ISRAEL JOURNAL OF MATHEMATICS 97 (1997), 317–345

LACUNARY ISOMORPHISM OF DECREASING SEQUENCES OF MEASURABLE PARTITIONS

BY

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ABSTRACT

The lacunary and orbital isomorphism problem is solved for a wide class of decreasing sequences of measurable partitions of Lebesgue spaces which are finitely isomorphic to the standard Bernoulli sequences.

Introduction

Let $(X, \mathcal{F}, m)$ be a Lebesgue space with $mX = 1$, and let

$$\varepsilon = \xi_0 \geq \xi_1 \geq \xi_2 \geq \cdots, \quad \xi = \{\xi_n\}_{n=0}^{\infty}$$

be a decreasing sequence of measurable partitions of X, where $\varepsilon = \varepsilon_X$ denotes the partitions of X into separate points.

The following natural isomorphism relations hold for such sequences.

Two decreasing sequences of measurable partitions (d.s.m.p.) $\xi = \{\xi_n\}$ and $\xi' = \{\xi'_n\}$, given on spaces $(X, \mathcal{F}, m)$ and $(X', \mathcal{F}', m')$, are called:

(i) isomorphic ($\xi \overset{I}{\sim} \xi'$), if $\Phi(\xi_n) = \xi'_n$ for all n, where Φ is an isomorphism $\Phi: X \to X'$,

(ii) finitely isomorphic ($\xi \overset{FI}{\sim} \xi'$), if for any n there exists an isomorphism $\Phi_n: X \to X'$ such that $\Phi_n(\xi_k) = \xi'_k$, $k = 1, 2, \ldots, n$,

(iii) lacunarily isomorphic ($\xi \overset{LI}{\sim} \xi'$), if there exist $n_1 < n_2 < n_3 < \cdots$ such that the subsequences $\{\xi_{n_k}\}_{k=1}^{\infty}$ and $\{\xi'_{n_k}\}_{k=1}^{\infty}$ are isomorphic,

Received August 8, 1994 and in revised form August 31, 1995

(iv) orbitally isomorphic ($\xi \overset{OI}{\sim} \xi'$), if the intersections

$$\theta(\xi) = \bigcap_{n=1}^{\infty} \xi_n, \quad \theta(\xi') = \bigcap_{n=1}^{\infty} \xi'_n$$

are isomorphic, where the partition $\theta(\xi)$ (not necessary measurable) is defined by

$$x \overset{\theta(\xi)}{\sim} y \Longleftrightarrow \exists n\colon x \overset{\xi_n}{\sim} y, \quad (x,y) \in X \times X.$$

It is obvious that

$$FI \Longleftarrow I \Longrightarrow LI \Longrightarrow OI.$$

A sequence ξ is called **ergodic** if the measurable intersection $\bigwedge_{n=0}^{\infty} \xi_n$ (i.e. the measurable hull of $\theta(\xi)$) is trivial.

The finite isomorphism problem was solved in [Gu]. The isomorphism problem for various classes of finitely isomorphic sequences was considered in [Ve_2], [Ve_3], [St], [Ru_1], [Ru_2], [Ru_3]. A new application to the measure change problem for Brownian motion has been recently discovered [DuFST].

A.M. Vershik proved a lacunary isomorphism theorem for the class of diadic homogeneous sequences [Ve_1] and this theorem was extended in [Ru_4] and [ViGo].

Many important applications of d.s.m.p. are connected with the orbital isomorphism problem for countable ergodic groups of non-singular transformations. Let G be such a group; the orbital partition $\theta(G)$ is defined as the partition of X into the orbits $Gx = \{Sx\colon S \in G\}$, $x \in X$. The group G is called **approximately finite** (AF) if $\theta(G) = \theta(\xi)$ for an appropriate d.s.m.p $\xi = \{\xi_n\}$, i.e.

$$C_{\theta(\xi)}(x) = \bigcup_{n=1}^{\infty} C_{\xi_n}(x) = Gx, \quad x \in X.$$

The orbital classification problem was solved for a measure preserving AF-group in [D_1], [D_2], for the AF-groups, containing a measure in [Kr_1], [Kr_2] and for a class of extensions of AF-groups containing a measure in [Ru_4].

It should be mentioned that if we consider the orbital isomorphism problem with respect to the *measure preserving* isomorphism, such orbital invariants as Krieger's associated flows (see [Kr_3]) are not complete orbital invariants of ergodic AF-groups.

We shall consider in this paper the following two problems:

A. When for ergodic d.s.m.p. does

$$FI + OI \Longrightarrow LI?$$

B. When are ergodic finitely isomorphic sequences orbitally isomorphic?

We prove a generalization of Vershik's lacunary isomorphism theorem for a wide class of finitely Bernoulli sequences of measurable partitions (see below). The theorem gives a positive answer to Problem A for this class. We also find a complete orbital invariant of finitely Bernoulli sequences. This is the associated modular action φ, introduced in [Ru_4], and thus we obtain a complete solution of Problem B.

The class of finitely Bernoulli (FB) sequences is defined as follows.

Let J_n, $n = 1, 2, \ldots$, be finite or countable sets, and let $\rho = \{\rho^{(n)}\}_{n=1}^{\infty}$ be a sequence of probability distributions

$$\rho^{(n)} = \{\rho_i^{(n)}\}_{i \in J_n}$$

on J_n, that is,

$$\sum_{i \in J_n} \rho_i^{(n)} = 1, \quad \rho_i^{(n)} > 0, \quad i \in J_n, \quad n = 1, 2, \ldots.$$

Consider the Cartesian product

$$(X_g, m_g) = \prod_{n=1}^{\infty} (J_n, \rho^{(n)}).$$

We define the standard Bernoulli sequence $\beta^{\rho} = \{\beta_n^{\rho}\}_{n=1}^{\infty}$, where β_n^{ρ} is the n-th tail partition of X_{ρ}, i.e.

$$x \overset{\beta_n^{\rho}}{\sim} y \Longleftrightarrow x_k = y_k, \quad k > n$$

for the points $x = \{x_k\}_{k=1}^{\infty}$ and $y = \{y_k\}_{k=1}^{\infty}$ of X_{ρ}.

A decreasing sequence of measurable partitions $\xi = \{\xi_n\}_{n=1}^{\infty}$ will be called **Bernoulli (finitely Bernoulli)** if it is isomorphic (finitely isomorphic) to the standard Bernoulli sequence $\beta^{\rho} = \{\beta_n^{\rho}\}_{n=1}^{\infty}$.

Denote these classes of d.s.m.p. by $B(\rho)$ and $FB(\rho)$, respectively.

Introduce the full group $[\xi]$ of $\xi \in FB(\rho)$, which consists of all non-singular transformations S of X, such that

$$C_{\theta(\xi)}(Sx) = C_{\theta(\xi)}(x)$$

for a.a. $x \in X$ and let

$$[\xi, m] = \{S \in [\xi] \colon m \circ S = m\}.$$

We will say that ξ satisfies Krieger's condition (K), if the group $[\xi, m]$ is ergodic. In this case the full group $[\xi]$ contains the measure m in Krieger's terminology [Kr$_1$].

For instance, β^ρ satisfies (K) if each distribution $\rho^{(n)}$ appears in the sequence $\rho = \{\rho^{(n)}\}_{n=1}^{\infty}$ infinitely many times (in particular, if $\rho^{(n)}$ does not depend on n).

If $\xi \in FB(\rho)$ satisfies (K), then β^ρ satisfies (K) too, but the converse is not true in general.

Denote by $\sigma(\xi)$ the measurable partition into ergodic components of the group $[\xi, m]$ and let

$$\pi_\sigma \cdot X \to X/\sigma(\xi) = X_\sigma$$

be the natural projection of (X, m) on the factor-space $X_\sigma = X/\sigma(\xi)$ with the factor-measure $m_\sigma = m/\sigma(\xi)$.

Let also $\triangle_\rho$ denote the subgroup of the group $\mathbb{R}_+^*$, generated by the set of ratios

$$\left\{ \frac{\rho_j^{(n)}}{\rho_i^{(n)}},\ i, j \in J_n,\ n = 1, 2, \ldots \right\}.$$

If $\xi \in FB(\rho)$ satisfies (K), the corresponding full group can have one of the following types:

(a) type II$_1$ if $\triangle_\rho = \{1\}$,

(b) type III$_\lambda$, $0 < \lambda < 1$, if $\triangle_\rho = \{\lambda^n, n \in \mathbb{Z}\}$,

(c) type III$_1$ if $\triangle_\rho$ is dense in $\mathbb{R}_+^*$.

It follows from [Kr$_1$], [Kr$_2$] that the group $\triangle_\rho$ is a complete orbital invariant for $\xi \in FB(\rho)$ satisfying the condition (K).

We will suppose in the sequel that the standard sequence β^ρ satisfies (K). Denote this condition by $\rho \in$ (K).

It seems to be a difficult problem to give a comprehensible sufficient and necessary condition (in terms of $\rho = \{\rho^{(n)}\}_{n=1}^{\infty}$) under which $\rho \in$ (K). The following statements show how wide is this class of ρ (see [Kr$_1$], [Kr$_2$]).

(1) If $\rho' \in$ (K) for a subsequence $\rho' = \{\rho^{(n_k)}\}_{k=1}^{\infty}$ of ρ and $\triangle_\rho \subset \triangle_{\rho'}$, then $\rho \in$ (K). However, $\rho' \in$ (K) does not imply $\rho \in$ (K) and $\rho \in$ (K) does not imply $\rho' \in$ (K) in general.

(2) If $m \in \mathbb{N}$ and $\rho' = \{\rho^{(m+n)}\}_{n=1}^{\infty}$, then $\rho \in$ (K) iff $\rho' \in$ (K) and $\triangle_\rho \subset \triangle_{\rho'}$.

(3) Suppose $\rho' = \{\rho^{(n'_k)}\}_{k=1}^{\infty}$ and $\rho'' = \{\rho^{(n''_k)}\}_{k=1}^{\infty}$ are two subsequences of ρ, such that

$$\mathcal{N}' \cup \mathcal{N}'' = \mathbb{N},\ \ \mathcal{N}' \cap \mathcal{N}'' = \emptyset,$$

where

$$\mathcal{N}' = \{n'_k,\ k \in \mathbb{N}\}, \quad \mathcal{N}'' = \{n''_k,\ k \in \mathbb{N}\}.$$

Then $\rho' \in$ (K) and $\rho'' \in$ (K) together imply $\rho \in$ (K).

The last statement holds also for any finite or infinite number of subsequences.

We can formulate now our main results.

THEOREM A: *Suppose the sequences $\xi \in FB(\rho)$ and $\xi' \in FB(\rho)$ are ergodic and the corresponding standard sequence β^ρ satisfies the condition* (K). *Then ξ and ξ' are lacunarily isomorphic iff they are orbitally isomorphic.*

THEOREM B: *Let β^ρ satisfy* (K). *Then:*

(1) *For any ergodic $\xi \in FB(\rho)$ there exists an ergodic measure preserving action*

$$\varphi_\xi\colon \triangle_\rho \to \mathcal{A}(X_\sigma, m_\sigma), \quad \sigma = \sigma(\xi)$$

of the group $\triangle_\rho$ on the space (X_σ, m_σ) such that

$$\varphi_\xi(\alpha)(\pi_\sigma(x)) = \pi_\sigma(Sx)$$

for $\alpha \in \triangle_\rho, S \in [\xi]$ and a.a. $x \in A_\alpha$, where

$$A_\alpha = \left\{ x \in X\colon \frac{dm(Sx)}{dm(x)} = \alpha \right\}.$$

(2) *Two ergodic sequences $\xi \in FB(\rho)$, $\xi' \in FB(\rho)$ are orbitally isomorphic iff their corresponding actions φ_ξ and $\varphi_{\xi'}$ are equivalent in the following sense: there exists an isomorphism $\Phi\colon X_\sigma \to X'_{\sigma'}$ such that*

$$\Phi \varphi_\xi(\alpha) \Phi^{-1} = \varphi_{\xi'}(\alpha)$$

for all $\alpha \in \triangle_\rho$.

(3) *For any ergodic m.p. action φ_0 of the group $\triangle_\rho$ on a Lebesgue space (X_0, m_0), there exists an ergodic sequence $\xi \in FB(\rho)$ such that the corresponding action φ_ξ is equivalent to φ_0.*

COROLLARY C: *A sequence $\xi \in FB(\rho)$ is lacunary isomorphic to the standard Bernoulli sequence β^ρ iff ξ satisfies the condition* (K).

The sequence $\rho = \{\rho^{(n)}\}$ and sequences ξ from the class $FB(\rho)$ are called **homogeneous**, if J_n are finite for all n and

$$\rho_i^{(n)} = |J_n|^{-1}, \quad i \in J_n.$$

In this case $\triangle_\rho = \{1\}$, $[\xi] = [\xi, m]$ and ξ satisfies the condition (K) iff ξ is ergodic. Thus the above corollary implies the following result of Vershik: any ergodic homogeneous sequence is lacunarily isomorphic to the correponding standard sequence [Ve_1], [Ve_3].

The main part of the proof of Theorems A and B is Theorem 3.1, which shows (provided that (K) holds) the existence of a special consistent sequence $\hat{\eta}_{n_1} \leq \hat{\eta}_{n_2} \leq \cdots$ for $\xi \in FB(\rho)$ such that

$$(\bigvee_{k=1}^{\infty} \hat{\eta}_{n_k}) \bigvee \sigma(\xi) = \varepsilon.$$

This result allows us to reduce our consideration to the so-called φ-extensions [Ru_4] and to complete the proof of Theorems A and B, using results of [Ru_4].

Section 2 contains some auxiliary results, in particular, a representation of $\xi \in FB(\rho)$ by extensions of the standard sequence β^ρ (cf. [Ru_1]).

Section 3 deals with the proof of the above-mentioned Theorem 3.1 and Section 4 contains the proofs of Theorems A and B.

1. Notation and terminology

Throughout the paper we consider only measure spaces which are Lebesgue spaces. We use the terms "homomorphism, isomorphism, automorphism" only for measure preserving mappings, and the term "non-singular transformation" means that the transformations leave quasiinvariant the considered measures.

We denote by $\mathcal{A}(X)$ the group of all invertible non-singular transformations of a Lebesgue space $(X, \mathcal{F}, m)$, and by $\mathcal{A}(X, m)$ the group of all automorphisms of $(X, \mathcal{F}, m)$, i.e.

$$\mathcal{A}(X, m) = \{S \in \mathcal{A}(X) \colon m \circ S = m\}.$$

We use terminology and results of Rokhlin's theory of measurable partitions of Lebesgue spaces (see [Ro_1], [Ro_2]). A more modern and detailed explanation of the theory can be found in [ViRuFe].

Let η be a partition of X into mutually disjoint sets $C \in \zeta$. The element of ζ containing a point x is denoted by $C_\zeta(x)$. The partition ζ is measurable iff there exists a measurable function $f \colon X \to \mathbb{R}$ such that

$$x \overset{\varepsilon}{\sim} y \text{ (i.e. } C_\zeta(x) = C_\zeta(y)) \iff f(x) = f(y).$$

Elements of ζ are considered as Lebesgue spaces $(C, \mathcal{F}^C, m^C)$, $C \in \zeta$, with the canonical system of conditional measures m^C, $C \in \zeta$. We shall denote also by $m(A|C)$ the conditional measures $m_\zeta^C(A \cap C)$ of a measurable set $A \in \mathcal{F}$ in the element C of ζ. Thus the function

$$X \ni x \to m(A|C_\zeta(x)) \in [0,1]$$

is measurable with respect to ζ and

$$mA = \int_X m(A|C_\zeta(x))dm(x), \quad A \in \mathcal{F}$$

or

$$mA = \int_{X_\zeta} m^C(A \cap C)dm_\zeta(C)$$

where $m_\zeta = m/\zeta$ is the factor-measure on the factor-space $X_\zeta = X/\zeta$ and $m_\zeta = m_X \circ \pi_\zeta^{-1}$ for the natural projection $\pi_\zeta \colon X \to X_\zeta$.

For a measurable partition ζ we denote by $\mathcal{F}(\zeta)$ the m-completion of the σ-algebra of all measurable ζ-sets.

We shall say that a set $A \in \mathcal{F}$ (a measurable partition ζ_1) is independent of ζ if the set A (the σ-algebra $\mathcal{F}(\zeta_1)$) is independent of the σ-algebra $\mathcal{F}(\zeta)$. We shall use the notations

$$A \perp \zeta, \quad \zeta_1 \perp \zeta$$

in this case. An independent complement of η is a measurable partition η_1 such that

$$\eta_1 \perp \eta, \quad \eta_1 \vee \eta = \varepsilon$$

where $\varepsilon = \varepsilon_X$ denotes the partition of X onto separate points ($\mathcal{F}(\varepsilon) = \mathcal{F}$).

We shall repeatedly use the following well known result.

LEMMA 1.1: *Let ζ be a measurable partition. Then the following conditions are equivalent:*

(1) *ζ admits an independent complement ζ_1,*
(2) *almost all elements of ζ are isomorphic among themselves (and to $(X_{\zeta_1}, m_{\zeta_1})$),*
(3) *the mapping*

$$\Phi \colon x \to (\pi_\zeta x, \pi_{\zeta_1} x), \quad x \in X$$

is an isomorphism of (X, m) onto the direct product $(X_\zeta \times X_{\zeta_1}, m_\zeta \times m_{\zeta_1})$ such that

$$\Phi\zeta = \pi_\zeta^{-1}\varepsilon_{X_\zeta}, \quad \Phi\zeta_1 = \pi_{\zeta_1}^{-1}\varepsilon_{X_{\zeta_1}}.$$

The totality of all independent complements of a measurable partition ζ is denoted by $IC(\zeta)$.

We shall write

$$\zeta_1 \perp \zeta_2 \pmod{\zeta}$$

if the partitions ζ_1 and ζ_2 are conditionally independent with respect to ζ; this means that

$$m(A \cap B | C_\zeta(x)) = m(A|C_\zeta(x))\, m(B|C_\zeta(x))$$

for all $A \in \mathcal{F}(\zeta_1)$, $B \in \mathcal{F}(\zeta_2)$ and a.a. $x \in X$

Let $G \subset \mathcal{A}(X)$ be a countable group of non-singular transformations of X. We denote by $\theta(G)$ the partition of X on the orbits $Gx = \{Sx, S \in G\}$, $x \in X$ of the group G, i.e. $C_{\theta(G)}(x) = Gx$. The corresponding full group $[G] = [\theta(G)]$ consists of all $S \in \mathcal{A}(X)$ such that $Sx \in Gx$ for a.a. x.

The partition $\theta(G)$ may be not measurable; its measurable hull coincides with the trivial partition ν in the case of an ergodic group G.

The equivalence relation

$$\mathcal{G}_{\theta(G)} = \mathcal{G} = \{(x,y) \in X \times X \colon x \overset{\theta(G)}{\sim} y\},$$

induced by the orbital partition $\theta(G)$, can be equipped with the canonical measure $\mu_{\mathcal{G}}$, such that $(\mathcal{G}, \mu_{\mathcal{G}})$ is measured discrete equivalence relation (see [FM]).

The full group of a measurable partition ζ is defined by

$$[\zeta] = \{S \in \mathcal{A}(X) \colon Sx \overset{\zeta}{\sim} x \text{ for a.a. } x \in X\}.$$

If almost all elements of ζ have discrete conditional measures, then there exists a countable subgroup G of $\mathcal{A}(X)$ such that $\theta(G) = \zeta$ and $[\zeta] = [\theta(G)]$.

The intersection $\theta(\xi)$ of a decreasing sequence $\xi = \{\xi_n\}_{n=1}^\infty$ of measurable partitions ξ_n is defined as the partition of X (not necessarily measurable) with the elements of the form

$$C_{\theta(\xi)}(x) = \bigcup_{n=1}^{\infty} C_{\xi_n}(x), \quad x \in X,$$

that is

$$x \overset{\theta(G)}{\sim} y \iff \exists n \colon x \overset{\xi_n}{\sim} y, \quad (x,y) \in X \times X.$$

If for all n the elements of ξ_n have discrete conditional measures, there exists a countable subgroup $G \subset \mathcal{A}(X)$ such that $\theta(\xi) = \theta(G)$. In this case the full group $[\xi] = [\theta(\xi)]$ of ξ coincides with $[G]$ and the groups G and $[G]$ are called **approximately finite** (AF). (See [D1], [D2], [Kr1], [Kr2], [Kr3], [FM].)

2. Constructing extensions

Throughout this section $\beta^\rho = \{\beta^\rho\}_{n=0}^\infty$ is a standard Bernoulli sequence satisfying the condition (K), i.e. $[\beta^\rho, m_\rho]$ is ergodic.

For $\xi \in FB(\rho)$ and $n \in \mathbb{N}$ we shall denote by

$$\mathcal{D}_n(\xi) = \mathcal{D}(\{\xi_k\}_{k=1}^n)$$

the totality of all finite sequences $\{\eta_k\}_{k=1}^n$ of the partitions

$$\eta_k = \{C_i^{(k)}, \quad i \in J_k\}$$

which satisfy for $k = 1, 2, \ldots, n$ the following conditions:

(i) $\eta_k \perp \xi_k$,
(ii) $\eta_k \vee \xi_k = \xi_{k-1}$, where $\xi_0 = \varepsilon$,
(iii) $mC_i^{(k)} = \rho_i^{(k)}$, $i \in J_k$.

We also shall denote by $\mathcal{D}(\xi)$ the totality of all sequences $\eta = \{\eta_k\}_{k=1}^\infty$ of partitions η_k satisfying (i), (ii), (iii) for all $k = 1, 2, \ldots$.

Proposition 2.1: *For $\xi \in FB(\rho)$ the classes $\mathcal{D}(\xi)$ and $\mathcal{D}_n(\xi)$, $n = 1, 2, \ldots$, are not empty.*

Proof: In the case when $\xi = \{\xi_n\}$ coincides with the standard sequence $\beta^\rho = \{\beta_n^\rho\}$, we can construct $\eta^0 = \{\eta_n^0\} \in \mathcal{D}(\beta^\rho)$ putting

$$c_i^{(n)} = \{x = \{i_k\}_{k=1}^\infty \in X_\rho : i_n = i\}$$

for all n and $i \in J_n$.

For arbitrary $\xi \in FB(\rho)$ we can construct the desired $\eta = \{\eta_k\}_{k=1}^n \in \mathcal{D}_n(\xi)$, by using any isomorphism between $\{\xi_k\}_{k=1}^n$ and $\{\beta_k^\rho\}_{k=1}^n$.

Moreover, for $\{\eta_k\}_{k=1}^n \in \mathcal{D}_n(\xi)$ we can find η_{n+1} such that $\{\eta_k\}_{k=1}^{n+1} \in \mathcal{D}_{n+1}(\xi)$ and so construct $\{\eta_k\}_{k=1}^\infty \in \mathcal{D}(\xi)$. ∎

With $\eta = \{\eta_n\}_{n=1}^\infty \in \mathcal{D}(\xi)$ we consider the measurable partitions

$$\hat{\eta}_n = \bigvee_{k=1}^n \eta_k \quad \text{and} \quad \hat{\eta} = \bigvee_{n=1}^\infty \eta_n.$$

Then $\hat{\eta}_n \in IC(\beta_n)$, i.e. $\hat{\eta}_n \perp \beta_n$ and $\hat{\eta}_n \vee \beta_n = \varepsilon$ for all n. We shall call the increasing sequence $\{\hat{\eta}_n\}_{n=0}^{\infty}$ the **consistent sequence** of independent complements of ξ_n, induced by $\eta \in \mathcal{D}(\xi)$.

All elements of the partition $\hat{\eta}$ have the following form:

$$\bigcap_{n=0}^{\infty} C_{i_n}^{(n)}, \quad i_n \in J_n, \quad C_i^{(n)} \in \eta_n$$

for an appropriate sequence $\{i_n\}_{n=1}^{\infty} \in X_\rho$. Define the mapping

$$\pi_\eta \colon X \to X_\rho$$

by putting $\pi_\eta(x) = \{i_n\}_{n=1}^{\infty}$ for $x \in \bigcap_{n=1}^{\infty} C_{i_n}^{(n)}$.

The orbital partition $\theta(\beta^\rho)$ coincides with the orbital partition of a countable ergodic group, which can be defined as follows:

Let

$$t_{ij}^{(k)}(\{i_n\}_{n=1}^{\infty}) = \{i'_n\}_{n=1}^{\infty} \in X_\rho,$$

where

$$i'_n = \begin{cases} j, & \text{if } n = k, \quad i_n = i, \\ i, & \text{if } n = k, \quad i_n = j, \\ i_n, & \text{otherwise.} \end{cases}$$

Then

$$\{t_{ij}^{(k)}, \quad k = 1, 2, \ldots, i, j \in J_k\}$$

is a family of non-singular invertible transformations of (X_ρ, m_ρ), and we denote by $G(\beta^\rho)$ the subgroup of $\mathcal{A}(X_\rho)$ generated by the above family, and by $G_n(\beta^\rho)$ the subgroup of $\mathcal{A}(X_\rho)$ generated by

$$\{t_{ij}^{(k)}, \ k \le n, \ i, j \in J_k\}.$$

For given $\eta = \{\eta_n\} \in \mathcal{D}(\xi)$, $\xi \in FB(\rho)$, we can introduce the groups

$$G(\xi, \eta), \ G_n(\xi, \eta), \ n = 1, 2, \ldots$$

generated by

$$\{T_{ij}^{(k)}, \quad k = 1, 2, \ldots, \ i, j \in J_k\}$$

and

$$\{T_{ij}^{(k)}, \ k = 1, 2, \ldots, n; \ i, j \in J_k\}$$

respectively, where the non-singular transformations $T_{ij}^{(k)} \in \mathcal{A}(X)$ are uniquely defined by the following properties:

$$\pi_\eta(T_{ij}^{(k)}x) = t_{ij}^{(k)}(\pi_\eta(x)), \quad x \in X$$

and $T_{ij}^{(k)} \in [\xi_k]$, i.e.

$$C_{\xi_k}(T_{ij}^{(k)}x) = C_{\xi_k}(x), \quad x \in X.$$

The next proposition follows directly from the above definitions.

PROPOSITION 2.2: *Let $\xi \in FB)\rho)$, $\eta \in \mathcal{D}(\xi)$. Then:*

(1) $\hat{\eta}_n \in IC(\xi_n), \hat{\eta}_n \nearrow \hat{\eta}$, $n \to \infty$, $\hat{\eta} = \pi_\eta^{-1}\varepsilon_{X_\rho}$.

(2) $\hat{\eta} \vee \xi_n = \varepsilon, \hat{\eta} \wedge \xi_n = \pi_\eta^{-1}\beta_n$,

$$\hat{\eta} \perp \xi_n (\operatorname{mod} \hat{\eta} \wedge \xi_n).$$

(3) $\theta(G(\xi,\eta)) = \theta(\xi)$, $\theta(G_n(\xi,\eta)) = \xi_n$, $\theta(G(\beta^\rho)) = \theta(\beta^\rho)$, $\theta(G_n(\beta^\rho)) = \beta_n$.

(4) $T\hat{\eta} = \hat{\eta}$ *for any* $T \in G(\xi,\eta)$, *and the factor transformation* $t = T/\hat{\eta} \in G(\beta^\rho)$ *satisfies:* $\pi_\eta \circ T = t \circ \pi_\eta$ *and*

$$\frac{dm(Tx)}{dm(x)} = \frac{dm_\rho(t(\pi_\eta(x)))}{dm_\rho(\pi_\eta(x))}$$

for a.a. $x \in X$.

From the last property (4) we have that the mappings

$$T|_C \colon C \to TC, \quad C \in \hat{\eta},$$

induced by $T \in G(\xi,\eta)$ on elements of $\hat{\eta}$, preserve the corresponding conditional measures m^C, $C \in \eta$. Hence the function

$$\varphi(x) = m^{C_{\hat{\eta}}(x)}(\{x\})$$

is invariant with respect to the group $G(\xi,\eta)$. Since ξ is ergodic the group $G(\xi,\eta)$ is ergodic too, and $\varphi(x)$ is constant a.e. Hence almost all (C, m^C), $C \in \hat{\eta}$, are isomorphic among themselves; they are homogeneous Lebesgue spaces.

Choosing an independent complement $\zeta \in IC(\hat{\eta})$, we can identify the space (X, m) with the direct product

$$(X_\rho \times Y,\ m_\rho \times m_Y)$$

where (Y, m_Y) is a homogeneous Lebesgue space which is isomorphic to $(X/\zeta,\ m/\zeta)$ and to almost all (C, m^C), $C \in \hat{\eta}$.

Under this identification we have

$$\hat{\eta} = \varepsilon_X \times \nu_Y$$

and the group $G(\xi, \eta)$ is represented as a Y-extension of $G(\beta^\rho)$ with an appropriate cocycle $f\colon \mathcal{G}_\rho \to \mathcal{A}(Y, m_Y)$, where $\mathcal{G}_\rho$ is the ergodic measured equivalence relation, corresponding to the orbital partition

$$\theta(\beta^\rho) = \theta(G(\beta^\rho)).$$

That is, $G(\xi, \eta)$ consists of all transformations of the form t_f, $t \in G(\beta^\rho)$, where

$$t_f(x, y) = (tx,\ f(tx, x)y), \quad (x, y) \in X_\rho \times Y.$$

The partitions $\{\xi_n\}$ and $\theta(\xi)$ may be described now as follows:

$$(x_1, y_1) \overset{\xi_n}{\sim} (x_2, y_2) \Longleftrightarrow x_1 \overset{\beta_n}{\sim} x_2, \quad y_2 = f(x_2, x_1)y_1,$$
$$(x_1, y_1) \overset{\theta(\xi)}{\sim} (x_2, y_2) \Longleftrightarrow x_1 \overset{\theta(\beta^\rho)}{\sim} x_2, \quad y_2 = f(x_2, x_1)y_1.$$

If these relations hold we say that the sequence $\xi = \{\xi_n\}$ (resp. $\theta(\xi)$) is the Y-extension of $\beta^\rho = \{\beta_n\}$ (resp. $\theta(\beta^\rho)$)) induced by the cocycle f.

It is clear that the skew products t_f are correctly defined also for all $t \in [\beta^\rho]$ and

$$\{t_f,\ t \in [\beta^\rho]\} = [\theta(\xi)] \cap N(\hat{\eta})$$

where $N(\hat{\eta}) = \{S \in \mathcal{A}(X)\colon S\hat{\eta} = \hat{\eta}\}$ is the normalizer of $\hat{\eta}$.

In particular we have the following:

Proposition 2.3: *Any ergodic sequence $\xi \in FB(\rho)$ can be represented as a Y-extension of the standard sequence β^ρ.*

We consider now properties of the partition $\sigma = \sigma(\xi)$. Recall that σ was defined as the partition into ergodic components of the group

$$[\xi, m] = [\xi] \bigcap \mathcal{A}(X, m).$$

Proposition 2.4: *Let β^ρ satisfy the condition* (K) *(i.e. $[\beta^\rho]$ contains m_ρ), $\xi \in FB(\rho)$ and $\eta = \{\eta_n\}_{n=0}^\infty \in \mathcal{D}(\xi)$, $\hat{\eta} = \vee_{n=1}^\infty \eta_n$. Then the partitions σ and $\hat{\eta}$ are independent, $\sigma \perp \hat{\eta}$.*

Proof: By condition (K) the full group $[\beta^\rho, m_\rho]$ is ergodic. We can assume that ξ is an extension of β^ρ with respect to the factor-mapping π_η. Let f be the corresponding cocycle.

Consider the extension

$$H \equiv \{t_f,\ t \in [\beta^\rho, m_\rho]\} \subset [\xi] \cap N(\hat{\eta})$$

of $[\beta^\rho, m_\rho]$. Since $t \in [\beta^\rho, m_\rho]$ and

$$f(x,y),\ (x,y) \in \mathcal{G}_\rho$$

are measure preserving, we have

$$H \subset [\xi, m].$$

Any measurable subset $A \in \mathcal{F}(\sigma)$ is invariant with respect to $[\xi, m]$ and, hence, with respect to H. The measurable function

$$\varphi_A(x) = m(A|C_{\hat{\eta}}(x))$$

is $\hat{\eta}$-measurable and H-invariant. Since the group $[\beta^\rho, m_\rho] = \{t/\hat{\eta},\ t \in H\}$ is ergodic, φ_A is constant for a.a. x. That is, A is independent with respect to $\hat{\eta}$. Thus $\sigma \perp \hat{\eta}$. ∎

COROLLARY 2.5: $IC(\sigma(\xi)) \neq \emptyset$.

Proof: By Propopsition 2.3 almost all elements of $\sigma(\xi)$ have continuous conditional measures. ∎

3. Special lacunary subsequences

The aim of this section is to prove the following theorem.

THEOREM 3.1: *Let $\xi \in FB(\rho)$; ξ is ergodic and β^ρ satisfying the condition* (K). *Then there exists a subsequence $\xi' = \{\xi_{n_k}\}_{k=1}^\infty$ of the sequence $\xi = \{\xi_n\}_{n=1}^\infty$ and a sequence*

$$\eta' = \{\eta'_k\} \in \mathcal{D}(\xi')$$

such that for $\hat{\eta}' = \bigvee_{k=1}^\infty \eta'_k$ the equality $\hat{\eta}' \vee \sigma(\xi) = \varepsilon$ holds.

Remark 3.2: (1) $[\xi] = [\xi']$ and, hence, $\sigma(\xi) = \sigma(\xi')$.

(2) $\hat{\eta} \perp \sigma(\xi)$ by Proposition 2.4. Thus the theorem states in fact that $\hat{\eta}' \in IC(\sigma(\xi))$.

(3) If ξ itself satisfies (K) we have $\sigma(\xi) = \nu$ and, hence, $\hat{\eta}' = \varepsilon$ by the above theorem. This means that $\xi' = \{\xi_{n_k}\}_{k=1}^{\infty}$ and $\beta' = \{\beta_{n_k}^{\rho}\}_{k=1}^{\infty}$ are isomorphic. Thus Corollary C is a direct consequence of Theorem 3.1.

For the proof of the theorem we need two lemmas. But first, introduce subsidiary partitions σ_n, γ_n, $n = 1, 2, \ldots$, in addition to $\sigma = \sigma(\xi)$.

Consider

$$u_n(x) = m^{C_{\xi_n}(x)}(\{x\}), \quad x \in X,$$

i.e. $u_n(x)$ is the conditional measure of the point x in the element $C_{\xi_n}(x)$ of ξ_n, which contains x. The function $u_n(x)$ is measurable and $u_n \colon X \to [0,1]$.

Put

$$\gamma_n = u_1^{-1}\varepsilon_{[0,1]} \quad \text{and} \quad \sigma_n = \gamma_n \vee \xi_n.$$

The following properties of γ_n and σ_n are direct consequences of the definition:

PROPOSITION 3.3:

(1) *σ_n is the smallest subpartition of ξ_n with homogeneous conditional measures*

$$[\sigma_n] = [\xi_n, m]; \quad \sigma_1 \geq \sigma_2 \geq \cdots; \quad \sigma_n \downarrow \sigma.$$

(2) *For any $\zeta \in IC(\xi_n)$, $\zeta \wedge \sigma_n = \gamma_n$ and $\zeta \perp \sigma_n (\mathrm{mod}\, \gamma_n)$.*

(3) *For a measurable $A \in \mathcal{F}$ the following conditions are equivalent:*
   - (a) *there exists $\zeta \in IC(\xi_n)$ such that $A \in \mathcal{F}(\zeta)$,*
   - (b) *the function $v_n(x) = m^{C_{\sigma_n}(x)}(\{x\})$ is γ_n-measurable.*

LEMMA 3.4: *Let $\varepsilon > 0$ and ζ be a finite partition such that $\zeta \perp \sigma$. Then there exist $E \subset X$, $n \in \mathbb{N}$ and $\eta_n \in IC(\xi_n)$ such that*

$$mE > 1 - \varepsilon, \quad \zeta|_E \leq \eta_n|_E$$

and, hence, $\zeta \overset{\varepsilon}{\leq} \eta_n$.

Proof: Let $\zeta = \{A_1, \ldots, A_p\}$, $mA_i > 0$, $\sum_{i=1}^{p} mA_i = 1$. Since $\zeta \perp \sigma$, we have

$$m(A_i | C_\sigma(x)) = mA_i, \quad i = 1, 2, \ldots, p$$

for a.a. $x \in X$.

Since $\sigma_n \downarrow \sigma$, we have also a.e. convergence of conditional measures:

$$u_{n,i}(x) \equiv m(A_i|C_{\sigma_n}(x)) \overset{n\to\infty}{\to} m(A_i|C_\sigma(x)) = mA_i$$

for $i = 1, 2, \ldots, p$.

Consider the function

$$a_n(x) = |C_{\sigma_n}(x)|, \quad x \in X,$$

i.e. $a_n(x)$ is the number of points in the element $C_{\sigma_n}(x)$ of σ_n.

The function a_n is γ_n-measurable and

$$1 \le a_n(x) < \infty$$

for a.a. x

If the condition (K) holds for β^p, almost all elements of the partition σ have continuous conditional measures (by Proposition 2.4). This implies with $\sigma_n \searrow \sigma$ that

$$a_n(x) \to +\infty \tag{2}$$

almost everywhere on X.

We can find γ_n-measurable functions $v_{n,i}(x)$ such that

$$|v_{n,i}(x) - mA_i| < \frac{1}{a_n(x)} \tag{3}$$

and

$$v_{n,i}(x) \in \left\{ \frac{k}{a_n(x)}, \ k = 1, 2, \ldots, a_n(x) \right\} \tag{4}$$

with

$$\sum_{i=1}^{p} v_{n,i}(x) = 1.$$

For a given $\varepsilon_1 > 0$, $\varepsilon_1 = \frac{\varepsilon}{2p+1}$, using (1), (2), (3), we can find n and a subset $E_1 \subset X$ with $mE_1 > 1 - \varepsilon_1$ such that

$$|u_{n,i}(x) - mA_i| < \varepsilon_1 \quad \text{and} \quad |v_{n,i}(x) - mA_i| < \varepsilon_1 \tag{5}$$

for $i = 1, 2, \ldots, p$ and $x \in E_1$.

By (4) and (5) we can choose a finite partition

$$\zeta' = \{B_1, \dots, B_p\}$$

such that

$$v_{n,i}(x) = m(B_i | C_{\sigma_n}(x)) \tag{6}$$

and

$$m(A_i \cap B_i | C_{\sigma_n}(x)) < 2\varepsilon_1$$

for all $i = 1, 2, \dots, p$ and $x \in E_1$.

Then

$$m(E_1 \cap (A_i \triangle B_i)) < 2\varepsilon_1$$

and

$$m\left(\bigcup_{i=1}^{p}(A_i \triangle B_i)\right) < (2p+1)\varepsilon_1 = \varepsilon.$$

Taking

$$E = E_1 - \left(\bigcup_{i=1}^{p}(A_i \triangle B_i)\right)$$

we get

$$mE > 1 - (2p+1)\varepsilon_1 = 1 - \varepsilon$$

and

$$\zeta|_E = \zeta'|_E. \tag{7}$$

Since the functions (6) are γ_n-measurable, we can apply Proposition 3.3(3) and find an independent complement $\eta_n \in IC(\xi_n)$ for ξ_n such that $\zeta' \leq \eta_n$ and, hence,

$$\zeta|_E \leq \eta_n|_E$$

on account of (7). ∎

Lemma 3.5: *Let $\varepsilon > 0$ and δ be a finite partition. Then there exist $n \in \mathbb{N}$, $\eta_n \in IC(\xi_n)$, $E \in \mathcal{F}$ with $mE > 1 - \varepsilon$ and a finite partition ζ such that*

$$\zeta \leq \sigma \quad \text{and} \quad \delta|_E \leq (\eta_n \vee \delta)|_E.$$

Proof: Under condition (K) the partition $\sigma = \sigma(\xi)$ has an independent complement $\bar{\zeta} \in IC(\sigma)$ (Corollary 2.5). Hence we can find increasing sequences of finite partitions $\{\zeta_n\}$ and $\{\bar{\zeta_n}\}$ such that

$$\zeta_n \le \sigma, \quad \bar{\zeta}_n \le \bar{\zeta}, \quad \zeta_n \uparrow \sigma, \quad \bar{\zeta}_n \uparrow \bar{\zeta}.$$

Then

$$\zeta_n \vee \bar{\zeta}_n \uparrow \sigma \vee \bar{\zeta} = \varepsilon, \quad n \to \infty$$

and any finite partition is "almost" measurable with respect to $\zeta_n \vee \bar{\zeta}_n$, if n is sufficiently large. More exactly, we can find $n_1 \in \mathbb{N}$ and a subset E_1 with $mE_1 > 1 - \varepsilon/2$ such that

$$\delta|_{E_1} \le (\zeta_{n_1} \vee \bar{\zeta}_{n_1})|_{E_1}.$$

Since $\bar{\zeta}_{n_1} \perp \sigma$, we can apply Lemma 3.4 and find $n > n_1$ and $\eta_n \in IC(\xi_n)$ such that

$$\bar{\zeta}_{n_1}|_{E_2} \le \eta_n|_{E_2}$$

for a suitable $E_2 \in \mathcal{F}$ with $mE_2 > 1 - \varepsilon/2$.

Then

$$(\zeta_{n_1} \vee \bar{\zeta}_{n_1})|_{E_2} \le (\zeta_{n_1} \vee \eta_n)|_{E_2}$$

and

$$\delta|_{E_1 \cap E_2} \le (\zeta_{n_1} \vee \eta_n)|_{E_1 \cap E_2}.$$

Taking $\zeta = \zeta_{n_1}$ and $E = E_1 \cap E_2$ ($mE > 1 - \varepsilon$) completes the proof. ∎

Proof of Theorem 3.1: Consider a sequence $\{A_k\}_{k=1}^{\infty}$ of $A_k \in \mathcal{F}$, which satisfies the following conditions:

(a) $\{A_k,\ k = 1, 2, \ldots\}$ is dense in $\mathcal{F}$ with respect to the semimetric

$$d(A, B) = m(A \triangle B), \quad A, B \in \mathcal{F}.$$

(b) Each A_k appears in the sequence $\{A_n\}_{n=1}^{\infty}$ infinitely many times.

Take $\varepsilon_k > 0$, $\varepsilon_k \to 0$, $k \to \infty$. We shall construct sequences

$$\{n_k\}_{k=1}^{\infty}, \quad \{\eta'_k\}_{k=1}^{\infty}, \quad \{\zeta_k\}_{k=1}^{\infty} \tag{8}$$

which satisfy for all $k = 1, 2, \ldots$ the following conditions:

(c) $n_1 < n_2 < \cdots < n_k$,

(d) $\{\eta'_i\}_{i=1}^k \in \mathcal{D}(\{\xi_{n_i}\}_{i=1}^k)$,

(e) ζ_k is a finite partition and $\zeta_k \le \sigma$,

(f) $A_k \overset{\varepsilon_k}{\in} \mathcal{F}(\zeta_k \vee \hat{\eta}'_k)$, where $\hat{\eta}'_k = \bigvee_{i=1}^k \eta'_i$.

If such sequences have been already constructed, we have

$$\{\eta'_k\}_{k=1}^\infty \in \mathcal{D}(\{\xi_{n_k}\}_{k=1}^\infty).$$

Further, using (b) we find infinitely many j such that

$$A_k \overset{\varepsilon_j}{\in} \mathcal{F}(\zeta_k \vee \hat{\eta}'_j)$$

and

$$A_k \overset{\varepsilon_j}{\in} \mathcal{F}(\sigma \vee \hat{\eta}'),\ \hat{\eta}' = \bigvee_{i=1}^\infty \eta'_i$$

since $\zeta_k \le \sigma$.

Using $\varepsilon_j \to 0,\ j \to \infty$, we have

$$A_k \in \mathcal{F}(\sigma \vee \hat{\eta}'), \quad k = 1, 2, \dots$$

and (a) implies

$$\mathcal{F} \subset \mathcal{F}(\sigma \vee \hat{\eta}').$$

Thus $\sigma \vee \hat{\eta}' = \varepsilon$ and $\{\eta'_k\}_{k=1}^\infty$ is the required sequence.

The sequences (8) will be constructed by induction on k.

For $k = 1$ we can use Lemma 3.5 with $\delta = \{A_1, X - A_1\}$. Thus we find $n_1 \in \mathbb{N}$, a finite partition ζ_1 and a partition η'_1 such that

$$\zeta_1 \le \sigma, \quad \eta'_1 \in IC(\xi_{n_1}), \quad A_1 \overset{\varepsilon_1}{\in} \mathcal{F}(\zeta_1 \vee \eta'_1).$$

This is the beginning of an induction.

Suppose that finite sequences

$$\{n_i\}_{i=1}^k,\ \{\eta'_1\}_{i=1}^k,\ \{\zeta_i\}_{i=1}^k,$$

which satisfy (c), (d), (e), (f) for a given k, have been constructed.

In order to find n_{k+1}, η'_{k+1} and ζ_{k+1}, consider the partition

$$\hat{\eta}'_k = \bigvee_{i=1}^k \eta'_i \in IC(\xi_{n_k}).$$

First we want to construct a finite partition δ and $D \in \mathcal{F}(\hat{\eta}'_k)$ such that

$$mD > 1 - \frac{\varepsilon_{k+1}}{2} \tag{9}$$

and

$$\delta \le \xi_{n_k}, \quad A_{k+1} \cap D \in \mathcal{F}(\delta \vee \hat{\eta}'_k). \tag{10}$$

To this end denote

$$\hat{\eta}'_k = \{C_1, C_2, \ldots\},$$

where

$$mC_s > 0, \quad s = 1, 2, \ldots, \quad \sum_s mC_s = 1.$$

Take s_0 such that

$$\sum_{s=1}^{s_0} mC_s > 1 - \frac{\varepsilon_{k+1}}{2},$$

put

$$D = \bigcup_{s=1}^{s_0} C_s$$

and let δ be the finite partition of X generated by the following sets:

$$B_s = \pi_{n_k}^{-1}(\pi_{n_k}(A_{k+1} \cap C_s)), \quad s = 1, 2, \ldots, s_0,$$

where π_{n_k} is the natural projection

$$\pi_{n_k} \colon X \to X/\xi_{n_k}.$$

The set B_s is the least ξ_{n_k}-set, containing $A_{k+1} \cap C_s$, and the set

$$A_{k+1} \cap D = \bigcup_{s=1}^{s_0} (B_s \cap C_s)$$

is measurable with respect to $\delta \vee \hat{\eta}'_k$.

Thus (9) and (10) hold. Consider, further, the sequence $\tilde{\xi} = \{\tilde{\xi}_n\}_{n=1}^{\infty}$ of the factor-partitions

$$\tilde{\xi}_n = \xi_{n_k+n}/\xi_{n_k}, \quad n = 1, 2, \ldots$$

on the factor-space $X/\xi_{n_k} = \pi_{n_k}(X) = \tilde{X}$ with the factor-measure $\tilde{m}$.

Then $\tilde{\xi} \in FB(\tilde{\rho})$, where

$$\tilde{\rho} = \{\rho^{(n_k+n)}\}_{n=1}^{\infty}$$

and $\beta^{\tilde{\rho}}$ satisfies the condition (K) as well as β^{ρ}.

We apply Lemma 3.5 for

$$\tilde{\xi} \in FB(\tilde{\rho}), \quad \tilde{\delta} = \delta/\xi_{n_k}, \quad \varepsilon' = \frac{\varepsilon_{k+1}}{2}$$

to find $m_0 \in \mathbb{N}$, $\tilde{\eta}_{m_0} \in IC(\tilde{\xi}_{m_0})$ and a finite partition $\tilde{\zeta}$ such that

$$\tilde{\zeta} \leq \sigma(\tilde{\xi}), \quad \tilde{\delta}|_{\tilde{E}} \leq (\tilde{\zeta} \vee \tilde{\eta}_{m_0})|_{\tilde{E}} \tag{11}$$

for a suitable $\tilde{E} \subset \tilde{X}$ with $\tilde{m}\tilde{E} > 1 - \varepsilon'$.

Put

$$n_{k+1} = n_k + m_0, \quad \eta'_{k+1} = \pi_{n_k}^{-1}\tilde{\eta}_{m_0}, \quad \zeta = \pi_{n_k}^{-1}\tilde{\zeta}.$$

We have from the construction $\delta = \pi_{n_k}^{-1}\tilde{\delta}$ and

$$\{\eta'_i\}_{i=1}^{k+1} \in \mathcal{D}(\{\xi_{n_i}\}_{i=1}^{k+1}),$$

and also from (11), that

$$\delta = \pi_{n_k}^{-1}\tilde{\delta}, \;\; \delta|_E \leq (\zeta \vee \eta'_{k+1})|_E \tag{12}$$

for $E = \pi_{n_k}^{-1}\tilde{E}$ with $mE > 1 - \varepsilon'$.

To find ζ_{k+1} we again consider the partition $\hat{\eta}'_k$. Since $\hat{\eta}'_k \in IC(\xi_{n_k})$, the mapping

$$\pi_{n_k}|_{C_s}\colon C_s \to \tilde{X} = X/\xi_{n_k}$$

is an isomorphism for any atom C_s of the partition $\hat{\eta}'_k$. This isomorphism transfers the restricted sequence $\xi|_{C_S}$ onto the sequence $\tilde{\xi}$ of factor-partitions $\tilde{\xi}_n$ and, hence, it transfers $\sigma(\xi|_{C_s})$ onto $\sigma(\tilde{\xi})$.

Then

$$\tilde{\zeta} \leq \sigma(\tilde{\xi}) \Longrightarrow \zeta|_{C_s} \leq \sigma(\xi|_{C_s})$$

and

$$\zeta|_{C_s} \leq \sigma(\tilde{\xi})|_{C_s}, \quad s = 1, 2, \ldots, \tag{13}$$

since

$$\sigma(\xi)|_C = \sigma(\xi|_C), \quad C \in \hat{\eta}'_k.$$

In spite of (13) the inequality $\zeta \leq \sigma$ (recall that $\sigma = \sigma(\xi)$) does not hold in general. But ζ is finite and we can take another finite partition ζ_{k+1} such that

$$\zeta_{k+1} \leq \sigma \tag{14}$$

and

$$\zeta|_{C_s} \leq \zeta_{k+1}|_{C_s}, \quad s = 1, 2, \ldots, s_0.$$

Then

$$(15) \qquad \zeta|_D \leq \zeta_{k+1}|_D, \quad D = \bigcup_{s=1}^{s_0} C_s$$

where $mD > 1 - \varepsilon'$, $\varepsilon' = \varepsilon_{k+1}/2$.

We get now from (10), (12) and (15) that $A_{k+1} \cap E \cap D \in \mathcal{F}((\delta \vee \hat{\eta}'_k)|_{E\cap D})$ and, for $\hat{\eta}'_{k+1} = \hat{\eta}'_k \vee \eta'_{k+1}$,

$$(\delta \vee \hat{\eta}'_k)|_{E\cap D} \leq (\zeta \vee \eta'_{k+1} \vee \hat{\eta}'_k)|_{E\cap D} \leq (\zeta_{k+1} \vee \hat{\eta}'_{k+1})|_{E\cap D}.$$

Since

$$m(E \cap D) > 1 - 2\varepsilon' = 1 - \varepsilon_{k+1}$$

we see that

$$A_{k+1} \overset{\varepsilon_{k+1}}{\in} \mathcal{F}(\zeta_{k+1} \vee \hat{\eta}'_{k+1}).$$

Thus the sequences

$$\{n_i\}_{i=1}^{k+1}, \quad \{\varepsilon'_i\}_{i=1}^{k+1}, \quad \{\zeta_{i+1}\}_{i=1}^{k+1}$$

satisfy the conditions (c), (d), (e), and (f). The induction is complete, so, as was shown earlier, the theorem is proved. ∎

4. The invariant φ_ξ and modular extensions

Throughout the section let ξ be an ergodic finitely Bernoulli sequence, $\xi \in FB(\rho)$, and we shall assume that the standard Bernoulli sequence β^ρ satisfies the condition (K), i.e. $[\beta^\rho, m_\rho]$ is ergodic ($\sigma(\beta^\rho) = \nu$).

We denote by $\triangle_\rho$ the countable subgroup of $\mathbb{R}_+^*$, generated by the ratios

$$\{\rho_i^{(n)}.(\rho_j^{(n)})^{-1}, \quad i, j \in J_n, \; n = 1, 2, \ldots\}$$

LEMMA 4.1: *The group $\triangle_\rho$ consists of all $a \in \mathbb{R}_+^*$ such that there exists a triple $u = (A, B, U)$, satisfying the following conditions:*

(a) $A \in \mathcal{F}$, $B \in \mathcal{F}$, $mA > 0$, $mB > 0$, $A \perp \sigma$, $B \perp \sigma$.

(b) *U is a non-singular invertible mapping $U\colon A \to B$ such that $UA = B$ and*

$$\frac{dm(Ux)}{dm(x)} = a$$

for a.a. $x \in A$.

(c) $U(\theta|_A) = \theta|_B$ *(where* $\theta = \theta(\xi)$ *and* $\sigma = \sigma(\xi)$*).*

Proof: Consider ξ in the form of an extension of β^ρ with $\eta = \{\eta_n\} \in \mathcal{D}(\xi)$ (see Proposition 2.2).

Since β^ρ satisfies the condition (K), $[\beta^\rho]$ contains m_ρ (see [Kr$_2$]).

Then for any $a \in \triangle_\rho$ and for any $A_0, B_0 \in \mathcal{F}$ satisfying

$$mB_0 = a \cdot mA_0 > 0$$

there exists $t \in G(\beta^\rho)$ such that

$$(t|_{A_0})(A_0) = B_0.$$

Take

$$A = \pi_\eta^{-1} A_0, \quad B = \pi_\eta^{-1} B_0, \quad U = T|_A,$$

where $T \in G(\xi, \eta)$ such that $t = T/\hat{\eta}$. Then $A \perp \sigma$, $B \perp \sigma$ (Proposition 2.4) and (A, B, U) is a desired triple.

Conversely, let (A, B, U) satisfy (a), (b), and (c). By (c) we can find $T \in G(\xi, \eta)$ and $A_1 \subset A$ with $mA_1 > 0$ such that $T|_{A_0} = U|_{A_0}$. Then for $t = T/\hat{\eta} \in G(\beta^\rho)$

$$\frac{dm_\rho(t(\pi_\eta(x)))}{dm(\pi_\eta(x))} = \frac{dm(Tx)}{dm(x)} = \frac{dm(Ux)}{dm(x)} = a$$

for $x \in A_0$ and

$$m_\rho \left\{ x_0 \in X_0 \colon \frac{dm_\rho(tx_0)}{dm_\rho(x_0)} = a \right\} > 0.$$

Hence $a \in \triangle_\rho$. ∎

Denote by $\mathcal{U}_a$ the totality of all triples $u = (A, B, U)$ satisfying the above conditions (a), (b), and (c) with fixed $a \in \triangle_\rho$ and

$$\mathcal{U} = \bigcup_{a \in \triangle_\rho} \mathcal{U}_a.$$

LEMMA 4.2: *For any* $u = (A, B, U) \in \mathcal{U}$,

$$U(\sigma|_A) = \sigma|_B.$$

Proof: If $S \in [\xi, m]|_A$, then $USU^{-1} \in [\xi]|_B$ by (c) and $USU^{-1} \in [\xi, m]|_A$ by (b). Hence $U([\xi, m]|_A)U^{-1} = [\xi, m]|_B$ and $U(\sigma|_A) = \sigma|_B$. ∎

LEMMA 4.3: *For any* $u = (A, B, U) \in \mathcal{U}_a$ *there exists* $\varphi_u(a) \in \mathcal{A}(X_\sigma, m_\sigma)$ *such that*

$$\pi_\sigma(Ux) = \varphi_u(a)(\pi_\sigma(x))$$

for a.a. $x \in A$.

Proof: Since $A \perp \sigma$ and $B \perp \sigma$,

$$\pi_\sigma(A) = \pi_\sigma(B) = X_\sigma,$$

$\varphi_u(a)$ is correctly defined by Lemma 4.2 and invertible, and for any measurable subset $E \subset X_\sigma$ we have

$$\begin{aligned} m_\sigma(\varphi_u(a)^{-1}E) &= m(\pi_\sigma^{-1}(\varphi_u(a)^{-1}E)) = \frac{1}{mA} m(\pi_\sigma^{-1}(\varphi_u(a)^{-1}E) \cap A) \\ &= \frac{1}{mA} m(U^{-1}(\pi_\sigma E \cap B)) = \frac{1}{mA}\frac{mA}{mB} m(\pi_\sigma^{-1}E \cap B) \\ &= m(\pi_\sigma^{-1}E) = m_\sigma(E). \end{aligned}$$

Hence $\varphi_u(a)$ is a m.p automorphism of (X_σ, m_σ). ∎

LEMMA 4.4: $u_1 \in \mathcal{U}_a,\ u_2 \in \mathcal{U}_a \Longrightarrow \varphi_{u_1}(a) = \varphi_{u_2}(a)$.

Proof: Let $u_1 = (A_1, B_1, U_1)$ and $u_2 = (A_2, B_2, U_2)$. Consider several cases.

CASE 1: $u_1 \le u_2$, that is,

$$A_1 \subset A_2, B_1 \subset B_2, U_1 = U_2|_{A_1}.$$

In this case the equality $\varphi_{u_1}(a) = \varphi_{u_2}(a)$ follows directly by definition.

CASE 2: $A_1 = A_2$. In this case, we consider partial transformation $U_1 \circ U_2^{-1}$, which preserves the measure m, because of condition (c). Then $u_3 = (B_2, A_2, U_1 \circ U_2^{-1})$ satisfies conditions (a), (b), and (c) with $a = 1$. Hence $\varphi_{u_3}(1) = \text{id}$ and $\varphi_{u_1}(a).\varphi_{u_2}(a)^{-1} = \text{id}$, i.e. $\varphi_{u_1}(a) = \varphi_{u_2}(a)$.

CASE 3: $mA_1 = mA_2$. Since $A_1 \perp \sigma, A_2 \perp \sigma$ there exists $T \in [\xi, m]$ such that $TA_2 = A_1$. Considering $u_1' = (A_2, B_1, U_1 \circ T|_{A_2})$, we see that $u_1' \in \mathcal{U}_a$ and $\varphi_{u_1}(a) = \varphi_{u_1'}(a)$. But $\varphi_{u_1'}(a) = \varphi_{u_2}(a)$ by case 2.

GENERAL CASE: For arbitrary u_1, u_2 belonging to $\mathcal{U}_a$ one can find $u_1' \le u_1$ and $u_2' \le u_2$ such that

$$u_1'(A_1', B_1', U_1'), \quad u_2' = (A_2', B_2', U_2'), \quad mA_1' = mA_2'.$$

Then

$$\varphi_{u_1}(a) = \varphi_{u_1'}(a) = \varphi_{u_2'}(a) = \varphi_{u_2}(a)$$

from the above. ∎

We can write now $\varphi(a)$ instead of $\varphi_u(a)$.

LEMMA 4.5: $\varphi(ab) = \varphi(a).\varphi(b)$.

Proof: Using the previous lemma, we can assume with no loss of generality that

$$\varphi(a) = \varphi_{u_1}(a), \quad \varphi(b) = \varphi_{u_2}(b)$$

and $A_2 = B_1$. But in this situation we take

$$\varphi(ab) = \varphi_{u_3}(ab), \quad u_3 = (A_1, B_2, U_2 \circ U_1)$$

and the required equality is obvious. ∎

We have thus constructed the action

$$\varphi\colon \triangle_\rho \ni a \to \varphi(a) \in \mathcal{A}(X_\sigma, m_\sigma)$$

of the group $\triangle_\rho$ on the factor-space (X_σ, m_σ).

We shall call φ the **modular action**, associated with ξ, and write $\varphi = \varphi_\xi$ to indicate the dependence on ξ.

It is easy to see that $\varphi_\xi(\triangle_\rho)$ is ergodic, because of the ergodicity of ξ.

Two actions φ_1, φ_2 of a group $\triangle$,

$$\varphi_i\colon \triangle \to \mathcal{A}(X_i, m_i), \quad i = 1, 2,$$

are called **equivalent** if there exists an isomophism $S\colon X_1 \to X_2$ such that

$$S\varphi_1(a)S^{-1} = \varphi_2(a)$$

for all $a \in \triangle$.

LEMMA 4.6: *The equivalence class of the action φ_ξ is an invariant of orbitally isomorphic sequences $\xi \in FB(\rho)$.*

Proof: Let $\xi \in FB(\rho)$, and $\xi' \in FB(\rho)$ be two ergodic sequences $\xi = \{\xi_n\}_{n=1}^\infty$ and $\xi' = \{\xi_n'\}_{n=1}^\infty$ defined on the spaces (X, m) and (X', m').

Denote $\sigma = \sigma(\xi)$, $\sigma' = \sigma(\xi')$ and let

$$\mathcal{U} = \bigcup_{a \in \triangle_\rho} \mathcal{U}_a, \quad \mathcal{U}' = \bigcup_{a \in \triangle_\rho} \mathcal{U}'_a$$

be the correponding classes of triples for ξ and ξ', respectively.

Suppose $S(\theta(\xi)) = \theta(\xi')$ for an isomorphism $S\colon X \to X'$. Then $S(\sigma) = \sigma'$, since $m \circ S^{-1} = m'$. Let $S^*\colon X_\sigma \to X'_{\sigma'}$ be the factor-isomorphism of S. For any triple $u = (A, B, U) \in \mathcal{U}_a, a \in \triangle_\rho$, the triple

$$u' = (SA, SB, SUS^{-1}) = (A', B', U')$$

belongs to $\mathcal{U}'_a$ and

$$\begin{aligned}\varphi_{u'}(a)S^*(\pi_{\sigma'}(x)) &= \varphi_{u'}(a)\pi_{\sigma'}(Sx) = \pi_{\sigma'}(U'Sx)\\ &= \pi_{\sigma'}(SUx) = S^*\pi_\sigma(Ux) = S^*(\varphi_u(a)(\pi_\sigma(x))\end{aligned}$$

for a.a. $x \in A$. Thus

$$\varphi_{u'}(a)S^* = S^*\varphi_n(a), \quad a \in \triangle_\rho$$

for appropriate $u \in \mathcal{U}_a$ and $u' \in \mathcal{U}'_a$ and hence

$$\varphi_{\xi'}(a)S^* = S^*\varphi_\xi(a), \quad a \in \triangle_\rho. \quad \blacksquare$$

MODULAR φ-EXTENSIONS. Consider an arbitrary ergodic m.p. action $\varphi\colon \triangle_\rho \to \mathcal{A}(Y, m_Y)$ on a space (Y, m_Y), and let r be the modular cocycle, $r\colon \mathcal{G}_\rho \to \mathbb{R}^*_+$, defined on the ergodic equivalence relation $\mathcal{G}_\rho = \mathcal{G}_{\beta^\rho}$ of the standard sequence β^ρ by

$$r(gx, x) = \frac{dm(gx)}{dm(x)}, \quad x \in X, \quad g \in [\beta^\rho].$$

Since $r(x, y) \in \triangle_\rho$ for a.a. $(x, y) \in \mathcal{G}_\rho$ we can introduce the cocycle

$$\varphi \circ r\colon \mathcal{G}_p \to \mathcal{A}(Y, m_Y)$$

and construct the Y-extension of β^ρ (see Section 2). Denote this Y-extension of β^ρ by $\xi^\varphi = \{\xi^\varphi_n\}^\infty_{n=i}$. We shall call ξ^φ the **modular** φ-extension of β^ρ.

PROPOSITION 4.7: *The associated modular action φ_{ξ^φ} of ξ^φ is equivalent to φ.*

Proof: In accordance with the definitions $\xi^\varphi = \{\xi_n^\varphi\}_{n=1}^\infty$ is defined on the space $(X_\rho \times Y, m_\rho \times m_Y)$ by

$$(x_1, y_1) \overset{\xi_n^\varphi}{\sim} (x_2, y_2) \Longleftrightarrow x_1 \overset{\beta_n}{\sim} x_2, y_2 = \varphi(r(x_2, x_1))(y_1).$$

Then

$$(x_1, y_1) \overset{\theta(\xi^\varphi)}{\sim} (x_2, y_2) \Longleftrightarrow (x_1, x_2) \in \mathcal{G}_\rho, y_2 = \varphi(r(x_2, x_1))(y_1)$$

and

$$(x_1, y_1) \overset{[\xi^\varphi, m]}{\sim} (x_2, y_2) \Longleftrightarrow (x_1, x_2) \in \mathcal{G}_\rho, r(x_1, x_2) = 1, \quad y_1 = y_2$$

with $m = m_\rho \times m_Y$.

Hence $\theta([\xi^\varphi, m]) \geq \nu_{X_\rho} \times \varepsilon_Y$ and $\sigma \geq \nu_{X_\rho} \times \varepsilon_Y$. On the other hand, the group $[\beta^\rho, m_\rho]$ is ergodic, and this implies $\sigma \leq \nu_{X_\rho} \times \varepsilon_X$. Thus $\sigma = \nu_{X_\rho} \times \varepsilon_Y$ and we can identify (X_σ, m_σ) with (Y, m_Y). Under this identification we have for

$$u = (A \times Y,\ B \times Y,\ U) \in \mathcal{U}_a, \quad a \in \triangle_\rho$$

and a.a. $(x, y) \in A \times Y$ that

$$\varphi_u(a)y = \pi_\sigma(U(x, y)) = \varphi(a)y$$

since

$$\frac{dm(U(x, y))}{dm(x, y)} = a.$$

Thus $\varphi_{\xi^\varphi} = \varphi$. ∎

As a consequence we have obtained the following classification of modular extensions.

Corollary 4.8: *For two ergodic measure preserving actions φ_1 and φ_2 the following conditions are equivalent:*

(1) *φ_1 and φ_2 are equivalent,*
(2) $\xi^{\varphi_1} \overset{I}{\sim} \xi^{\varphi_2}$,
(3) $\xi^{\varphi_1} \overset{LI}{\sim} \xi^{\varphi_2}$,
(4) $\xi^{\varphi_1} \overset{QI}{\sim} \xi^{\varphi_2}$.

Proof: (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) is obvious and (4) $\Longrightarrow$ (1) because of Lemma 4.6 and Proposition 4.7. ∎

PROPOSITION 4.9: *Let ξ be ergodic, $\xi \in FB(\rho)$ and let β^ρ satisfy the condition* (K). *Then:*

(1) *ξ is isomorphic to a modular extension of β^ρ iff there exists $\eta \in \mathcal{D}(\xi)$ such that $\sigma(\xi) \vee \hat{\eta} = \varepsilon$.*

(2) *ξ is lacunarily (and hence orbitally) isomorphic to the φ_ξ-extension of β_ρ.*

Proof: (1) If $\sigma(\xi) \vee \hat{\eta} = \varepsilon$, then $\hat{\eta} \in IC(\sigma(\xi))$ by Proposition 2.4, and we can identify (X, m) with the direct product $(X_\rho \times X_\sigma, m_\rho \times m_\sigma)$ under the isomorphism

$$X \ni x \to (\pi_\eta x, \pi_\sigma x) \in X_\rho \times X_\sigma$$

where π_η is the factor-mapping, described in Proposition 2.2 and π_σ is the natural projection.

Under this identification

$$\sigma(\xi) = \nu_{X_\rho} \times \varepsilon_{X_\sigma}, \quad \hat{\eta} = \varepsilon_{X_\rho} \times \nu_{X_\sigma}$$

and there exists a cocycle $f\colon \mathcal{G}_\rho \to \mathcal{A}(X_\sigma, m_\sigma)$ such that

$$(x_1, y_1) \overset{\xi_n}{\sim} (x_2, y_2) \Longleftrightarrow (x_1 \overset{\beta_n^\rho}{\sim} x_2), \quad y_2 = f(x_2, x_1) y_1$$

almost everywhere on $X_\rho \times X_\sigma$.

Consider now for all $i, j \in J_n$ and $n \in \mathbb{N}$ the triples

$$u_{ij}^{(n)} = (C_i^{(n)}, C_j^{(n)}, T_{ij}^{(n)}) \in \mathcal{U}_{a_{ij}^{(n)}}, \quad a_{ij}^{(n)} = \frac{\rho_j^{(n)}}{\rho_i^{(n)}}$$

where

$$C_i^{(n)} = \pi_\eta^{-1} c_i^{(n)}, \quad c_i^{(n)} = \{x_1 = \{i_k\}_{k=1}^\infty \in X_\rho\colon i_n = i\},$$

and

$$t_{ij}^{(n)}\colon c_i^{(n)} \to c_j^{(n)}, \quad T_{ij}^{(n)}\colon C_i^{(n)} \to C_j^{(n)}$$

are defined as in Section 2.

Then for a.a. $(x, y) \in C_i^{(n)}$ we have

$$(T_{ij}^{(n)}(x, y) = (t_{ij}^{(n)} x, f(t_{ij}^{(n)} x, x) y) = (t_{ij}^{(n)} x, \varphi_\xi(a_{ij}^{(n)}) y)$$

and

$$f(tx, x) = \varphi_\xi(r(tx, x)) \quad \text{a.e.}$$

holds for $t = t_{ij}$ and a.a. $x \in c_i^{(n)}$, because of $r(t_{ij}^{(n)} x, x) = a_{ij}^{(n)}$.

Thus $f = \varphi_\xi \circ r$, i.e. ξ is a φ_ξ-extension of β^ρ.

The inverse statement is obvious.

(2) By Theorem 3.1, for any ergodic $\xi \in FB(\rho)$ one can find a subsequence $\xi' = \{\xi_{n_k}\}_{k=1}^\infty$ and $\eta' = \{\eta'_k\}_{k=1}^\infty \in \mathcal{D}(\xi')$ such that $\sigma \vee \hat{\eta}' = \varepsilon$, where $\sigma = \sigma(\xi) = \sigma(\xi')$ and $\hat{\eta}' = \bigvee_{k=1}^\infty \eta'_k$.

Hence ξ' is isomorphic to the φ-extension of $\{\beta^\rho_{n_k}\}_{k=1}^\infty$ (by part (1)), and ξ itself is lacunarily isomorphic to the φ_ξ-extension of β^ρ (here $\varphi_\xi = \varphi_{\xi'}$). ∎

We have now got all parts of the proof of Theorems A and B.

Part (1) of Theorem B follows from Lemmas 4.1–4.5.

The "only if" part of Theorem B, part (2) follows from Lemma 4.6.

The "if" part of Theorem B, part (2) follows from Corollary 4.8 and Proposition 4.9(2).

Part (3) of Theorem B follows from Proposition 4.7.

Theorem A follows from Proposition 4.9(2) and Corollary 4.8.

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