

Suppose now that $k \geq 2$ and let $J_{01}J_{10} \neq 0$. If $u \in J_{01}, v \in J_{10}$ are such that $uv \neq 0$, then there exists $e_{ii} \in M_k(F)$ such that $ue_{ii}v \neq 0$ for some $i \in \{1, \ldots, k\}$. Let $j \neq i$ and, as in the proof of Lemma 3, pick matrix units $a_1, \ldots, a_{2k-1} \in M_k(F)$ such that $St_{2k-1}(a_1, \ldots, a_{2k-1}) = e_{ij}$. Then, since $M_k(F)u = vM_k(F) = 0$, we obtain

$$\begin{aligned} &St_{2k+1}(ue_{ii}, a_1, \ldots, a_{2k-1}, e_{ji}v) \\ &= ue_{ii}St_{2k-1}(a_1, \ldots, a_{2k-1})e_{ji}v + f(a_1, \ldots, a_{2k-1}, e_{ji}vue_{ii}) \end{aligned}$$

where f is a suitable multilinear polynomial in $a_1, \ldots, a_{2k-1}, e_{ji}vue_{ii}$. Since $ue_{ii}St_{2k-1}(a_1, \ldots, a_{2k-1})e_{ji}v = ue_{ii}e_{ij}e_{ji}v = ue_{ii}v$ is a non-zero element of $J_{01}J_{10} \subseteq J_{00}$, $vu \in J_{10}J_{01} \subseteq J_{11}$ and $J_{00} \cap J_{11} = 0$, it follows that $St_{2k+1} \notin Id(R)$. In case $J_{01}J_{10} = 0$ and $J_{10}J_{01} \neq 0$, let $u \in J_{10}, v \in J_{01}$ be such that $uv \neq 0$. There exist $e_{ii}, e_{jj} \in M_k(F)$ such that $e_{ii}uve_{jj} \neq 0$. Since $e_{ii}uve_{jj} = e_{ik}e_{ki}e_{ii}uve_{jj}e_{jk}e_{kj}$, by replacing u with $e_{ki}u$ and v with ve_{jk}, we may assume that $e_{kk}u = u$ and $ve_{kk} = v$.

We now compute

$$St_{2k+1}(e_{11}, e_{12}, e_{22}, \ldots, e_{k-1,k}, e_{kk}, u, v) = 2e_{1k}uv \neq 0$$

since $vu \in J_{01}J_{10} = 0$. Hence $St_{2k+1} \notin Id(R)$ in this case. Similarly one can show that the same conclusion holds in case $J_{10}J_{00} \neq 0$ or $J_{00}J_{01} \neq 0$.

We next show that $c_{k^2+2} \notin Id(R)$. Suppose, for instance, that $J_{01}J_{10} \neq 0$ and let $u \in J_{01}, v \in J_{10}, uv \neq 0$. As in the proof of Lemma 4, there exists an ordered basis $\{b_1, \ldots, b_{k^2}\}$ of A consisting of all matrix units e_{ij}, and $a_0, \ldots, a_{k^2} \in A$ such that

$$a_0 c_{k^2}(b_1, \ldots, b_{k^2}; a_1, \ldots, a_{k^2-1})a_{k^2} = e_{11}.$$

We may assume, as shown above, that $ue_{11}v \neq 0$; hence

$$c_{k^2+2}(u, b_1, \ldots, b_{k^2}, v; a_0, a_1, \ldots, a_{k^2}) = ue_{11}v \neq 0$$

and $c_{k^2+2} \notin Id(R)$. The other cases are proved similarly. ∎

4. Asymptotics for the standard identities

In this section we shall prove our main results about the standard identity and its asymptotics. We start by examining a finite dimensional algebra of a special type

Lemma 6: *Let $R = A + J(R)$ where $A = M_k(F)$ and R satisfies S_{2k+1} or C_{k^2+2}. If $J_{01}J_{10} = J_{10}J_{01} = J_{10}J_{00} = J_{00}J_{01} = 0$, then $\mathrm{var}(R) = \mathrm{var}(A_1 \oplus A_2 \oplus J_{00})$ where $A_1 = A + J_{10}$ and $A_2 = A + J_{01}$.*

Proof: Clearly $Id(R) \subseteq Id(A_1 \oplus A_2 \oplus J_{00})$. Let now $f = f(x_1, \ldots, x_n)$ be a multilinear polynomial such that $f \notin Id(R)$.

Suppose first that

$$f \in Id(A + J_{11} + J_{10}) \cap Id(A + J_{11} + J_{01}) \cap Id(J_{00})$$

and let $b_1, \ldots, b_n \in R$ be such that $f(b_1, \ldots, b_n) \neq 0$. We may assume by linearity that $b_1, \ldots, b_n$ belong to $A \cup J_{10} \cup J_{01} \cup J_{11} \cup J_{00}$. By the assumption, $b_1, \ldots, b_n$ do not belong, at the same time, to $A \cup J_{11} \cup J_{10}$ or to $A \cup J_{11} \cup J_{01}$ or to J_{00}. Thus there exist b_i, b_j, $i \neq j$, such that one of the following three possibilities occurs: (1) $b_i \in J_{10}$ and $b_j \in J_{01}$. (2) $b_i \in J_{10}$ and $b_j \in J_{00}$. (3) $b_i \in J_{01}$ and $b_j \in J_{00}$. Since the J_{kl}'s are A-bimodules, $J_{01}J_{10} = J_{10}J_{01} = J_{10}J_{00} = J_{00}J_{01} = 0$ and, by Lemma 2, $J_{01}J_{00} = J_{00}J_{10} = J_{00}J_{11} = J_{11}J_{00} = J_{01}J_{01} = J_{10}J_{10} = 0$; we have that each of the above three cases leads to $b_{\sigma(1)} \cdots b_{\sigma(n)} = 0$ for all $\sigma \in S_n$. Thus $f \in Id(R)$, contrary to the assumption.

We have proved that $Id(R) \supseteq Id(A+J_{11}+J_{10}) \cap Id(A+J_{11}+J_{01}) \cap Id(J_{00})$. If we prove that $Id(A+J_{11}+J_{10}) = Id(A+J_{10})$ and $Id(A+J_{11}+J_{01}) = Id(A+J_{01})$, we would get that $Id(R) \supseteq Id(A_1) \cap Id(A_2) \cap Id(J_{00})$ and the proof would be complete.

In order to prove that $Id(A + J_{11} + J_{10}) = Id(A + J_{10})$, suppose that there exists $f(x_1, \ldots, x_n) \notin Id(A+J_{11}+J_{10})$ and let f be multilinear. Since $J_{11} = AN$, A commutes with N and N is commutative by Lemma 4, we have that for all $b_1, \ldots, b_m \in A + J_{11} + J_{10}$, $a \in A, d \in N$

$$b_1 \cdots b_k a d b_{k+1} \cdots b_m = d b_1 \cdots b_k a b_{k+1} \cdots b_m.$$

It follows that if $b_1, \ldots, b_n \in A \cup J_{11} \cup J_{10}$ are such that $f(b_1, \ldots, b_n) \neq 0$, then we can write

$$f(b_1, \ldots, b_n) = d' f(b'_1, \ldots, b'_n)$$

for some $d' \in N$, $b'_1, \ldots, b'_n \in A \cup J_{10}$. Thus $f \notin Id(A + J_{10})$ and

$$Id(A + J_{11} + J_{10}) = Id(A + J_{10})$$

follows. Similarly, one can show that $Id(A + J_{11} + J_{01}) = Id(A + J_{01})$. This completes the proof of the lemma. ∎

For $m \geq 1$ let us denote by $\mathrm{var}(St_m)$ the variety of associative algebras defined by the standard identity St_m. Also, for $k, l \geq 1$, let us denote by $M_{k \times l}(F)$ the algebra of $(k+l) \times (k+l)$ matrices having the last l rows and the last k columns equal to zero.

THEOREM 2:

(1) $\mathrm{var}(St_{2k}) = \mathrm{var}(M_k(F) \oplus B)$ *for some finite dimensional algebra* B *such that* $\exp(B) < k^2$. *In particular*

$$c_n(St_{2k}) \simeq c_n(M_k(F)).$$

(2) $\mathrm{var}(St_{2k+1}) = \mathrm{var}(M_{k \times 2k}(F) \oplus M_{2k \times k}(F) \oplus B)$ *for some finite dimensional algebra* B *with* $\exp(B) < k^2$. *In particular*

$$c_n(St_{2k+1}) \simeq c_n(M_{k \times 2k}(F) \oplus M_{2k \times k}(F)).$$

Proof: We know by [3] that $\exp(St_{2k}) = \exp(St_{2k+1}) = k^2$. Also, by [11, Theorem 2.3], for any $q \geq 1$, $\mathrm{var}(St_q)$ is generated by a finite dimensional algebra. Thus, by Corollary 1, there exists a finite number of finite dimensional reduced algebras $B_1, \ldots, B_t$ (see Definition 1) and a finite dimensional algebra D such that

$$\mathrm{var}(St_q) = \mathrm{var}(B_1 \oplus \cdots \oplus B_t \oplus D) \tag{1}$$

and $\exp(B_1) = \cdots = \exp(B_t) = k^2$, $\exp(D) < k^2$.

The strategy of the proof will be as follows. We shall first analyze the structure of a finite dimensional reduced algebra R such that $St_q \in Id(R)$. We shall next split the proof into the two cases $q = 2k$ and $q = 2k+1$. In each case we shall plug into (1) the structure of each reduced algebra and we will deduce the desired result.

Let R be a finite dimensional reduced algebra satisfying $St_q \equiv 0$ ($q = 2k$ or $q = 2k+1$). Write $R = A_1 \oplus \cdots \oplus A_m + J$ where $A_1, \ldots, A_m$ are simple subalgebras and $J = J(R)$. By the definition of reduced algebra, $A_1 J A_2 J \cdots J A_m \neq 0$. Hence by [8] (see also [7]), R contains a subalgebra isomorphic to the upper block triangular matrix algebra

$$UT(d_1, \ldots, d_m) = \begin{pmatrix} M_{d_1}(F) & & & * \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & \cdots & 0 & M_{d_m}(F) \end{pmatrix}$$

where $A_i \cong M_{d_i}(F)$ for all $i = 1, \ldots, m$ and

$$\exp(R) = \exp(UT(d_1, \ldots, d_m)) = d_1^2 + \cdots + d_m^2.$$

Clearly $UT(d_1, \ldots, d_m)$ does not satisfy St_{2d-1} where $d = d_1 + \cdots + d_m$. Hence $UT(d_1, \ldots, d_m)$ satisfies St_q for $q = 2k$ or $q = 2k+1$ only if $d_1 + \cdots + d_m \leq k$. On the other hand, $d_1^2 + \cdots + d_m^2 < k^2$ for any $m > 1$. It follows that if R is a reduced algebra with $\exp(R) = k^2$ satisfying $St_q \equiv 0$ ($q = 2k$ or $q = 2k+1$), then $R \cong M_k(F) + J$.

We now split the proof into two cases according as $q = 2k$ or $q = 2k - 1$.

Case 1: Suppose that $q = 2k$ and let $R = A + J$ be a reduced algebra as above where $A = M_k(F)$. Write $J = J_{00} + J_{01} + J_{10} + J_{11}$ as in Lemma 2. Since $St_{2k} \in Id(R)$, by Lemmas 3 and 4, $J_{10} + J_{01} = 0$ and $J_{11} = AN$ where N is commutative. Since $AJ_{00} = J_{00}A = J_{11}J_{00} = J_{00}J_{11} = 0$ and $J_{10} = J_{01} = 0$, we obtain that $R = (A + J_{11}) \oplus J_{00}$ with $J_{00} \subseteq J$ a nilpotent ideal of R. Moreover, by Lemma 2, $A + J_{11} = A + AN \cong A \otimes_F N^\sharp$ where $N^\sharp$ is the algebra obtained from N by adjoining a unit element. Since $N^\sharp$ is commutative, it follows that $A + J_{11}$ and A satisfy the same identities. Thus $\mathrm{var}(R) = \mathrm{var}(A \oplus J_{00})$.

We have proved that if R is any reduced algebra such that $St_{2k} \in Id(R)$ and $\exp(R) = k^2$, then $\mathrm{var}(R) = \mathrm{var}(M_k(F) \oplus J_{00})$ with J_{00} a finite dimensional nilpotent algebra. But then, by recalling the decomposition given in (1), we get that $\mathrm{var}(St_{2k}) = \mathrm{var}(M_k(F) \oplus D')$ where D' is a finite dimensional algebra with $\exp(D') < \exp(M_k(F)) = k^2$.

CASE 2: Suppose now that $q = 2k+1$ and let $R = A + J$ be a reduced algebra with $A = M_k(F)$ and R satisfies St_{2k+1}. In this case, by Lemma 5, we get that $J_{01}J_{10} = J_{10}J_{01} = J_{10}J_{00} = J_{00}J_{01} = 0$. Hence, by Lemma 6, $\mathrm{var}(R) = \mathrm{var}((A + J_{10}) \oplus (A + J_{01}) \oplus J_{00})$ and J_{00} is a nilpotent algebra. Now, the left A-module J_{10} is isomorphic to $t \geq 1$ copies of a left ideal of $M_k(F)$. Since $J_{10}A = J_{10}J_{10} = 0$ and $A = M_{k\times k}(F)$, then $A + J_{10}$ as an F-algebra is isomorphic to $M_{k\times(k+t)}(F)$ and it is clear that $A + J_{10}$ has the same identities as $M_{k\times 2k}(F)$. Similarly, one shows that $A + J_{01}$ satisfies the same identities as $M_{2k\times k}(F)$.

We have proved that if R is any reduced algebra such that $St_{2k+1} \in Id(R)$, then $\mathrm{var}(R) = \mathrm{var}(M_{k\times 2k}(F) \oplus M_{k\times 2k}(F) \oplus J_{00})$ with J_{00} a finite dimensional nilpotent algebra. By invoking the decomposition given in (1), we get that

$$\mathrm{var}(St_{2k+1}) = \mathrm{var}(M_{k\times 2k}(F) \oplus M_{2k\times k}(F) \oplus B)$$

where $\exp(B) \leq k^2 - 1$. ∎

COROLLARY 3: *$c_n(St_{2k}) \simeq \alpha n^{(1-k^2)/2} k^{2n}$ and*

$$\frac{\alpha}{k^2} n^{(3-k^2)/2} k^{2n} \lesssim c_n(St_{2k+1}) \lesssim \frac{2\alpha}{k^2} n^{(3-k^2)/2} k^{2n}$$

where

$$\alpha = \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \left(\frac{1}{2}\right)^{\frac{1}{2}(k^2+1)} \cdot 1!2!\cdots(k-1)!k^{\frac{1}{2}(k^2+4)}.$$

Proof: The precise asymptotics for $c_n(St_{2k})$ follow from the relation $c_n(St_{2k}) \simeq c_n(M_k(F))$ and the result in [15] where the asymptotics for $c_n(M_k(F))$ were computed explicitly.

Let $I_1 = Id(M_{k\times 2k}(F))$ and $I_2 = Id(M_{2k\times k}(F))$. From [10] it follows that $I_1 = Id(M_k(F))\cdot F\langle X\rangle$, $I_2 = F\langle X\rangle \cdot Id(M_k(F))$ and $c_n(I_1) = c_n(I_2) = nc_{n-1}(M_k(F))$. Since $\langle St_{2k+1}\rangle_T = I_1 \cap I_2 \cap I_3$ where, by Theorem 2, I_3 is a T-ideal with $c_n(I_3) \lesssim (k^2-1)^n$, we get

$$c_n(I_1) \leq c_n(St_{2k+1}) \leq c_n(I_1) + c_n(I_2) + c_n(I_3) \simeq 2c_n(I_1).$$

The second part of the corollary now follows from the asymptotics of $c_n(M_k(F))$. ∎

5. Asymptotics for the Capelli identities

In [13] it was shown that $\exp(C_{m+1}) = m, m-1, m-2$ or $m-3$.

Also, $\exp(C_{m+1}) = m$ if and only if $m = k^2$ is a square and, if m is not a square, $\exp(C_{m+1}) = m-1$ if and only if $m-1$ is a square or the sum of two squares.

In this section we shall study $\mathrm{var}(C_{m+1})$ and the asymptotics of the corresponding codimensions in case m is a square or $m-1$ is a square or the sum of two squares. By the above, this will imply the cases $\exp(C_{m+1}) = m$ or $m-1$.

We start with the easy case $\exp(C_{m+1}) = m$. In fact we have

THEOREM 3: *Let $m = k^2$. Then $\mathrm{var}(C_{m+1}) = \mathrm{var}(M_k(F) \oplus B)$ for some finite dimensional algebra B such that $\exp(B) < k^2$.*

Proof: By [11, Theorem 2.3], $\mathrm{var}(C_{k^2+1})$ is generated by a finite dimensional algebra and, by [3], $\exp(C_{k^2+1}) = k^2$. Thus, by Corollary 1, there exist finite dimensional algebras $B_1, \ldots, B_t, D$ such that

$$\mathrm{var}(C_{k^2+1}) = \mathrm{var}(B_1 \oplus \cdots \oplus B_t \oplus D) \tag{2}$$

where $B_1, \ldots, B_t$ are reduced and $\exp(B_1) = \cdots = \exp(B_t) = k^2$, $\exp(D) < k^2$. We next analyze the structure of any such reduced algebra.

Let R be a finite dimensional reduced algebra such that $C_{k^2+1} \in Id(R)$ and $\exp(R) = k^2$. As in the proof of the previous theorem, R contains a subalgebra of the type $UT(d_1, \ldots, d_t)$ with $\exp(R) = d_1^2 + \cdots + d_t^2 = k^2$. It is not difficult to show (see also [13]) that $UT(d_1, \ldots, d_t)$ does not satisfy $c_{d_1^2+\cdots+d_t^2+t-1}$. Hence $t = 1, d_1 = k$ and we may write $R = A + J$ where $A = M_k(F), J = J(R)$. Since by Lemma 3, $J_{01} = J_{10} = 0$, then $J = J_{00} + J_{11}$ and, as noted in the proof of Theorem 2, we have that $R = (A + J_{11}) \oplus J_{00}$ with J_{00} a nilpotent ideal of R. Also, by Lemma 4, $J_{11} = AN$ where N is a nilpotent commutative subalgebra centralizing A in R. Hence, as in the proof of Theorem 2, A and $A + J_{11}$ have the same identities and $\mathrm{var}(R) = \mathrm{var}(A \oplus J_{00})$ follows.

By the decomposition given in (2), it follows that $\mathrm{var}(C_{k^2+1})$ is generated by $M_k(F) \oplus B$ where B is a finite dimensional algebra and $\exp(B) < k^2$. ∎

COROLLARY 4:

$$c_n(C_{k^2+1}) \simeq c_n(St_{2k}) \simeq \alpha n^{(1-k^2)/2} k^{2n}$$

where α is the constant of Corollary 3.

We now start to study the case $\exp(C_{m+1}) = m - 1$.

LEMMA 7: *Let $R = A + J$ where $A = M_k(F)$ and $J = J(R)$. If $R \in \mathrm{var}(C_{m+1})$ where $m - 1 = k^2$, then*

$$\mathrm{var}(R) = \mathrm{var}(M_{k\times 2k}(F) \oplus M_{2k\times k}(F) \oplus B)$$

with B a finite dimensional algebra such that $\exp(B) < m - 1$.

Proof: By Lemmas 4 and 5, we have that $J_{01}J_{10} = J_{10}J_{01} = J_{10}J_{00} = J_{00}J_{01} = 0$ and $J_{11} = AN$ where N is commutative and centralizes A. By Lemma 6, $\mathrm{var}(R) = \mathrm{var}(A_1 \oplus A_2 \oplus J_{00})$ where $A_1 = A + J_{10}$ and $A_2 = A + J_{01}$. As in the proof of Theorem 2, we see that $\mathrm{var}(A_1) = \mathrm{var}(M_{k\times 2k}(F))$ and $\mathrm{var}(A_2) = \mathrm{var}(M_{2k\times k}(F))$ and the conclusion follows. ∎

In the next two lemmas we examine the case when $m - 1$ is the sum of two squares and R is a finite dimensional algebra whose semisimple part has only two simple components.

LEMMA 8: *Suppose $m - 1 = k_1^2 + k_2^2$ and let $R = A \oplus B + J$ where $A = M_{k_1}(F), B = M_{k_2}(F)$ and $J = J(R)$. If $R \in \mathrm{var}(C_{m+1})$ and $AJB \neq 0$, then $\mathrm{var}(R) = \mathrm{var}(A_1 \oplus A_2 \oplus D)$ where $A_1 = A + B + AJA + BJB + AJB$, $A_2 = A + B + AJA + BJB + BJA$ and $\exp(D) < m - 1$.*

Proof: We first claim that

$$BJAJB = 0 \quad \text{and} \quad AJBJA = 0. \tag{3}$$

Suppose, by contradiction, that there exist elements $x, y \in J$ such that $BxAyB \neq 0$. If 1_A and 1_B denote the unit elements of A and B respectively, then we may assume that $1_Bx = x1_A = x$ and $1_Ay = y1_B = y$. Hence

$$Ax = xB = yA = By = 0. \tag{4}$$

As in the proof of Lemma 3, pick matrices

$$u_1, \ldots, u_{k_1^2}, a_0, a_1, \ldots, a_{k_1^2-1} \in A,\ v_1, \ldots, v_{k_2^2}, b_0, b_1, \ldots, b_{k_2^2-1} \in B$$

with the property that

$$xa_0u_1a_1u_2\cdots a_{k_1^2-1}u_{k_1^2}1_Ayb_0v_1b_1v_2\cdots b_{k_2^2-1}v_{k_2^2} \neq 0$$

and any non-trivial permutation of $u_1, \ldots, u_{k_1^2}, v_1, \ldots, v_{k_2^2}$ in the above product is zero. Then from (4) it follows that

$$c_{k_1^2+k_2^2+2}(x, u_1, \ldots, u_{k_1^2}, y, v_1, \ldots, v_{k_2^2}; a_0, a_1, \ldots, a_{k_1^2-1}, 1_A, b_0, b_1, \ldots, b_{k_2^2-1}) \neq 0,$$

a contradiction. The second equality in (3) is proved similarly.

Next we show that if $xAyB \neq 0$, for some $x, y \in J$, then we may take $x \in AxA \subseteq AJA$. In fact, since B is a unitary algebra, $Bx \neq 0$ implies $BxAyB \neq 0$, and this contradicts (3). Hence $Bx = 0$. If also $Ax = 0$, then, as above, it would follow that $c_{k_1^2+k_2^2+2}$ is not an identity for R. Therefore $Ax \neq 0$ and $x \in AJA$.

Similarly, it can be proved that if $AxBy \neq 0$, for some $x, y \in J$, then we may take $y \in BJB$.

Consider now a non-zero product of the type

$$a_1 d_1 \cdots a_{m-1} d_{m-1} a_m$$

where $a_1, \ldots, a_m \in A \cup B$ and $d_1, \ldots, d_{m-1} \in J$. Then, by (3), either $a_1, \ldots, a_m \in A$ or $a_1, \ldots, a_m \in B$ or there exists $1 \leq k \leq m$ such that $a_1, \ldots, a_{k-1} \in A$, $a_k, \ldots, a_m \in B$ or $a_1, \ldots, a_{k-1} \in B$, $a_k, \ldots, a_m \in A$.

Similarly, if

$$d_0 a_1 d_1 \cdots a_{m-1} d_{m-1} a_m \neq 0$$

where $a_1, \ldots, a_{k-1} \in A$, $a_k, \ldots, a_m \in B$ and $d_0, d_1, \ldots, d_{m-1} \in J$, then, by what we proved above, we may take $d_0 \in AJA$.

We next show that

$$\operatorname{var}(R) = \operatorname{var}(A_1 \oplus A_2 \oplus A_3 \oplus A_4)$$

where $A_3 = A + J$ and $A_4 = B + J$. Since $\exp(A_3 \oplus A_4) = \max\{k_1^2, k_2^2\} < k_1^2 + k_2^2$, this will complete the proof of the lemma.

Let $f \notin Id(R)$, a multilinear polynomial, and suppose that $f \in Id(A_3) \cap Id(A_4)$. Since $AB = BA = 0$, in order to obtain a non-zero evaluation of f we must substitute at least one element $y \in J$ such that $AyB \neq 0$ or $ByA \neq 0$. As we remarked in the proof of (4), the element y can be taken in AJB or BJA, respectively. Taking into account the relation (3) and the above discussion, it follows that all the other variables must be evaluated in $A \cup B \cup AJA \cup BJB$. Thus either $f \notin Id(A_1)$ or $f \notin Id(A_2)$. This proves that

$$Id(R) \supseteq Id(A_1) \cap Id(A_2) \cap Id(A_3) \cap Id(A_4).$$

The other inclusion is obvious and the proof of the lemma is complete. ∎

LEMMA 9: *Let $m - 1 = k_1^2 + k_2^2$ and let $A_1, A_2 \in \operatorname{var}(C_{m+1})$ be the algebras defined in the previous lemma. Then there exist finite dimensional algebras D_1 and D_2 such that*

$$\operatorname{var}(A_1) = \operatorname{var}(UT(k_1, k_2) \oplus D_1), \quad \operatorname{var}(A_2) = \operatorname{var}(UT(k_2, k_1) \oplus D_2)$$

and $\exp(D_1), \exp(D_2) < m-1$.

Proof: Consider the algebra $A' = A + AJA$. By Lemma 2 we have that $AJA = (AJA)_{11} = AN \cong A \odot N$ for some nilpotent algebra N. Suppose first that N is non-commutative. Then, by Lemma 4, $C_{k_1^2+2} \not\subseteq Id(A')$. If $AJAJB \neq 0$, as in the proof of Lemma 8, we can prove that $c_{k_1^2+k_2^2+2} \notin Id(A_1)$, a contradiction. Hence $AJAJB = 0$. Since $AB = BA = 0$, we obtain that

$$\mathrm{var}(A_1) = \mathrm{var}((A + B + AJB + BJB) \oplus (A + AJA))$$

and $A+AJA$ has exponent less than $m-1$. In case N is commutative, $Id(AJA) \subseteq Id(A)$ implies that $\mathrm{var}(A_1) = \mathrm{var}(A + B + AJB + BJB)$.

By applying the same arguments to the summand BJB we obtain that $\mathrm{var}(A_1) = \mathrm{var}((A + B + AJB) \oplus D)$, for some finite dimensional algebra D with $\exp(D) < m-1$.

Now, as it was shown in [8], the algebra $A + B + AJB$ contains a subalgebra isomorphic to $UT(k_1, k_2)$. Hence $Id(A + B + AJB) \subseteq Id(UT(k_1, k_2))$. On the other hand, by [8], $Id(UT(k_1, k_2)) = I_1 I_2$ where $I_1 = Id(M_{k_1}(F))$ and $I_2 = Id(M_{k_2}(F))$. Since $(AJB)^2 = 0$, it is easy to see that if $f_1 \in I_1$ and $f_2 \in I_2$ then $f_1 f_2 \equiv 0$ is an identity for $A + B + AJB$. Hence $\mathrm{var}(A + B + AJB) = \mathrm{var}(UT(k_1, k_2))$ and the conclusion of the lemma follows for the algebra A_1. The proof for the algebra A_2 is obtained by making the obvious changes. ∎

THEOREM 4: *Let $m \neq k^2$ and suppose that $m-1$ is a square or the sum of two squares. Then*

$$\mathrm{var}(C_{m+1}) = \mathrm{var}(A \oplus B \oplus D)$$

where D is a finite dimensional algebra with $\exp(D) < m-1$,

$$A = \begin{cases} M_{r\times 2r}(F) \oplus M_{2r\times r}(F) & \text{if } m-1 = r^2 \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

and

$$B = \begin{cases} 0 & \text{if } m-1 \text{ is not the sum of two squares,} \\ \bigoplus_{s^2+t^2=m-1} UT(s,t) & \text{otherwise.} \end{cases}$$

Proof: By [11, Theorem 2.3], $\mathrm{var}(C_{m+1})$ is generated by a finite dimensional algebra. As in the proof of Theorem 2, by invoking Corollary 1, we need only examine finite dimensional reduced algebras in $\mathrm{var}(C_{m+1})$. We then apply Lemmas 7, 8 and 9 to complete the proof of the theorem. ∎

LEMMA 10: *Let $k^2 = s^2 + t^2$. Then*

$$\sqrt{n} c_n(M_{k\times 2k}(F)) \simeq \beta c_n(UT(s,t)),$$

for some constant β.

Proof: Since by [15], $c_n(M_k(F)) \simeq \alpha n^{(1-k^2)/2} k^{2n}$, by [10] we obtain that

$$c_n(M_{k\times 2k}(F)) = n c_{n-1}(M_k(F)) \simeq \alpha n^{(1-k^2)/2+1} k^{2n}.$$

On the other hand, by [8], $c_n(UT(s,t)) \simeq \beta n^{(1-s^2-t^2)/2+3/2}(s^2+t^2)^n$. ∎

COROLLARY 5: *Let $m \neq k^2$ and suppose that $m-1$ is a square or the sum of two squares. If $m-1$ is the sum of two squares, then there exist non-zero constants $\alpha_1 \leq \alpha_2$ such that*

$$\alpha_1 n^a (m-1)^n \lesssim c_n(C_{m+1}) \lesssim \alpha_2 n^a (m-1)^n$$

where $a = (5-m)/2$. If $m-1 = k^2$ is not the sum of two squares, then

$$\alpha n^b (m-1)^n \lesssim c_n(C_{m+1}) \lesssim 2\alpha n^b (m-1)^n$$

where $b = (4-m)/2$ and α is the constant of Corollary 3.

Proof: If $m-1$ is the sum of two squares then, by applying the previous lemma to the conclusion of Theorem 4, we can write

$$c_n(C_{m+1}) \simeq c_n(A_1 \oplus \cdots \oplus A_r)$$

where $A_1, \ldots, A_r$ are algebras of the type $UT(s,t)$ with $s^2+t^2 = m-1$. Hence

$$\max\{c_n(A_1), \ldots, c_n(A_r)\} \lesssim c_n(C_{m+1}) \lesssim c_n(A_1) + \cdots + c_n(A_r)$$

and the first part of the proof follows from the asymptotic equality (see [8])

$$\begin{aligned} c_n(UT(s,t)) &\simeq \alpha(s,t) n^{(1-s^2-t^2)/2+3/2}(s^2+t^2)^n \\ &= \alpha(s,t) n^{(5-m)/2}(m-1)^n, \end{aligned} \tag{5}$$

where $\alpha(s,t)$ is a constant depending on s and t.

Suppose now that $m-1 = k^2$ is not the sum of two squares. In this case, by Theorem 4, $c_n(C_{m+1}) \simeq c_n(M_{k\times 2k}(F) \oplus M_{2k\times k}(F))$ and the asymptotics for this sequence were estimated in Corollary 3 for the standard identity St_{2k+1}. ∎

We remark that the constants α_1 and α_2 in Corollary 5 can be computed using the relations (5). Namely,

$$\alpha_1 = \max\{\alpha(s,t) \mid s^2+t^2=m-1\}, \quad \alpha_2 = \sum_{s^2+t^2=m-1} \alpha(s,t)$$

where

$$\alpha(s,t) = \frac{\alpha(s)\alpha(t)}{s^{s^2-1}t^{t^2-1}(s^2+t^2)^{(4-s^2-t^2)/2}}$$

(see [8]) and

$$\alpha(k) = \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1}\left(\frac{1}{2}\right)^{\frac{1}{2}(k^2+1)} \cdot 1!2!\cdots(k-1)!k^{\frac{1}{2}(k^2+4)}.$$

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