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THE CHOQUET SIMPLEX OF INVARIANT MEASURES FOR MINIMAL FLOWS

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ABSTRACT

The set of invariant measures of a compact dynamical system is well known to be a nonempty compact metrizable Choquet simplex. It is shown that all such simplices are realized already for the class of minimal flows. Moreover, sufficient is the class of 0–1 Toeplitz flows. Previously, it is proved that the set of invariant measures of the regular Toeplitz flows contains homeomorphic copies of all metric compacta.

I. Introduction

The list of various examples of minimal dynamical systems which are not uniquely ergodic is long. We mention here only several of them. In 1952, Oxtoby [12] presented a construction of a minimal flow with exactly two ergodic measures. In 1969, Jacobs and Keane [7] introduced Toeplitz sequences, which include the example of Oxtoby and gave rise to further investigation of non-uniquely ergodic minimal flows (e.g. [10]). An example of a flow, ergodic measures of which form a closed arc, was given in 1981 by Katznelson and Weiss [8]. In [13], Williams gave a construction of a Toeplitz flow in Σ^Z with ergodic measures corresponding bijectively to the elements of the compact set Σ. Based on this construction, in [1] we give an example of a 0–1 Toeplitz flow with noncompact set of ergodic measures.

In this paper we show that the set of invariant measures of a minimal flow may have the affine-topological structure of an arbitrary metrizable compact Choquet simplex. Moreover, all such simplices are obtained for the class of 0–1 Toeplitz flows. In particular, by the result of Choquet and Haydon (e.g. [5]), the set of ergodic measures may have the topological structure of an arbitrary Polish space.

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II. Preliminaries

By a dynamical system (flow) we will mean a pair (X, S), where X is a compact metric space and S is a homeomorphism of X onto itself. A nonempty closed subset Y of X is called minimal if Y is invariant (i.e. $SY = Y$) and it has no proper closed invariant subsets. It is well known that then Y is equal to the orbit closure $\bar{O}(x) = \{S^j x : j \in Z\}^-$ of any $x \in Y$. Clearly, different minimal subsets of (X, S) are disjoint.

A probability Borel measure μ on X is called invariant if $\mu = S\mu$, where $S\mu$ is given by $S\mu(B) = \mu(S^{-1}B)$. An invariant measure is called ergodic if every invariant Borel set has the measure 0 or 1. The set of all invariant measures of (X, S) will be denoted by $M(X, S)$. The set $M(X, S)$ is well known to be a Choquet simplex and the ergodic measures are its extreme points. We will consider the weak* topology in $M(X, S)$ given by the convergence on continuous real-valued functions on X. In this setting $M(X, S)$ is compact. If $M(X, S)$ is a singleton then (X, S) is called uniquely ergodic. If the flow $(\bar{O}(x), S)$ is uniquely ergodic then we call x uniquely ergodic and by x^* we will denote the invariant measure of this flow.

Let $p = (p_n)$ be an increasing sequence of natural numbers. An element x in (X, S) is called p-generic whenever the weak* limit $\mu = \lim p_n^{-1} \sum_{i=0}^{p_n - 1} S^i \delta_x$ exists, where δ_x denotes the point measure at x. Then $\mu \in M(X, S)$. For $p_n \equiv n$ the (p_n)-generic points are called generic. If μ is an ergodic measure then μ-almost every $x \in X$ is generic for μ (see [4]). Let X be zero-dimensional and $\mathcal{U}$ be a family of clopen sets such that $\{S^j U : j \in Z,\ U \in \mathcal{U}\}$ is a base for X. If the trajectory of an element $x \in X$ visits every $U \in \mathcal{U}$ with p-density, i.e. if the limits

$$p(U) = \lim_n p_n^{-1} |\{0 \le j < p_n : S^j x \in U\}|$$

exist, then x is p-generic for an invariant measure μ such that $\mu(U) = p(U)$ for every $U \in \mathcal{U}$.

We establish an additional notation for the 0–1 shift flow, i.e. for $X = \{0,1\}^Z$ and S given by $S\eta(j) = \eta(j+1)$ $(j \in Z)$. Suppose Δ is some space and $\{a_j : j \in Z\}$ a fixed sequence of elements of Δ. Given a function $f : \Delta \to \{0,1\}$, by $\underline{f} \in \{0,1\}^Z$ we will denote the sequence $\underline{f}(j) = f(a_j)$. If $\underline{f}$ is uniquely ergodic for the shift flow, then the appropriate measure will be denoted by $\underline{f}^*$. If now h is a mapping of some set K into $\{0,1\}^\Delta$ then the mapping $\underline{h} : K \to \{0,1\}^Z$ is given by

$\underline{h}(x) = \underline{h}(\underline{x})$. If, moreover, each $\underline{h}(x)$ is uniquely ergodic then h^* is defined on K by $h^*(x) = (h(x))^*$.

III. Toeplitz sequences

As in [13], for any sequence $\eta \in \Sigma^Z$ (Σ compact) and any $p \in N$ we denote

$$\mathrm{Per}_p(\eta) = \{j \in Z : \eta(j) = \eta(k) \text{ whenever } j = k \bmod p\}$$

and

$$\mathrm{Aper}(\eta) = Z \setminus \bigcup_{p \in N} \mathrm{Per}_p(\eta).$$

A sequence η is called Toeplitz if $\mathrm{Aper}(\eta) = \varnothing$. It is known that the orbit closure of a Toeplitz sequence in the shift flow is minimal (see [7]). A sequence η for which $Z = \bigcup_{n \in N} \mathrm{Per}_{2^n}(\eta)$ will be called dyadic Toeplitz. In this section we will consider the case of $\Sigma = \{0,1\}$.

Consider the compact monothetic group Δ_2 of dyadic integers (see [6]). The element $\theta = (1,0,0,0\dots)$ is a topological generator of Δ_2, i.e. $\{j\theta : j \in Z\}$ is dense in Δ_2. Topologically Δ_2 can be identified with the classical Cantor set C by sending $t = (t_1, t_2, \dots)$ to the real number $\sum_{i=1}^{\infty} (2t_i/3^i)$. The natural ordering of C corresponds to the lexicographical order in Δ_2. The points $j\theta$ ($j \in Z$) are unilateral cluster points in Δ_2. Let $f: \Delta_2 \to \{0,1\}$ be continuous at each $j\theta$. As in [2, sect. 2], it follows that $\underline{f}$ given by $\underline{f}(j) = f(j\theta)$ is a dyadic Toeplitz sequence (see also [9]).

Remark 1. Consider an interval $(r,s) = \{t : r < t < s\}$ in Δ_2. Clearly, if neither r nor s is in $Z\theta$ then the indicator $\mathbf{1}_{(r,s)}$ is continuous at each $j\theta$. In the other case, it becomes continuous at each $j\theta$ if (r,s) is modified by attaching, if necessary, one or both endpoints.

Denote $t^0 = 2\theta = (0,1,0,0,0,\dots)$, $t^1 = (1,0,1,0,1,0,\dots)$, $t^2 = -\theta = (1,1,1,\dots)$.

Definition 1. A function $f: \Delta_2 \to \{0,1\}$ is said to have a core if $f = 0$ on $[t^0, t^1)$ and $f = 1$ on $(t^1, t^2]$.

Theorem 1. *Suppose f and g both have the core and are continuous at each $j\theta$. If the sequences $\underline{f}$ and $\underline{g}$ belong to the same minimal orbit closure, then $\underline{f} = \underline{g}$.*

Proof. Assume that $\underline{g} = \lim S^{n_k} \underline{f}$. Choose a subnet n_α of n_k such that $n_\alpha \theta$ converges to some s and the functions f_{n_α} defined by $f_{n_\alpha}(t) = f(n_\alpha \theta + t)$ converge pointwise on Δ_2 to a function g'. Since $\underline{f}_{n_\alpha} = S^{n_\alpha} \underline{f}$, we obtain $\underline{g}' = \underline{g}$. Further, we have $g'(t) = f(s + t)$ whenever $s + t$ is a continuity point of f. Since f is continuous at each $j\theta$, it remains to show that $s = (0,0,\dots)$, the neutral element of Δ_2. Suppose, to the contrary, that there exists the least number $m \geq 1$ for which $s_m = 1$. If m is even, fix $t = (1,0,1,0,\dots,1,0,0,0,\dots)$, where t_{m-1} is the last digit 1. Clearly, t is of the form $j\theta$, so $g'(t) = g(t)$. Since g has the core and $t \in [t^0, t^1)$, $g(t) = 0$. On the other hand, the first m digits of $s + t$ are $1,0,1,0,\dots,1,1$, hence $s + t \in (t^1, t^2]$. Thus, $s + t$ is a continuity point of f, and $f(s+t) = 1$. We have obtained $0 = g'(t) = f(s+t) = 1$, a contradiction. For an odd $m > 1$ a similar argument works with $t = (1,0,1,0,\dots,1,0,1,1,1,\dots)$, where t_{m+1} is the last digit 0. Analogously, by taking $t = t^0$ for s starting with the digits 1,0, and $t = (1,1,0,0,0,\dots)$ for s starting with 1,1, we eliminate the remaining case $m = 1$, which ends the proof of the theorem.

The quasi-uniform convergence in $\{0,1\}^Z$ is given by the Weyl pseudometric

$$D_W(\eta,\eta') = \lim_L \sup_k L^{-1} |\{0 \leq j < L : \eta(k+j) \neq \eta'(k+j)\}|.$$

It follows from Lemma 5 in [2] that for Riemann integrable functions f on Δ_2 the convergence in $L_1(\lambda)$, where λ is the normalized Haar measure on Δ_2, implies the quasi-uniform convergence of the sequences $\underline{f}$.

A Toeplitz sequence η is called regular if the supremum over $p \in N$ of the densities of the sets $\mathrm{Per}_p(\eta)$ equals 1. If the set of discontinuities of $f : \Delta_2 \to \{0,1\}$ is disjoint with $Z\theta$ and has the λ measure 0, then $\underline{f}$ is regular (see [2, Remark 1]). Every regular Toeplitz sequence is uniquely ergodic and the quasi-uniform convergence of uniquely ergodic sequences implies the weak* convergence of their measures ([7], see also [2, Theorem 2]).

Denote by T_c the set of all functions $f : \Delta_2 \to \{0,1\}$ such that

(1) f is Riemann integrable;

(2) the set of discontinuities of f is disjoint with $Z\theta$ and it has the λ measure 0;

(3) f has the core.

We endow T_c with the convergence in $L_1(\lambda)$ norm. By T_c^* we will mean the set $f^* : f \in T_c$ endowed with the weak* topology. Notice that, by Theorem 1 and the remarks on quasi-uniform convergence, $f \to f^*$ is a one-to-one continuous mapping between T_c and T_c^*.

Theorem 2. *For every compact metric space K there exists a homeomorphic embedding h of K into T_c.*

Proof. Fix some $t^3 \in (0, t^0) \subset \Delta_2$, not of the form $j\theta$. Then $\lambda[0, t^3) = a > 0$. Let $\{a_n : n \geq 1\}$ be a family of strictly positive continuous functions, which separates the points of K and such that $\Sigma a_n(x) \equiv a$ on K. For $x \in K$ we partition $[0, t^3)$ into consecutive intervals $A_n(x)$ such that each indicator $\mathbf{1}_{A_{2n}(x)}$ is continuous at each $j\theta$ (see Remark 1) and $\lambda(A_n(x)) = a_n(x)$. Now, let $f_x : \Delta_2 \to \{0,1\}$ be the function with the core defined on $[0, t^0)$ as the indicator of the set $\bigcup A_{2n}(x)$. Since the endpoints of the $A_{2n}(x)$'s accumulate only at t^3, f_x is continuous at each $j\theta$. The set of discontinuities of f_x is countable, hence $f_x \in T_c$. Clearly, the mapping $h : x \to f_x$ is continuous in $L_1(\lambda)$. We show that it is 1-1. Let $x \neq y$ in K and let n be the least number for which $a_n(x) \neq a_n(y)$, say $a_n(x) < a_n(y)$. Then $f_x \neq f_y$ everywhere on the (nonempty) interior of $A_n(y) \cap A_{n+1}(x)$, which ends the proof.

Corollary 1. *The mapping h^* is a homeomorphic embedding of K into T_c^*.*

IV. Affine embedding

First we establish some more notation. For a block $b = b(0)b(1)\cdots b(n-1)$ ($b \in \{0,1\}^n$), by $|b|$ we denote its length n, and for $0 \leq m < m' \leq |b|$, by $b[m,m')$ we will mean the block $b(m)b(m+1)\cdots b(m'-1)$. If blocks b, b_0 satisfy $|b| \leq |b_0|$ then we denote by $F_{b_0}^*(b)$ the frequency at which b occurs in b_0, i.e.

$$F_{b_0}^*(b) = |b_0|^{-1} |\{m \in N : 0 \leq m \leq |b_0| - |b|,\ b_0[m, m+|b|) = b\}|.$$

Moreover, for $0 \leq j < k$, we define

$$F_{b_0}^{**}(b,k,j) = |b_0|^{-1} |\{m \in N : 0 \leq m \leq |b_0| - |b|,\ m = j \bmod k,\ b_0[m, m+|b|) = b\}|.$$

If $|b| > |b_0|$ then we let $F_{b_0}^*(b) = F_{b_0}^{**}(b,k,j) = 0$. Clearly,

$$F_{b_0}^*(b) = \sum_{j=0}^{k-1} F_{b_0}^{**}(b,k,j) \qquad \text{for every } b_0,\ b \text{ and } k.$$

Let v^* be the weight of blocks given by $v^*(b) = 2^{-|b|}$. Fix a sequence $c_k > 0$ such that $\sum_{k \text{ odd}} c_k = 1$. Let v^{**} be the weight of the triples (b,k,j) with k odd and $0 \leq j < k$, given by $v^{**}(b,k,j) = 2^{-|b|} c_k$. It is easily seen that $\int F_{b_0}^* \, dv^* = \int F_{b_0}^{**} \, dv^{**} < 1$ for every block b_0. We define two metrics on the set of blocks:

$$d^*(b_1,b_2) = \int |F^*_{b_1} - F^*_{b_2}|\, dv^* \quad \text{and} \quad d^{**}(b_1,b_2) = \int |F^{**}_{b_1} - F^{**}_{b_2}|\, dv^{**}.$$

Clearly $d^* \le d^{**}$.

By a cylinder U_b we will mean the set $\{\eta \in \{0,1\}^Z : \eta[0,|b|) = b\}$. For a shift invariant measure μ on $\{0,1\}^Z$ we write $F^*_\mu(b) = \mu(U_b)$ and $F^{**}_\mu(b,k,j) = k^{-1}\mu(U_b)$. Clearly $\int F^*_\mu\, dv^* = \int F^{**}_\mu\, dv^{**} = 1$. Since invariant measures are distinguished by its values on cylinders U_b, we can extend the domain of the metrics d^* and d^{**} to comprise $M(\{0,1\}^Z, S)$ by simply replacing one or both b_i's by μ_i's in the defining formulas for d^* and d^{**}. It is easy to see that $d^*(\mu_1,\mu_2) = d^{**}(\mu_1,\mu_2)$, and $d^*(\mu_n,\mu) \to 0$ iff $\mu_n \to \mu$ in the weak* topology. It is also easily verified that, for large n, the measure $n^{-1}\sum_{i=k}^{k+n-1} S^i\delta_\eta$ ($\eta \in \{0,1\}^Z$) is of small d^*-distance from the block $\eta[k,k+n)$. In particular, this implies that if (b_n) is a sequence of blocks appearing in some subshift (Y,S), such that $|b_n| \to \infty$, then there exists a subsequence $b_{n'}$ convergent in d^* to a measure $\mu \in M(Y,S)$. On the other hand, for every ergodic measure μ on Y, there exists a sequence (b_n) of blocks in Y with $d^*(b_n,\mu) \to 0$; in fact, take $b_n = \eta[0,n)$, where η is generic for μ.

Definition 2. By $M_0(Y,S)$ we denote the set of all invariant measures μ of (Y,S) for which

$$d^*(b_n,\mu) \to 0 \Rightarrow d^{**}(b_n,\mu) \to 0,$$

for every sequence (b_n) of blocks appearing in Y.

An easy proof of the following lemma is omitted.

Lemma 1. *For every regular dyadic Toeplitz sequence η we have $\eta^* \in M_0(\bar{O}(\eta),S)$. In particular, $d^{**}(\eta[0,n),\eta^*) \to 0$.*

Another observation will be useful.

Lemma 2. *For $i = 1,2,\ldots,n$ let b_i be some blocks of the same length m and μ_i some invariant measures on $\{0,1\}^Z$. If $d^{**}(b_i,\mu_i) \le \epsilon$ for each i then $d^{**}(b_0, n^{-1}\sum_{i=1}^n \mu_i) \le 2\epsilon$, where b_0 is the block of the length nm obtained by the concatenation of the blocks $b_1,b_2,\ldots,b_n$.*

Proof. Observe that

$$F^{**}_{b_0}(b,k,j) \ge n^{-1}[F^{**}_{b_1}(b,k,j) + F^{**}_{b_2}(b,k,(j-m)_k) + \cdots + F^{**}_{b_n}(b,k,(j-(n-1)m)_k)].$$

Since $F^{**}_{\mu_1}(b,k,j)$ does not depend on j, we have

$$\int \left| n^{-1} \sum_{i=1}^{n} F^{**}_{b_i}(b,k,(j-(i-1)m)_k) - n^{-1} \sum_{i=1}^{n} F^{**}_{\mu_i}(b,k,j) \right| dv^{**}(b,k,j)$$

$$= \int \left| n^{-1} \sum_{i=1}^{n} F^{**}_{b_i}(b,k,(j-(i-1)m)_k) \right.$$

$$\left. - n^{-1} \sum_{i=1}^{n} F^{**}_{\mu_i}(b,k,(j-(i-1)m)_k) \right| dv^{**}(b,k,j)$$

$$\leq n^{-1} \sum_{i=1}^{n} \int |F^{**}_{b_i}(b,k,(j-(i-1)m)_k)$$

$$- F^{**}_{\mu_i}(b,k,(j-(i-1)m)_k)| \, dv^{**}(b,k,j)$$

$$= n^{-1} \sum_{i=1}^{n} d^{**}(b_i,\mu_i) \leq \epsilon.$$

Since $\int F^{**}_{\mu_i} \, dv^{**} = 1$, the obtained inequality implies in particular that

$$\int n^{-1} \sum_{i=1}^{n} F^{**}_{b_i}(b,k,(j-(i-1)m)_k) \, dv^{**}(b,k,j) \geq 1 - \epsilon.$$

Thus, by the first inequality in the proof we obtain

$$\int \left| F^{**}_{b_0}(b,k,j) - n^{-1} \sum_{i=1}^{n} F^{**}_{b_i}(b,k,(j-(i-1)m)_k) \right| dv^{**}(b,k,j) \leq \epsilon,$$

and finally

$$d^{**}\left(b_0, n^{-1} \sum_{i=1}^{n} \mu_i\right) = \int \left| F^{**}_{b_0} - n^{-1} \sum_{i=1}^{n} F^{**}_{\mu_i} \right| dv^{**} \leq 2\epsilon,$$

which ends the proof.

Some more remarks on the structure of Δ_2 will be necessary. Let $H_m = \{t \in \Delta_2 : t_1 = t_2 = \cdots = t_m = 0\}$. Then $H_m = \{2^m j\theta : j \in Z\}^-$ and H_m is a clopen subgroup of Δ_2 homeomorphically isomorphic to Δ_2. For the map $t \to t + \theta$, Δ_2 can be viewed as a 2^m-tower:

$$H_m \to H_m + \theta \to \cdots \to H_m + (2^m - 1)\theta.$$

DEFINITION 3. For $i = 1,2,\ldots,n$ fix $a_i \geq 0$, $\sum_{i=1}^n a_i = 1$ and $f_i \in T_c$. Let $\{A_i : i = 1,2,\ldots,n\}$ be the partition of H_m into consecutive intervals with $\lambda(A_i) = 2^{-m}a_i$ (modified as in Remark 1). We define

$$\mathrm{MIX}_m(a_1 f_1, a_2 f_2, \ldots, a_n f_n) = \sum_{i=1}^{n} \sum_{j=0}^{2^m-1} \mathbf{1}_{A_i+j\theta} f_i.$$

The following remarks are easy to verify:

(4) $\mathrm{MIX}_m(a_1 f_1, a_2 f_2, \ldots, a_n f_n) \in T_c$;

(5) the sequence $\underline{\mathrm{MIX}}_m(a_1 f_1, a_2 f_2, \ldots, a_n f_n)$ equals $\underline{f}_i$ on a subset of Z of density a_i, built of intervals of the form $[k2^m,(k+1)2^m)$.

Recall that a convex metrizable subset K of a topological vector space is called a Choquet simplex if each $x \in K$ is a barycenter of a unique probability measure μ_x supported by the set $\operatorname{ex} K$ of the extreme points of K. The map $\mu_x \to x$, mapping the probability measures on $\operatorname{ex} K$ onto K, is then affine, continuous and bijective. If, in addition, $\operatorname{ex} K$ is compact then this map is an affine homeomorphism, in which case K is called a Bauer simplex. It is known that every compact metrizable Choquet simplex is affinely homeomorphic to the intersection of a decreasing sequence of metrizable Bauer simplices ([3, Thm 9]).

Let B be a Bauer simplex and let h^* be some homeomorphism of B into T_c^*. Since $h^*(\operatorname{ex} B)$ is a compact extreme subset of the Choquet simplex of all shift invariant measures on $\{0,1\}^Z$, it follows that $C = \overline{\operatorname{conv}}\, h^*(\operatorname{ex} B)$ is a Bauer simplex. The formula $h_A^*(x) = \int_{\operatorname{ex} B} h^*(y)\, d\mu_x(y)$ defines an affine homeomorphism of B onto C.

LEMMA 3. *Let B be a metric Bauer simplex and let $h : B \to T_c$ be a homeomorphic embedding. Then for every $\epsilon > 0$ there exists a homeomorphic embedding $h_B : B \to T_c$ and $m \in N$ such that*

(i) *$d^*(h_B^*(x), h_A^*(x)) \leq 5\epsilon$ for all $x \in B$;*

(ii) *for each $x \in B$ and $k \in Z$ the block $b = \underline{h}_B(x)\,[k2^m + 1,(k+1)2^m)$ appears in $\underline{h}(y)$ for some $y \in \operatorname{ex} B$ and $d^{**}(b, h^*(y)) \leq \epsilon$.*

PROOF. Choose a finite subset $\{e_1, e_2, \ldots, e_n\}$ of $\operatorname{ex} B$ such that for every $x \in B$ there exists y in $F = \operatorname{conv}\{e_1, e_2, \ldots, e_n\}$ with $d^*(h_A^*(x), h_A^*(y)) \leq \epsilon$. Since d^* is a convex metric, it follows from Michael's Theorem [11] that there exists a continuous mapping $\tau : B \to F$ such that $d^*(h_A^*(x), h_A^*(\tau(x))) \leq \epsilon$. Since F is again a Bauer simplex, we have $\tau(x) = \sum_{i=1}^n a_i(x) e_i$, where the functions a_i are uniquely determined and continuous. By the regularity of the Toeplitz sequences $\underline{h}(e_i)$ and by Lemma 1, it is not hard to see that there exists $m \in N$ such that for each i and each block b, with $|b| = 2^m - 1$, appearing in $\underline{h}(e_i)$ we have $d^{**}(b, h^*(e_i)) \le \epsilon$. We are in a position to define h_B. For $x \in B$ let

$$g(x) = \mathrm{MIX}_m(a_1(x)h(e_1), a_2(x)h(e_2), \ldots, a_n(x)h(e_n)).$$

Let also $g'(x): H_m \to \{0,1\}$ be given by $g'(x)(t) = h(x)(2^{-m}t)$ (multiplication by 2^{-m} gives a homeomorphism of H_m onto Δ_2). Finally define $h_B(x)$ on Δ_2 by

$$h_B(x)(t) = \begin{cases} g'(x)(t) & \text{for } t \in H_m, \\ g(x)(t) & \text{otherwise.} \end{cases}$$

It is clear that $h_B(x)$ satisfies (1) and (2). Also, since $H_m \subset [0, t^0)$, $h_B(x)$ satisfies (3), so $h_B(x) \in T_c$. By the continuity of the functions a_i and g', it is seen that h_B is a continuous mapping. Also, since g' is 1-1, such is h_B, so h_B is a homeomorphic embedding of B into T_c. Observe that if $b = \underline{h}_B(x)[k2^m + 1, (k+1)2^m)$ for some $x \in B$, $k \in Z$ then $b = \underline{g}(x)[k2^m + 1, (k+1)2^m)$ and, by (5), b appears in $\underline{h}(e_{i(k)})$ for some $1 \le i(k) \le n$. Since $|b| = 2^m - 1$ and by the choice of m, we have $d^{**}(b, h^*(e_{i(k)})) \le \epsilon$ and (ii) is proved. Now, if m is chosen large enough, we have $d^{**}(b, b[1, 2^m)) \le \epsilon$ for every block b of the length 2^m. Hence, $d^{**}(\underline{h}_B(x)[k2^m, (k+1)2^m), h^*(e_{i(k)})) \le 2\epsilon$ and, by Lemma 2, for $r \in N$,

$$d^{**}\left(\underline{h}_B(x)[0, r2^m), \sum_{i=1}^{n} r_i r^{-1} h^*(e_i)\right) \le 4\epsilon,$$

where $r_i = |\{1 \le k \le r : i(k) = i\}|$. Passing to the limit in r and using (5) and Lemma 1 we obtain that $d^{**}(h_B^*(x), \sum_{i=1}^n a_i(x)h^*(e_i)) \le 4\epsilon$. By the definitions of τ and h_A^*, $\sum_{i=1}^n a_i(x)h^*(e_i) = h_A^*(\tau(x))$. Since $d^{**}(h_A^*(x), h_A^*(\tau(x))) \le \epsilon$ the assertion (i) is proved.

THEOREM 3. *Let K be a compact metrizable Choquet simplex. Then there exists a closed invariant set Y in $(\{0,1\}^Z, S)$ such that $M(Y,S)$ coincides with $M_0(Y,S)$ and it is affinely homeomorphic to K.*

PROOF. We can represent K as an intersection of decreasing metric Bauer simplices (B_n, d). Let h_1 be the mapping h of Theorem 2 applied to B_1. Let also $\epsilon_1 = 1$. Suppose, for some $n \ge 2$, we have already defined a homeomorphic embedding $h_{n-1}: B_{n-1} \to T_c$ and $\epsilon_{n-1} > 0$, so that for each $x \in B_{n-1}$

$$d^*(h_{n-1}^*(x), h_{A,n-1}^*(x)) \le 10\epsilon_{n-1}, \tag{6}$$

where $h_{A,n-1}^*$ is the affine extension of $h_{n-1}^* | \mathrm{ex}\, B_{n-1}$.

Since $h^*_{A,n-1}$ is a homeomorphic embedding, we can choose $\epsilon_n \le \epsilon_{n-1}/2$ such that

$$d(x,y) \ge \epsilon_{n-1} \Rightarrow d^*(h^*_{A,n-1}(x), h^*_{A,n-1}(y)) \ge 61\epsilon_n \qquad (x,y \in B_{n-1}). \tag{7}$$

We let $h_n = h_B | B_n$, where h_B is the mapping of Lemma 3 applied to $B = B_{n-1}$, $h = h_{n-1}$ and $\epsilon = \epsilon_n$. So h^*_n is a homeomorphic embedding and it is $5\epsilon_n$-approximated by $h^*_{A,n-1}$. For $x \in B_n$, let μ_x denote the measure on $\operatorname{ex} B_n$ representing x. We have

$$d^*\left(h^*_n(x), \int_{\operatorname{ex} B_n} h^*_n(y)\, d\mu_x(y)\right) \le d^*(h^*_n(x), h^*_{A,n-1}(x))$$
$$+ d^*\left(\int_{\operatorname{ex} B_n} h^*_{A,n-1}(y)\, d\mu_x(y), \int_{\operatorname{ex} B_n} h^*_n(y)\, d\mu_x(y)\right) \le 10\epsilon_n,$$

since $h^*_{A,n-1}$ is affine and d^* is a convex metric. We have proved formula (6) for n. Moreover,

$$\begin{aligned} d^*(h^*_{n-1}(x), h^*_n(x)) &\le d^*(h^*_{n-1}(x), h^*_{A,n-1}(x)) + d^*(h^*_{A,n-1}(x), h^*_n(x)) \\ &\le 10\epsilon_{n-1} + 5\epsilon_n \le 12.5\epsilon_{n-1}; \end{aligned} \tag{8}$$

$$\begin{aligned} &\text{if } d(x,y) \ge \epsilon_{n-1} \text{ then } d^*(h^*_n(x), h^*_n(y)) \\ &\ge d^*(h^*_{A,n-1}(x), h^*_{A,n-1}(y)) - 10\epsilon_n \ge 51\epsilon_n. \end{aligned} \tag{9}$$

By (8), the mappings h^*_n converge uniformly on $K = \bigcap B_n$ to a continuous mapping h^*. For $x \in K$, we have $d^*(h^*_n(x), h^*(x)) \le \sum_{i=n}^{\infty} 12.5\epsilon_i \le 25\epsilon_n$. Now, by (9), $d(x,y) \ge \epsilon_{n-1} \Rightarrow d^*(h^*(x), h^*(y)) \ge 51\epsilon_n - 50\epsilon_n = \epsilon_n$, which proves that h^* is a homeomorphic embedding. By (6), h^* is the uniform limit of $h^*_{A,n}$ and hence it is affine. Let

$$Y = \bigcap_{n \ge 1} \left(\bigcup_{i \ge n} \bigcup_{x \in K} \bar{O}(\underline{h}_i(x)) \right)^{-}.$$

We will show that $h^*(K) = M(Y,S) = M_0(Y,S)$. First observe that every measure $h^*(x)$ $(x \in K)$, being a limit of the $h^*_n(x)$'s, is invariant and carried by Y. To prove the converse, it suffices to show that every ergodic measure on Y is in $h^*(K)$. We show this for each measure being a d^*-limit of blocks appearing in Y. Let μ be such a measure. Fix some n and $r \in N$, and let b be a block appearing in Y such that $d^*(b,\mu) < \epsilon_n$ and $|b| = r2^m$, where m is the number of Lemma 3 applied, as previously, to $B = B_{n-1}$, $h = h_{n-1}$ and $\epsilon = \epsilon_n$. By the definition of Y, the block b appears in $\underline{h}_i(x)$ for some $i \ge n$ and $x \in K$. Now, by the inductive construction of h_i and by (ii) of Lemma 3, it is seen that $\underline{h}_i(x)$ can be divided into blocks b_k of length 2^m such that each $b_k[1,2^m)$ appears in $\underline{h}_{n-1}(y)$ for some $y \in \operatorname{ex} B_{n-1}$, and $d^{**}(b_k[1,2^m), h^*_{n-1}(y)) \le \epsilon_n$. If m is large enough we also have $d^{**}(b_k, h^*_{n-1}(y)) \le 2\epsilon_n$. For large r, b is of small d^{**}-distance from a concatenation of r blocks b_k. Thus, by Lemma 2,

$$d^{**}(b, \operatorname{conv} h^*_{n-1}(\operatorname{ex} B_{n-1})) \le 5\epsilon_n, \quad \text{i.e. } d^{**}(b, h^*_{A,n-1}(x_n)) \le 5\epsilon_n,$$

for some $x_n \in B_{n-1}$. Since $d^* \le d^{**}$, we obtain $d^*(\mu, h^*_{A,n-1}(x_n)) \le 6\epsilon_n$. Passing to the uniform limit in n we conclude that $\mu = h^*(\lim x_n) \in h^*(K)$. Finally, since $d^{**} = d^*$ for measures, we also obtain $d^{**}(b,\mu) \le 11\epsilon_n$, which proves that $\mu \in M_0(Y,S)$ and the proof is completed.

V. Passing to a Toeplitz flow

In this section we present the construction of a Toeplitz sequence $\mathfrak{z}$ with $M(\bar{O}(\mathfrak{z}),S)$ affinely homeomorphic to a given Choquet simplex K. In view of Theorem 3, we assume that K is represented as $M(Y,S) = M_0(Y,S)$ for a certain 0–1 subshift (Y,S). We let Σ be the two-point compactification of Z, say $\Sigma = Z \cup \{-\infty, \infty\}$. The construction below is based on that of [13, sec. 4].

CONSTRUCTION. Let b_n be a sequence of 0–1 blocks appearing in Y with $|b_n| = \prod_{i=1}^n (2^i - 1)$ and such that each block appearing in Y initiates some of the b_n's. Let c_n be a copy of b_n obtained by replacing the symbols 0 and 1 by $-n$ and n, respectively. We define the sequence $\mathfrak{z} \in \Sigma^Z$ by induction. In the first step we let $\mathfrak{z}(j) = c_1$ for $j = 0 \bmod 4$ ($|c_1| = 1$).

After the nth step, $\mathfrak{z}$ is defined on a periodic subset of Z, with the period $p_n = \prod_{i=1}^n 2^{i+1}$, leaving $p'_n = \prod_{i=1}^n (2^{i+1} - 1)$ unoccupied positions in each period. In the $(n+1)$st step we let $p_{n+1} = \prod_{i=1}^{n+1} 2^{i+1}$. If n is odd then we fill out all the p'_n unoccupied positions in each of the p_n-intervals $[0,p_n) + kp_{n+1}$ with the consecutive values of c_{n+1} (notice that $|c_{n+1}| = p'_n$). In case of n even we fill out in the same way the p_n-intervals $[-p_n, 0) + kp_{n+1}$. We obtain a p_{n+1}-periodic sequence with $p'_{n+1} = \prod_{i=1}^{n+1} (2^{i+1} - 1)$ unoccupied positions in each period. In this manner $\mathfrak{z}$ is defined as a dyadic Toeplitz sequence in Σ^Z. The following properties of $\mathfrak{z}$ and its orbit closure will be useful:

(10) the density of $\pm n$ in $\mathfrak{z}$ equals $\beta_n = p'_{n-1}/p_n \le 2^{-(n+1)}$, hence the supremum of the densities of $\operatorname{Per}_{2^n}(\mathfrak{z})$ equals $d(\mathfrak{z}) = \sum_{n \ge 1} \beta_n < 1$, so $\mathfrak{z}$ is not regular;

(11) by minimality of $\bar{O}(\mathfrak{z})$ it is easy to see that for $\omega \in \bar{O}(\mathfrak{z})$,

$$\operatorname{Aper}(\omega) = \{j \in Z : \omega(j) = \pm\infty\}.$$

For the sequence $p = (p_n)$ the p-density of a set $A \subset Z$ is given by

$$D_p(A) = \lim_n p_n^{-1} |A \cap [0, p_n)|$$

provided the limit exists. By the same argument as in Lemma 1 in [1] and in Lemma 4.2 in [13], we have

(12) the set $W = \{\omega \in \bar{O}(\mathfrak{z}) : \mathrm{Aper}(\omega)$ is two-sided infinite and $D_p(\mathrm{Aper}(\omega)) = 1 - d(\mathfrak{z})\}$ is of measure 1 for every measure in $M(\bar{O}(\mathfrak{z}), S)$.

For $\omega \in W$ and $m \in N$ it is easily seen that

$$D_p\{j \in Z : m \leq |\omega(j)| < \infty\} = \sum_{n \geq m} \beta_n. \tag{13}$$

For $\epsilon > 0$, by n_ϵ we denote such a natural number that $\sum_{n \geq n_\epsilon} \beta_n \leq \epsilon$. For any finite block c over Σ we denote by $I(c)$ the number of infinite symbols in c and we define $\varphi(c)$ to be the 0–1 block such that $|\varphi(c)| = I(c)$, and the consecutive values of $\varphi(c)$ are 0 or 1 according as the consecutive infinite symbols in c are $-\infty$ or ∞, respectively. If c has no infinite symbols then $\varphi(c) = \emptyset$. Analogously, φ is defined on Σ^Z; for $\omega \in \bar{O}(\mathfrak{z})$, $\varphi(\omega)$ is determined by $\omega | \mathrm{Aper}(\omega)$. Now, the equality

$$\varphi(W) = Y \tag{14}$$

has exactly the same proof as Lemma 4.3 in [13] (the differences between the construction of $\mathfrak{z}$ and that of η in [13, sec. 4] are easily seen to be inessential).

Lemma 4. *An element $\omega \in W$ is p-generic for a measure $\mu \in M(\bar{O}(\mathfrak{z}), S)$ if and only if each block c over Σ appears in ω with p-density. The p-density is then equal to $\mu(U_c)$.*

Proof. For a given block c over Σ and $\epsilon > 0$ define

$$U_{c,\epsilon} = \{\omega \in \Sigma^Z : \omega(i) = c(i) \text{ if } c(i) \text{ is finite, } \omega(i) \in [-\infty, -n_{\epsilon/I(c)}]$$
$$\text{if } c(i) = -\infty,\ \omega(i) \in [n_{\epsilon/I(c)}, \infty] \text{ if } c(i) = \infty\}.$$

Clearly $U_{c,\epsilon}$ is a clopen cylinder. Now, let $\omega \in W$ and let $S^j(\omega) \in U_{c,\epsilon} \setminus U_c$. Then at least one of the infinite symbols in c is replaced in $\omega[j, j + |c|)$ by a finite symbol with modulus at least $n_{\epsilon/I(c)}$. Since, by (13), all such symbols in ω have the p-density at most $\epsilon/I(c)$, we have proved that the upper p-density at which the trajectory of ω visits $U_{c,\epsilon} \setminus U_c$ is at most ϵ. Now, let $\omega \in W$ be generic for μ. Fix a block c over Σ. Since $U_{c,\epsilon}$ is clopen, it is visited by the trajectory of ω with the p-density $\mu(U_{c,\epsilon})$. Both upper and lower p-densities of visiting U_c are now between $\mu(U_{c,\epsilon})$ and $\mu(U_{c,\epsilon}) - \epsilon$, hence the p-density of c in ω exists and equals $\lim_{\epsilon\to 0}\mu(U_{c,\epsilon}) = \mu(U_c)$. Conversely, suppose that the p-densities of all the blocks in ω exist. We ought to show that the trajectory of ω visits every cylinder U of the form $\{\omega \in \Sigma^Z : \omega(i) \in A_i,\ i = 1,2,\ldots,n\}$ with all A_i clopen in Σ. Every such U can, for given $\epsilon > 0$, be obtained as a disjoint union of finitely many cylinders U_{c_k} with $I(c_k) = 0$ and at most $2n$ cylinders $U_{c'_j,\delta}$ with $\delta < \epsilon/2n$. Thus the upper and lower p-densities at which the trajectory of ω visits U are between D and $D + \epsilon$, where D is the joint p-density of the blocks c_k and c'_j in ω, which ends the proof.

REMARK 2. In the notion of the foregoing proof we have obtained that the function $\mu \to \mu(U)$ on $M(\bar{O}(\mathfrak{z}),S)$ is a uniform limit of the functions $\mu \to \mu(U_\epsilon)$, where $U_\epsilon = \bigcup_k U_{c_k} \cup \bigcup_j U_{c'_j}$.

For a block c over Σ, we denote by $m(c)$ the maximal modulus of the finite symbols in c.

LEMMA 5. *Let* $\nu \in M(Y,S)$ *be such that* $W_\nu \neq \emptyset$, *where*

$$W_\nu = \{\omega \in W : \varphi(\omega[0,p_n)) \text{ converges in } d^* \text{ to } \nu\}.$$

Then every $\omega \in W_\nu$ *is p-generic for the same measure* μ *(depending only on* ν*).*

PROOF. Let $\omega \in W_\nu$. By Lemma 4, it suffices to show that each block c over Σ appears in ω with a p-density depending only on ν. First, let c contain no infinite symbols. Then c appears in $\mathfrak{z}$ periodically and hence it appears with the same p-density; write $D(c)$, in each element of $\bar{O}(\mathfrak{z})$. Now consider a block c appearing in ω and satisfying

$$\varphi(c) = b \neq \emptyset; \tag{15}$$

(16) the last symbol of c is the only one in c of modulus $m(c)$.

Denote by $j_0 + k_0 p_{m(c)}$ ($k_0 \in Z$, $0 \le j_0 < p_{m(c)}$) a position of $c(0)$ in some appearance of c in ω. Recall the construction of $\mathfrak{z}$ and observe that in ω (as in $\mathfrak{z}$) the symbols $\pm n$ fill all the positions unoccupied by smaller symbols in p_n-periodically repeating intervals of length p_{n-1} (called p_{n-1}-intervals). Since in c there appears an infinite symbol, we conclude that whenever c appears in ω, its last symbol is the first $\pm m(c)$ in such a $p_{m(c)}$-periodically repeating $p_{m(c)-1}$-interval. Thus all appearances of c in ω differ by multiples of $p_{m(c)}$, and $c(0)$ occupies the position $j_0 \bmod p_{m(c)}$, while the first symbol $\pm\infty$ in c occupies also a constant position $j \bmod p_{m(c)}$. Since $\mathrm{Aper}(\omega)$ has either 0 or $p'_{m(c)}$ elements in each $p_{m(c)}$-interval, the corresponding appearances of b in $\varphi(\omega)$ differ by multiples of $p'_{m(c)}$, say $b(0)$ occupies there a position j' mod $p'_{m(c)}$.

Conversely, by periodicity of finite symbols in ω, it is not hard to see that c appears in ω whenever b appears in $\varphi(\omega)$ at a position j' mod $p'_{m(c)}$. In conclusion, for given $n \in N$, c appears in $\omega[0,p_n)$ as many times as b does in $\varphi(\omega[0,p_n))$ at the position j' mod $p'_{m(c)}$. Since $p'_{m(c)} = \sum_{i=1}^{m(c)} (2^{i+1} - 1)$ is an odd number, the frequency of b in $\varphi(\omega[0,p_n))$ at the position j' mod $p'_{m(c)}$ is denoted by $F^{**}_{\varphi(\omega[0,p_n))}(b,p'_{m(c)},j')$. By the assumption of this Lemma, $\varphi(\omega[0,p_n))$ converge in d^* to $\nu \in M(Y,S) = M_0(Y,S)$, hence the convergence holds in d^{**}. In particular,

$$F^{**}_{\varphi(\omega[0,p_n))}(b,p'_{m(c)},j') \underset{n}{\to} p'^{-1}_{m(c)}\nu(U_b).$$

Now, since $\omega \in W$,

(17) the appearance of c in ω has the p-density $p'^{-1}_{m(c)}\nu(U_b)(1 - d(\mathfrak{z}))$.

To see that the family of the blocks satisfying (15) and (16) is "rich enough" we show that for every block c appearing in $\bar{O}(\mathfrak{z})$, relatively to $\bar{O}(\mathfrak{z})$, U_c is a finite disjoint union of cylinders over blocks satisfying (15) and (16). In fact, whenever c appears in some $\omega \in \bar{O}(\mathfrak{z})$, it can be extended to the right until the first finite symbol of modulus m greater than $m(c)$. So obtained block c' satisfies both (15) and (16), and $m(c') = m$. Now, U_c relatively to $\bar{O}(\mathfrak{z})$, U_c is equal to the disjoint union of $U_{c'}$ over all such extensions c' of c. Recall the construction of $\mathfrak{z}$ and observe that between any two symbols of different moduli n_1 and n_2, say $n_1 < n_2$ in $\mathfrak{z}$, there appears a symbol of modulus $n_1 - 1$. By a standard argument for coordinatewise limit, the same is valid in each $\omega \in \bar{O}(\mathfrak{z})$ even for $n_2 = \infty$. So, for each above extension c' of c we obtain that $m(c') = m(c) + 1$, otherwise between $\pm\infty$ and the last symbol $\pm m(c')$ in c' a symbol of modulus $m(c') - 1$ would be missing. By the periodicity of finite symbols, $\bar{O}(\mathfrak{z})$ admits only finitely many such extensions c' of c, which ends the proof.

THEOREM 4. *There exists an affine homeomorphism between $M(Y,S)$ and $M(\bar{O}(\mathfrak{z}),S)$.*

PROOF. First observe that the set $\{\nu \in M(Y,S) : W_\nu \neq \emptyset\}$ contains all the ergodic measures on (Y,S). In fact, let $\eta \in Y$ be generic for such a measure ν. By (14), we can choose $\omega \in W$ with $\varphi(\omega) = \eta$, and consequently $\omega \in W_\nu$. The desired affine homeomorphism will be defined as follows: $\Phi(\nu) = \mu$, where

$$\mu(U_c) = D(c) \qquad \text{for } c \text{ without infinite symbols,}$$

$$\mu(U_c) = p'^{-1}_{m(c)}\nu(U_{\varphi(c)})(1 - d(\mathfrak{z})) \qquad \text{for } c \text{ satisfying (15) and (16).}$$

Since for every block c, U_c is a finite union of cylinders over blocks satisfying the above two cases, the formula determines each $\mu(U_c)$. At the same time the uniqueness of $\Phi(\nu)$ is guaranteed. By Lemma 5, we recognize (see (17)) that if $W_\nu \neq \varnothing$ then $\Phi(\nu)$ exists in $M(\bar{O}(\mathfrak{z}),S)$, and hence, by an affine extension, the same is true for each $\nu \in M(Y,S)$. It is also seen that the function $\nu \to \Phi(\nu)(U_c)$ is continuous for each block c, which, by Remark 2, implies the continuity of Φ. Now, since $\varphi(c)$ with c satisfying (15) and (16) runs through all the 0–1 blocks appearing in Y (see (14)), it is seen that Φ is 1-1. It only remains to show that each ergodic measure μ on $\bar{O}(\mathfrak{z})$ is in the image of Φ.

Since $\mu(W) = 1$, we can choose ω both generic for μ and contained in W. Now, it suffices to show that $\omega \in W_\nu$ for some $\nu \in M(Y,S)$. To this end fix a 0–1 block b and $\epsilon > 0$. Each appearance of b in $\varphi(\omega)$ corresponds to an appearance in ω of a block c such that $\varphi(c) = b$ and both ending symbols of c are infinite. Now, every such c can, in a unique way, be extended in ω to the right until the first finite symbol $\pm m$ with $m > m(c)$. Denote the so obtained block by c'. Clearly there are finitely many possible so obtained blocks c' with $m \leq n_\epsilon$, and, by Lemma 4, the appearance of all such blocks in ω has a p-density. Since $\omega \in W$, the appearance of the remaining blocks c' (with $m > n_\epsilon$) has upper density at most ϵ. We have proved that the joint appearance of all such blocks c' in ω has a p-density. Again, since $\omega \in W$, we conclude that $F^*_{\varphi(\omega[0,p_n))}(b)$ converges in n, which implies that $\varphi(\omega[0,p_n))$ converges in d^* to some measure $\nu \in M(Y,S)$, and hence $\omega \in W_\nu$, completing the proof.

In the sequel we show that Σ in Theorem 4 may be replaced by $\{0,1\}$. The method used here is a modification of that in [1, sec. 3].

LEMMA 6. *There exists a* 0–1 *dyadic Toeplitz sequence* $\mathfrak{z}^\circ$ *such that the flow* $(\bar{O}(\mathfrak{z}^\circ),S)$ *is topologically conjugate to* $(\bar{O}(\mathfrak{z}),S)$.

PROOF. Define a map $^\circ\colon \Sigma^Z \to \{0,1\}^Z$ by $\omega^\circ(j) = 2(\operatorname{sign}\omega(j) - 1/2)$. Clearly $^\circ$ is continuous and it commutes with S. It remains to show that $^\circ$ is 1-1 on $\bar{O}(\mathfrak{z})$. Fix $\omega^\circ \in \bar{O}(\mathfrak{z}^\circ)$ and $j \in Z$. Since the value of $\omega^\circ(j)$ determines sign $\omega(j)$, it suffices to show that we are able to recover $|\omega(j)|$ using ω°. For given $n \in N$ consider the values $\omega^\circ(j + kp_n)$ for $k \in Z$. If $|\omega(j)| \leq n$ then these values are constant for k. Conversely, the above fails if $|\omega(j)| > n$. In fact, in this case j is one of the positions unoccupied by the p_n-periodic part of ω, say each $j + kp_n$ is the ith such position in the kth p_n-interval $(k \in Z)$. For some block b_{n_0} used in the construction of $\mathfrak{z}$ $(n_0 \geq n)$ we have $b_{n_0}(i) = 1 - \omega^\circ(j)$ (we need not consider the trivial case of

Y consisting of a constant sequence). Thus, in the p_n-interval filled out (in the n_0th step) with $b_{n_0}[0,p_n')$ we find $\omega^\circ(j+kp_n) = 1 - \omega^\circ(j)$. We have proved that

$$|\omega(j)| = \sup\{n \in N : \omega^\circ(j+kp_n) = \omega^\circ(j) \text{ for all } k \in Z\},$$

and we are done.

We are in a position to state our main result, the proof of which is now a combination of Theorem 3, Theorem 4 and Lemma 6.

THEOREM 5. *For every compact metric Choquet simplex K there exists a* 0–1 *dyadic Toeplitz flow whose set of invariant measures is affinely homeomorphic to K.*

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