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# THE AUTOMORPHISM GROUP OF THE GAUSSIAN MEASURE CANNOT ACT POINTWISE

BY

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ABSTRACT

Classical ergodic theory deals with measure (or measure class) preserving actions of locally compact groups on Lebesgue spaces. An important tool in this setting is a theorem of Mackey which provides spatial models for Boolean $G$-actions. We show that in full generality this theorem does not hold for actions of Polish groups. In particular there is no Borel model for the Polish automorphism group of a Gaussian measure. In fact, we show that this group as well as many other Polish groups do not admit any nontrivial Borel measure preserving actions.

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## Introduction

Our motivation is threefold: invariant measures; Borel liftings; Gaussian measures.

*Invariant measures:* By a famous theorem of A. Weil, if a Polish group $G$ admits a $\sigma$-finite invariant measure then $G$ is locally compact (see Appendix B).

Nonetheless, even if $G$ is not locally compact, a homogeneous space of $G$ might even admit a finite invariant measure. For example, the group $G$ of $\mu$-preserving homeomorphisms of the Cantor set $\Omega$, will act transitively on $\Omega$ for a suitable choice of $\mu$ (see, for example, [8]). We show that this never happens for some classes of Polish groups $G$ (for instance, the full unitary group of a separable Hilbert space), except for the trivial case: a measure concentrated on fixed points.

*Borel liftings:* Let $G$ be a closed subgroup of the Polish group of all invertible measure preserving transformations of (say) $[0,1]$ with Lebesgue measure. An element $g \in G$ is an equivalence class of maps $[0,1] \to [0,1]$ rather than a single map; thus, $g(x)$ is defined only almost everywhere. Can we define $g(x)$ everywhere? More exactly: can we lift the mod0 action to a Borel action? We give a general criterion for lifting, and a negative answer for some classes of groups including the Gaussian case.

*Gaussian measures:* Every Euclidean space carries its standard Gaussian measure. However, a separable Hilbert space $H$ (over $\mathbb{R}$) does not. The standard Gaussian process over $H$ is a linear isometry between $H$ and the subspace spanned by a sequence of i.i.d. $N(0,1)$ random variables in $L_2$ over a probability space. Can we implement each point of (some version of) the probability space as a function on $H$ or another superstructure over $H$? Some well-known such constructions are 'isotropic', that is, invariant under the full orthogonal group of $H$. Others give a standard probability space. We show that these two desirable properties exclude each other. Two proofs are given, one via 'invariant measures', the other via 'Borel liftings'.

Having thus stated our goals in outline let us be more precise. Traditionally, ergodic theory is treated within the context of locally compact groups acting on standard Lebesgue probability spaces. However, it is often the case that one has to deal with near-actions (see definition below) or merely with an action of the group on a measure algebra (i.e., the Borel algebra modulo sets of measure zero) and it is then desirable to find a standard Lebesgue model, or even better, a Polish (=complete metric, second countable space) or a compact model where the group acts continuously.

Recall that a **Borel action** of $G$ on a Borel space $(X,\mathcal{X})$ is a Borel map $G\times X\to X$ (denote it just $(g,x)\mapsto gx$) satisfying the two conditions, $ex=x$ and $g(hx)=(gh)x$ for all $g,h\in G$ and all $x\in X$. Such an object is called also a **Borel $G$-space**.

*Definition 0.1* (see Zimmer [34, Def. 3.1]): Let $G$ be a Polish group and $(X,\mathcal{X},\mu)$ a standard Borel space with a probability measure $\mu$. By a **near-action** of $G$ on $(X,\mathcal{X},\mu)$ we mean a Borel map $G\times X\to X, (g,x)\mapsto gx$ with the following properties:

(i) With $e$ the identity element of $G$, $ex=x$ for almost every $x$.
(ii) For each pair $g,h\in G$, $g(hx)=(gh)x$ for almost every $x$ (where the set of points $x\in X$ of measure one where this equality holds may depend on the pair $g,h$).
(iii) Each $g\in G$ preserves the measure $\mu$.

Let $\mathrm{Aut}(X)=\mathrm{Aut}(X,\mathcal{X},\mu)$ be the Polish group of all equivalence classes of invertible measure preserving transformations $X\to X$,

with the neighborhood basis at the identity formed by sets of the form

$$N(A,\varepsilon)=\{T\in\mathrm{Aut}(X):\mu(A\Delta TA)<\varepsilon\},$$

for $A\in\mathcal{X}$ and $\varepsilon>0$. What we would like to show next is that the following three notions are equivalent.

(I) A near-action of $G$ on $(X,\mathcal{X},\mu)$.
(II) A continuous homomorphism from $G$ to $\mathrm{Aut}(X)$.
(III) A Boolean action of $G$ on $(X,\mathcal{X},\mu)$, that is, a continuous homomorphism from $G$ to the automorphism group of the associated measure algebra.

Given a near-action of $G$, it is easy to check that the natural mapping from $G$ to $\mathrm{Aut}(X)$ defines a measurable mapping. That it is a homomorphism follows from the defining property of being a near-action, and since, as is well known, measurable homomorphisms of Polish groups are continuous, we get (II) from (I). To go in the other direction, we must construct from a continuous homomorphism of $G$ into $\mathrm{Aut}(X)$, a near-action of $G$ on $(X,\mathcal{X},\mu)$.

For this we need to define a section on equivalence classes of Borel measurable functions where the equivalence relation is that of equality $\mu$ a.e. Let $(X,\mathcal{X})$ and $(Y,\mathcal{Y})$ be standard Borel spaces and $\mu$ a probability measure on $(X,\mathcal{X})$. Then the set $L_0(X,Y)=L_0((X,\mathcal{X},\mu),(Y,\mathcal{Y}))$ of all equivalence classes (mod0 with respect to $\mu$) of Borel (or just $\mu$-measurable) maps $X\to Y$ is also a standard

Borel space; its $\sigma$-algebra is generated by functions $f \mapsto \mu(A \cap f^{-1}(B))$ for $A \in \mathcal{X}$, $B \in \mathcal{Y}$.

There exists a (highly non-unique) Borel map $V\colon L_0(X,Y) \times X \to Y$ such that for every $f \in L_0(X,Y)$ the function $x \mapsto V(f,x)$ belongs to the equivalence class $f$. For example, assuming $X = Y = (0,1)$ (with the usual Borel $\sigma$-algebra and Lebesgue measure $\mu$), we may take

$$V(f,x) = \limsup_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(x_1)dx_1.$$

The Polish group $\mathrm{Aut}(X)$ is a Borel subspace of $L_0(X,X)$. Returning now to our situation, let $\phi$ denote a continuous homomorphism of $G$ into $\mathrm{Aut}(X)$. Composing $\phi$ with the restriction of $V$ above to $\mathrm{Aut}(X) \times X$ gives us a Borel mapping from $G \times X$ to $X$, and one checks easily that the properties for being a near-action are satisfied. Thus (II) implies (I).

Finally, $\mathrm{Aut}(X)$ may be thought of as the automorphism group of the measure algebra $\mathrm{MALG}(X, \mathcal{X}, \mu) = (\bar{\mathcal{X}}, \bar{\mu})$, where $\bar{\mathcal{X}}$ is $\mathcal{X}$ modulo nullsets, and $\bar{\mu}$ the corresponding measure; automorphisms of the measure algebra must preserve Boolean operations and $\bar{\mu}$. Thus (II) and (III) are equivalent.

This completes the discussion of the equivalence of the various notions of a near-action.

In contrast to near-actions, we define the notion of spatial action.

*Definition 0.2:* Let $G$ be a Polish group. By a **spatial $G$-action** we mean a Borel action of $G$ on a standard Lebesgue space $(X, \mathcal{X}, \mu)$ such that each $g \in G$ preserves the measure $\mu$. We say that two spatial actions are isomorphic, if there exists a measure preserving one-to-one map between two $G$-invariant subsets of full measure in the corresponding spaces which intertwines the $G$-actions (the same two sets for all $g \in G$).

Every spatial action is also a near-action. In that case the spatial action will be called a **spatial model** of the near-action (or the corresponding Boolean action). The question is: when does a given near-action admit a spatial model?

Rohlin, in [29], when discussing $\mathbb{R}$-flows, distinguishes between these two notions calling our near-actions continuous flows and the spatial actions measurable flows. He notes there that the theories of these two notions are not equivalent. Indeed, already J. von Neumann in his foundational work [24] was aware of these distinctions and in footnote 13 writes that he hopes to provide a proof that every near-flow has a continuous spatial model.

We recall that a Polish $G$-space is a Polish space $X$ together with a continuous action $G \times X \to X$ of a Polish group $G$. Such an action will be called a Polish action. If, in addition, $X$ is compact, then it is a compact Polish $G$-space.

Every Polish action is also a Borel action. In that case the Polish action will be called a **Polish model** of the Borel action.

We have the following classical theorems, due to Mackey, Varadarajan and Ramsay ([30, Th. 3.2], [19], [28, Th. 3.3] and [31]; see also [32] and [5]).

THEOREM 0.3: *Let $G$ be a locally compact second countable topological group.*

(a) *Every near-action (or Boolean action) of $G$ admits a spatial model.*

(b) *Every spatial action of $G$ admits a Polish model.*

A powerful generalization to Polish groups of Theorem 0.3(b), given in [3, Th. 5.2.1], is crucial for our work.

THEOREM 0.4 (Becker and Kechris):

(a) *Every Borel action of a Polish group admits a Polish model.*

(b) *Every Borel $G$-space is embedded (as a $G$-invariant Borel subset) into a compact Polish $G$-space.*

Item (b) above follows from [3, Th. 2.6.6] (which in turn utilizes a theorem of Beer [4]).

In the present work we show that a full generalization to Polish groups of the first part of Theorem0.3 is not possible. Many near-actions (or Boolean actions) of Polish groups admit no spatial models.

In Section 1 we recall the definition of Lévy groups, a class of groups which includes $\mathrm{Aut}(X, \mathcal{X}, \mu)$ and $U(H)$, the unitary group on an infinite-dimensional Hilbert space. We show that these groups admit no non-trivial spatial actions, and discuss some further examples. In Section 2 we take up the general question of finding criteria for a near-action to admit a spatial model, or more generally a spatial factor. We give a necessary and sufficient condition for this in terms of $G$-continuous functions. These are those functions $f \in L^\infty(\mu)$ with the property that $f \circ g_n$ converges to $f$ in $L^\infty(\mu)$ norm whenever $g_n \to e$, the unit element in $G$.

In order to apply the criterion of Section 2 we introduce in Section 3 whirly actions which may be defined as near-actions of $G$ on $(X, \mathcal{X}, \mu)$ such that for all sets $A, B \in \mathcal{X}$ of positive measure, every neighborhood of $e$ contains $g$ with $\mu(A \cap gB) > 0$. We show that whirly actions have no non-constant $G$-continuous functions and verify some easy examples of whirly actions.

In Section 4 we return to the natural near-action of the Polish orthogonal group $G = O(H)$, when $H$ is the "first chaos" Gaussian Hilbert space, and show that this near-action is whirly, thus giving the promised second proof of the non-existence of a spatial model for this action.

Appendix A collects elementary proofs of the Lévy property for many well known Lévy families. In Appendix B we provide a proof of the fact, mentioned at the beginning of the Introduction, that a Polish group which admits a Borel $\sigma$-finite invariant measure is locally compact.

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## 1. Lévy groups admit no spatial actions

The phenomenon of concentration of mass was first considered by E. Borel in his "law of large numbers", where it is manifested in the family of "discrete cubes" $\{0,1\}^n$ equipped with Hamming distance and counting measure. In the 1930's P. Lévy studied the concentration phenomenon for the family of Euclidean spheres. Then about 1970 V. Milman in [20] revitalized the area when he discovered a new proof of Dvoretzky's theorem using the concentration phenomenon on spheres. In the work [11] M. Gromov and V. Milman considered applications of the concentration phenomenon in topological dynamics. In particular, the notion of a Lévy family is introduced in [11]. See [23] and [27] for further details on the history of this subject.

Let $(X_n, d_n, \mu_n)$, $n = 1, 2, 3 \ldots$ be a family of metric spaces with probability measures $\mu_n$. Call such a family a **Lévy family** if the following condition is satisfied. If $A_n \subset X_n$ is a sequence of subsets such that $\liminf \mu_n(A_n) > 0$, then for any $\varepsilon > 0$, $\lim \mu(B_\varepsilon(A_n)) = 1$, where $B_\varepsilon(A)$ is the $\varepsilon$ neighborhood of $A$.

A Polish group $G$ is a **Lévy group** if there exists a family of compact subgroups $K_n \subset K_{n+1}$ such that the group $F = \bigcup_{n \in \mathbb{N}} K_n$ is dense in $G$ and the corresponding family $(K_n, d, m_n)$ is a Lévy family; here $m_n$ is the normalized Haar measure on $K_n$, and $d$ is a right-invariant compatible metric on $G$ (the choice of $d$ does not matter). Using left-invariant metrics instead, we get an equivalent definition (just apply the map $g \mapsto g^{-1}$).

Here is a list of some Polish groups well-known to be Lévy groups. Refer to Milman [21], [22], Gromov and Milman [11], Glasner [7], Pestov [27] and Giordano and Pestov [27]; also see Appendix A for more details.

- The full unitary group $U(H)$ (of a separable Hilbert space $H$); and the full orthogonal group $O(H)$ (of a separable Hilbert space $H$ over $\mathbb{R}$),

where both groups are equipped with the strong operator topology.

- The dense subgroup of $U(H)$ consisting of all unitary (or orthogonal) $U$ such that $\operatorname{tr}((1-U)^*(1-U)) < \infty$.
- The group $\operatorname{Aut}(X)$ (mentioned in the Introduction).
- The commutative (moreover, monothetic) group $L_0([0,1], S^1)$ of all (equivalence classes of) measurable functions $[0,1] \to S^1$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

THEOREM 1.1: *Every spatial action of a Lévy group is trivial; i.e., the set of fixed points is of full measure.*

*Proof:* By Theorem 0.4(b), every Borel $G$-space is embedded into a compact Polish $G$-space. Therefore, it suffices to prove the theorem for a continuous action of $G$ on a metrizable compact space $X$ and a $G$-invariant Borel probability measure $\mu$. We will see that $G$ acts trivially on $\operatorname{supp}\mu$ (the support of the measure).

The action is a continuous homomorphism from $G$ to the Polish group $\operatorname{Homeo}(X)$ of all homeomorphisms of $X$ (as noted in [27, p. 427]). We equip $X$ with a compatible metric $\rho$, and $\operatorname{Homeo}(X)$ with the compatible right-invariant metric $(f,g) \mapsto \max_{x\in X} \rho(f(x), g(x))$. Now the homomorphism is uniformly continuous, provided that $G$ is also equipped with a right-invariant metric (which will be assumed).

The family $(g \mapsto g\cdot x)_{x\in X}$ of maps $G \to X$ is equicontinuous. By [11, 2.1], it sends the Lévy family $(m_n)$ of measures on $G$ to a Lévy family $(m_n \cdot x)$ of measures on $X$, uniformly in $x \in X$. In other words: for all $x_1, x_2, \ldots \in X$ the family $(m_n \cdot x_n)$ is Lévy.

By [11, 2.4] the family of measures is degenerate in the sense that

$$\min_{y\in X} \int_X \rho(\cdot, y)\, d(m_n \cdot x) = \min_{y\in X} \int_{K_n} \rho(g\cdot x, y)\, dm_n(g) \to 0$$

for $n \to \infty$, uniformly in $x \in X$. The proof is simple. Assuming the contrary and using compactness, we choose $x_k \in X$ and $n_k \to \infty$ such that measures $m_{n_k} \cdot x_k$ converge (weakly) to some measure $\nu$ on $X$ satisfying $\min_{y\in X} \int_X \rho(\cdot, y)\, d\nu > 0$, which means that the support of $\nu$ contains at least two points. Every open set $A \subset X$ such that $\nu(A) > 0$ satisfies $\nu(B_\varepsilon(A)) = 1$ for all $\varepsilon > 0$; here $B_\varepsilon(A)$ is the closed $\varepsilon$-neighborhood of $A$. We get a contradiction by choosing $A$ such that some points of the support of $\nu$ belong to $A$ and some do not belong to the closure of $A$.

For each $n$ we introduce the subspace $H_n \subset L_2(\mu)$ of all $K_n$-invariant functions, and the corresponding orthogonal projection $Q_n$,

$$Q_n f(x) = \int_X f d(m_n \cdot x) = \int_{K_n} f(g \cdot x) dm_n(g).$$

For $f \in C(X) \subset L_2(X)$ the degeneracy of measures gives

$$\int |f(g \cdot x) - (Q_n f)(x)|^2 dm_n(g) \to 0$$

for $n \to \infty$, uniformly in $x$.

We see this as follows. Denoting by $y_{n,x}$ the minimizer of $\int_X \rho(\cdot, y) d(m_n \cdot x)$ we have

$$\begin{aligned}\int |f(g \cdot x) - (Q_n f)(x)|^2 dm_n(g) &= \min_{a \in \mathbb{R}} \int |f(\cdot) - a|^2 d(m_n \cdot x) \\ \leq \int |f(\cdot) - f(y_{n,x})|^2 d(m_n \cdot x) &\leq \int_{B_\varepsilon(y_{n,x})} + \int_{X \setminus B_\varepsilon(y_{n,x})} \\ \leq \max_{B_\varepsilon(y_{n,x})} |f(\cdot) - f(y_{n,x})|^2 &+ \left( \max_X |f(\cdot) - f(y_{n,x})|^2 \right) \frac{1}{\varepsilon} \int_X \rho(\cdot, y_{n,x}) d(m_n \cdot x).\end{aligned}$$

Taking $\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \sup_{x \in X} (\ldots)$ we get 0.

On the other hand, the integral

$$\int |f(g \cdot x) - (Q_n f)(x)|^2 d\mu(x)$$

does not depend on $g \in K_n$ and is equal to $\|f - Q_n f\|^2$. Therefore

$$\|f - Q_n f\|^2 = \iint |f(g \cdot x) - (Q_n f)(x)|^2 dm_n(g) d\mu(x) \to 0$$

for $n \to \infty$. However, the inclusion $K_n \subset K_{n+1}$ implies $H_n \supset H_{n+1}$ and $\|f - Q_n f\| \leq \|f - Q_{n+1} f\|$. So, $\|f - Q_n f\| = 0$. It means that $f$ is $K_n$-invariant, that is, $f(x) = f(g \cdot x)$ for all $x \in \operatorname{supp} \mu$ and all $f \in C(X)$. Thus, $g \cdot x = x$ for all such $x$ and all $g \in \bigcup K_n$, therefore all $g \in G$. ∎

QUESTION 1.2: *Can a Lévy group admit a nontrivial nonsingular (that is, preserving a measure class) Borel action?*

*Remark 1.3:* The basic idea in the proof of Theorem 1.1 is derived from Gromov and Milman [11] where they show that Lévy groups have the fixed point on compacta property. The question arises whether every group with the fixed point property does not admit a nontrivial spatial measure preserving Borel action. Now it was shown by Pestov [26] that the Polish group $G = \mathrm{Aut}(\mathbb{Q}, <)$ of order preserving permutations of the rational numbers, equipped with the topology of pointwise convergence (with respect to the *discrete* topology on $\mathbb{Q}$), has the fixed point on compacta property (or is extremely amenable). However it is easy to see that this group also acts ergodically by homeomorphisms on the "$\mathbb{Q}$-Bernoulli system" $(\Omega, \mathcal{F}, \mu)$. Here $\Omega = \{1, -1\}^{\mathbb{Q}}$,$\mu$ is the product measure $\mu = (1/2, 1/2)^{\mathbb{Q}}$ and $G$ acts on "configurations" $\omega \in \Omega$ by permuting the indices. We therefore conclude that some Polish groups with the fixed point property can have nontrivial spatial actions.

*Remark 1.4:* Recall that a topological group $G$ is **amenable** if each compact $G$-space admits a $G$-invariant probability measure. Using this definition of amenability and the fact that every compact group is amenable, it is easy to deduce that every Lévy group is amenable. Now if $G$ is a Polish Lévy group and $(X, G)$ is a compact $G$-space, then by amenability of $G$ there is a $G$-invariant probability measure $\mu$ on $X$. It can be shown that $(X, G)$ is represented as an inverse limit of a directed system of metrizable $G$-spaces $\{(X_\alpha, G)\}$. Let $\mu_\alpha$ be the image of $\mu$ on $X_\alpha$, then apply Theorem 1.1 to deduce that $\mathrm{supp}(\mu_\alpha)$ is a closed nonempty collection of fixed points. It is now easy to conclude that the support of $\mu$, $\mathrm{supp}(\mu)$, is a nonempty closed subset of $X$ consisting of fixed point. Thus the Gromov–Milman theorem that every Lévy group has the fixed point on compacta property follows from Theorem 1.1. Of course this is a rather circumventive way of proving it.

*Remark 1.5:* The following application of Theorem 1.1 was pointed out to us by V. Pestov. Some years ago he and M. Cowling conjectured that every invariant mean on the unitary group $U(H)$ is contained in the weak* closed convex hull of the multiplicative invariant means. Now for any topological group $G$ the above statement holds iff every $G$-invariant measure on the greatest ambit $\mathcal{S}(G)$ of $G$ (i.e., the Gelfand space of the Banach algebra $BLUC(G)$ of bounded left uniformly continuous functions on $G$) is supported on the set of fixed points. Again using the fact that for a Polish Lévy group $\mathcal{S}(G)$ is an inverse limit of a directed system of metrizable $G$-spaces, we deduce from Theorem 1.1 that *every invariant mean on a Polish Lévy group is contained in the weak* closed convex hull of the multiplicative invariant means.*

*Remark 1.6:* Note that, for example, the group $S_\infty$ of permutations of $\mathbb{N}$ (with the topology of pointwise convergence) is a nonlocally compact Polish subgroup of $\mathcal{U}(H)$ which admits nontrivial measure preserving spatial actions.

*Remark 1.1:* There are well known examples of Polish groups $G$ which do not admit any weakly continuous linear representations on a Banach space; see, e.g., [13] and [2]. In the latter Banaszcyk provides, for every infinite-dimensional normed space $E$, examples of the form $G = E/K$ where $K \subset E$ is a discrete subgroup. It is easy to see that any such group is moreover monothetic. Of course, such "strongly exotic groups" as they are called by Herer, Christensen and Banaszcyk cannot admit even a nontrivial near-action. Moreover, every nonsingular near-action (preserving a measure class rather than a measure) leads, by a standard construction, to a unitary representation. Thus these strongly exotic groups cannot admit nontrivial *nonsingular* near-actions.

By Theorem 1.1, a nontrivial near-action of a Lévy group cannot admit a spatial model. An important example is the automorphism group of an infinite-dimensional Gaussian measure. Up to isomorphism, the relevant probability space is the product $(\mathbb{R}^\infty, \gamma^\infty)$ of countably many copies of $(\mathbb{R}, \gamma)$, where $\gamma$ is the standard one-dimensional Gaussian measure (normal distribution). The space (so-called **first chaos**) of all measurable linear functionals on $(\mathbb{R}^\infty, \gamma^\infty)$ is $l_2$. The action of the full orthogonal group $O(l_2)$ on measurable linear functionals is well-known to be induced by its near-action on $(\mathbb{R}^\infty, \gamma^\infty)$, which is what we mean by the automorphism group of the Gaussian measure. (In this sense, $O(l_2)$ is a closed subgroup of $\mathrm{Aut}(\mathbb{R}^\infty, \gamma^\infty)$; see also Section 4.)

COROLLARY 1.8: *The near-action of the automorphism group of the Gaussian measure admits no spatial model.*

*Remark 1.9:* In [32, Theorem 4] A. Vershik states: There is no measurable realization of the group $U(H)$, that is, there is no set of full measure that is invariant under all $u \in U(H)$. This would yield Corollary 1.6, however the proof given there appears to us to be incomplete.

*Remark 1.10:* Another proof of Corollary 1.8 uses the Lévy group $L_0([0,1], S^1)$ rather than $O(l_2)$. The latter group contains (an isomorphic copy of) the former group as a closed subgroup; see, e.g., Lemańczyk, Parreau and Thouvenot [17]. The near-action of (the copy of) $L_0([0,1], S^1)$ on $(\mathbb{R}^\infty, \gamma^\infty)$ is nontrivial; by Theorem 1.1 it cannot admit a spatial model, which implies Corollary 1.8.

*Remark 1.11:* About the meaning of Corollary 1.8. Almost all points of $(\mathbb{R}^\infty, \gamma^\infty)$ do not belong to $l_2$ and therefore cannot be interpreted as continuous linear functionals on $l_2$. One could hope for interpreting them as another superstructures over $l_2$ (say, densely defined discontinuous linear functionals) that form a Borel $G$-space ($G$ being the symmetry group). Corollary 1.8 shows that it is impossible. Maybe, Borel measurability could be weakened (say, to universal measurability)? We do not know. Some related ideas can be found in [6, Prop. E.2] and [14, Example 1.27].

## 2. Which actions admit spatial models?

In this section we enhance our understanding of the lifting problem by relating it to a notion of $G$-continuity of functions which is reminiscent of the classical notion of a rigid action in ergodic theory.

*Definition 2.1:* Having a near-action (or Boolean action) of $G$ on $(\mathcal{X}, \mu)$ we say that $f \in L^\infty(\mu)$ is $G$-**continuous**, if $f \circ g_n$ converges to $f$ in $L^\infty(\mu)$ norm whenever $g_n \to e$.

The collection $\mathcal{A}(G)$ of all $G$-continuous functions is a $G$-invariant closed subalgebra of $L^\infty(\mu)$.

THEOREM 2.2: *A near-action admits a spatial model if and only if there exists a sequence of $G$-continuous functions that generates the $\sigma$-algebra (equivalently: separates points).*

*Proof:* Suppose first that we have a spatial model, that is, a Borel $G$-space with an invariant measure. By Theorem 0.4(b) this Borel $G$-space can be embedded into a compact Polish $G$-space $X$ (with an invariant measure). The continuous functions on $X$ form a separable Banach space and a dense sequence in $C(X)$ will provide a sequence of $G$-continuous functions in $L^\infty(\mu)$ which separates points.

Conversely, suppose there exists a sequence $\{f_n : n \in \mathbb{N}\} \subset L^\infty(\mu)$ of $G$-continuous functions that generates the $\sigma$-algebra. Let $G_0 \subset G$ be a countable dense subgroup of $G$. Let $A \subset L^\infty(\mu)$ be the smallest closed $G_0$-invariant subalgebra containing $\{f_n : n \in \mathbb{N}\}$ and the constant functions. Clearly $A$ is a separable subalgebra and the fact that $G_0$ is dense in $G$ implies that $A$ is in fact $G$-invariant.

Let $Y$ be the compact metric Gelfand space of $A$. (Thus the elements of $Y$ are the multiplicative linear functionals of norm one on $A$ and the map $A \cong$

$C(Y)$, $f \mapsto \hat{f}$, where $\hat{f}(y) = y(f)$, is an isometric isomorphism of Banach algebras.) Then, for each $g \in G$, the linear action $f \mapsto f \circ g$ of $g$ on $A$ defines a homeomorphism $g : Y \to Y$ and $\widehat{f \circ g} = \hat{f} \circ g$. If $y_n \to y$ in $Y$ and $g_n \to e$ in $G$ are convergent sequences, then for every $\hat{f} \in C(Y)$ we have

$$\begin{aligned} |\hat{f}(g_n y_n) - \hat{f}(y)| &\le |\hat{f}(g_n y_n) - \hat{f}(y_n)| + |\hat{f}(y_n) - \hat{f}(y)| \\ &\le \|\widehat{f \circ g_n} - \hat{f}\| + |\hat{f}(y_n) - \hat{f}(y)|, \end{aligned}$$

hence $\lim_{n\to\infty} |\hat{f}(g_n y_n) - \hat{f}(y)| = 0$. It follows that $\lim_{n\to\infty} g_n y_n = y$ and we conclude that the action of $G$ on $Y$ is topological.

The linear functional $\mu$: $A \to \mathbb{R}, f \mapsto \int f d\mu$ defines a probability measure $\nu$ on $Y$ and the dynamical system $(Y, \mathcal{Y}, \nu, G)$, where $\mathcal{Y}$ is the Borel $\sigma$-algebra on $Y$, yields a Boolean action $(\mathcal{Y}, \nu, G)$ which is isomorphic to the given Boolean action. We conclude that $(Y, \mathcal{Y}, \nu, G)$ is a spatial model as required. ∎

*Remark 2.3:* In general, when we do not assume that the $G$-continuous functions on the near action $(X, \mathcal{X}, \mu, G)$ separate points, we can still consider the smallest $\sigma$-algebra $\mathcal{D} \subset \mathcal{X}$ with respect to which all the functions in $\mathcal{A}(G)$ are measurable and then the closed subspace of $L^2(\mu)$ consisting of $\mathcal{D}$-measurable functions. This subspace defines a factor near-action and it is clear that this factor is the largest factor which admits a spatial model.

Theorem 1.1 together with Remark 2.3 yield the following:

COROLLARY 2.4: *For a Lévy group $G$, an ergodic near-action admits only constants as $G$-continuous functions.*

It is an interesting fact that a seemingly weaker condition already implies $G$-continuity. To see this we first need a lemma.

LEMMA 2.5: *Let $X$ be a Polish space, $f$: $X \to L^1(\mu)$ a continuous map such that the image $f(X)$ is contained in $L^\infty(\mu)$ and that as a subset of the Banach space $L^\infty(\mu)$ it is separable. Then $f$ treated as a map $X \to L^\infty(\mu)$ is continuous at every point of some dense $G_\delta$ subset of $X$.*

*Proof:* Every closed ball in $L^\infty(\mu)$ is a closed subset of $L^1(\mu)$. We choose $x_1, x_2, \ldots \in X$ such that $f(x_k)$ are $L^\infty$-dense in $f(X)$. We consider closed balls $B_{n,r}$ in $L^\infty(\mu)$ of radius $r$ centered at $f(x_n)$. Their inverse images $f^{-1}(B_{n,r})$ are closed in $X$, and $\bigcup_n f^{-1}(B_{n,r}) = X$ (for every $r > 0$). Denoting by $U_{n,r}$ the interior of $f^{-1}(B_{n,r})$ we observe that $\bigcup_n U_{n,r}$ is a dense open set in $X$ and $\bigcap_r \bigcup_n U_{n,r}$ is a dense $G_\delta$ set (Baire's theorem).

If $y \in U_{n,r}$ and $y_k \to y$, then $f(y) \in B_{n,r}$ and $f(y_k) \in B_{n,r}$ for large $k$, therefore $\limsup_k \|f(y_k) - f(y)\|_\infty \le 2r$. So, if $y \in \bigcap_r \bigcup_n U_{n,r}$ and $y_k \to y$, then $\limsup_k \|f(y_k) - f(y)\|_\infty = 0$. ∎

PROPOSITION 2.6: *A function $f \in L^\infty(\mu)$ is $G$-continuous if and only if its $G$-orbit is a separable subset of $L^\infty(\mu)$.*

*Proof:* The necessity is easy to see. The sufficiency follows from Lemma 2.5, applied to the map $G \to L^\infty(\mu)$, $g \mapsto f \circ g$. By homogeneity, its continuity at a single point implies continuity everywhere. ∎

## 3. Whirly actions

Often one can use the necessary and sufficient condition of Theorem 2.2 to verify directly that a given near-action has no spatial model. This is done most easily by the following notion which will guarantee that a near-action admits only constants as $G$-continuous functions.

*Definition 3.1:* A near-action of $G$ on $(X, \mathcal{X}, \mu)$ is **whirly**, if for all sets $A, B \in \mathcal{X}$ of positive measure, for almost all $g$ in $G$ with respect to Baire category, $\mu(A \cap gB) > 0$.

*Definition 3.2* (equivalent to 3.1)*:* A near-action of $G$ on $(X, \mathcal{X}, \mu)$ is **whirly**, if for all sets $A, B \in \mathcal{X}$ of positive measure, every neighborhood of $e$ (the unit of $G$) contains $g$ such that $\mu(A \cap gB) > 0$.

Clearly, 3.2 follows from 3.1 (since a neighborhood cannot be Baire-negligible). On the other hand, 3.1 follows from 3.2, since $\mu(A \cap gB)$ is a continuous function of $g$, therefore the set $V(A, B) = \{g : \mu(A \cap gB) > 0\}$ is open. Its closure contains $e$. The same holds for the set $V(gA, B) = gV(A, B)$, which shows that $V(A, B)$ is dense in $G$.

PROPOSITION 3.3:

(a) *If a near-action is whirly, then all $G$-continuous functions are constants.*
(b) *A whirly action has no spatial model; moreover, such an action cannot have nontrivial spatial factors.*

*Proof:* (a) Assume that a $G$-continuous function $f \in L^\infty(\mu)$ is non-constant; then the sets $A = f^{-1}((-\infty, a))$ and $B = f^{-1}((b, +\infty))$ are of positive measure, provided that $a < b$ are chosen appropriately. All sufficiently small $g \in G$ (that is, close enough to $e$) satisfy $\|f - f \circ g^{-1}\|_\infty < b - a$, therefore $\mu(A \cap gB) = 0$ and the action cannot be whirly.

(b) This follows from part 1 and Theorem 2.2. The claim about the factors follows from Remark 2.3. ∎

*Remark 3.4:* Here is yet another equivalent definition. A near-action of $G$ on $(X, \mathcal{X}, \mu)$ is whirly, iff for every set $A \in \mathcal{X}$ of positive measure and every neighborhood $U$ of $e$ in $G$,

$$\mu(UA) = 1;$$

here $UA$ means $\bigcup_n (g_n A)$ where $(g_n)$ is a dense sequence in $U$ (its choice does not matter mod0). *Proof:* $\mu((UA) \cap B) > 0 \iff \exists n \mu((g_n A) \cap B) > 0 \iff \exists g \in U \mu((gA) \cap B) > 0 \iff \exists g \in U \mu(A \cap g^{-1}B) > 0$.

We will next describe some applications of Proposition 3.3. Our first application will be to the natural near-action (on $X$) of the group $G = \mathrm{Aut}(X)$ of the automorphisms of the Lebesgue space $(X, \mathcal{X}, \mu)$. We have already seen that this action has no spatial model since $G$ is a Lévy group. There is, however, a more direct proof; we simply verify that the action is whirly. To this end recall that a neighborhood of the identity in $G$ is given by a finite measurable partition of $X$ into sets $\{P_1, P_2, \ldots, P_N\}$ and $\varepsilon > 0$ as

$$U = \{S \in G : \sum_{j=1}^{N} \mu(P_j \bigtriangleup SP_j) < \varepsilon\}.$$

For any sets $A, B \in \mathcal{X}$ of positive measure, if $A_0 \subset A, B_0 \subset B$ are measurable, disjoint and have the same measure $\mu(A_0) = \mu(B_0) < \varepsilon/2$, and $S$ is defined to be a measure preserving transformation which is the identity on $X \setminus (A_0 \cup B_0)$ and interchanges $A_0$ with $B_0$, then $S \in U$ and it satisfies $\mu(A \cap SB) > 0$.

The same kind of argument can be given for many subgroups of $G$ and their natural near-action on $X$. For example, we can start with any countable subgroup $\Gamma \subset G$ that acts ergodically on $X$. The **full group** of this action, $[\Gamma]$, consists of all the measure preserving transformations $T \in G = \mathrm{Aut}(X, \mathcal{X}, \mu)$ such that for $\mu$ a.e. $x \in X$, $Tx \in \Gamma x$.

PROPOSITION 3.5: *The near-action of* $[\Gamma]$ *on* $X$ *is whirly.*

*Proof:* The argument given above for the entire group $G$ works here as well, almost verbatim. The only place where some change is needed is when we choose the transformation $S$; this time it should be in $[\Gamma]$. Now suppose we are given the open set $U = U(P_0, P_1, \ldots, P_N; \varepsilon)$ and two positive measure sets $A, B \in X$. If $\mu(A \cap B) > 0$ there is nothing left to show. Otherwise, the ergodicity of the $\Gamma$-action guarantees the existence of a $\gamma \in \Gamma$ with $\mu(A \cap \gamma B) > 0$. Set $A_1 = A \cap \gamma B$ and choose any $A_0 \subset A_1$ with $0 < \mu(A_0) < \varepsilon/2$. We let $B_0 = \gamma^{-1} A_0$. The transformation $S$ is now defined as the identity on $X \setminus (A_0 \cup B_0)$ and it interchanges $A_0$ with $B_0$ by means of $\gamma$ and $\gamma^{-1}$. Clearly $S \in [\Gamma]$, $S \in U$ and it satisfies $\mu(A \cap SB) > 0$. ∎

*Remark 3.6:* Let us note that, clearly, for every dense subgroup $H$ of $G = \mathrm{Aut}(X, \mathcal{X}, \mu)$ the action of $H$ on $X$ is whirly. Moreover, it can be shown that in the notation of the previous discussion, the group $[\Gamma]$ is dense in $G$ whenever $\Gamma$ acts ergodically on $X$. However, proving the latter assertion requires a considerably more elaborate argument than the direct proof we provided in Proposition 3.5. In addition to the topology of convergence in measure on $G = \mathrm{Aut}(X, \mathcal{X}, \mu)$ one can consider the much stronger *uniform topology* given by the metric $d_u(S, T) = \mu\{x \in X : Sx \neq Tx\}$.

Although with respect to this topology $G$ is a non-Polish topological group, the subgroup $[\Gamma]$ is a closed Polish subgroup (see Hamachi–Osikawa [12], Lemma 53, or observe directly that for a countable generating collection of measurable partitions $\mathcal{P} = \{P = (P_1, \ldots, P_n)\}$ of $X$ and a fixed enumeration $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$, the set of elements

$$\{h(P; k_1, k_2, \ldots, k_n) \in [\Gamma] : P \in \mathcal{P},\ (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n\},$$

where

$$h(P; k_1, k_2, \ldots, k_n) \restriction P_j = \gamma_{k_j},\ j = 1, 2, \ldots, n,$$

is a countable dense subset of $[\Gamma]$). We now note that for both $G$ and $[\Gamma]$, our proofs show that their actions on $(X, \mathcal{X}, \mu)$ are in fact whirly with respect to the uniform topology. Again we are indebted to V. Pestov for pointing this out.

## 4. The automorphism group of the Gaussian measure

We now turn back to the full orthogonal group $G = O(l^2)$ acting on $(\mathbb{R}^\infty, \gamma^\infty)$ as explained before Corollary 1.8. Thus we let $\zeta_1, \zeta_2, \ldots : X \to \mathbb{R}$ be i.i.d. $N(0,1)$ random variables defined as the coordinate functions on the space of sequences $X = \mathbb{R}^\infty$ equipped with its Borel $\sigma$-algebra $\mathcal{X}$ and the Gauss measure $\mu = \gamma^\infty$. We identify $l^2$ with the closed linear subspace $H \subset L^2(\mu)$ generated by the functions $\zeta_1, \zeta_2, \ldots$; namely, $(c_1, c_2, \ldots) \in l^2$ with $c_1\zeta_1 + c_2\zeta_2 + \ldots \in H$. The near-action is given by

$$(c'_1\zeta_1 + c'_2\zeta_2 + \cdots) = (c_1\zeta_1 + c_2\zeta_2 + \cdots) \circ g \quad \text{whenever } (c'_1, c'_2, \dots) = g(c_1, c_2, \dots)$$

for $(c_1, c_2, \dots) \in l^2$, $g \in O(l^2)$; we call it the automorphism group of the Gaussian measure.

THEOREM 4.1: *The near-action of the automorphism group of the Gaussian measure is whirly.*

The following technical lemma is important for the proof. Basically, it states that a remote perturbation of a finite-dimensional condition forces the conditional probability to be strictly positive. Note that the relation $\mathbb{P}(A|\dots) > 0$ may be written as $\operatorname{sgn}\mathbb{P}(A|\dots) = 1$, using the (discontinuous) sign function.

LEMMA 4.2: *Let $A \subset X$ be a measurable set of positive probability, and $\alpha \in (0, \pi/2)$. Then*

(a) $\operatorname{sgn}\mathbb{P}(A|\zeta_1 \cos\alpha + \zeta_n \sin\alpha) \to 1$ *in probability, for $n \to \infty$;*

(b) *for each $m = 1, 2, \dots$*

$$\operatorname{sgn}\mathbb{P}(A|\zeta_1 \cos\alpha + \zeta_n \sin\alpha, \dots, \zeta_m \cos\alpha + \zeta_{n+m-1} \sin\alpha) \to 1 \quad \text{for } n \to \infty$$

*in probability.*

*Proof:* (a) We introduce functions $f_n \colon \mathbb{R} \to [0,1]$, $g_n \colon \mathbb{R}^2 \to [0,1]$ by

$$\mathbb{P}(A|\zeta_1 \cos\alpha + \zeta_n \sin\alpha) = f_n(\zeta_1 \cos\alpha + \zeta_n \sin\alpha),$$
$$\mathbb{P}(A|\zeta_1, \zeta_n) = g_n(\zeta_1, \zeta_n),$$

and a set $B \subset \mathbb{R}$ by

$$B = \{x_1 : \mathbb{P}(A|\zeta_1 = x_1) > 0\}.$$

(These $f_n, g_n, B$ are treated mod0, of course.) We have

$$\mathbb{P}(A|\zeta_1, \zeta_n) \to \mathbb{P}(A|\zeta_1)$$

in probability (for $n \to \infty$). On $B \times \mathbb{R}$ we get

$$\operatorname{sgn} g_n \to 1 \quad \text{in measure,}$$

with respect to $\gamma \times \gamma$, where $\gamma = N(0,1)$ is the one-dimensional Gaussian measure. However, any equivalent (that is, mutually absolutely continuous) finite measure on $B \times \mathbb{R}$ may be used equally well.

Taking into account that $f_n$ results from $g_n$ by integration (along straight lines orthogonal to the unit vector $(\cos\alpha, \sin\alpha)$) we get

$$\mathrm{sgn} f_n(u) \geq \operatorname{ess\,sup}_x \mathrm{sgn} g_n\Big(x, \frac{u - x\cos\alpha}{\sin\alpha}\Big).$$

The map

$$(x, u) \mapsto \Big(x, \frac{u - x\cos\alpha}{\sin\alpha}\Big)$$

of $B \times \mathbb{R}$ to itself sends the measure to an equivalent measure. So,

$$\mathrm{sgn} g_n\Big(x, \frac{u - x\cos\alpha}{\sin\alpha}\Big) \to 1$$

in measure, which implies $\mathrm{sgn} f_n \to 1$ in measure (with respect to $\gamma$).

(b) The same as before, but $\mathbb{R}$ is replaced by $\mathbb{R}^m$, $\mathbb{R}^2$ by $\mathbb{R}^{2m}$, $\zeta_1$ by $(\zeta_1, \ldots, \zeta_m)$ and $\zeta_n$ by $(\zeta_n, \ldots, \zeta_{n+m-1})$. ∎

*Proof of Theorem 4.1:* Let $A \in \mathcal{X}$ be a set of positive measure, and $U$ a neighborhood of $e$ in $G$; by Remark 3.4 it is sufficient to prove that $\mu(UA) = 1$. Of course, $UA$ is treated as in Remark 3.4 (and the same about $ZA$ for any $Z \subset G$).

Ergodicity of $G$ ensures that $\mu(GA) = 1$. Applying the same argument to conditional measures we get (almost everywhere)

$$\mathbb{P}(G_m A \big| \zeta_1, \ldots, \zeta_m) \geq \mathrm{sgn}\mathbb{P}(A \big| \zeta_1, \ldots, \zeta_m),$$

where $G_m = \{g \in G : g\zeta_1 = \zeta_1, \ldots, g\zeta_m = \zeta_m\}$. For $m$ large enough we have $G_m \subset U$, therefore

$$\mathbb{P}(UA \big| \zeta_1, \ldots, \zeta_m) \geq \mathrm{sgn}\mathbb{P}(A \big| \zeta_1, \ldots, \zeta_m).$$

However, there is nothing special in $\zeta_1, \ldots, \zeta_m$; by the $O(l^2)$-invariance, the same holds for $\xi_1 = \zeta_1\cos\alpha + \zeta_n \sin\alpha, \ldots, \xi_m = \zeta_m \cos\alpha + \zeta_{n+m-1}\sin\alpha$ provided that $n > m$ and the corresponding subgroup $G_{m,n,\alpha} = \{g \in G : g\xi_1 = \xi_1, \ldots, g\xi_m = \xi_m\}$ is contained in $U$. We choose $m$ so large and $\alpha$ so small that $G_{m,n,\alpha} \subset U$ for every $n > m$ (this is possible since for every $h \in H$ its distance from the span of $\xi_1, \ldots, \xi_m$ tends to 0 uniformly in $n$ for $m \to \infty$, $\alpha \to 0$). We have

$$\mathbb{P}(UA \big| \xi_1, \ldots, \xi_m) \geq \mathrm{sgn}\mathbb{P}(A \big| \xi_1, \ldots, \xi_m)$$

for all $n > m$. For $n \to \infty$ the right-hand side converges to 1 in probability (therefore, in $L^1$) by Lemma 4.2. Taking the expectation we get $\mu(UA) = 1$. ∎

## Appendix A

Measure concentration (that is, the property of being a Lévy family or group) is proven for various cases by a number of methods [23], [16]. Strong results need complicated proofs involving advanced methods (Riemann geometry, representation theory, etc.). More elementary arguments give weaker results which are satisfactory for many topological applications such as the ones we needed in Section 1. This appendix collects elementary (complete) proofs for many Lévy families.

A.1. Consider Gaussian measures $\gamma_\sigma^n$ on $\mathbb{R}^n$,

$$\gamma_\sigma^n(dx) = (2\pi)^{-n/2}\sigma^{-n}\exp\Big(-\frac{|x|^2}{2\sigma^2}\Big)dx;$$

note that $\gamma_\sigma^n(\mathbb{R}^n) = 1$ and $\int |x|^2\gamma_\sigma^n(dx) = n\sigma^2$. We claim that $(\mathbb{R}^n, d_n, \gamma_{\sigma_n}^n)$ is a Lévy family whenever the positive numbers $\sigma_n$ satisfy $\sigma_n \to 0$; here $d_n(x,y) = |x-y|$ is the usual Euclidean metric on $\mathbb{R}^n$. (Only the case $\sigma_n = n^{-1/2}$ will be used.)

According to the well-known relation between Lévy families and Lipschitz functions [16, Sect. 1.3], it suffices to prove the inequality

$$\int f^2 d\gamma_\sigma^n \le \sigma^2\|f\|_{\mathrm{Lip}}^2$$

for all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $\int f d\gamma_\sigma^n = 0$ and

$$\|f\|_{\mathrm{Lip}} = \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|} < \infty.$$

Here is a proof. We introduce functions $\phi, u\colon \mathbb{R}^n \times (0,\sigma^2) \to \mathbb{R}$,

$$\phi(x,t) = (2\pi)^{-n/2}t^{-n/2}\exp\Big(-\frac{|x|^2}{2t}\Big),$$
$$u(x,t) = \int f(y)\phi(y-x,\sigma^2-t)dy;$$

they satisfy the (famous) partial differential equations

$$\Big(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\Big)\phi(x,t) = 0, \quad \Big(\frac{\partial}{\partial t} + \frac{1}{2}\Delta\Big)u(x,t) = 0;$$

here $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. Note that $u(0,0+) = \int f d\gamma_\sigma^n = 0$ and $u(x,\sigma^2-) = f(x)$. It remains to prove the inequality

$$\int u^2(x,t)\phi(x,t)dx \le t$$

for $0 < t < \sigma^2$, assuming $\|f\|_{\mathrm{Lip}} \le 1$, that is, $|\nabla u(x,t)| \le 1$ for all $x, t$; here $\nabla$ is the gradient (in $x$). For $t \to 0+$ the integral tends to $u^2(0,0) = 0$. We have

$$\frac{d}{dt} \int u^2(x,t)\phi(x,t)dx = \int \Big( -u\Delta u + \frac{1}{2}\Delta u^2 \Big) \phi dx = \int |\nabla u|^2 \phi dx \le 1,$$

which completes the proof.

See also [16, pp. 42, 49].

A.2. Euclidean spheres $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ are a Lévy family, since a random point of $S^{n-1}$ can be obtained from a Gaussian random vector $\xi \in \mathbb{R}^n$ distributed $\gamma^n_{1/\sqrt{n}}$ by the normalization map $\xi \mapsto \frac{\xi}{|\xi|}$; the map belongs to Lip(2) as far as $|\xi| \ge 1/2$. The other case, $|\xi| < 1/2$, may be ignored, since its probability tends to 0 for $n \to \infty$. The argument works also when the radius $r_n$ of the sphere is not just 1 but satisfies $r_n = o(\sqrt{n})$. See also [11, 2.1, 2.3], [16, Prop. 2.10], [10, $3\frac{1}{2}$.24].

An alternative, comparably elementary way to A.1 and A.2 is first proving A.2 via the spectral gap of the Laplace operator on the sphere [11, 4.2(a)], [16, Th. 3.1 and p. 49] and then deriving A.1 from A.2 [16, p. 28].

A.3. The Stiefel manifolds $W^n_2 = \{(x_1, x_2) \in S^{n-1} \times S^{n-1} : \langle x_1, x_2 \rangle = 0\}$ are a Lévy family, since a random point of $W^n_2$ can be obtained from a $2n$-dimensional Gaussian random vector $(\xi_1, \xi_2) \in \mathbb{R}^n \oplus \mathbb{R}^n$ distributed $\gamma^n_{1/\sqrt{n}} \otimes \gamma^n_{1/\sqrt{n}}$ by normalization, subsequent orthogonalization $(\xi_1, \xi_2) \mapsto (\xi_1, \xi_2 - \langle \xi_2, \xi_1 \rangle \xi_1)$ and normalization again. The Lipschitz property is ensured as far as $|\langle \xi_1, \xi_2 \rangle| \le \varepsilon_2 |\xi_1||\xi_2|$ and $|\xi_1|, |\xi_2| \in [1 - \varepsilon_2, 1 + \varepsilon_2]$, where $\varepsilon_2$ is an appropriate absolute constant. The other case may be ignored, since its probability tends to 0 for $n \to \infty$. The orthonormalization commutes with the natural action of $O(n)$; thus, $O(n)$-invariance of the Gaussian measure ensures $O(n)$-invariance of the measure on $W^n_2$. The same argument (with $\varepsilon_k$ in place of $\varepsilon_2$) works for

$$W^n_k = \{(x_1, \dots, x_k) \in (S^{n-1})^k : \langle x_i, x_j \rangle = 0 \text{ for } 1 \le i < j \le k\}.$$

See [21], [22]. The proof in [11, 3.3] is somewhat less elementary.

A.4. The full orthogonal group $O(H)$ of a separable infinite-dimensional Hilbert space $H$ over $\mathbb{R}$ is a Lévy group.

*Proof:* We equip $O(H)$ with a left-invariant metric and consider the subgroups $O(n) = \{g \in O(H) : ge_{n+1} = e_{n+1}, ge_{n+2} = e_{n+2}, \dots\}$ where $e_1, e_2, \dots$ are a chosen orthonormal basis of $H$. Let $A_n \subset O(n)$, $\liminf m_n(A_n) > 0$. Given $\varepsilon > 0$, we take $k$ and $\delta$ such that

$$\|ge_1 - g'e_1\|^2 + \dots + \|ge_k - g'e_k\|^2 < \delta^2 \Rightarrow d(g, g') < \varepsilon$$

for all $g, g' \in O(H)$. The maps $T_n\colon O(n) \to W_k^n$, $T_n(g) = (ge_1, \dots, ge_k)$ satisfy

$$\mathrm{dist}(T_n(g), T_n(g')) < \delta \Rightarrow d(g, g') < \varepsilon$$

for all $g, g' \in O(H)$. Therefore

$$B_\varepsilon(A_n) \supset T_n^{-1}(B_\delta(T_n(A_n))).$$

It remains to apply A.3.

A.5. The commutative Polish group $L_0([0,1], S^1)$ is a Lévy group. It consists of all equivalence classes of measurable functions $[0,1] \to S^1$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and is in fact monothetic [7]. The following proof of its Lévy property is basically an extract from [16, Sect. 1.6, 4.1]. See also [16, p. 31] and [7].

Let $G$ be a commutative Polish group with a compatible invariant metric $d$, and $\mu$ a Borel probability measure on $G$; we define

$$\mathrm{VarLip}(\mu) = \sup\{\sqrt{\textstyle\int f^2 d\mu} \mid \|f\|_{\mathrm{Lip}} \le 1, \textstyle\int f d\mu = 0\}.$$

Clearly,

$$\mathrm{VarLip}(\mu) \le \operatorname{diam} \operatorname{supp}(\mu)$$

(the diameter of the support). It is easy to see that

$$\mathrm{VarLip}(\mu) = \sup_f \frac{\|f^2 * \mu - (f * \mu)^2\|_{\sup}^{1/2}}{\|f\|_{\mathrm{Lip}}},$$

where $(f * \mu)(x) = \int f(x-y)\mu(dy)$, $\|f\|_{\sup} = \sup_{x \in G} |f(x)|$, and the squares are taken pointwise. For any two measures $\mu, \nu$

$$\begin{aligned}
&\|f^2 * \mu * \nu - (f * \mu * \nu)^2\|_{\sup} \\
&\quad \le \|(f^2 * \mu - (f * \mu)^2) * \nu\|_{\sup} + \|(f * \mu)^2 * \nu - (f * \mu * \nu)^2\|_{\sup} \\
&\quad \le \|f^2 * \mu - (f * \mu)^2\|_{\sup} + \mathrm{VarLip}^2(\nu)\|f * \mu\|_{\mathrm{Lip}}^2 \\
&\quad \le \mathrm{VarLip}^2(\mu)\|f\|_{\mathrm{Lip}}^2 + \mathrm{VarLip}^2(\nu)\|f\|_{\mathrm{Lip}}^2,
\end{aligned}$$

thus

$$\mathrm{VarLip}(\mu * \nu) \le \sqrt{\mathrm{VarLip}^2(\mu) + \mathrm{VarLip}^2(\nu)}.$$

The argument is applied to $G = L_0((0,1), S^1)$ as follows. We choose the $L_1$-metric

$$d(x, y) = \int_0^1 |x(t) - y(t)| dt.$$

For each $n$ we consider the $n$-dimensional compact subgroup

$$K_n = K_{n,1} + \cdots + K_{n,n} \subset G$$

where $K_{n,m}$ is the one-dimensional group of functions constant on $(\frac{m-1}{n}, \frac{m}{n})$ and equal to 1 on $(0,1) \setminus (\frac{m-1}{n}, \frac{m}{n})$. The corresponding invariant measures are related by

$$m_{K_n} = m_{K_{n,1}} * \cdots * m_{K_{n,n}}.$$

However,

$$\operatorname{VarLip}(m_{K_{n,m}}) \le \operatorname{diam}(K_{n,m}) \le 2/n,$$

therefore $\operatorname{VarLip}^2(m_{K_n}) \le n \cdot (2/n)^2 \to 0$ for $n \to \infty$. It remains to use the relation between Lévy families and Lipschitz functions mentioned in A.1.

A.6. The Polish group $G = \operatorname{Aut}([0,1])$ is a Lévy group. It consists of all equivalence classes of invertible transformations $[0,1] \to [0,1]$ preserving Lebesgue measure. Its Lévy property may be proven by the argument of A.5, generalized to an arbitrary (not just commutative) Polish group $G$ with a compatible right-invariant metric $d$ (that is, $d(g_1 h, g_2 h) = d(g_1, g_2)$). Still,

$$\operatorname{VarLip}^2(\mu * \nu) \le \operatorname{VarLip}^2(\mu) + \operatorname{VarLip}^2(\nu)$$

where $\operatorname{VarLip}(\mu)$ is *defined* as $\sup_f \|f^2 * \mu - (f * \mu)^2\|_{\sup}^{1/2} / \|f\|_{\mathrm{Lip}}$, $f * \mu$ is defined by $(f * \mu)(x) = \int f(xy^{-1}) \mu(dy)$, and $\mu * \nu$ is defined by $\int f d(\mu * \nu) = \int f(xy) \mu(dx) \nu(dy)$. However, the inequality $\operatorname{VarLip}(\mu) \le \operatorname{diam} \operatorname{supp}(\mu)$ need not hold, since the map $y \mapsto xy^{-1}$ need not be isometric. If the metric $d$ is bi-invariant (that is, $d(g_1 h, g_2 h) = d(g_1, g_2) = d(hg_1, hg_2)$), then $\operatorname{VarLip}(\mu) \le \operatorname{diam} \operatorname{supp}(\mu)$.

We apply the argument to the group $S_n$ of all permutations of $\{1, \ldots, n\}$ equipped with the Hamming metric

$$d(g, h) = \frac{\#\{k : g(k) \ne h(k)\}}{n}.$$

Its invariant measure $m_{S_n}$ is the convolution of $n$ measures, each concentrated on transpositions (that is, $g$ such that $d(g, e) \le 2/n$). Indeed, $S_{n-1}$ is naturally embedded into $S_n$, and $m_{S_n} = m_{S_{n-1}} * \mu$ where $\mu$ is distributed uniformly on transpositions of $n$ and $k$ for $k = 1, \ldots, n$. So, $\operatorname{VarLip}^2(m_{S_n}) \le n \cdot (4/n)^2 \to 0$ for $n \to \infty$.

It remains to note that there exists a natural embedding of the inductive limit group $\mathbf{S} = \lim_{n \to \infty} S_{2^n}$ as a dense subgroup of $G = \operatorname{Aut}([0,1])$. Here the group $S_{2^n}$ is embedded into $S_{2^{n+1}}$ as the subgroup of permutations $\hat{\sigma}$ of $\{0,1,\ldots,2^{n+1}-1\}$ of the form $\hat{\sigma}(2k) = 2\sigma(k)$ and $\hat{\sigma}(2k+1) = 2\sigma(k)+1$, $k = 0,1,\ldots,2^n-1$, $\sigma \in S_{2^n}$. Then, for each $n$, the group $S_{2^n}$ is identified as the subgroup of $G = \mathrm{Aut}([0,1])$ which consists of the transformations permuting the $2^n$ dyadic sub-intervals of $[0,1]$ by translations. It can be easily seen that the restriction of the uniform metric $d_u(S,T) = \mu\{x \in X : Sx \neq Tx\}$ on $G$ to $S_{2^n}$ is the Hamming metric. Thus with respect to this metric and using the estimation for $\mathrm{VarLip}^2(m_{S_n})$ we see that $\mathbf{S}$ is a Lévy group. Since the identity map from $(\mathbf{S}, d_u)$ to $G$ is continuous and since $\mathbf{S}$ is dense in $G$, we can finally conclude that also $G$ is a Lévy group. See [9] and also [16, Corollary 4.3].

## Appendix B

The theorem stated below is well known and widely used. However, it seems that complete proofs are not easily found. The proof we provide is from Oxtoby [25] where it is attributed to Ulam.

THEOREM B.1: *A Polish topological group $G$ which admits a Borel $\sigma$-finite (either right or left) invariant measure class is locally compact.*

*Proof:* Let $\mu$ be a $\sigma$-finite measure on $G$ such that

$$\mu(B) > 0 \iff \mu(gB) > 0, \quad \forall \text{ measurable } B, \forall g \in G.$$

Let $A$ be a measurable subset of $G$ with $0 < \mu(A) < \infty$. Since every Borel measure on a Polish space is regular, there exists a compact set $K \subset A$ with $0 < \mu(K) < \infty$. Let $H < G$ be the subgroup of $G$ which is generated by $L = K \cup K^{-1}$. Clearly $H = \bigcup\{L^n : n \in \mathbb{N}\}$ is a $\sigma$-compact group. If $G/H$ is uncountable, then there are uncountably many distinct cosets of $H$ in $G$ and, in particular, uncountably many pairwise disjoint translations of $K$. This, however, contradicts the $\sigma$-finiteness of $\mu$ and we conclude that $G/H$ is countable. Now Baire's theorem implies that $\mathrm{int}L^n \neq \emptyset$ for some $n$ and we conclude that $G$ is locally compact. ∎

*Remark B.2:* In [18, theorem 7.1] Mackey proved a more general theorem. He showed that if $G$ is an analytic Borel group (no topology is given) which admits an invariant $\sigma$-finite measure class, then there exists a unique locally compact topology on $G$ whose Borel structure is the given one and under which $G$ is a topological group. His proof relies on Weil's theorem, [33]. Finally, we note that in [1] A. D. Alexandroff proved a related result (he is not assuming $\sigma$-finiteness of the invariant measure but, instead, the existence of an open set with finite positive measure).

## References


[1] A. D. Alexandroff, *On groups with an invariant measure*, Doklady Akademii Nauk SSSR **34** (1942), 5-9.

[2] W. Banaszczyk, *On the existence of exotic Banach–Lie groups*, Mathematische Annalen **264** (1983), 485–493.

[3] H. Becker and A. S. Kechris, *The descriptive set theory of Polish groups*, London Mathematical Society Lecture Notes **232**, Cambridge University Press, 1996.

[4] G. Beer, *A Polish topology for the closed subsets of a Polish space*, Proceedings of the American Mathematical Society **113** (1991), 1123–1133.

[5] A. Danilenko, *Point realization of Boolean actions of countable inductive limits of locally compact groups*, Matematicheskaya Fizika, Analiz, Geometriya **7** (2000), 35–48.

[6] R. M. Dudley, *Real Analysis and Probability*, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1989.

[7] E. Glasner, *On minimal actions of Polish groups*, Topology and Its Applications **85** (1998), 119–125.

[8] E. Glasner and B. Weiss, *The universal minimal system for the group of homeomorphisms of the Cantor set*, Fundamenta Mathematicae **176** (2003), 277–289.

[9] T. Giordano and V. Pestov, *Some extremely amenable groups*, Comptes Rendus de l'Académie des Sciences, Paris **334** (2002), 273–278.

[10] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, Basel, 1999.

[11] M. Gromov and V. D. Milman, *A topological application of the isoperimetric inequality*, American Journal of Math **105** (1983), 843–854.

[12] T. Hamachi and M. Osikawa, *Ergodic groups of automorphisms and Krieger's theorems*, Seminar on Mathematical Sciences, 3. Keio University, Department of Mathematics, Yokohama, 1981.

[13] W. Herer and J. P. R. Christensen, *On the existence of pathological submeasures and the construction of exotic topological groups*, Mathematische Annalen **213** (1975), 203–210.

[14] S. Janson, *Gaussian Hilbert Spaces*, Cambridge University Press, 1997.

[15] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics **156**, Springer-Verlag, Berlin, 1991.

[16] M. Ledoux, *The Concentration of Measure Phenomenon,* American Mathematical Society, Providence, RI, 2001.

[17] M. Lemańczyk, F. Parreau and J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self-joinings are Gaussian,* Fundamenta Mathematicae **164** (2000), 253–293.

[18] G. W. Mackey, *Borel structure in groups and their duals,* Transactions of the American Mathematical Society **85** (1957), 134–165.

[19] G. W. Mackey, *Point realizations of transformation groups,* Illinois Journal of Mathematics **6** (1962), 327–335.

[20] V. D. Milman, *A new proof of the theorem of A. Dvoretzky on sections of convex bodies,* Functional Analysis **5** (1971), 28–37.

[21] V. D. Milman, *Asymptotic properties of functions of several variables that are defined on homogeneous spaces,* Soviet Mathematics Doklady **12** (1971), 1277–1281; translated from Doklady Akademii Nauk SSSR **199** (1971), 1247–1250.

[22] V. D. Milman, *On a property of functions defined on infinite-dimensional manifolds,* Soviet Mathematics Doklady **12** (1971), 1487–1491; translated from Doklady Akademii Nauk SSSR **200** (1971), 781–784.

[23] V. D. Milman, *The heritage of P. Lévy in geometrical functional analysis,* Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987), Asterisque **157–158** (1988), 273–301.

[24] J. von Neumann, *Zur Operatorenmethode in der klassischen Mechanik,* Annals of Mathematics **33** (1932), 587–642.

[25] J. C. Oxtoby, *Invariant measures in groups which are not locally compact,* Transactions of the American Mathematical Society **60** (1946), 215–237.

[26] V. Pestov, *On free actions, minimal flows, and a problem by Ellis,* Transactions of the American Mathematical Society **350** (1998), 4149–4165.

[27] V. Pestov, *Topological groups: where to from here?,* Topology Proceedings **24** (1999), 421–502.

[28] A. Ramsay, *Virtual groups and group actions,* Advances in Mathematics **6** (1971), 253–322.

[29] V. A. Rohlin, *Selected topics from the metric theory of dynamical systems,* Advances in Mathematics (Russian) Uspehi Matematicheskii Nauk (N.S.) **4** (1949), 57–128; American Mathematical Society Translations, Series 2, **49** (1966), 171–240.

[30] V. S. Varadarajan, *Groups of automorphisms of Borel spaces,* Transactions of the American Mathematical Society **109** (1963), 191–220.

[31] V. S. Varadarajan, *Geometry of Quantum Theory,* Vol. II, Van Nostrand, Princeton, 1970.

[32] A. M. Vershik, *Measurable realizations of automorphism groups and integral representations of positive operators*, (Russian) Sibirskii Matematicheskii Zhurnal **28** (1987), 52–60; English translation: Siberian Mathematical Journal **28** (1987), 36–43.

[33] A. Weil, *L'intègration dans les groupes topologiques et ses applications*, Hermann et Cie, Paris, 1940.

[34] R. J. Zimmer, *Extensions of ergodic group actions*, Illinois Journal of Mathematics **20** (1976), 373–409.