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A REMARK ON ALMOST EXTENSIONS OF LIPSCHITZ FUNCTIONS

BY

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ABSTRACT

We present a simpler proof of a result of J. Bourgain on almost extensions of functions satisfying a Lipschitz condition on δ-nets.

Theorem 1 of the article [B] is:

Let $S = \{\|x\| = 1\}$ be the unit sphere of the n-dimensional normed space E, and let $\mathcal{E}_\delta$ be a δ-net in S. Assume that Y is a Banach space and that $f\colon \mathcal{E}_\delta \to Y$ is a Lipschitz map with constant L. Let $\tau > C\delta$. Then there is a map $\bar{f}\colon E \to Y$ satisfying

$$\|f(x) - \bar{f}(x)\| \le \tau L \quad \text{for } x \in \mathcal{E}_\delta,$$
$$\|\bar{f}\|_{\text{lip}} \le C(1 + \delta\tau^{-1}n)L.$$

(C is a universal numerical constant.)

The proof in [B] uses ingenious and delicate arguments. The purpose of this note is to give a transparent and natural proof of the crucial step of the theorem. To make this article self-contained, we include a complete proof of Bourgain's theorem.

We begin with some notation.

Let E be a normed space of dimension n, let K be a convex subset of E, and fix $\tau > 0$. Denote by $B(x, \tau)$ the open ball of radius τ in E centered at x, and

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put $K_\tau = \bigcup_{x\in K} B(x,\tau)$, the τ-neighborhood of K. Denote by μ the Lebesgue measure on E normalized by $\mathrm{Vol}(B(0,\tau)) = 1$. We denote by χ the indicator function of $B(0,\tau)$. Let Y be a normed space. For a mapping $f\colon K \to Y$ its Lipschitz constant on K is

$$\|f\|_{\mathrm{lip}(K)} = \sup_{x_1,x_2\in K} \frac{\|f(x_1)-f(x_2)\|}{\|x_1-x_2\|}.$$

The result we prove is the following

PROPOSITION: *Fix $0<\delta, L<\infty$. Let K be a convex subset of E, and assume that the map $f\colon K_\tau \to Y$ satisfies*

$$(*) \qquad \|f(x_1)-f(x_2)\| \le L(\|x_1-x_2\|+\delta) \quad \text{for all } x_1,x_2\in K_\tau.$$

Then the map $g = f*\chi$, which is well defined from K into Y, satisfies

$$(**) \qquad \|g\|_{\mathrm{lip}(K)} \le L\Big(1+\frac{\delta n}{2\tau}\Big).$$

Proof: Consider two points in K. By translating K we may assume that one of them is the origin and denote the other one by x. Put $B = B(0,\tau)$ and $B_x = B(x,\tau)$. Then $g(0)-g(x) = \int_B f - \int_{B_x} f = \int_M f - \int_{M'} f$, where $M = B \setminus B_x$, $M' = B_x \setminus B$.

Consider the family $\{L_t = t+\mathbb{R}\cdot x\}_{t\in E}$ of straight lines parallel to x. Each L_t intersects M and M' in intervals of the same length (which does not exceed $\|x\|$, and is exactly $\|x\|$ in case L_t intesects $B\cap B_x$).

Let $c\colon M \to M'$ be the transformation which translates each interval $L_t\cap M$ onto the interval $L_t \cap M'$ along L_t (Cavalieri transformation).

Fix a hyperplane $Z\subset E$ such that $x\notin Z$, and let P be the projection from E to Z along x, and $w(t)$ $(t\in E)$ be the length of $L_t\cap B$. The Lebesgue measure on E is the product of any two appropriately normalized measures on Z and on the line $\mathbb{R}\cdot x$. We normalize the measure on $\mathbb{R}\cdot x$ by setting the measure of $[0,x]$ to be equal to $\|x\|$. This and the choice $\mu(B)=1$ determine the Lebesgue measure, ν, on Z, and we have

$$\int_{P(B)} w(z)\,d\nu(z) = \mathrm{Vol}\, B = 1.$$

Consider B as the disjoint union of two sets ("wide" and "narrow"):

$$W = \{t\in B \mid L_t\cap B\cap B_x \ne \emptyset\},$$
$$N = \{t\in B \mid L_t\cap B\cap B_x = \emptyset\} \quad \text{(clearly, } N\subset M).$$

Thus the length of the intersection $L_t \cap M$ is $\|x\|$ for $t \in W$ and $w(t)$ for $t \in N$. Also

$$\|t - c(t)\| = \begin{cases} w(t), & t \in W \cap M, \\ \|x\|, & t \in N. \end{cases}$$

Now, we can estimate $\|g\|_{\mathrm{lip}(S)}$:

$$\begin{aligned}
\|g(0) - g(x)\| &= \|\int_M f d\mu - \int_{M'} f d\mu\| = \|\int_M (f - f \circ c) d\mu\| \\
&\le \int_M \|f - f \circ c\| d\mu \\
&\le L \int_M (\|t - c(t)\| + \delta) d\mu(t) = L\big(\delta \cdot \operatorname{Vol} M + \int_M \|t - c(t)\| d\mu(t)\big).
\end{aligned}$$

(The second equality holds because c is measure preserving from M onto M', and the last inequality follows from the condition $(*)$ of the proposition.)

The last integral will be estimated separately on $W \cap M$ and on N:

$$\begin{aligned}
\int_{W\cap M} \|t - c(t)\| d\mu(t) &= \int_{W\cap M} w(t) d\mu(t) \\
&= \int_{P(W\cap M)} \|x\| \cdot w(t) d\nu(t) \\
&= \|x\| \int_{P(W\cap M)} w(t) d\nu(t) \\
&= \|x\| \cdot \operatorname{Vol} W.
\end{aligned}$$

(The second equality follows from the normalization of the measures and from the fact that for each t, $W \cap M$ intersects L_t in an interval of length $\|x\|$.)

$$\int_N \|t - c(t)\| d\mu(t) = \int_N \|x\| d\mu(t) = \|x\| \cdot \operatorname{Vol} N.$$

Hence

$$\int_M \|t - c(t)\| d\mu(t) = \|x\| \cdot (\operatorname{Vol} W + \operatorname{Vol} N) = \|x\| \operatorname{Vol} B = \|x\|.$$

We also observe that for $\|x\| < 2\tau$

$$\operatorname{Vol} M = \frac{1}{2} \operatorname{Vol}(B \Delta B_x) \le \frac{1}{2\tau^n}\Big[\Big(\tau + \frac{\|x\|}{2}\Big)^n - \Big(\tau - \frac{\|x\|}{2}\Big)^n\Big] = \frac{n}{2\tau}\|x\| + \theta_{\tau,n}(\|x\|),$$

where $\theta_{\tau,n}(\|x\|) = o(\|x\|)$.

(Indeed, $B \Delta B_x \subset B(x/2, \tau + \|x\|/2) \setminus B(x/2, \tau - \|x\|/2)$.)

Combining all the estimates, we obtain

$$\|g(x) - g(0)\| \leq L(\delta \operatorname{Vol} M + \int_M \|t - c(t)\| d_\mu(t)) \leq L\Big(1 + \frac{\delta n}{2\tau}\Big)\|x\| + o(\|x\|)$$

and the result follows from the convexity of μ (indeed, for $h\colon [0,1] \to Y$, if

$$\limsup_{s \to 0} \frac{\|h(r+s) - h(r)\|}{s} \leq C \quad \text{for any } 0 \leq r \leq 1,$$

then clearly $\|h(1) - h(0)\| \leq C$). This proves the proposition.

Remark: The formula for $\operatorname{Vol} M$ actually yields estimates for g even when K is not convex.

We now reproduce Bourgain's argument for the final part of the proof of his theorem.

Denote the points of $\mathcal{E}_\delta$ by $\{x_p\}_1^N$. Using a translation in Y we may assume that

$$\|f\|_\infty := \max_{1 \leq p \leq N} \|f(x_p)\| \leq 2\|f\|_{\text{lip}} = 2L.$$

Consider a partition of unity $\{\varphi_p\}$ on S subordinated to the covering $\{B(x_p, 2\delta) \cap S\}_1^N$. Define $f_1\colon S \to Y$ by

$$f_1(x) = \sum_1^N f(x_p)\varphi_p(x).$$

Then $\|f_1\|_\infty \leq \|f\|_\infty \leq 2L$, and $\|f_1(x) - f(x)\| \leq 2\delta \cdot L$ for any $x \in \mathcal{E}_\delta$.

For arbitrary $z_1, z_2 \in S$,

$$\begin{aligned}\|f_1(z_1) - f_1(z_2)\| &\leq \max\{\|f(x_1) - f(x_2)\| \mid x_i \in B(z_i, 2\delta) \cap \mathcal{E}_\delta, i = 1, 2\} \\ &\leq L(\|z_1 - z_2\| + 4\delta).\end{aligned}$$

Direct computation shows that if we extend f_1 to all of E by putting

$$\tilde{f}(x) = \alpha(\|x\|) \cdot f_1\Big(\frac{x}{\|x\|}\Big) \qquad (\tilde{f}(0) = 0),$$

where $\alpha(t) = \max(1 - |t - 1|, 0)$ $(t \in \mathbb{R}_+)$, then f satisfies

$$\|\tilde{f}(z) - \tilde{f}(x)\| \leq 4L(\|z - x\| + \delta)$$

for any $x, z \in E$.

Finally, we use the Proposition. Normalize the Lebesgue measure in E by $\mathrm{Vol}(B(0,\tau/8)) = 1$, let χ be the indicator function of $B(0,\tau/8)$, and set $\bar{f} = \tilde{f} * \chi$. Fix any $x \in \mathcal{E}_\delta$, then $\tilde{f}(x) = f_1(x)$, hence

$$\begin{aligned}\|\bar{f}(x) - f(x)\| &\leq \|\bar{f}(x) - \tilde{f}(x)\| + \|f_1(x) - f(x)\| \\ &\leq 4L(\tau/8 + \delta) + L \cdot 2\delta = L(\tau/2 + 6\delta) < \tau L,\end{aligned}$$

if we choose $\tau > 12\delta$.

By the Proposition (with τ replaced by $\tau/8$, L by $4L$, and with $K = E$ — the whole space),

$$\|\bar{f}\|_{\mathrm{lip}} \leq 4L\Big(1 + \frac{\delta n}{2(\tau/8)}\Big) \leq 16L\Big(1 + \frac{\delta n}{\tau}\Big),$$

as demanded.

Remark: In Bourgain's argument $\bar{f}$ is taken to be $\bar{f} = \tilde{f} * K_\tau$ for K_τ with special properties, rather than $\bar{f} = \tilde{f} * \chi$. Our proposition shows that one can use the simplest convolution (averaging).

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References

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