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IMPLICIT FUNCTIONS FROM TOPOLOGICAL VECTOR SPACES TO BANACH SPACES

BY

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ABSTRACT

We prove implicit function theorems for mappings on topological vector spaces over valued fields. In the real and complex cases, we obtain implicit function theorems for mappings from arbitrary (not necessarily locally convex) topological vector spaces to Banach spaces.

Introduction

In this article, we prove implicit function theorems (and generalizations) for mappings from topological vector spaces over valued fields to Banach spaces. Our main results can be summarized as follows. Let $(\mathbb{K}, |.|)$ be a (non-trivial, not necessarily complete) valued field, E and F be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ and $V \subseteq F$ be open subsets, and $f\colon U \times V \to F$ be a C^1-map. Let $(x_0, y_0) \in U \times V$ such that $d_2 f(x_0, y_0, \bullet) \in \mathrm{GL}(F)$. Then, given the respective hypotheses stated in the first four columns of the following table, there exists an open neighbourhood $Q \subseteq U$ of x_0 and an open neighbourhood $B \subseteq V$ of y_0 such that, for every $x \in Q$, there is a unique element $\beta(x) \in B$ such that $f(x, \beta(x)) = f(x_0, y_0)$, and the mapping $\beta\colon Q \to B$ so obtained has the property

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shown in the last column:

$\mathbb{K}$	E	F	f	β
$\mathbb{R}$	arbitrary	$\dim F < \infty$	C^k	C^k
$\mathbb{R}$	arb.	Banach	C^{k+1}	C^k
$\mathbb{R}$	arb.	Banach	$\mathbb{R}$-analytic	$\mathbb{R}$-analytic
$\mathbb{C}$	arb.	Banach	$\mathbb{C}$-analytic	$\mathbb{C}$-analytic
local	arb.	$\dim F < \infty$	C^1	C^0
local	metrizable	$\dim F < \infty$	C^k	C^k
complete ultram.	metrizable	Banach	C^{k+1}	C^k
$\mathbb{K} \subseteq \mathbb{R}$	metrizable	Banach	C^2	C^1
arb.	arb.	Banach	C^2	C^0
arb.	Banach	Banach	SC^1	SC^1
complete	Banach	Banach	SC^k	SC^k

Here $k \in \mathbb{N} \cup \{\infty\}$, and C^k-maps are understood in the sense of [2], where a differential calculus over arbitrary non-discrete topological fields is developed. A map between open subsets of real locally convex spaces is C^k in the former sense if and only if it is C^k in the sense of Michal–Bastiani (i.e., a Keller C^k_c-map [29]). Keller's C^∞_c-maps are a popular class of maps, which have been used as the basis of infinite-dimensional Lie theory by many authors (see, e.g., [13], [21], [38], [39], [41], [53]). The symbol SC^k refers to k times strictly differentiable mappings, as defined below.

Our results were inspired by Hiltunen's implicit function theorems, which he formulated in the setting of C^k_Π-maps on locally convex spaces (see [24] for the real case, [25] for the complex case). In contrast to that paper, we are working throughout in the realm of topology; no recourse to convergence structures is necessary. Furthermore, we are able to work over valued fields other than $\mathbb{R}$ or $\mathbb{C}$, and need not assume that the domains be locally convex. Cf. also [47, La.1 and Rem.1] for related results in the convenient setting of analysis. While we strive to allow non-Banach domains for Banach space-valued functions of interest, other generalizations of the implicit function theorem aim to replace Banach space-valued functions by functions into more general spaces. A classical example is the Nash–Moser theorem (for maps into tame Fréchet spaces) (see [22] and the references given there). For further generalizations in this direction see [37] (complex analytic case) and [26], where also applications of generalized implicit function theorems to well-posedness of partial differential equations are described.

THE BASIC IDEA. Our approach is based on the classical idea that every implicit function theorem has an underlying "inverse function theorem with parameters" (cf. [45] and [47]). For example, in the case of a C^k-map $f\colon U \times V \to F$, where U is a subset of a real topological vector space and $F = \mathbb{R}^n$, given $(x_0, y_0) \in U \times V$ with $d_2 f(x_0, y_0, \bullet)$ invertible, after shrinking U we interpret f as a family $(f_x)_{x \in U}$ of mappings $f_x := f(x, \bullet)\colon V \to F$ between open subsets of the finite-dimensional space F, to which the classical inverse function theorem applies, and then show that $\psi\colon (x, y) \mapsto f_x^{-1}(y)$ makes sense on some open neighbourhood of $(x_0, f(x_0, y_0))$ and defines a C^k-map there. Then $y(x) = \psi(x, f(x_0, y_0))$ gives a C^k-solution to the equation $f(x, y) = f(x_0, y_0)$.

Local convexity does not play a role here, and we are also able to tackle the ultrametric case. This is much more difficult, since the absence of a fundamental theorem of calculus and mean value theorem in this setting makes it necessary to discuss continuous extensions to iterated difference quotient maps, rather than the mere existence and continuity of higher differentials. To get from the Banach case to implicit functions on open subsets of metrizable topological vector spaces over a complete ultrametric field $\mathbb{K}$, we exploit the fact that a map on a metrizable space is C^k if and only if all of its compositions with smooth maps from $\mathbb{K}^{k+1}$ to the space are C^k [2]. This result can be seen as an adaptation of ideas from the convenient differential calculus of Frölicher, Kriegl and Michor ([10], [31]) to non-archimedian analysis. A similar argument has also been used in [47, proof of La. 1] to establish smoothness of implicit functions in the convenient sense.

Applications. It is clear that a generalization of such a central and basic result as the implicit function theorem has immediate applications.

- In [16], our ultrametric inverse function theorem with parameters in a Fréchet space is used to prove that the inversion map $\mathrm{Diff}(M) \to \mathrm{Diff}(M)$, $\gamma \mapsto \gamma^{-1}$ of the diffeomorphism group of a paracompact, finite-dimensional smooth manifold over a local field is smooth. Also, composition being smooth, $\mathrm{Diff}(M)$ is a Lie group.[1]
- Let A be a commutative, unital, associative, locally convex and complete complex topological algebra whose group of units is open and whose inversion map is continuous. In [3], our implicit function theorem for analytic mappings from complex locally convex spaces to Banach spaces is used to prove a sufficient condition for the existence of a solution $b \in A^m$ to an equation $f[a, b] = 0$ given by multi-variable holomorphic functional calculus, where $f\colon \Omega \to \mathbb{C}^m$ is a holomorphic function on an open subset

1 Cf. also [35], [36] for certain diffeomorphism groups for $\operatorname{char}\mathbb{K} = 0$ (mainly considred as topological groups).

$\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^m$ and $a \in A^n$.

- Our inverse function theorem for SC^k-maps between ultrametric Banach spaces is used in [18] to construct stable manifolds for dynamical systems over ultrametric fields, using Irwin's method (see [27] for the real case). Also, pseudo-stable manifolds can be tackled (see [28], resp., [19]). This facilitates the calculation of the scale $s_G(\alpha)$ (as introduced in [51], [52]) of suitable automorphisms α of finite-dimensional Lie groups G over a local field $\mathbb{K}$ of positive characteristic, and a diffeomorphic decomposition $U_\alpha M_\alpha U_{\alpha^{-1}}$ of an open subset of G into contraction groups and a Levi factor (see [20]).[2] These results, in turn, can be used to construct finite-dimensional smooth $\mathbb{K}$-Lie groups not admitting a $\mathbb{K}$-analytic Lie group structure compatible with the given topological group structure [14].
- The quantitative information provided by our inverse function theorem (with and without parameters) is also used essentially in [15], to establish a C^k-compatible $\mathbb{K}$-analytic Lie group structure on each finite-dimensional C^k-Lie group over a local field $\mathbb{K}$ of characteristic 0.

STRUCTURE OF THE ARTICLE. The article is structured as follows. Having described the precise setting of differential calculus (Section 1), we present our results in the real and complex case. We do not try to be self-contained here, but rather re-use the standard Inverse Mapping Theorem for real Banach spaces and its corollaries (as proved in [32]), which should be well-known to most readers, to get to the point as quickly as possible.

In Section 3, we recall the notion of a strictly differentiable mapping from an open subset of a normed vector space over a valued field $\mathbb{K}$ to a polynormed $\mathbb{K}$-vector space (cf. [6]). We show that any strictly differentiable map is C^1, and we show that every C^2-map from an open subset of a normed $\mathbb{K}$-vector space to a polynormed $\mathbb{K}$-vector space is strictly differentiable. In Section 4, we specialize to locally compact $\mathbb{K}$. In this case, a map from an open subset of a finite-dimensional $\mathbb{K}$-vector space to a polynormed $\mathbb{K}$-vector space is C^1 if and only if it is strictly differentiable, if and only if it is "locally uniformly differentiable" (an *a priori* even stronger differentiability property). In Section 5, we introduce k times strictly differentiable mappings (SC^k-maps). Any such map is C^k, and, conversely, we show that every C^{k+1}-map from an open subset of a normed vector space over a valued field $\mathbb{K}$ to a polynormed $\mathbb{K}$-vector space is SC^k.

Recall from [43, Ex. 26.6] that there is a map $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ from the p-adic integers to the p-adic numbers which is totally differentiable at each $x \in \mathbb{Z}_p$,

2 Compare [11] and [48] for the case of characteristic 0, which is much easier.

with $f'(x) = 1$ (whence $f'(x)$ is invertible and $f'\colon \mathbb{Z}_p \to \mathbb{Q}_p$ continuous), but such that f is not injective on any 0-neighbourhood. Thus, in the ultrametric case, an inverse function theorem cannot be based on the mere existence and continuity of differentials. In contrast, the SC^k-property is well-adapted to inverse and implicit function theorems. An inverse function theorem for once strictly differentiable maps between open subsets of Banach spaces over complete valued fields is well-known (see [6, 1.5.1], where no proofs are given and where all Banach spaces over ultrametric fields are assumed ultrametric). In the real case, strict differentiability facilitates refined implicit function theorems for maps between Banach spaces ([33], [34], [30, Thm. 6.3.6]). Higher order differentiability in the finite-dimensional ultrametric case has been discussed in [1] for implicit functions, in [46] (with merely partial proofs) for inverse functions. Getting beyond these known facts, using an inductive argument which goes back and forth between inverse functions and implicit functions, we establish the Inverse and Implicit Function Theorems for SC^k-maps between open subsets of Banach spaces over complete valued fields (Section 7). Combining these results with parameter-dependent Newton approximation (from Section 6) and the tools of differential calculus on metrizable spaces outlined above, we obtain an Implicit Function Theorem for maps from metrizable topological vector spaces to (not necessarily ultrametric) Banach spaces over complete ultrametric fields (see Section 8). To illustrate the use of our results, two applications are sketched in Section 9 (smoothness of inversion in diffeomorphism groups; existence of stable manifolds).

In an appendix, we show that, in the real case, every k times continuously Fréchet differentiable map is an SC^k-map. For $k = 1$, the converse also holds [6, 2.3.3].

All results are formulated in a way which transports as much useful information as possible. For example, instead of formulating mere implicit function theorems, we explicitly spell out "inverse function theorems with parameters." We also provide information concerning the size of images of balls. In this sense, the results include "quantitative inverse function theorems."

Note added in proofs: In the meantime, the results could be strengthened further (see arXiv.math.FA/0511218).

1. Differential calculus over topological fields

In this article, we are working in the setting of differential calculus over non-discrete topological fields developed in [2]. In this section, we briefly recall basic definitions and facts.

Unless stated otherwise, in this section $\mathbb{K}$ denotes a non-discrete topological field. All topological vector spaces are assumed Hausdorff. Before we define C^k-maps, we need an efficient notation for the domains of certain mappings associated with C^k-maps.

Definition 1.1: If E is a topological $\mathbb{K}$-vector space and $U \subseteq E$ an open subset, we define $U^{[0]} := U$ and

$$U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{K}: x + ty \in U\},$$

which is an open subset of the topological $\mathbb{K}$-vector space $E \times E \times \mathbb{K}$. Having defined $U^{[j]}$ inductively for a natural number $j \geq 1$, we set $U^{[j+1]} := (U^{[j]})^{[1]}$.

In particular, $E^{[1]} = E \times E \times \mathbb{K}$, $E^{[2]} = E \times E \times \mathbb{K} \times E \times E \times \mathbb{K} \times \mathbb{K}$, etc.

Definition 1.2: Let E and F be topological $\mathbb{K}$-vector spaces, and $f: U \to F$ be a mapping, defined on an open subset $U \subseteq E$. We say that f is **of class** $C^0_{\mathbb{K}}$ if f is continuous, we set $f^{[0]} := f$ and call $f^{[0]}$ the 0-th extended difference quotient map of f. If f is continuous and there exists a continuous mapping $f^{[1]}: U^{[1]} \to F$ such that

$$(1) \quad \frac{1}{t}(f(x+ty) - f(x)) = f^{[1]}(x, y, t) \quad \text{for all } (x, y, t) \in U^{[1]} \text{ such that } t \neq 0,$$

we say that f is **of class** $C^1_{\mathbb{K}}$, and call $f^{[1]}$ the (first) **extended difference quotient map** of f. Here $f^{[1]}$ is uniquely determined, as $\mathbb{K}$ is non-discrete. Recursively, having defined $C^j_{\mathbb{K}}$-maps and j-th extended difference quotient maps for $j = 0, \ldots, k-1$ for some natural number $k \geq 2$, we call f a mapping **of class** $C^k_{\mathbb{K}}$ if f is of class $C^{k-1}_{\mathbb{K}}$ and $f^{[k-1]}$ is of class $C^1_{\mathbb{K}}$. In this case, we define the k-th extended difference quotient map of f via

$$f^{[k]} := (f^{[k-1]})^{[1]}: U^{[k]} \to F.$$

The mapping f is **of class** $C^\infty_{\mathbb{K}}$ (or $\mathbb{K}$**-smooth**) if it is of class $C^k_{\mathbb{K}}$ for all $k \in \mathbb{N}_0$. If $\mathbb{K}$ is understood, we simply write C^k instead of $C^k_{\mathbb{K}}$, and we call f smooth or of class C^∞ if it is $\mathbb{K}$-smooth.

1.3. For example, every continuous linear mapping $\lambda: E \to F$ is smooth, with $\lambda^{[1]}(x, y, t) = \lambda(y)$ for all $(x, y, t) \in E \times E \times \mathbb{K}$. If V, W and F are topological $\mathbb{K}$-vector spaces and $\beta: V \times W \to F$ is a continuous bilinear map, then β is smooth, with

$$\beta^{[1]}((v, w), (v', w'), t) = \beta(v, w') + \beta(v', w) + t\beta(v', w')$$

for all $v, v' \in V$, $w, w' \in W$, and $t \in \mathbb{K}$ (cf. [2]).

1.4. If $k \geq 2$, then a map f is of class $C^k_{\mathbb{K}}$ if and only if f is of class $C^1_{\mathbb{K}}$ and $f^{[1]}$ is of class $C^{k-1}_{\mathbb{K}}$; in this case, $f^{[k]} = (f^{[1]})^{[k-1]}$ [2, Rem. 4.2].

1.5. Given a $C^1_{\mathbb{K}}$-map $f\colon U \to F$ as before, we define

$$df(x,v) := \lim_{0 \neq t \to 0} \frac{1}{t}(f(x+tv) - f(x)) = f^{[1]}(x,v,0)$$

for $(x,v) \in U \times E$. Then $df\colon U \times E \to F$ is continuous, being a partial map of $f^{[1]}$, and it can be shown that the "differential" $df(x,\bullet)\colon E \to F$ of f at x is a continuous $\mathbb{K}$-linear map, for each $x \in U$ [2, Prop. 2.2]. If f is of class C^2, we define a continuous mapping $d^2f\colon U \times E^2 \to F$ via

$$d^2f(x,v_1,v_2) := \lim_{t \to 0} \frac{1}{t}(df(x+tv_2,v_1) - df(x,v_1)) = f^{[2]}((x,v_1,0),(v_2,0,0),0).$$

Similarly, if f is of class $C^k_{\mathbb{K}}$, we obtain continuous maps $d^jf\colon U \times E^j \to F$ for all $j \in \mathbb{N}_0$ such that $j \leq k$ (where $d^0f := f$). It can be shown that $d^jf(x,\bullet)\colon E^j \to F$ is a symmetric j-linear map [2, La. 4.8].

Our discussion of implicit function theorems in the real case will be made easy by the following fact ([2, Prop. 7.4]):

PROPOSITION 1.6: *Let E be a real topological vector space, F a locally convex real topological vector space, $U \subseteq E$ an open subset, $f\colon U \to F$ a map, and $k \in \mathbb{N}_0 \cup \{\infty\}$. Then f is of class $C^k_{\mathbb{R}}$ if and only if it is a C^k-map in the sense of Michal–Bastiani, i.e., f is continuous, the differentials $d^jf\colon U \times E^j \to F$ described in 1.5 exist for all $j \in \mathbb{N}$ such that $j \leq k$,*[3] *and are continuous.*

In more general situations, it is necessary to work with the functions $f^{[j]}$, since the differentials alone do not encode enough information. For example, $d^jf = 0$ for all $j \geq 2$ if $\mathbb{K}$ is a non-discrete topological field of characteristic 2 and f any smooth function on $\mathbb{K}$ (cf. [2, Thm. 5.4]). Even worse, injective smooth functions $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ are known to exist whose derivative vanishes identically [43, Exercise 29.G].

1.7 (CHAIN RULE). If E, F, and H are topological $\mathbb{K}$-vector spaces, $U \subseteq E$ and $V \subseteq F$ are open subsets, and $f\colon U \to V \subseteq F$, $g\colon V \to H$ are mappings of class C^k, then also the composition $g \circ f\colon U \to H$ is of class C^k. If $k \geq 1$, we have $(f(x), f^{[1]}(x,y,t), t) \in V^{[1]}$ for all $(x,y,t) \in U^{[1]}$, and

$$(g \circ f)^{[1]}(x,y,t) = g^{[1]}(f(x), f^{[1]}(x,y,t), t). \tag{2}$$

3 That is to say, all limits occurring in the recursive definition of the differentials d^jf exist.

In particular, $d(g\circ f)(x,y) = dg(f(x), df(x,y))$ for all $(x,y) \in U \times E$ [2, Prop. 3.1 and 4.5].

We recall from [2, La. 4.9] that being of class C^k is a local property.

LEMMA 1.8: *Let E and F be topological $\mathbb{K}$-vector spaces, and $f\colon U \to F$ be a mapping, defined on an open subset U of E. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. If there is an open cover $(U_i)_{i\in I}$ of U such that $f|_{U_i}\colon U_i \to F$ is of class C^k for each $i \in I$, then f is of class C^k.*

Definition 1.9: A **valued field** is a field $\mathbb{K}$, together with an absolute value $|.|\colon \mathbb{K} \to [0,\infty[$ (see [50]); we require furthermore that the absolute value be non-trivial (meaning that it gives rise to a non-discrete topology on $\mathbb{K}$). An **ultrametric field** is a valued field $(\mathbb{K}, |.|)$ whose absolute value satisfies the ultrametric inequality

$$|x+y| \leq \max\{|x|, |y|\} \quad \text{for all } x, y \in \mathbb{K}.$$

Locally compact, totally disconnected, non-discrete topological fields will be referred to as **local fields**.

Remark 1.10: It is well-known that every local field $\mathbb{K}$ admits an ultrametric absolute value defining its topology [49]. Fixing such an absolute value on $\mathbb{K}$, we can consider $\mathbb{K}$ as an ultrametric field.

Remark 1.11: Note that we do not require that valued fields (nor ultrametric fields) be complete (with respect to the metric induced by the absolute value). Whenever our results depend on completeness properties of the ground field, we will state these explicitly.

1.12. Recall that a topological vector space E over an ultrametric field $\mathbb{K}$ is called **locally convex** if every zero-neighbourhood of E contains an open $\mathbb{O}$-submodule of E, where $\mathbb{O} := \{t \in \mathbb{K}\colon |t| \leq 1\}$ is the valuation ring of $\mathbb{K}$. Equivalently, E is locally convex if and only if its vector topology is defined by a family of ultrametric continuous seminorms $\gamma\colon E \to [0,\infty[$ on E (cf. [40] for more information). Let $\mathbb{K}$ be a valued field. We call a topological $\mathbb{K}$-vector space **polynormed** if its vector topology is defined by a family of continuous seminorms (which need not be ultrametric when $\mathbb{K}$ is an ultrametric field). This terminology deviates from the one in Bourbaki [6], where only polynormed vector spaces over ultrametric fields are considered whose topology arises from a family of continuous *ultrametric* seminorms, and which therefore are precisely the locally convex spaces over such fields in our terminology. Ultrametric seminorms are called "ultra-semi-norms" in [6] and [7]. Occasionally, we shall write $\|.\|_\gamma$ for a continuous seminorm γ.

1.13. A **Banach space** over a valued field $\mathbb{K}$ is a normed $\mathbb{K}$-vector space $(E, \|.\|)$ (see [7, Ch. I, §1, no. 2]) which is complete in the metric associated with $\|.\|$. Given a normed $\mathbb{K}$-vector space $(E, \|.\|)$, a point $x \in E$ and $r > 0$, we let $B_r^E(x) := \{y \in E\colon \|y - x\| < r\}$ be the open ball of radius r around x. We write $B_r(x) := B_r^E(x)$ when no confusion is possible. $\overline{B}_r(x) := \{y \in E\colon \|y - x\| \leq r\}$ denotes the corresponding closed ball.

1.14. We shall not presume that normed spaces (nor Banach spaces) over ultrametric fields be ultrametric, unless saying so explicitly. For example, $\ell^1(\mathbb{Q}_p)$ is a non-ultrametric (and non-locally convex) Banach space over $\mathbb{Q}_p$.

The following fact ([2, Thm. 12.4]) is essential for our discussion of implicit function theorems over ultrametric fields:

PROPOSITION 1.15: *Let $(\mathbb{K}, |.|)$ be either $\mathbb{R}$, equipped with the usual absolute value, or an ultrametric field. Let E and F be topological $\mathbb{K}$-vector spaces (which need not be locally convex when $\mathbb{K} = \mathbb{R}$), $f\colon U \to F$ be a mapping, defined on a non-empty open subset $U \subseteq E$, and $k \in \mathbb{N}_0$. If E is metrizable, then the following conditions are equivalent:*

(a) *f is a mapping of class $C^k_\mathbb{K}$.*

(b) *The composition $f \circ c\colon \mathbb{K}^{k+1} \to F$ is of class $C^k_\mathbb{K}$, for every smooth mapping $c\colon \mathbb{K}^{k+1} \to U$.*

In particular, f is smooth if and only if $f \circ c$ is smooth, for every $k \in \mathbb{N}$ and every smooth map $c\colon \mathbb{K}^k \to U$.

Finally, let us recall the basics of real and complex analytic mappings.

Definition 1.16 ([4, Defn. 5.6])*:* Let E be a complex topological vector space, F be a locally convex complex topological vector space, $U \subseteq E$ be an open subset, and $f\colon U \to F$ be a map. Then f is called **complex analytic** if it is continuous and, for every $x \in U$, there exists a zero-neighbourhood $V \subseteq E$ such that $x + V \subseteq U$ and continuous homogeneous polynomials $\beta_n\colon E \to F$ of degree $n \in \mathbb{N}_0$ such that

$$f(x+h) = \sum_{n=0}^{\infty} \beta_n(h) \quad \text{for all } h \in V$$

as a pointwise limit.

Real analytic mappings are defined as follows:

Definition 1.17: Let E be a real topological vector space, F be a locally convex real topological vector space, $U \subseteq E$ be open, and $f\colon U \to F$ be a map. Then f is called **real analytic** if it extends to a complex analytic mapping $V \to F_{\mathbb{C}}$, defined on some open neighbourhood V of U in $E_{\mathbb{C}}$.

Remark 1.18: Real analyticity of a mapping $f\colon U \to F$ is a local property in the sense that real analyticity of $f|_{U_j}$ for an open cover $(U_j)_{j\in J}$ of U entails real analyticity of f. In fact, if f is real analytic locally, then for every $x \in U$ we find an open, balanced zero-neighbourhood $W_x \subseteq E$ such that $x + W_x \subseteq U$ and $f|_{x+W_x} = g_x|_{x+W_x}$ for some complex analytic mapping $g_x\colon V_x := (x + W_x) + iW_x \to F_{\mathbb{C}}$. Given $x, y \in U$, we have $V_{x,y} := V_x \cap V_y = ((x + W_x) \cap (y + W_y)) + i(W_x \cap W_y)$, where $W_x \cap W_y$ is a balanced open 0-neighbourhood and thus connected. Hence, the connected components of $V_{x,y}$ are of the form $C + i(W_x \cap W_y)$, where C is a connected component of $(x + W_x) \cap (y + W_y)$. Since g_x and g_y coincide on C, they coincide on $C + i(W_x \cap W_y)$ by the Identity Theorem [4, Prop. 6.6II]. Thus $g_x|_{V_{x,y}} = g_y|_{V_{x,y}}$ for all $x, y \in U$, whence $g := \bigcup_{x\in U} g_x\colon \bigcup_{x\in U} V_x \to F_{\mathbb{C}}$ is a well-defined complex analytic mapping extending f.

As a consequence of [2, Prop. 7.7 and La. 10.1], we have:

LEMMA 1.19: *Every real or complex analytic map $f\colon U \to F$ as before is of class $C^\infty_{\mathbb{R}}$.*

For the next observation, see [2, Prop. 7.7]:

LEMMA 1.20: *Let E be a complex topological vector space, F be a locally convex complex topological vector space, $U \subseteq E$ be open, and $f\colon U \to F$ be a map. Then f is complex analytic if and only if f is of class $C^\infty_{\mathbb{C}}$, if and only if f is of class $C^\infty_{\mathbb{R}}$ and $df(x, \bullet)\colon E \to F$ is complex linear for each $x \in U$.*

The Chain Rule for $C^\infty_{\mathbb{C}}$-functions readily entails that compositions of composable complex analytic (resp., real analytic) mappings are complex analytic (resp., real analytic).

2. Generalized implicit function theorem in the real and complex case

We begin with a preparatory result, providing *continuous* implicit functions in very general situations. A mapping from an open subset of a normed real vector space to a real locally convex space will be called an FC^k-map if it is k times continuously differentiable in the Fréchet sense (cf. [6, 2.3.1]).

PROPOSITION 2.1: *Let $k \in \mathbb{N} \cup \{\infty\}$, P be a topological space, E a real Banach space, and $U \subseteq E$ an open subset. Suppose that $f\colon P \times U \to E$ is a continuous function such that*

(i) *$f_p := f(p, \bullet)\colon U \to E$ is of class FC^k, for all $p \in P$, and*
(ii) *the map $P \times U \to L(E)$, $(p,x) \mapsto f_p'(x) := d(f_p)(x, \bullet) = d_2 f(p,x,\bullet)$ is continuous, where $L(E)$ is equipped with the operator norm.*

Let $(p,x) \in P \times U$, and suppose that $A := f_p'(x) \in \mathrm{GL}(E) := L(E)^\times$. Let $0 < a < 1 < b$ be given. Then there exists an open neighbourhood $Q \subseteq P$ of p and $r > 0$ such that $B := B_r^E(x) \subseteq U$ and the following holds:

(a) *$f_q(B)$ is open in E, for each $q \in Q$, and $\phi_q\colon B \to f_q(B)$, $\phi_q(y) := f_q(y) = f(q,y)$ is an invertible FC^k-map, with inverse $(\phi_q)^{-1}\colon f_q(B) \to B$ of class FC^k.*
(b) *For all $q \in Q$, $y \in B$, and $s \in]0, r - \|y - x\|]$, we have*

$$f_q(y) + A.B_{as}(0) \subseteq f_q(B_s(y)) \subseteq f_q(y) + A.B_{bs}(0). \tag{3}$$

(c) *$W := \bigcup_{q \in Q}(\{q\} \times f_q(B))$ is an open subset of $P \times E$ and the mapping $\psi\colon W \to B$, $\psi(q,v) := \phi_q^{-1}(v)$ is continuous. Furthermore, the map $\theta\colon Q \times B \to W$, $\theta(q,y) := (q, f(q,y))$ is a homeomorphism, with inverse given by $\theta^{-1}(q,v) = (q, \psi(q,v))$.*
(d) *$Q \times (f_p(x) + A.B_\delta(0)) \subseteq W$ for some $\delta > 0$.*

In particular, for each $q \in Q$, there is a unique element $\beta(q) \in B$ such that $f(q, \beta(q)) = f(p,x)$, and the mapping $\beta\colon Q \to B$ so obtained is continuous.

Proof: In view of hypotheses (i) and (ii), we find an open neighbourhood $Q_1 \subseteq P$ of p and $R > 0$ such that $B_R(x) \subseteq U$ and the following holds:

1. $f_q'(y) \in \mathrm{GL}(E)$ for all $q \in Q_1$ and $y \in B_R(x)$;
2. $\|f_{q_1}'(y_1)^{-1} f_{q_2}'(y_2) - \mathbf{1}\| < \frac{1}{2}$ for all $q_1, q_2 \in Q_1$ and $y_1, y_2 \in B_R(x)$;
3. $\|f_q'(y)^{-1}(f_q'(y_1) - f_q'(y_2))\| \leq 1 - \sqrt{a}$ for all $q \in Q_1$ and $y, y_1, y_2 \in B_R(x)$;
4. $\|A^{-1} f_q'(y)\| < b$ and $\|f_q'(y)^{-1} A\| \leq 1/\sqrt{a}$ for all $q \in Q_1$ and $y \in B_R(x)$.

Choose $r \in]0, R/2[$. There is an open neighbourhood $Q \subseteq Q_1$ of p such that

$$\|A^{-1}.(f_q(x) - f_p(x))\| < ar/2 \quad \text{for all } q \in Q. \tag{4}$$

Define $\delta := ar/2$. We claim that all assertions of the proposition are satisfied with Q, r, and δ as just defined.

(a) Given $q \in Q$, we consider the map $h\colon B \to E$, $h(y) := A^{-1} f_q(y)$. Then $\|h'(y) - \mathbf{1}\| = \|A^{-1} f_q'(y) - \mathbf{1}\| < \frac{1}{2}$ by 2., whence h is injective. In fact, if

$y_1 \neq y_2 \in B$, then

$$\begin{aligned}\|h(y_2) - h(y_1)\| &\geq \|y_2 - y_1\| - \|h(y_2) - y_2 - (h(y_1) - y_1)\| \\ &= \|y_2 - y_1\| - \left\| \int_0^1 (h'(y_1 + t(y_2 - y_1)) - \mathbf{1}).(y_2 - y_1)dt \right\| \\ &\geq \frac{1}{2}\|y_2 - y_1\| > 0.\end{aligned}$$

Hence also $f_q|_B = A \circ h$ is injective, and since $f_q'(y) \in \mathrm{GL}(E)$ for all $y \in B$ (by 1.) and f_q is FC^k, the standard Inverse Function Theorem [32, Thm. I.5.2] shows that $f_q(B)$ is open in E and $\phi_q^{-1}\colon f_q(B) \to B$ an FC^k-map, where $\phi_q := f_q|_B^{f_q(B)}$.

(b) Let $y \in B$, $s \in]0, r - \|y - x\|]$, and $q \in Q$. Given $z \in B_s(y)$, we have

$$\max\{\|A^{-1}.f_q'(y + t(z - y))\|\colon t \in [0,1]\} < b$$

by 4., entailing that

$$\|A^{-1}.(f_q(z) - f_q(y))\| = \left\| \int_0^1 A^{-1}.f_q'(y + t(z - y)).(z - y)dt \right\| < bs.$$

Thus $f_q(z) \in f_q(y) + A.B_{bs}(0)$, verifying the second inclusion in (3).

To tackle the first inclusion in (3), let y, s, q be as before; we want to apply [32, La. I.5.4] (with ρ, s' as below playing the role of r, s) to the FC^1-map

$$g\colon B_s(0) \to E, \quad g(v) := f_q'(y)^{-1}(f_q(y + v) - f_q(y)).$$

Let $z \in B_{as}(0)$. Then $z' := f_q'(y)^{-1}.A.z \in B_{as/\sqrt{a}}(0) = B_{s\sqrt{a}}(0)$, by 4. If we can find $w \in B_s(0)$ such that $g(w) = z'$, then $y + w \in B_s(y)$ and

$$f_q(y + w) = f_q'(y).g(w) + f_q(y) = f_q(y) + A.z,$$

showing that $f_q(y) + A.z \in f_q(B_s(y))$, as desired. Now, since $\|z'\| < s\sqrt{a}$, we find $\rho \in]\|z'\|/\sqrt{a}, s[$. Then $\|z'\| < \rho\sqrt{a} = (1 - s')\rho$ with $s' := 1 - \sqrt{a}$. As $\rho < s$, we have $\overline{B}_\rho(0) \subseteq B_s(0)$. Also

$$\|g'(v_2) - g'(v_1)\| = \|f_q'(y)^{-1}.(f_q'(y + v_2) - f_q'(y + v_1))\| \leq 1 - \sqrt{a} = s'$$

for all $v_1, v_2 \in \overline{B}_\rho(0)$, by 3. Hence [32, La. I.5.4] provides $w \in \overline{B}_\rho(0) \subseteq B_s(0)$ such that $g(w) = z'$, as required. Thus, the first inclusion in (3) holds.

(c) Let $q \in Q$, $y \in B$, and $\varepsilon \in]0, r - \|y - x\|]$. There is an open neighbourhood $V \subseteq Q$ of q such that $f(q, y) - f(q', y) \in A.B_{a\varepsilon/2}(0)$ for all $q' \in V$. Given $q' \in V$ and $z \in f_q(y) + A.B_{a\varepsilon/2}(0) \subseteq f_{q'}(y) + A.B_{a\varepsilon}(0)$, by (b) there exists

$w \in B_\varepsilon(y) \subseteq B$ such that $\phi_{q'}(w) = f_{q'}(w) = z$, whence $(q', z) \in W$ and $w = \phi_{q'}^{-1}(z) \in B_\varepsilon(y)$ (which will be useful later). Thus $Y := V \times (f_q(y) + A.B_{a\varepsilon/2}(0))$ is an open neighbourhood of $(q, f_q(y))$, contained in W. We conclude that W is open. Furthermore, for all (q', z) in the open neighbourhood Y of $(q, f_q(y))$, we have $\psi(q', z) = \phi_{q'}^{-1}(z) \in B_\varepsilon(y) = B_\varepsilon(\psi(q, f_q(y)))$. The continuity of ψ follows. Now, apparently θ is continuous and is a bijection whose inverse has the asserted form. Hence, the map ψ being continuous, also θ^{-1} is continuous.

(d) Let $q \in Q$. Using (b) with $y := x$, $s := r$ gives $f_q(x) + A.B_{ar}(0) \subseteq f_q(B)$. Hence $f_p(x) + A.B_\delta(0) = f_p(x) + A.B_{ar/2}(0) = (f_p(x) - f_q(x)) + f_q(x) + A.B_{ar/2}(0) \subseteq A.B_{ar/2}(0) + f_q(x) + A.B_{ar/2}(0) = f_q(x) + A.B_{ar}(0) \subseteq f_q(B)$, exploiting (4) for the first inclusion. Therefore $\{q\} \times (f_p(x) + A.B_\delta(0)) \subseteq W$. The final conclusion is clear; we have $\beta(q) = \psi(q, f(p,x)) = \phi_q^{-1}(f(p,x))$. ∎

Lemma 2.2: *Let P be an open subset of a real topological vector space Z, E be a real Banach space, $U \subseteq E$ be open, $f\colon P \times U \to E$ be a map, and $k \in \mathbb{N} \cup \{\infty\}$. If f is of class C^{k+1} or if E is finite-dimensional and f is of class C^k, then hypotheses* (i) *and* (ii) *of Proposition* 2.1 *are satisfied. Furthermore, the mapping*

$$h\colon P \times U \to L(E), \quad h(p,x) := f_p'(x) = df((p,x),(0,\bullet))$$

is of class C^{k-1}.

Proof: *If E is finite-dimensional* and f is of class C^k, then $f_p := f(p,\bullet)\colon U \to E$ is a C^k-map between open subsets of a finite-dimensional space, hence k times continuously partially differentiable in the traditional sense, and thus an FC^k-map, as is well-known. Let $e_1, \ldots, e_n$ be a basis of E. The mappings $P \times U \to E$, $(p,x) \mapsto f_p'(x).e_j = d(f_p)(x, e_j) = df((p,x),(0,e_j))$ being of class C^{k-1} for $j = 1, \ldots, n$, we readily deduce that $P \times U \to L(E) \cong M_n(\mathbb{R})$, $(p,x) \mapsto f_p'(x)$ is C^{k-1} and hence continuous.

If E is infinite-dimensional and f is of class C^{k+1}, then, for each $p \in P$, $f_p := f(p,\bullet)\colon U \to E$ is a C^{k+1}-map (in the Michal–Bastiani sense) between open subsets of Banach spaces and therefore an FC^k-map (see [29, Cor. 2.7.2 and p. 110], or, for a direct proof, [17, appendix]). The continuous linear map $\lambda\colon E \to Z \times E$, $\lambda(y) := (0,y)$ gives rise to a continuous linear (and hence smooth) map

$$L(\lambda, E)\colon L(Z \times E, E) \to L(E), \quad A \mapsto A \circ \lambda,$$

where $L(Z \times E, E)$ is equipped with the topology of uniform convergence on bounded sets. The mapping h can be written as the composition $h = L(\lambda, E) \circ f'$, where

$$f' \colon P \times U \to L(Z \times E, E), \quad f'(p,x) := df((p,x), \bullet)$$

is of class C^{k-1} by [17, Prop. 2.1] (which remains valid for non-locally convex domains, with identical proof). Hence h is C^{k-1} (and thus continuous). ∎

THEOREM 2.3 (Generalized Implicit Function Theorem): *Let $k \in \mathbb{N} \cup \{\infty\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, Z be a topological $\mathbb{K}$-vector space, $P \subseteq Z$ an open subset, E a Banach space over $\mathbb{K}$, $U \subseteq E$ an open subset, and $f \colon P \times U \to E$ a map.*

We consider two situations:

(i) *$\mathbb{K} = \mathbb{R}$ and f is of class $C^{k+1}_{\mathbb{R}}$; or $\mathbb{K} = \mathbb{R}$, E is finite-dimensional, and f is of class $C^k_{\mathbb{R}}$; respectively:*

(ii) *f is $\mathbb{K}$-analytic.*

Let $(p,x) \in P \times U$, and suppose that $A := f'_p(x) \in \mathrm{GL}(E)$, where $f_p := f(p, \bullet)$. Furthermore, let $0 < a < 1 < b$ be given. Then there exists an open neighbourhood $Q \subseteq P$ of p and $r > 0$ such that $B := B_r(x) \subseteq U$ and the following holds:

(a) *$f_q(B)$ is open in E, for each $q \in Q$, and $\phi_q \colon B \to f_q(B)$, $\phi_q(y) := f_q(y) = f(q,y)$ is an invertible FC^k-map (resp., $\mathbb{K}$-analytic map), whose inverse $(\phi_q)^{-1} \colon f_q(B) \to B$ is of class FC^k (resp., $\mathbb{K}$-analytic).*

(b) *For all $q \in Q$, $y \in B$, and $s \in]0, r - \|y - x\|]$, we have*

$$f_q(y) + A.B_{as}(0) \subseteq f_q(B_s(y)) \subseteq f_q(y) + A.B_{bs}(0).$$

(c) *$W := \bigcup_{q \in Q}(\{q\} \times f_q(B))$ is open in $Z \times E$, and the map $\psi \colon W \to B$, $\psi(q,v) := \phi_q^{-1}(v)$ is of class $C^k_{\mathbb{R}}$ (resp., $\mathbb{K}$-analytic). Furthermore, the map $\theta \colon Q \times B \to W$, $\theta(q,y) := (q, f(q,y))$ is a $C^k_{\mathbb{R}}$-diffeomorphism (resp., a $\mathbb{K}$-analytic diffeomorphism), with inverse given by $\theta^{-1}(q,v) = (q, \psi(q,v))$.*

(d) *$Q \times (f_p(x) + A.B_\delta(0)) \subseteq W$ for some $\delta > 0$.*

In particular, for each $q \in Q$ there is a unique element $\beta(q) \in B$ such that $f(q, \beta(q)) = f(p,x)$, and the mapping $\beta \colon Q \to B$ so obtained is of class $C^k_{\mathbb{R}}$ (resp., $\mathbb{K}$-analytic).

Proof: Let f, p, x, a, and b be given as described in the theorem. Then hypotheses (i) and (ii) of Proposition 2.1 are satisfied, by Lemma 2.2. We let Q, r, B, W, ψ, θ, and δ be as described in Proposition 2.1. Then (b) and (d) of the theorem hold by Proposition 2.1 (b) and (d), and in view of Part(a) of the proposition, apparently Part(a) of the theorem will hold if we can establish(c). We already know that W is open, and we know that ψ is continuous.

Let us assume first that we are in the situation of (i), and prove by induction on $j \in \mathbb{N}$, $j \leq k$ that ψ is of class $C^j_{\mathbb{R}}$. If $j = 1$, let $(q, v) \in W$ and $(q_1, v_1) \in Z \times E$. Set $y := \phi_q^{-1}(v)$. There exists $\tau > 0$ such that $(q, v) +]-\tau, \tau[(q_1, v_1) \subseteq W$. Define $g\colon]-\tau, \tau[\times U \to \mathbb{R} \times E$ via $g(t, u) := (t, f(q + tq_1, u))$. Then g is a mapping between open subsets of the Banach space $\mathbb{R} \times E$ which is of class $C^k_{\mathbb{R}}$ if E (and thus $\mathbb{R} \times E$) is finite-dimensional, otherwise of class $C^{k+1}_{\mathbb{R}}$. In any case, g is an FC^k-map and thus an FC^1-map, and apparently $g'(0, y)$ is invertible, as it can be considered as a lower triangular 2×2 block matrix with invertible diagonal entries $\mathrm{id}_{\mathbb{R}}$ and $f_q'(y)$. By the classical Inverse Function Theorem for Banach spaces ([32], Theorem I.5.2), we find $0 < \sigma \leq \min\{\tau, r - \|y - x\|\}$ such that g restricts to an FC^1-diffeomorphism h from $]-\sigma, \sigma[\times B_\sigma(y)$ onto an open subset $S \subseteq \mathbb{R} \times E$. There is $0 < \kappa < \sigma$ and an open neighbourhood $V \subseteq E$ of v such that $]-\kappa, \kappa[\times V \subseteq S$. Then apparently $h^{-1}(t, w) = (t, \phi^{-1}_{q+tq_1}(w)) = (t, \psi(q + tq_1, w))$ for all $(t, w) \in]-\kappa, \kappa[\times V$, entailing that $]-\kappa, \kappa[\times V \to E$, $(t, w) \mapsto \psi(q + tq_1, w)$ is an FC^1-map. After shrinking κ, we may assume that $v +]-\kappa, \kappa[v_1 \subseteq V$. Then, by the preceding, $c\colon]-\kappa, \kappa[\to E$, $t \mapsto \psi(q + tq_1, v + tv_1)$ is of class $C^1_{\mathbb{R}}$, and thus $d\psi((q, v), (q_1, v_1)) = c'(0)$ exists. Since $f(q + tq_1, c(t)) = v + tv_1$, the Chain Rule and the Rule on Partial Derivatives give

$$\begin{aligned} v_1 &= \frac{d}{dt}\Big|_{t=0} f(q + tq_1, c(t)) \\ &= d_1 f(q, \psi(q, v); q_1) + d_2 f(q, \psi(q, v); d\psi((q, v), (q_1, v_1))). \end{aligned}$$

Now $d_2 f(q, \psi(q, v); \bullet) = f_q'(\psi(q, v))$ is invertible by Proposition 2.1(a). Thus

$$\begin{aligned} d\psi((q, v), (q_1, v_1)) &= f_q'(\psi(q, v))^{-1}.(v_1 - d_1 f(q, \psi(q, v); q_1)) \\ &= \varepsilon(f_q'(\psi(q, v))^{-1}, v_1 - d_1 f(q, \psi(q, v); q_1)) \end{aligned} \tag{5}$$

for all $(q, v) \in W$ and $(q_1, v_1) \in Z \times E$, where $\varepsilon\colon L(E) \times E \to E$ is the bilinear evaluation map, which is continuous since E is normed. Now ε, ψ, $d_1 f$, inversion $\iota\colon \mathrm{GL}(E) \to \mathrm{GL}(E)$ and the mapping

$$P \times U \to L(E), \quad h(s, u) := f_s'(u) \tag{6}$$

being continuous, we deduce from (5) that $d\psi\colon W \times Z \times E \to E$ is continuous, whence ψ is of class $C^1_{\mathbb{R}}$. Similarly, if $1 < j < k$ and ψ is of class $C^j_{\mathbb{R}}$ by induction hypothesis, using that ε (being continuous bilinear), ι (cf. [16]) and the map in (6) (by Lemma 2.2) are of class $C^j_{\mathbb{R}}$, we deduce from (5) and the Chain Rule that $d\psi$ is of class $C^j_{\mathbb{R}}$. Thus ψ is $C^1_{\mathbb{R}}$ with $d\psi$ of class $C^j_{\mathbb{R}}$, and hence ψ is of class $C^{j+1}_{\mathbb{R}}$ (cf. [12]). Thus, the assertions of the theorem hold in situation (i).

We may pass to the complex analytic and real analytic cases using standard ideas from the finite-dimensional case (cf. [9, (10.2.4)]). Indeed, if $\mathbb{K} = \mathbb{C}$ and f is complex analytic, then f is of class $C^\infty_{\mathbb{R}}$ in particular, and thus ψ is of class $C^\infty_{\mathbb{R}}$, by what has already been shown. Equation(5) shows that $d\psi(w, \bullet)\colon Z \times E \to Z \times E$ is complex linear, for all $w \in W$. With Lemma 1.20, we deduce that ψ is complex analytic. Finally, assume that $\mathbb{K} = \mathbb{R}$ and assume that f is real analytic. Equip $E_{\mathbb{C}} \cong E \times E$ with the maximum norm. Given $(q, z) \in W$, set $y := \psi(q, z)$. There is a complex analytic function $F\colon Y \to E_{\mathbb{C}}$, defined on an open neighbourhood Y of $P \times U$ in $Z_{\mathbb{C}} \times E_{\mathbb{C}}$, such that $F|_{P\times U} = f$. Then $d_2F(q, y; \bullet) = f'_q(y)_{\mathbb{C}}$ being invertible, by the complex analytic case of the theorem just established, there exist open neighbourhoods $P_1 \subseteq Z_{\mathbb{C}}$ of q and $r_1 > 0$ such that $P_1 \times B_1 \subseteq Y$ with $B_1 := B^{E_{\mathbb{C}}}_{r_1}(y)$, and such that $F(p_1, \bullet)|_{B_1}$ is a complex analytic diffeomorphism onto an open set, for each $p_1 \in P_1$, and also $W_1 := \bigcup_{p_1 \in P_1} \{p_1\} \times F(\{p_1\} \times B_1)$ is open in $Z_{\mathbb{C}} \times E_{\mathbb{C}}$, and $\psi_1\colon W_1 \to E_{\mathbb{C}}$, $\psi_1(p_1, v_1) := (F(p_1, \bullet)|_{B_1})^{-1}(v_1)$ is complex analytic. Then $Q \cap P_1$ and $B \cap B_1$ are open neighbourhoods of q and y in Z and E, respectively. The map θ being a homeomorphism onto the open set W, we deduce that

$$W_2 := \bigcup_{s \in Q \cap P_1} (\{s\} \times f_s(B \cap B_1)) = \theta((Q \cap P_1) \times (B \cap B_1))$$

is open in $Z \times E$. Then $\psi_1\colon W_1 \to E_{\mathbb{C}}$ is a complex analytic map on an open neighbourhood of W_2 in $Z_{\mathbb{C}} \times E_{\mathbb{C}}$ which extends $\psi|_{W_2}$, and thus $\psi|_{W_2}$ is real analytic. Being real analytic locally by the preceding, ψ is real analytic (Remark 1.18). ∎

3. Strict differentiability

We now leave the framework of real and complex analysis and turn to differential calculus over arbitrary valued fields. In order to be able to prove implicit function theorems, we require a differentiability property which is stronger than being C^1, namely "strict differentiability." In this section, we recall the definition of strictly differentiable mappings from open subsets of normed $\mathbb{K}$-vector spaces to polynormed $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is a valued field. We show that every strictly differentiable mapping is of class C^1, and we show that, conversely, every mapping of class C^2 from an open subset of a normed space to a polynormed $\mathbb{K}$-vector space is strictly differentiable.

Definition 3.1: Let $\mathbb{K}$ be a valued field, E be a normed $\mathbb{K}$-vector space, F a polynormed $\mathbb{K}$-vector space, $U \subseteq E$ be open, and $f\colon U \to F$ be a map. Given $x \in U$, we say that f is **strictly differentiable at** x if there exists a continuous linear map $f'(x) \in L(E,F)$ such that, for every $\varepsilon > 0$ and continuous seminorm γ on F, there exists $\delta > 0$ such that

$$\|f(z) - f(y) - f'(x).(z-y)\|_\gamma < \varepsilon \|z-y\|$$

for all $y, z \in U$ such that $\|z-x\| < \delta$ and $\|y-x\| < \delta$. The map f is called **strictly differentiable** if it is strictly differentiable at each $x \in U$.

It is clear that $f'(x)$ is uniquely determined in the preceding situation.

If E is a normed vector space over a valued field $\mathbb{K}$, and F a polynormed $\mathbb{K}$-vector space, we equip the space $L(E,F)$ of continuous $\mathbb{K}$-linear mappings $E \to F$ with the topology of uniform convergence on bounded subsets of E. This topology makes $L(E,F)$ a polynormed $\mathbb{K}$-vector space, whose vector topology arises from the family of continuous seminorms

$$\|A\|_\gamma := \sup\{\|A.v\|_\gamma \cdot \|v\|^{-1} \colon 0 \neq v \in E\} \in [0,\infty[$$

(cf. [42], p. 59), where γ ranges through the continuous seminorms on F. If also F is normed, with norm γ, then $L(E,F)$ is normable; its vector topology arises from the operator norm $\|.\| := \|.\|_\gamma$.

LEMMA 3.2: *Let $\mathbb{K}$ be a valued field, E be a normed $\mathbb{K}$-vector space, F a polynormed $\mathbb{K}$-vector space, $U \subseteq E$ be open, and $f\colon U \to F$ be a strictly differentiable map. Then f is of class C^1, we have $f'(x) = df(x,\bullet)$ for all $x \in U$, and the mapping*

$$f'\colon U \to L(E,F), \quad x \mapsto f'(x)$$

is continuous.

Proof: Directional derivatives. Given $x \in U$ and $y \in E$, let us show that the directional derivative $df(x,y)$ exists, and is given by $f'(x).y$. For $y = 0$ this is trivial. If $0 \neq y \in E$, there exists $r > 0$ such that $x + ty \in U$ for all $0 \neq t \in B_r(0) \subseteq \mathbb{K}$. By strict differentiability of f in x, for every continuous seminorm γ on F we have

$$\left\|\frac{1}{t}(f(x+ty) - f(x)) - f'(x)y\right\|_\gamma = \|y\| \cdot \frac{\|f(x+ty) - f(x) - f'(x)ty\|_\gamma}{\|ty\|} \to 0$$

as $t \to 0$, showing that $df(x,y) := \lim_{0\neq t\to 0} \frac{1}{t}(f(x+ty) - f(x)) = f'(x).y$ indeed.

f' is continuous. In fact, given $\varepsilon > 0$ and a continuous seminorm γ on F, we find $\delta > 0$ such that $\|f(z) - f(y) - f'(x).(z-y)\|_\gamma \leq \varepsilon\|z-y\|$ for all $y, z \in U$ such that $\|y-x\| < \delta$ and $\|z-x\| < \delta$. Let $y \in U$ such that $\|y-x\| < \delta$. Then, given $0 \neq u \in E$ we have $y + tu \in U$ and $\|y+tu-x\| < \delta$ for all $t \in \mathbb{K}^\times$ sufficiently close to 0, entailing that

$$\begin{aligned}\|(f'(y) - f'(x)).u\|_\gamma &= \lim_{t\to 0}\left\|\frac{1}{t}(f(y+tu) - f(y)) - f'(x).u\right\|_\gamma \\ &= \|u\| \cdot \lim_{t\to 0}\frac{\|f(y+tu) - f(y) - f'(x).(tu)\|_\gamma}{\|tu\|} \leq \|u\| \cdot \varepsilon.\end{aligned}$$

As a consequence, $\|f'(y) - f'(x)\|_\gamma \leq \varepsilon$ for all $y \in U$ such that $\|y-x\| < \delta$, showing that $f'\colon U \to L(E,F)$ is continuous.

f is of class $C^1_{\mathbb{K}}$. Note first that f is continuous. In fact, given $x \in U$, $\varepsilon > 0$, and a continuous seminorm γ on F, we find $\delta > 0$ as in Definition 3.1. Pick $0 < \rho \leq \delta$ such that $\rho \cdot (\varepsilon + \|f'(x)\|_\gamma) < \varepsilon$. Then, for all $y \in B_\rho(x) \cap U$, we estimate

$$\begin{aligned}\|f(y) - f(x)\|_\gamma &\leq \|f(y) - f(x) - f'(x).(y-x)\|_\gamma + \|f'(x).(y-x)\|_\gamma \\ &\leq \varepsilon \cdot \|y-x\| + \|f'(x)\|_\gamma \cdot \|y-x\| < \varepsilon.\end{aligned}$$

We deduce that f is continuous.

Next, let $W := \{(x,y,t) \in U^{[1]}\colon t \neq 0\}$. Define $g\colon U^{[1]} \to F$ via $g(x,y,t) := \frac{1}{t}(f(x+ty) - f(x))$ for $(x,y,t) \in W$, $g(x,y,0) := f'(x).y$ for $(x,y) \in U \times E$. Then $g|_W$ is continuous since f is continuous, and $g|_{U\times E\times\{0\}}\colon U \times E \times \{0\} \to F$ is continuous since f' and the evaluation map $L(E,F) \times E \to F$ are continuous (the space E being normed). Hence g will be continuous if we can show that $g(x_\alpha, y_\alpha, t_\alpha) \to g(x,y,0)$, for every net $((x_\alpha, y_\alpha, t_\alpha))_{\alpha\in I}$ in W which converges to some $(x,y,0) \in U^{[1]}$. Since $\|y_\alpha\| \leq \|y\| + 1$ eventually, we may assume without loss of generality that $\|y_\alpha\| \leq \|y\| + 1$ for all α. If $y \neq 0$, then $y_\alpha \neq 0$ eventually, whence we may furthermore assume in this case that $y_\alpha \neq 0$ for all α. Then, for a given α, we either have $y_\alpha = 0$ (in which case $y = 0$ by the preceding): then

$$\|g(x_\alpha, y_\alpha, t_\alpha) - g(x,y,0)\| = \|0 - 0\| = 0;$$

or we have $y_\alpha \neq 0$, in which case

$$\begin{aligned}
&\|g(x_\alpha, y_\alpha, t_\alpha) - g(x, y, 0)\|_\gamma \\
&= \left\| \frac{1}{t_\alpha}(f(x_\alpha + t_\alpha y_\alpha) - f(x_\alpha)) - f'(x).y \right\|_\gamma \\
&\leq \|y_\alpha\| \cdot \frac{\|f(x_\alpha + t_\alpha y_\alpha) - f(x_\alpha) - f'(x).t_\alpha y_\alpha\|_\gamma}{|t_\alpha| \cdot \|y_\alpha\|} + \|f'(x).(y_\alpha - y)\|_\gamma \\
&\leq (\|y\| + 1) \cdot \frac{\|f(x_\alpha + t_\alpha y_\alpha) - f(x_\alpha) - f'(x).t_\alpha y_\alpha\|_\gamma}{\|t_\alpha y_\alpha\|} + \|f'(x)\|_\gamma \cdot \|y_\alpha - y\|,
\end{aligned}$$

where the first term tends to 0 as α increases since f is strictly differentiable at x, and the second term tends to 0 for trivial reasons. ∎

We want to show that every C^2-map is strictly differentiable. The proof hinges on symmetry properties of $f^{[1]}$ and $f^{[2]}$, as described in the following lemma:

LEMMA 3.3: *Let E and F be topological vector spaces over a non-discrete topological field $\mathbb{K}$, and $f: U \to F$ be a map, defined on an open subset of E.*

(a) *If f is C^1, $t \in \mathbb{K}^\times$, and $(x, y, s) \in E \times E \times \mathbb{K}$ such that $(x, y, ts) \in U^{[1]}$, then also $(x, ty, s) \in U^{[1]}$, and*

$$tf^{[1]}(x, y, ts) = f^{[1]}(x, ty, s). \tag{7}$$

(b) *If f is C^2, $t \in \mathbb{K}^\times$, $x, x_1, y, y_1 \in E$ and $s, s_1, s_2 \in \mathbb{K}$ such that*

$$((x, y, ts), (x_1, y_1, ts_1), ts_2) \in U^{[2]},$$

then also $((x, t^2y, s/t), (tx_1, t^3y_1, s_1), s_2) \in U^{[2]}$, and

$$t^3 f^{[2]}((x, y, ts), (x_1, y_1, ts_1), ts_2) = f^{[2]}((x, t^2y, s/t), (tx_1, t^3y_1, s_1), s_2). \tag{8}$$

Proof: (a) Since $x + (ts)y = x + s(ty)$, it is obvious that $(x, ty, s) \in U^{[1]}$ if and only if $(x, y, ts) \in U^{[1]}$. In this case, we have

$$tf^{[1]}(x, y, ts) = \frac{1}{s}(f(x + tsy) - f(x)) = f^{[1]}(x, ty, s)$$

provided $s \neq 0$; if $s = 0$, then $f^{[1]}(x, ty, s) = f^{[1]}(x, ty, 0) = df(x, ty) = tdf(x, y) = tf^{[1]}(x, y, 0) = tf^{[1]}(x, y, ts)$. Thus (7) is established.

(b) Let $t \in \mathbb{K}^\times$, $x, y, x_1, y_1 \in E$, and $s, s_1, s_2 \in \mathbb{K}$ such that

$$((x, y, ts), (x_1, y_1, ts_1), ts_2) \in U^{[2]}.$$

Then $(x, y, ts) \in U^{[1]}$ and hence also $(x, t^2y, s/t) \in U^{[1]}$, by Part (a). If $s_2 = 0$, this entails that $((x, t^2y, s/t), (tx_1, t^3y_1, s_1), s_2) \in U^{[2]}$. If $s_2 \neq 0$, we calculate:

$$\begin{aligned}
f^{[2]}&((x,y,ts),(x_1,y_1,ts_1),ts_2)\\
&= (f^{[1]})^{[1]}((x,y,ts),(x_1,y_1,ts_1),ts_2)\\
&= \frac{1}{ts_2}(f^{[1]}(x+ts_2x_1, y+ts_2y_1, \underbrace{ts+t^2s_1s_2}_{=t^2(s/t+s_1s_2)}) - f^{[1]}(x,y,\underbrace{ts}_{=t^2\cdot s/t}))\\
&= \frac{1}{t^3s_2}(f^{[1]}(x+ts_2x_1, t^2y+t^3s_2y_1, s/t+s_1s_2) - f^{[1]}(x,t^2y,s/t))\\
&= \frac{1}{t^3}f^{[2]}((x,t^2y,s/t),(tx_1,t^3y_1,s_1),s_2),
\end{aligned}$$

showing that $((x, t^2y, s/t), (tx_1, t^3y_1, s_1), s_2) \in U^{[2]}$ and (8) holds, when $s_2 \neq 0$. Here, we used (a) to pass to the third line. Letting $s_2 \neq 0$ tend to 0, by the continuity of the functions involved (8) remains valid for $s_2 = 0$. ∎

Cf. also [16, La. 6.8] for a (less explicit) result concerning $f^{[k]}$ for arbitrary k.

We are now in the position to prove:

PROPOSITION 3.4: *Let $\mathbb{K}$ be a valued field, E be a normed $\mathbb{K}$-vector space, F a polynormed $\mathbb{K}$-vector space, $U \subseteq E$ be open, and $f\colon U \to F$ be a mapping of class C^2. Then f is strictly differentiable.*

Proof: Given $x \in U$, let us show that f is strictly differentiable at x. For all $y, z \in U$, we have

(9)
$$\begin{aligned}
f(z)-&f(y) - f'(x).(z-y)\\
&= f(z) - f(y) - f'(y).(z-y) + f'(y).(z-y) - f'(x).(z-y)\\
&= f^{[1]}(y, z-y, 1) - f^{[1]}(y, z-y, 0) + f^{[1]}(y, z-y, 0) - f^{[1]}(x, z-y, 0)\\
&= f^{[2]}((y, z-y, 0), (0,0,1), 1) + f^{[2]}((x, z-y, 0), (y-x, 0, 0), 1).
\end{aligned}$$

Let us have a closer look at the individual terms. For each $t \in \mathbb{K}^\times$, we have

(10)
$$\begin{aligned}
f^{[2]}((y, z-y, 0), (0,0,1), 1) &= f^{[2]}((y, t^2\cdot \frac{1}{t^2}(z-y), \frac{1}{t}\cdot 0), (t\cdot 0, t^3\cdot 0, 1), 1)\\
&= t^3 f^{[2]}((y, \frac{1}{t^2}(z-y), 0), (0,0,t), t),
\end{aligned}$$

by Lemma 3.3(b). The map $f^{[2]}((x, \bullet, 0), (y-x, 0, 0), 1) = f'(y) - f'(x)\colon E \to F$ is linear. Furthermore, we have

(11) $$f^{[2]}((x, z-y, 0), (y-x, 0, 0), 1) = sf^{[2]}\Big((x, z-y, 0), \Big(\frac{1}{s}(y-x), 0, 0\Big), s\Big)$$

for all $s \in \mathbb{K}^\times$, by Lemma 3.3(a). Let $\varepsilon > 0$ and γ be a continuous seminorm on F. Since $f^{[2]}((x,0,0),(0,0,0),0) = d^2 f(x,0,0) = 0$, in view of the continuity of $f^{[2]}$ and openness of $U^{[2]}$, there is $\delta > 0$ such that $((u,v,0),(w,0,a),b) \in U^{[2]}$ and

$$\begin{gathered}\|f^{[2]}((u,v,0),(w,0,a),b)\|_\gamma < \varepsilon \\ \text{for } u \in B_\delta^E(x), \quad v,w \in B_\delta^E(0), \quad a,b \in B_\delta^{\mathbb{K}}(0).\end{gathered} \tag{12}$$

We may assume that $\delta \leq 1$. Pick $\rho \in \mathbb{K}^\times$ such that $|\rho| < 1$, and $s \in \mathbb{K}^\times$ such that $|s| \leq \delta|\rho|^2/2$. Define $r := \min\{|s|\delta, \frac{1}{8}\delta^3|\rho|^6\}$. Let $y, z \in B_r^E(x) \subseteq U$ such that $y \neq z$. Then there is a unique integer $k \in \mathbb{Z}$ such that

$$|\rho|^{k+1} \leq \sqrt{\frac{\|z-y\|}{\delta}} < |\rho|^k.$$

Set $t := \rho^k$. Then $\|t^{-2}(z-y)\| = |\rho|^{-2k}\|z-y\| < \delta$,

$$|t| = |\rho|^k \leq \frac{1}{|\rho|} \cdot \sqrt{\frac{\|z-y\|}{\delta}} \leq \frac{1}{|\rho|} \cdot \sqrt{\frac{2r}{\delta}} \leq \frac{1}{|\rho|}\sqrt{\frac{1}{4}\frac{\delta^3|\rho|^6}{\delta}} = \frac{\delta|\rho|^2}{2} < \delta,$$

and

$$\begin{aligned}\frac{|t|^3}{\|z-y\|} &= |\rho|^{k-2}\frac{|\rho|^{2k+2}}{\|z-y\|} \leq \frac{|\rho|^{k-2}}{\delta} \leq \frac{1}{\delta|\rho|^3}\sqrt{\frac{\|z-y\|}{\delta}} \leq \frac{1}{\delta|\rho|^3}\sqrt{\frac{2r}{\delta}} \\ &\leq \frac{1}{\delta|\rho|^3}\sqrt{\frac{1}{4}\delta^2|\rho|^6} = \frac{1}{2}.\end{aligned}$$

Hence, using (10) and (12) (with $u := y$),

$$\begin{aligned}\frac{\|f^{[2]}((y,z-y,0),(0,0,1),1)\|_\gamma}{\|z-y\|} &= \frac{|t|^3}{\|z-y\|}\|f^{[2]}((y,\frac{1}{t^2}(z-y),0),(0,0,t),t)\|_\gamma \\ &< \varepsilon/2.\end{aligned} \tag{13}$$

Using (11) with s as just chosen, we obtain

$$\begin{aligned}&\frac{\|f^{[2]}((x,z-y,0),(y-x,0,0),1)\|_\gamma}{\|z-y\|} \\ &= \frac{|s|}{\|z-y\|}\|f^{[2]}((x,z-y,0),(\frac{1}{s}(y-x),0,0),s)\|_\gamma \\ &= \frac{|s|\cdot|\rho|^{2k}}{\|z-y\|}\|f^{[2]}((x,\rho^{-2k}(z-y),0),(\frac{1}{s}(y-x),0,0),s)\|_\gamma < \varepsilon/2.\end{aligned} \tag{14}$$

Indeed, $|s| < \delta$ by choice of s, $|s|^{-1}\|y-x\| < |s|^{-1}r \leq \delta$, and $|\rho|^{-2k}\|z-y\| < \delta$, whence $\|f^{[2]}((x,\rho^{-2k}(z-y),0),(\frac{1}{s}(y-x),0,0),s)\|_\gamma < \varepsilon$ holds, by (12). Also

$|s||\rho|^{2k}\|z-y\|^{-1} \le |s|/\delta|\rho|^2 \le \frac{1}{2}$. By (9), (13) and (14), we have

$$\frac{\|f(z)-f(y)-f'(x).(z-y)\|_\gamma}{\|z-y\|} < \varepsilon$$

for all $z \neq y \in B_r(x) \subseteq U$. Thus f is strictly differentiable at x. ∎

A variant of Proposition 3.4 involving parameters will be needed later:

LEMMA 3.5: *Let $(\mathbb{K}, |.|)$ be a valued field, E be a normed $\mathbb{K}$-vector space, $U \subseteq E$ an open subset, F be a polynormed $\mathbb{K}$-vector space, Z a topological $\mathbb{K}$-vector space, and $P \subseteq Z$ be open. Let $f: P \times U \to F$ be a $C^2_{\mathbb{K}}$-map. Then the map*

$$(15) \qquad h: P \times U \to L(E,F), \quad (p,x) \mapsto f'_p(x)$$

is continuous, where $f_p := f(p, \bullet)$ and $f'_p(x) := d(f_p)(x, \bullet)$.

For all $p \in P$, $x \in U$, $\varepsilon > 0$ and continuous seminorm γ on F, there exists a neighbourhood Q of p in P and $r > 0$ such that

$$(16) \qquad \frac{\|f_q(z)-f_q(y)-f'_q(x).(z-y)\|_\gamma}{\|z-y\|} < \varepsilon \quad \text{for all } q \in Q \text{ and } y \neq z \in B_r(x) \cap U.$$

Proof: Let $p \in P$, $x \in U$, $\varepsilon > 0$, and γ be a continuous seminorm on F. Since

$$g: P \times U^{[2]} \to F, \quad (q,z) \mapsto (f_q)^{[2]}(z)$$

is continuous as a partial map of $f^{[2]}$ and $g(p,(x,0,0),(0,0,0),0) = d^2(f_p)(x,0,0) = 0$, there exists $\delta > 0$ and an open neighbourhood Q of p in P such that $((u,v,0),(w,0,a),b) \in U^{[2]}$ and

$$\|(f_q)^{[2]}((u,v,0),(w,0,a),b)\|_\gamma = \|g(q,(u,v,0),(w,0,a),b)\|_\gamma < \varepsilon,$$

for all $q \in Q$, $u \in B^E_\delta(x)$, $v,w \in B^E_\delta(0)$, and $a,b \in B^{\mathbb{K}}_\delta(0)$. For each fixed $q \in Q$, replacing f with f_q in the preceding proof, we find r (independent of q) as described there and can repeat the estimates verbatim, to obtain (16).

To see that h is continuous, let $p \in P$, $x \in U$, $\varepsilon > 0$ and a continuous seminorm γ on F be given. Since $f^{[2]}(((p,x),(0,0),0),((0,0),(0,0),0),0) = d^2f((p,x),(0,0),(0,0)) = 0$ and $f^{[2]}$ is continuous, there is $\delta > 0$ and a balanced zero-neighbourhood $V \subseteq Z$ such that

$$\|f^{[2]}(((p,x),(0,u),0),((v,z),(0,0),0),t)\|_\gamma \le \varepsilon$$

for all $u,z \in B^E_\delta(0)$, $v \in V$, and $t \in B^{\mathbb{K}}_\delta(0)$. Pick $\rho \in \mathbb{K}^\times$ such that $|\rho| < 1$, and pick $t \in \mathbb{K}^\times$ such that $|t| \le \delta|\rho|$. Define $r := \delta|t|$. We claim that

$$(17) \qquad \|f'_q(y) - f'_p(x)\|_\gamma \le \varepsilon$$

for all $q \in P \cap (p + tV)$ and $y \in B_r(x) \cap U$. To see this, let $0 \neq u \in E$. There is a unique $k \in \mathbb{Z}$ such that $|\rho|^{k+1} \leq \|u\|/\delta < |\rho|^k$. Then, using Lemma 3.3(a) and linearity in u,

$$\begin{aligned}
&\frac{\|(f'_q(y) - f'_p(x)).u\|_\gamma}{\|u\|} \\
&= \frac{\|f^{[2]}(((p,x),(0,u),0),((q-p,y-x),(0,0),0),1)\|_\gamma}{\|u\|} \\
&= \frac{|t||\rho|^k}{\|u\|}\|f^{[2]}(((p,x),(0,\rho^{-k}u),0),((\frac{1}{t}(q-p),\frac{1}{t}(y-x)),(0,0),0),t)\|_\gamma \\
&\leq \varepsilon,
\end{aligned}$$

whence (17) holds. Hence h is continuous, which completes the proof. ∎

4. Uniform differentiability

For mappings on open subsets of finite-dimensional topological vector spaces over locally compact topological fields with values in polynormed spaces, the results of the preceding section can be strengthened substantially: such a mapping is of class C^1 if and only if it is strictly differentiable, if and only if it is "locally uniformly differentiable."

Definition 4.1: Suppose that $(\mathbb{K}, |.|)$ is a valued field, $(E, \|.\|)$ a normed $\mathbb{K}$-vector space, F a polynormed $\mathbb{K}$-vector space, $U \subseteq E$ an open subset, and $f\colon U \to F$ a map. Then f is called **uniformly differentiable** if there exists a function $f'\colon U \to L(E,F)$ such that, for every $\varepsilon > 0$ and continuous seminorm γ on F, there exists $\delta > 0$ with the following property: for all $x, y, z \in U$ such that $\|y - x\| < \delta$, $\|z - x\| < \delta$ and $y \neq z$, we have

$$\frac{\|f(z) - f(y) - f'(x).(z-y)\|_\gamma}{\|z - y\|} < \varepsilon.$$

We call f **locally uniformly differentiable** if every $x \in U$ has an open neighbourhood $V \subseteq U$ such that $f|_V$ is uniformly differentiable.

Remark 4.2: It is apparent from the definitions that every locally uniformly differentiable mapping is strictly differentiable.

Remark 4.3: Strengthening Lemma 3.2, it is easy to see that $f'\colon U \to L(E,F)$ is uniformly continuous, for every uniformly differentiable map $f\colon E \supseteq U \to F$. As we shall not need this fact, the simple proof is omitted.

LEMMA 4.4: *Let $\mathbb{K}$ be a locally compact, non-discrete topological field, and $|.|$ be an absolute value on $\mathbb{K}$ defining its topology. Let E be a finite-dimensional normed $\mathbb{K}$-vector space, F be a polynomed $\mathbb{K}$-vector space, $U \subseteq E$ be open, and $f\colon U \to F$ be a mapping of class C^1. Then f is locally uniformly differentiable. If, furthermore, U is compact, then f is uniformly differentiable.*

Proof: (cf. [43], Exercise 28E(ii) when $E = F = \mathbb{K}$). Pick $0 \neq \rho \in \mathbb{K}$ such that $|\rho| < 1$. Let $V \subseteq U$ be an open subset with compact closure $\overline{V} \subseteq U$. Define $f'\colon U \to L(E,F)$, $f'(x) := df(x, \bullet) = f^{[1]}(x, \bullet, 0)$. Given $\varepsilon > 0$ and a continuous seminorm γ on F, consider the continuous function

$$g\colon U^{[1]} \to F, \quad g(x,y,t) := f^{[1]}(x,y,t) - f^{[1]}(x,y,0).$$

Then $\overline{V} \times \overline{B^E_{1/|\rho|}(0)} \times \{0\} \subseteq U \times E \times \{0\} \subseteq U^{[1]}$ is a compact subset on which g vanishes identically. Using a compactness argument, we find $\sigma > 0$ such that $\overline{V} \times \overline{B^E_{1/|\rho|}(0)} \times B^{\mathbb{K}}_\sigma(0) \subseteq U^{[1]}$ and such that $\|g(x,y,t)\|_\gamma < \varepsilon/2$ for all $(x,y,t) \in \overline{V} \times \overline{B^E_{1/|\rho|}(0)} \times B^{\mathbb{K}}_\sigma(0)$. Let $e_1, \ldots, e_n$ be a basis of E, and $e_1^*, \ldots, e_n^* \in E'$ be its dual basis.

Given $A \in L(E,F)$, for each $0 \neq v \in E$ we have

$$\frac{\|A.v\|_\gamma}{\|v\|} = \frac{\|\sum_{i=1}^n e_i^*(v) A.e_i\|_\gamma}{\|v\|} \leq \sum_{i=1}^n \frac{|e_i^*(v)|}{\|v\|} \cdot \|A.e_i\|_\gamma \leq \sum_{i=1}^n \|e_i^*\| \cdot \|A.e_i\|_\gamma.$$

Thus

$$\|A\|_\gamma \leq \sum_{i=1}^n \|e_i^*\| \cdot \|A.e_i\|_\gamma \quad \text{for all } A \in L(E,F). \tag{18}$$

Let $i \in \{1, \ldots, n\}$. The mapping $\overline{V} \to F$, $x \mapsto df(x, e_i)$ being continuous and thus uniformly continuous, we find $\delta_i > 0$ such that

$$\|df(y, e_i) - df(x, e_i)\|_\gamma < \varepsilon/(2n\|e_i^*\|)$$

for all $x, y \in \overline{V}$ such that $\|x - y\| < \delta_i$. Define $\delta := \frac{1}{2}\min\{\sigma, \delta_1, \ldots, \delta_n\}$. By (18) and the choice of δ_i, we have $\|df(y, \bullet) - df(x, \bullet)\|_\gamma < \varepsilon/2$ for all $x, y \in \overline{V}$ such that $\|x - y\| < 2\delta$.

Let $x, y, z \in V$ be given such that $y \neq z$, $\|y - x\| < \delta$, and $\|z - x\| < \delta$. There exists $k \in \mathbb{Z}$ such that $|\rho|^{k+1} \leq \|z - y\| < |\rho|^k$. We set $s := \rho^{k+1}$. Then

$\|\frac{1}{s}(z-y)\| < 1/|\rho|$, $|s| = |\rho|^{k+1} \leq \|z-y\| < 2\delta \leq \sigma$, and $\|z-y\| < 2\delta$. Thus

$$\begin{aligned}
&\frac{\|f(z)-f(y)-f'(x).(z-y)\|_\gamma}{\|z-y\|} \\
&\leq \frac{\|f(z)-f(y)-f'(y).(z-y)\|_\gamma}{\|z-y\|} + \frac{\|(f'(y)-f'(x)).(z-y)\|_\gamma}{\|z-y\|} \\
&= \frac{|s|}{\|z-y\|} \cdot \left\| \frac{1}{s}(f(z)-f(y)) - f'(y).\frac{1}{s}(z-y) \right\|_\gamma + \frac{\varepsilon}{2} \\
&\leq \left\| f^{[1]}\Big(y, \frac{1}{s}(z-y), s\Big) - f^{[1]}\Big(y, \frac{1}{s}(z-y), 0\Big) \right\|_\gamma + \frac{\varepsilon}{2} \\
&= \left\| g\Big(y, \frac{1}{s}(z-y), s\Big) \right\|_\gamma + \frac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}$$

Hence $f|_V$ is uniformly differentiable. The first assertion readily follows. For the second assertion, choose $V := U$ in the first part of the proof. ∎

We also need a variant of Lemma 4.4 involving parameters.

LEMMA 4.5: *Let $\mathbb{K}$ be a locally compact, non-discrete topological field, and $|.|$ be an absolute value on $\mathbb{K}$ defining its topology. Let E be a finite-dimensional normed $\mathbb{K}$-vector space, $U \subseteq E$ be open, F be a polynormed $\mathbb{K}$-vector space, and P be a topological space. Let $f\colon P \times U \to F$ be a continuous mapping such that $f_p := f(p, \bullet)\colon U \to F$ is of class $C^1_{\mathbb{K}}$ for all $p \in P$, and such that the mapping*

$$P \times U^{[1]} \to F, \quad (p, y) \mapsto (f_p)^{[1]}(y)$$

is continuous. Let $p \in P$ and $u \in U$ be given. Then, for every $\varepsilon > 0$ and continuous seminorm γ on F, there is a neighbourhood Q of p in P and $\delta > 0$ such that

$$\frac{\|f_q(z) - f_q(y) - f_q'(u).(z-y)\|_\gamma}{\|z-y\|} < \varepsilon$$

for all $q \in Q$ and $y, z \in B_\delta(u) \cap U$ such that $y \neq z$, where $f_q'(u) := d(f_q)(u, \bullet)$.

Proof: Let $\varepsilon > 0$ and γ be given. Pick $0 \neq \rho \in \mathbb{K}$ such that $|\rho| < 1$. Let $V \subseteq U$ be an open neighbourhood of u with compact closure $\overline{V} \subseteq U$. Consider the continuous mapping

$$g\colon P \times U^{[1]} \to F, \quad g(q, x, y, t) := f_q^{[1]}(x, y, t) - f_q^{[1]}(x, y, 0).$$

Then $\{p\} \times \overline{V} \times \overline{B^E_{1/|\rho|}(0)} \times \{0\} \subseteq P \times U \times E \times \{0\} \subseteq P \times U^{[1]}$ is a compact subset on which g vanishes identically. Using a compactness argument, we find

$\sigma > 0$ and a neighbourhood P_0 of p in P such that $\overline{V} \times \overline{B^E_{1/|\rho|}(0)} \times B^{\mathbb{K}}_\sigma(0) \subseteq U^{[1]}$ and such that

$$\|g(q,x,y,t)\|_\gamma < \varepsilon/2 \quad \text{for all } (q,x,y,t) \in P_0 \times \overline{V} \times \overline{B^E_{1/|\rho|}(0)} \times B^{\mathbb{K}}_\sigma(0).$$

Let $e_1, \ldots, e_n$ be a basis of E, and $e_1^*, \ldots, e_n^*$ be its dual basis. Using the compactness of $\overline{V}$, we find a neighbourhood $Q \subseteq P_0$ of p and $\kappa > 0$ such that $\|df_q(z, e_i) - df_q(y, e_i)\|_\gamma < \varepsilon/2n\|e_i^*\|$ for all $q \in Q$, $i \in \{1, \ldots, n\}$, and all $y, z \in \overline{V}$ such that $\|z - y\| < \kappa$. Define $\delta := \min\{\sigma/2, \kappa/2\}$. Re-using the estimates from the proof of Lemma 4.4, we see that the assertion of the lemma is satisfied for Q and δ. ∎

5. Strict differentiability of higher order

Generalizing the standard notion of (once) strictly differentiable mappings, in this section we define and discuss k times strictly differentiable mappings on open subsets of normed vector spaces over valued fields.

Definition 5.1: Let $\mathbb{K}$ be a valued field, E be a normed $\mathbb{K}$-vector space, F be a polynormed $\mathbb{K}$-vector space, and $U \subseteq E$ be an open subset. A function $f\colon U \to F$ is called an $\boldsymbol{SC^0}$**-map** if it is continuous; it is called an $\boldsymbol{SC^1}$**-map** if it is strictly differentiable (and hence C^1 in particular). Inductively, having defined SC^k-maps for some $k \in \mathbb{N}$ (which are C^k in particular), we call f an $\boldsymbol{SC^{k+1}}$**-map** if it is an SC^k-map and the map $f^{[k]}\colon U^{[k]} \to F$ is SC^1, where $E^{[k]}$ is equipped with the maximum norm. The map f is $\boldsymbol{SC^\infty}$ if it is an SC^k-map for all $k \in \mathbb{N}_0$.

Remark 5.2: In other words, f is SC^k if and only if f is C^k and $f^{[j]}\colon U^{[j]} \to F$ is strictly differentiable for all $j \in \mathbb{N}_0$ such that $j < k$. It follows from this and 1.4 that f is SC^k if and only if f is SC^1 and $f^{[1]}$ is SC^{k-1}.

Remark 5.3: If $f\colon E \supseteq U \to F$ is of class C^{k+1} in the preceding situation, then f is an SC^k-map. In fact, for every $j \in \mathbb{N}_0$ such that $j < k$, the map $f^{[j]}$ is of class C^{k+1-j}, where $k + 1 - j \geq 2$, and hence strictly differentiable by Proposition 3.4.

Remark 5.4: A mapping from an open subset of a finite-dimensional $\mathbb{K}$-vector space to a polynormed $\mathbb{K}$-vector space over a locally compact, non-discrete topological field $\mathbb{K}$ is of class C^k if and only if it is an SC^k-map, by a simple induction based on Lemma 3.2, Lemma 4.4, and Remark 5.2.

Compositions of composable SC^k-maps are SC^k.

PROPOSITION 5.5: *Let $\mathbb{K}$ be a valued field, E and F be normed $\mathbb{K}$-vector spaces, G be a polynormed $\mathbb{K}$-vector space, and $U \subseteq E$, $V \subseteq F$ be open subsets. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and suppose that $f\colon U \to V \subseteq F$ and $g\colon V \to G$ are SC^k-maps. Then $g \circ f\colon U \to G$ is an SC^k-map.*

Proof: We may assume that $k < \infty$; the proof is by induction. The case $k = 0$ is trivial. The case $k = 1$ is known (cf. [6, 1.3.1], where however no proof is given), and can be shown as follows: Given $x \in U$, let γ be a continuous seminorm on G, and $\varepsilon > 0$. Set

$$\varepsilon' := \min\left\{\frac{1}{2(\|f'(x)\|+1)}, \frac{1}{2(\|g'(f(x))\|_\gamma + 1)}\right\} < 1.$$

By strict differentiability of g at $f(x)$, there exists $r > 0$ such that

$$\|g(z) - g(y) - g'(f(x)).(z-y)\|_\gamma \le \varepsilon' \|z - y\| \quad \text{for all } y, z \in B_r^F(f(x)) \cap V.$$

By strict differentiability of f at x and continuity of f at x, there exists $\delta > 0$ such that

$$\|f(z) - f(y) - f'(x).(z-y)\| \le \varepsilon'\|z-y\| \quad \text{for all } z, y \in B_\delta^E(x) \cap U,$$

and $f(y) \in B_r^F(f(x))$ for all $y \in B_\delta^E(x) \cap U$. Then

$$\begin{aligned}
&\|g(f(z)) - g(f(y)) - g'(f(x)).f'(x).(z-y)\|_\gamma \\
&\le \|g(f(z)) - g(f(y)) - g'(f(x)).(f(z) - f(y))\|_\gamma \\
&\quad + \|g'(f(x)).(f(z) - f(y) - f'(x).(z-y)\|_\gamma \\
&\le \varepsilon' \underbrace{\|f(z) - f(y)\|}_{\le (\|f'(x)\| + \varepsilon')\cdot\|z-y\|} + \varepsilon'\|g'(f(x))\|_\gamma \cdot \|z-y\| \\
&\le \varepsilon\|z-y\|
\end{aligned}$$

for all $y, z \in B_\delta^E(x) \cap U$, whence $g \circ f$ is strictly differentiable at x, with differential $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$.

Induction step. Assume that $2 \le k \in \mathbb{N}$, and suppose that the assertion is correct when k is replaced with $k-1$. Let f and g be SC^k-maps, as above. By the preceding, $g \circ f$ is an SC^1-map. We also know that

$$(19) \qquad (g \circ f)^{[1]}(x, y, t) = g^{[1]}(f(x), f^{[1]}(x, y, t), t) \quad \text{for all } (x, y, t) \in U^{[1]},$$

where $f^{[1]}$ and $g^{[1]}$ are SC^{k-1}-maps (cf. (2)). Now f being an SC^k-map and thus an SC^{k-1}-map, and the continuous linear map $E \times E \times \mathbb{K} \to E$, $(x, y, t) \mapsto x$

being an SC^{k-1}-map, by induction the composition $U^{[1]} \to F$, $(x,y,t) \mapsto f(x)$ is an SC^{k-1}-map. Also $U^{[1]} \to \mathbb{K}$, $(x,y,t) \mapsto t$ is an SC^{k-1}-map, being the restriction of a continuous linear map. As a consequence,

$$\widehat{T}(f)\colon U^{[1]} \to V^{[1]}, \quad (x,y,t) \mapsto (f(x), f^{[1]}(x,y,t), t)$$

is an SC^{k-1}-map, its components being SC^{k-1} (cf. [2], proof of La. 4.4). But thus $(g \circ f)^{[1]} = g^{[1]} \circ \widehat{T}(f)$ is an SC^{k-1}-map, by the induction hypothesis. Now $g \circ f$ being SC^1 with $(g \circ f)^{[1]}$ being SC^{k-1}, we deduce that $g \circ f$ is an SC^k-map (see Remark 5.2). ∎

6. Newton approximation with parameters

In this section, we discuss Newton approximation and Newton approximation with parameters, as the basis for our inverse function theorems (resp., implicit function theorems) for valued fields. Our first lemma can also be considered as a Lipschitz Inverse Function Theorem.

LEMMA 6.1 (Newton approximation): *Let $(E, \|.\|)$ be a Banach space over a valued field $(\mathbb{K}, |.|)$. Let $r > 0$, $x \in E$, and $f\colon B_r(x) \to E$ be a mapping. We suppose that there exists $A \in \mathrm{GL}(E) := L(E)^\times$ such that*

$$(20) \qquad \sigma := \sup\left\{\frac{\|f(z) - f(y) - A.(z-y)\|}{\|z-y\|} : y, z \in B_r(x), y \neq z\right\} < \frac{1}{\|A^{-1}\|}.$$

Then the following holds:

(a) *Let $a := 1 - \sigma\|A^{-1}\| \in]0,1]$ and $b := 1 + \sigma\|A^{-1}\| \in [1,2[$. Then*

$$(21) \quad a\|z-y\| \leq \|A^{-1}.f(z) - A^{-1}.f(y)\| \leq b\|z-y\| \quad \textit{for all } y, z \in B_r(x).$$

For every $y \in B_r(x)$ and $s \in]0, r - \|y-x\|]$, we have

$$(22) \qquad f(y) + A.B_{as}(0) \subseteq f(B_s(y)) \subseteq f(y) + A.B_{bs}(0).$$

In particular, f has open image and is a homeomorphism onto its image.

(b) *If $(\mathbb{K}, |.|)$ is an ultrametric field in particular and $(E, \|.\|)$ an ultrametric Banach space, then $A^{-1} \circ f\colon B_r(x) \to E$ is isometric. For each $y \in B_r(x)$ and $s \in]0, r]$, we have $B_s(y) \subseteq B_r(x)$ and*

$$(23) \qquad f(B_s(y)) = f(y) + A.B_s(0).$$

Proof (cf. [43, Lemma 27.4] when $E = \mathbb{K}$ is a complete ultrametric field. Compare also [30, Theorem 6.3.6] for a related result in the real case):

(a) Given $y, z \in B_r(x)$, we have

$$\begin{aligned}\|A^{-1}.f(z) - A^{-1}.f(y)\| &= \|A^{-1}.(f(z) - f(y) - A.(z-y)) + z - y\| \\ &\leq \|A^{-1}\| \cdot \|f(z) - f(y) - A.(z-y)\| + \|z-y\| \\ &\leq (\sigma\|A^{-1}\| + 1)\|z-y\| = b\|z-y\|\end{aligned}$$

and

$$\begin{aligned}\|z-y\| &= \|A^{-1}.(f(z) - f(y) - A.(z-y)) - (A^{-1}.f(z) - A^{-1}.f(y))\| \\ &\leq \|A^{-1}\| \cdot \|f(z) - f(y) - A.(z-y)\| + \|A^{-1}.f(z) - A^{-1}.f(z)\| \\ &\leq \sigma\|A^{-1}\| \cdot \|z-y\| + \|A^{-1}.f(z) - A^{-1}.f(y)\|,\end{aligned}$$

whence (21) holds. As a consequence of (21), $A^{-1} \circ f$ and hence also f is injective and a homeomorphism onto its image. Now suppose that $y \in B_r(x)$ and $s \in]0, r - \|y-x\|]$. By the preceding, we have $f(B_s(y)) \subseteq f(y) + A.B_{bs}(0)$. To see that $f(y) + A.B_{as}(0) \subseteq f(B_s(y))$, let $c \in f(y) + A.B_{as}(0)$. There exists $t \in]0,1[$ such that $c \in f(y) + A.\overline{B}_{tas}(0)$. Given $z \in \overline{B}_{st}(y)$, we define

$$g(z) := z - A^{-1}.(f(z) - c).$$

Then $g(z) \in \overline{B}_{st}(y)$, since

$$\begin{aligned}\|g(z) - y\| &\leq \underbrace{\|z - y - A^{-1}.f(z) + A^{-1}.f(y)\|}_{\leq \|A^{-1}\|\sigma\|z-y\| \leq \|A^{-1}\|\sigma st} + \underbrace{\|A^{-1}.c - A^{-1}.f(y)\|}_{\leq ats} \\ &\leq (\|A^{-1}\|\sigma + a)st = st.\end{aligned}$$

Thus $g(\overline{B}_{st}(y)) \subseteq \overline{B}_{st}(y)$. The map $g\colon \overline{B}_{st}(y) \to \overline{B}_{st}(y)$ is a contraction, since

$$\begin{aligned}\|g(u) - g(v)\| &= \|u - v - A^{-1}.(f(u) - f(v))\| \\ &\leq \|A^{-1}\| \cdot \|f(u) - f(v) - A.(u-v)\| \\ &\leq \sigma \cdot \|A^{-1}\| \cdot \|u-v\|\end{aligned} \tag{24}$$

for all $u, v \in \overline{B}_{st}(y)$, where $\|A^{-1}\|\sigma < 1$. By Banach's Contraction Theorem ([43], p. 269), there exists a unique element $z_0 \in \overline{B}_{st}(y)$ with $g(z_0) = z_0$ and hence $f(z_0) = z_0$. This completes the proof of (a).

(b) Let $(\mathbb{K}, |.|)$ be an ultrametric field now and $(E, \|.\|)$ be an ultrametric Banach space. For all $y, z \in B_r(x)$ such that $y \neq z$, we have the inequalities $\|A^{-1}.f(z) - A^{-1}.f(y) - (z-y)\| \leq \|A^{-1}\| \cdot \|f(z) - f(y) - A.(z-y)\| < \|z-y\|$, using (20) to obtain the final inequality. Hence, the norm $\|.\|$ being ultrametric, we must have $\|A^{-1}.f(z) - A^{-1}.f(y)\| = \|z-y\|$. Thus $A^{-1} \circ f$ is in fact isometric.

If $y \in B_r(x)$ and $s \in]0,r]$, then $B_s(y) \subseteq B_r(y) = B_r(x)$, as $\|.\|$ is ultrametric. The map $A^{-1} \circ f$ being isometric, we have $f(B_s(y)) = A.(A^{-1} \circ f)(B_s(y)) \subseteq A.B_s(A^{-1}.f(y)) = f(y) + A.B_s(0)$. If $c \in f(y) + A.B_s(0)$ is given, define

$$g(z) := z - A^{-1}.(f(z) - c) \quad \text{for } z \in B_s(y).$$

Then

$$\begin{aligned}\|g(z) - y\| &= \|(z-y) - (A^{-1}.f(z) - A^{-1}.f(y)) + A^{-1}.(c - f(y))\| \\ &\leq \max\{\|z-y\|, \|A^{-1}.f(z) - A^{-1}.f(y)\|, \|A^{-1}.(c - f(y))\|\} < s\end{aligned}$$

for $z \in B_s(y)$, whence $g(z) \in B_s(y)$. The map $g\colon B_s(y) \to B_s(y)$ is a contraction, by the calculation from (24). Recall that, the norm on E being ultrametric, the open ball $B_s(y)$ is also closed and therefore complete in the induced metric. By Banach's Contraction Theorem ([43], p. 269), there is a unique element $z_0 \in B_s(y)$ such that $g(z_0) = z_0$ and thus $f(z_0) = c$. The proof is complete. ∎

More generally, we shall need Newton approximation with parameters.

LEMMA 6.2 (Newton approximation with parameters): *Let $(E, \|.\|)$ be a Banach space over a valued field $(\mathbb{K}, |.|)$, and P be a topological space. Let $r > 0$, $x \in E$, and $f\colon P \times B \to E$ be a continuous mapping, where $B := B_r^E(x)$. Given $p \in P$, we abbreviate $f_p := f(p, \bullet)\colon B \to E$. We suppose that there exists $A \in \mathrm{GL}(E) := L(E)^\times$ such that*

(25)
$$\sigma := \sup\left\{\frac{\|f_p(z) - f_p(y) - A.(z-y)\|}{\|z-y\|} : p \in P,\, y, z \in B,\, y \neq z\right\} < \frac{1}{\|A^{-1}\|}.$$

Then $f_p(B)$ is open in E and $f_p|_B$ is a homeomorphism onto its image, for each $p \in P$. The set $W := \bigcup_{p \in P}\{p\} \times f_p(B)$ is open in $P \times E$, and $\psi\colon W \to E$, $\psi(p,z) := (f_p|_B^{f_p(B)})^{-1}(z)$ is continuous. Furthermore, the map $\theta\colon P \times B \to W$, $\theta(p,y) := (p, f(p,y))$ is a homeomorphism, with inverse given by $\theta^{-1}(p,z) = (p, \psi(p,z))$.

Proof: By Lemma 6.1, applied to f_p, the set $f_p(B)$ is open in E and $f_p|_B$ a homeomorphism onto its image. Define $a := 1 - \sigma\|A^{-1}\|$. Let us show openness of W and continuity of h. If $(p,z) \in W$, there exists $y \in B$ such that $f_p(y) = z$. Let $\varepsilon \in]0, r - \|y - x\|]$ be given. There is an open neighbourhood Q of p in P such that $f(q,y) \in f(p,y) + A.B_{a\varepsilon/2}(0)$ for all $q \in Q$, by continuity of f. Then, as a consequence of Lemma 6.1(a), Eqn. (22),

$$f_q(B_\varepsilon(y)) \supseteq f(q,y) + A.B_{a\varepsilon}(0) \supseteq f(p,y) + A.B_{a\varepsilon/2}(0) = z + A.B_{a\varepsilon/2}(0).$$

By the preceding, $Q \times (z + A.B_{a\varepsilon/2}(0)) \subseteq W$, whence W is a neighbourhood of (p,z). Furthermore, $\psi(q,z') = (f_q)^{-1}(z') \in B_\varepsilon(y) = B_\varepsilon((f_p)^{-1}(z)) = B_\varepsilon(\psi(p,z))$ for all (q,z') in the neighbourhood $Q \times (z + A.B_{a\varepsilon/2}(0))$ of (p,z). Thus W is open and ψ is continuous. The assertions concerning θ follow immediately. ∎

7. Inverse and implicit function theorems for SC^k-maps between Banach spaces

In this section, we prove an inverse function theorem and an implicit function theorem for SC^k-maps between Banach spaces over complete valued fields, which parallel the classical theorems for continuously Fréchet differentiable mappings in the real case.

We begin with an Inverse Function Theorem for mappings strictly differentiable at a point (cf. also [6, 1.5.1]), strictly differentiable maps, and locally uniformly differentiable maps.

PROPOSITION 7.1: *Let $(E, \|.\|)$ be a Banach space over a valued field $(\mathbb{K}, |.|)$, $U \subseteq E$ be an open subset, $x \in U$, and $f\colon U \to E$ be a mapping which is strictly differentiable at x (resp., strictly differentiable, resp., locally uniformly differentiable), with $A := f'(x) \in \mathrm{GL}(E)$. Let $a, b \in \mathbb{R}$ be given such that $0 < a < 1 < b$. Then there exists $r > 0$ such that $B := B_r(x) \subseteq U$ and*

- (a) *$a\|z-y\| \leq \|A^{-1}.f(z) - A^{-1}.f(y)\| \leq b\|z-y\|$ for all $y, z \in B$, whence $f|_B$ is injective in particular;*
- (b) *$f(y) + A.B_{as}(0) \subseteq f(B_s(y)) \subseteq f(y) + A.B_{bs}(0)$ for all $y \in B$ and $s \in]0, r - \|y\|]$, whence, in particular, $f(B)$ is an open subset of E;*
- (c) *the mapping $g := (f|_B^{f(B)})^{-1}\colon f(B) \to B$ is strictly differentiable at $f(x)$ (resp., strictly differentiable, resp., uniformly differentiable).*

If $(\mathbb{K}, |.|)$ is an ultrametric field and $(E, \|.\|)$ an ultrametric Banach space, then r can be chosen such that (c) *holds but* (a) *and* (b) *may be replaced with:*

- (a)′ *The mapping $A^{-1} \circ f|_B$ is isometric.*
- (b)′ *For all $y \in B$ and $s \in]0, r]$, we have $f(B_s(y)) = f(y) + A.B_s(0)$. Thus, if A is an isometry, then in fact $f(B_s(y)) = B_s(f(y))$.*

Proof: Let $A := f'(x)$ and

$$c := \min\left\{\frac{b-1}{\|A^{-1}\|}, \frac{1-a}{\|A^{-1}\|}\right\}.$$

Then $1 - c\|A^{-1}\| \geq a$, $1 + c\|A^{-1}\| \leq b$, and $c < 1/\|A^{-1}\|$. The map f being strictly differentiable at x, there exists $r > 0$ such that $B_r(x) \subseteq U$ and

$$\frac{\|f(z) - f(y) - A.(z-y)\|}{\|z-y\|} \leq c \quad \text{for all } z, y \in B_r(x) \text{ such that } z \neq y.$$

Thus hypothesis (20) of Lemma 6.1 is satisfied; parts (a), (b), (a)$'$ and (b)$'$ directly follow from that lemma.

(c) Assume that f is locally uniformly differentiable, or strictly differentiable. Since $\mathrm{GL}(E)$ is open in $L(E)$, $f'(x) \in \mathrm{GL}(E)$, and $f'|_B$ is continuous, after shrinking r we may assume that $f'(B_r(x)) \subseteq \mathrm{GL}(E)$. Inversion in $\mathrm{GL}(E)$ being continuous, after shrinking r further we may also assume that $\|f'(y)^{-1}\| \leq \|A^{-1}\| + 1$ for all $y \in B_r(x)$.

Assume that f is locally uniformly differentiable. After shrinking r further if necessary, we may assume that f is uniformly differentiable on $B := B_r(x)$. Abbreviate $g := (f|_B^{f(B)})^{-1}$ and define $g' \colon f(B) \to L(E)$ via $g'(y) := f'(g(y))^{-1}$. Given $\varepsilon > 0$, due to uniform differentiability of $f|_B$ there exists $\delta \in]0, r]$ such that

$$\text{(26)} \qquad \frac{\|f(w) - f(v) - f'(u).(w-v)\|}{\|w-v\|} < \frac{a\varepsilon}{(\|A^{-1}\|+1)\|A^{-1}\|}$$

for all $u, v, w \in B$ such that $\|v - u\| < \delta$, $\|w - u\| < \delta$, and $v \neq w$. Set $\delta' := a\delta/\|A^{-1}\|$. Let $u', v', w' \in f(B)$ such that $v' \neq w'$, $\|v' - u'\| < \delta'$, and $\|w' - u'\| < \delta'$. Then $u := g(u')$, $v := g(v')$, and $w := g(w')$ are elements of B such that $v \neq w$,

$$\begin{aligned}\|v - u\| &\leq a^{-1}\|A^{-1}.f(v) - A^{-1}.f(u)\| = a^{-1}\|A^{-1}.(v' - u')\| \\ &\leq a^{-1}\|A^{-1}\| \cdot \|v' - u'\| < a^{-1}\|A^{-1}\|\delta' = \delta\end{aligned}$$

(using part(a)), and similarly $\|w - u\| < \delta$ and $\|w - v\| \leq a^{-1}\|A^{-1}\| \cdot \|w' - v'\|$. Using (26) and part(a), we obtain the following estimates:

$$\begin{aligned}&\frac{\|g(w') - g(v') - g'(u').(w' - v')\|}{\|w' - v'\|} \\ &= \frac{\|w - v\|}{\|w' - v'\|} \cdot \frac{\|w - v - f'(u)^{-1}.(f(w) - f(v))\|}{\|w - v\|} \\ &\leq \|f'(u)^{-1}\| \cdot \frac{\|w - v\|}{\|w' - v'\|} \cdot \frac{\|f(w) - f(v) - f'(u).(w - v)\|}{\|w - v\|} \\ &< (\|A^{-1}\| + 1) \cdot a^{-1} \cdot \|A^{-1}\| \cdot \frac{a\varepsilon}{(\|A^{-1}\|+1)\|A^{-1}\|} = \varepsilon.\end{aligned}$$

Thus g is uniformly differentiable.

If f is strictly differentiable at x, we see along the preceding lines (holding however $u := x$ and $u' := f(x)$ fixed now) that $g := (f|_B^{f(B)})^{-1}$ is strictly differentiable at $f(x)$, where $B := B_r(x)$.

If f is strictly differentiable on all of B, because $f'(y) \in \mathrm{GL}(E)$ for all $y \in B$, we may apply the preceding proof just as well when x is replaced with y; thus g is strictly differentiable at each $z = f(y) \in f(B)$. ∎

Isometries are encountered frequently in the ultrametric case. If $\mathbb{K}$ is a valued field and $(E, \|.\|)$ a normed $\mathbb{K}$-vector space, we let $\mathrm{Iso}(E, \|.\|)$ denote the group of all bijective linear isometries of E. If the norm on E is understood, we simply write $\mathrm{Iso}(E)$. We have:

LEMMA 7.2: *If $(E, \|.\|)$ is an ultrametric Banach space over an ultrametric field $\mathbb{K}$, then $\mathrm{Iso}(E)$ is open in $\mathrm{GL}(E) = L(E)^\times$.*

Proof: If $A \in L(E)$ and $\|A\| < 1$, then $\|Ax\| < \|x\|$ and hence $\|(\mathbf{1}+A)x\| = \|x\|$ for all $0 \neq x \in E$, whence $\mathbf{1} + A$ is an isometry. Furthermore, using Neumann's series we see that $\mathbf{1} + A$ is invertible, with inverse $(\mathbf{1} + A)^{-1} = \sum_{k=0}^{\infty}(-1)^k A^k$. Thus $\mathbf{1} + A \in \mathrm{Iso}(E)$ for all $A \in L(E)$ such that $\|A\| < 1$, entailing that $\mathrm{Iso}(E)$ is open in $\mathrm{GL}(E)$. ∎

Let $\mathbb{K}$ be a valued field. An SC^k**-diffeomorphism** is an invertible SC^k-map $f\colon U \to V$ between open subsets of normed $\mathbb{K}$-vector spaces, such that f^{-1} is an SC^k-map.

THEOREM 7.3 (Inverse Function Theorem for SC^k-maps): *Let E be a Banach space over a valued field $(\mathbb{K}, |.|)$, $k \in \mathbb{N} \cup \{\infty\}$, $U \subseteq E$ be an open subset, and $f\colon U \to E$ be an SC^k-map. If $k > 1$, we assume that $\mathbb{K}$ is complete.*

Suppose that $f'(x) := df(x, \bullet) \in \mathrm{GL}(E)$ for some $x \in U$. Then there exists $r > 0$ such that $B := B_r(x) \subseteq U$, the set $f(B)$ is open in E, and $f|_B^{f(B)}$ is an SC^k-diffeomorphism.

For our inductive proof of the Inverse Function Theorem, we need the Implicit Function Theorem, which we formulate as an "Inverse Function Theorem with Parameters."

THEOREM 7.4 (Implicit Function Theorem for SC^k-maps): *Let $(\mathbb{K}, |.|)$ be a valued field, $k \in \mathbb{N} \cup \{\infty\}$, Z and F be Banach spaces over $\mathbb{K}$, $P \subseteq Z$ and $U \subseteq F$ be open subsets, $f\colon P \times U \to F$ be an SC^k-map, and $(p, x) \in P \times U$ be a point such that $A := f_p' = d_2 f(p, x, \bullet) = df((p,x),(0,\bullet)) \in \mathrm{GL}(F)$; here $f_p := f(p, \bullet)\colon U \to F$, and $E := Z \times F$ is equipped with the maximum norm,*

$\|(q,y)\| := \max\{\|q\|,\|y\|\}$ *for all* $q \in Z$, $y \in F$. *If* $k > 1$, *we assume that* $\mathbb{K}$ *is complete. Let* $a, b \in \mathbb{R}$ *be given such that* $0 < a < 1 < b$. *Then there exists an open neighbourhood* $Q \subseteq P$ *of* p *and* $r > 0$ *such that* $B := B_r^F(x) \subseteq U$ *and the following holds:*

(a) $f_q(B)$ *is open in* F, *for each* $q \in Q$, *and* $\phi_q \colon B \to f_q(B)$, $\phi_q(y) := f_q(y) = f(q,y)$ *is an* SC^k*-diffeomorphism.*

(b) *For all* $q \in Q$, $y \in B$, *and* $s \in]0, r - \|y-x\|]$, *we have*

$$f_q(y) + A.B_{as}(0) \subseteq f_q(B_s(y)) \subseteq f_q(y) + A.B_{bs}(0).$$

(c) $W := \bigcup_{q\in Q}(\{q\} \times f_q(B))$ *is open in* $Z \times F$, *and* $\psi \colon W \to B$, $\psi(q,v) := \phi_q^{-1}(v)$ *an* SC^k*-map. Furthermore, the map* $\theta \colon Q \times B \to W$, $\theta(q,y) := (q, f(q,y))$ *is an* SC^k*-diffeomorphism, with inverse given by* $\theta^{-1}(q,v) = (q, \psi(q,v))$.

(d) $Q \times (f(p,x) + A.B_\delta(0)) \subseteq W$ *for some* $\delta > 0$.

In particular, for each $q \in Q$ *there is a unique element* $\beta(q) \in B$ *such that* $f(q,\beta(q)) = f(p,x)$, *and the mapping* $\beta \colon Q \to B$ *so obtained is of class* SC^k.

If $(\mathbb{K}, |.|)$ *is ultrametric here and* $(F, \|.\|)$ *an ultrametric Banach space, then* r *can be chosen such that* (a)–(d) *can be replaced with the following stronger assertions:*

(a)′ $f_q(B) = f(p,x) + A.B_r(0) =: V$, *for each* $q \in Q$, *and* $\phi_q \colon B \to V$, $\phi_q(y) := f(q,y)$ *is an* SC^k*-diffeomorphism.*

(b)′ $\phi_q(B_s(y)) = \phi_q(y) + A.B_s(0)$ *for all* $q \in Q$, $y \in B$ *and* $s \in]0, r]$.

(c)′ $\psi \colon W := Q \times V \to B$, $\psi(q,v) := \phi_q^{-1}(v)$ *is an* SC^k*-map, and the map* $\theta \colon Q \times B \to Q \times V = W$, $\theta(q,y) := (q, f(q,y))$ *is an* SC^k*-diffeomorphism, with inverse given by* $\theta^{-1}(q,v) = (q, \psi(q,v))$.

Proof of Theorems 7.3 and 7.4: We proceed in various steps.

7.5. *If the Inverse Function Theorem for* SC^k*-mappings is correct for some* $k \in \mathbb{N} \cup \{\infty\}$, *then also the Implicit Function Theorem holds for* SC^k*-maps.*

In fact, suppose that Z, F, P, U, (p,x), E, $0 < a < 1 < b$, an SC^k-map $f \colon P \times U \to F$, and A are given as described in Theorem 7.4. Define

$$c := \min\left\{\frac{b-1}{\|A^{-1}\|}, \frac{1-a}{\|A^{-1}\|}\right\} < \frac{1}{\|A^{-1}\|}.$$

Then $1 - c\|A^{-1}\| \geq a$ and $1 + c\|A^{-1}\| \leq b$. Since f is strictly differentiable at (p,x), there exists $r > 0$ such that $B_r^E(p,x) = B_r^Z(p) \times B_r^F(x) \subseteq P \times U$,

$$\frac{\|f(q_2,y_2) - f(q_1,y_1) - f'(p,x).(q_2-q_1, y_2-y_1)\|}{\|(q_2-q_1, y_2-y_1)\|} \leq c$$

for all $(q_1, y_1) \neq (q_2, y_2) \in B_r^E(p,x)$, and $\|f_q'(y) - A\| < c$ for all $(q,y) \in B_r^f(p,x)$, since $f_q'(y) = f'(q,y)(0,\bullet)$ depends continuously on (q,y) (cf. Lemma 3.2 and proof of Lemma 2.2). Set $Q := B_r^Z(p)$ and $B := B_r^F(x)$. Then $f_q'(y) \in \mathrm{GL}(F)$ for all $q \in Q$ and $y \in B$. In fact, $\|A^{-1}f_q'(y) - \mathbf{1}\| \leq \|A^{-1}\|\|f_q'(y) - A\| < \|A^{-1}\|c < 1$, whence $A^{-1}f_q'(y)$ is invertible and hence so is $f_q'(y)$. Then

(27)
$$\frac{\|f_q(z) - f_q(y) - A.(z-y)\|}{\|z-y\|} = \frac{\|f(q,z) - f(q,y) - f'(p,x).(0,z-y)\|}{\|(0,z-y)\|} \leq c$$

for all $q \in Q$ and $y \neq z \in B$, where $c < 1/\|A^{-1}\|$. Thus Lemma 6.2 applies to $f|_{Q \times B}$, showing that $f_q(B)$ is open in E and $\phi_q := f_q|_B^{f_q(B)}$ a homeomorphism onto its image, for each $q \in Q$; the set $W := \bigcup_{q \in Q}\{q\} \times f_q(B)$ is open in $P \times F$ and hence in $Z \times F$, and $\psi \colon W \to B$, $\psi(q,z) := \phi_q^{-1}(z)$ is continuous; the map $\theta \colon Q \times B \to W$, $\theta(q,y) := (q, f(q,y))$ is a homeomorphism, with inverse given by $\theta^{-1}(q,z) = (q, \psi(q,z))$. Furthermore, in view of (27), Lemma 6.1 applies to $f_q|_B$ for all $q \in Q$, whence (b) holds. By the SC^k-case of the Inverse Function Theorem, $\phi_q \colon B \to f_q(B)$ is an SC^k-diffeomorphism, for all $q \in Q$. Thus (a) holds. To complete the proof of (c), note that the homeomorphism $\theta \colon Q \times B \to W$ is an SC^k-map, whose differential $\theta'(q,y)$ at any given point $(q,y) \in Q \times B$ can be interpreted as an upper triangular 2×2-block matrix with id_Z and $f_q'(y)$ on the diagonal, entailing that $\theta'(q,y)$ is invertible. Hence, by the Inverse Function Theorem for SC^k-maps, θ restricts to an SC^k-diffeomorphism (onto the image) on some open neighbourhood of (q,y), entailing that θ^{-1} is an SC^k-map on some open neighbourhood of $\theta(q,y)$. Thus θ is an SC^k-diffeomorphism. Since $\theta^{-1}(q,z) = (q, \psi(q,z))$ for all $(q,z) \in W$, we readily deduce that ψ is an SC^k-map, thus completing the proof of (c).

(d) is easily established: we set $\delta := ar/2$. After shrinking Q, we may assume that $\|f(q,x) - f(p,x)\| < \delta$ for all $q \in Q$. Then, using (b) with $y := x$ and $s := r$, we see that $\{q\} \times f_q(B) \supseteq \{q\} \times (f_q(x) + A.B_{ar}(0)) \supseteq \{q\} \times (f_p(x) + A.B_\delta(0))$, for all $q \in Q$. Thus (d) holds. The assertion concerning β is then obvious.

In the special case where $\mathbb{K}$ is an ultrametric field and F an ultrametric Banach space, we establish (a)–(c) as just described, choosing however Q so small that $\|f(q,x) - f(p,x)\| < r$ for all $q \in Q$. Then $f_q(B) = f_q(y) + A.B_r(0) = f_p(y) + A.B_r(0) =: V$ for all $q \in Q$, and hence $W = Q \times V$. In fact, in view of (27), we can apply Lemma 6.1(b) to the map $f_q|_B$, and then use that B is an additive subgroup of F. Furthermore, again by Lemma 6.1(b), $f_q(B_s(y)) = f_q(y) + A.B_s(0)$ for all $q \in Q$, $y \in B$, and $s \in]0,r]$. Thus (a)′–(c)′ hold.

7.6. *Suppose that the assertion of the Inverse Function Theorem for SC^k-maps* *is valid for all $k \in \mathbb{N}$. Then it is also valid for $k = \infty$.*

In fact, let E, U, $x \in U$, and an SC^∞-map $f\colon U \to E$ be given as described in Theorem 7.3. By the SC^1-version of Theorem 7.3, we find $r > 0$ such that $B := B_r(x) \subseteq U$, $f(B)$ is open in E and such that $f|_B^{f(B)}$ is an invertible SC^1-map, with inverse $g := (f|_B^{f(B)})^{-1}\colon f(B) \to B$ of class SC^1. Then g is of class SC^∞. In fact, let $k \in \mathbb{N}$. Given any $y' \in f(B)$, set $y := g(y') \in B$. Then $f'(y)$ is invertible and thus, by the SC^k-case of Theorem 7.3, there exists a neighbourhood $V \subseteq B$ of y such that $(f|_V^{f(V)})^{-1} = g|_{f(V)}^V$ is of class SC^k, where $f(V)$ is a neighbourhood of $f(y) = y'$. Thus g is locally SC^k and thus an SC^k-map. As $k \in \mathbb{N}$ was arbitrary, g is an SC^∞-map.

7.7. In view of 7.5 and 7.6, in order to establish Theorem 7.3 and Theorem 7.4, it suffices to establish the SC^k-case of Theorem 7.3 for all $k \in \mathbb{N}$. This we accomplish by induction. For $k = 1$, the assertion of Theorem 7.3 is covered by Proposition 7.1.

7.8 (Induction step). Suppose that $2 \leq k \in \mathbb{N}$ is given, and suppose that the SC^{k-1}-case of the Inverse Function Theorem holds. Let E, U, x and an SC^k-map $f\colon U \to E$ be as described in Theorem 7.3. Let $r > 0$ and $B := B_r(x)$ be as described in the SC^{k-1}-case of the theorem. Thus $f(B)$ is open in E and $f|_B^{f(B)}$ is an invertible SC^k-map, whose inverse $g := (f|_B^{f(B)})^{-1}$ is an SC^{k-1}-map. After replacing f with $f|_B$, we may assume without loss of generality that $U = B$. Set $V := f(U)$. Since $g\colon V \to U$ is an SC^{k-1}-map, it is clear that $g^{[1]}\colon V^{[1]} \to E$ is SC^{k-1} on the open subset $\{(y,z,t) \in V^{[1]}\colon t \neq 0\}$ of $V^{[1]}$. Thus $g^{[1]}$ will be an SC^{k-1}-map (and thus g an SC^k-map) if we can show that, for every $(y_0, z_0) \in V \times E$, the mapping $g^{[1]}$ is SC^{k-1} on some open neighbourhood of $(y_0, z_0, 0)$ in $V^{[1]}$. To this end, we observe first that $f \circ g = \mathrm{id}_V$ and the Chain Rule entail that $f^{[1]}(g(y), g^{[1]}(y,z,t), t) = z$ for all $(y,z,t) \in V^{[1]}$. There are open neighbourhoods $W_1 \subseteq E$ of $g(y_0)$, $W_2 \subseteq E$ of $g^{[1]}(y_0, z_0, 0)$, and $W_3 \subseteq \mathbb{K}$ of 0 such that $W_1 \times W_2 \times W_3 \subseteq U^{[1]}$. Next, we find an open neighbourhood $P = P_1 \times P_2 \times P_3 \subseteq V^{[1]}$ of $(y_0, z_0, 0)$ such that $g(P_1) \subseteq W_1$, $g^{[1]}(P) \subseteq W_2$, and $P_3 \subseteq W_3$. By the preceding, the SC^{k-2}-map $\beta := g^{[1]}|_P^{W_2}\colon P \to W_2$ satisfies

$$h((y,z,t), \beta(y,z,t)) = 0 \quad \text{for all } (y,z,t) \in P, \tag{28}$$

where $h\colon P \times W_2 \to E$ is the SC^{k-1}-map $h((y,z,t),w) := f^{[1]}(g(y), w, t) - z$. Since $h((y_0, z_0, 0), w) = f'(g(y_0)).w - z_0$ is affine-linear in w, the differential of h with respect to the w-variable satisfies $A := d_2h((y_0, z_0, 0), \beta(y_0, z_0, 0); \bullet) = f'(g(y_0)) \in \mathrm{GL}(E)$. Since β is a continuous solution to the equation (28), we deduce from the SC^{k-1}-case of the Implicit Function Theorem 7.4 (which holds in view of the induction hypothesis and 7.5) that β is SC^{k-1} on some open neighbourhood of $(y_0, z_0, 0)$, as we set out to show. This completes the proof of Theorems 7.3 and 7.4. ∎

Remark 7.9: If $k > 1$, in the preceding induction step we encounter a map β defined on $P \subseteq V^{[1]} \subseteq E^{[1]}$. In order that $E^{[1]} = E \times E \times \mathbb{K}$ be a Banach space (so that the implicit function theorem can be applied), we need that $\mathbb{K}$ is complete.

8. Ultrametric implicit function theorem

We are now in the position to prove a generalized implicit function theorem for mappings from open subsets of metrizable topological vector spaces over complete ultrametric fields to Banach spaces over such fields.

THEOREM 8.1 (Ultrametric Implicit Function Theorem): *Let $(\mathbb{K}, |.|)$ be a complete ultrametric field, $k \in \mathbb{N} \cup \{\infty\}$, Z be a metrizable topological $\mathbb{K}$-vector space, and E be a Banach space over $\mathbb{K}$. Let $P \subseteq Z$ and $U \subseteq E$ be open subsets, and $f: P \times U \to E$ be a map. We assume that at least one of the following conditions is satisfied:*

(i) *$\mathbb{K}$ is locally compact, E is finite-dimensional, and f is of class $C^k_{\mathbb{K}}$.*

Or:

(ii) *f is of class $C^{k+1}_{\mathbb{K}}$.*

We abbreviate $f_q := f(q, \bullet): U \to E$ for $q \in P$. Suppose that $(p, x) \in P \times U$ is given such that $A := f'_p(x) := d_2 f(p, x, \bullet) := df((p, x), (0, \bullet)) \in \mathrm{GL}(E)$. Let $a, b \in \mathbb{R}$ be given such that $0 < a < 1 < b$. Then there exists an open neighbourhood $Q \subseteq P$ of p and $r > 0$ such that $B := B_r(x) \subseteq U$ and the following holds:

(a) *$f_q(B)$ is open in E, for each $q \in Q$, and $\phi_q: B \to f_q(B)$, $\phi_q(y) := f_q(y) = f(q, y)$ is an SC^k-diffeomorphism.*

(b) *For all $q \in Q$, $y \in B$, and $s \in]0, r - \|y - x\|]$, we have*

$$f_q(y) + A.B_{as}(0) \subseteq f_q(B_s(y)) \subseteq f_q(y) + A.B_{bs}(0).$$

(c) *$W := \bigcup_{q \in Q}(\{q\} \times f_q(B))$ is open in $Z \times E$, and the map $\psi: W \to B$, $\psi(q, v) := \phi_q^{-1}(v)$ is C^k. Furthermore, the map $\theta: Q \times B \to W$, $\theta(q, y) := (q, f(q, y))$ is a $C^k_{\mathbb{K}}$-diffeomorphism, with inverse $\theta^{-1}(q, v) = (q, \psi(q, v))$.*

(d) *$Q \times (f_p(x) + A.B_\delta(0)) \subseteq W$ for some $\delta > 0$.*

In particular, for each $q \in Q$ there is a unique element $\beta(q) \in B$ such that $f(q, \beta(q)) = f(p, x)$, and the mapping $\beta: Q \to B$ so obtained is of class $C^k_{\mathbb{K}}$.

If $(E, \|.\|)$ is an ultrametric Banach space here, then Q and r can be chosen such that (a)–(d) *can be replaced with the following stronger assertions:*

(a)′ $f_q(B) = f(p,x) + A.B_r(0) =: V$, *for each* $q \in Q$, *and* $\phi_q\colon B \to V$, $\phi_q(y) := f(q,y)$ *is an* SC^k*-diffeomorphism.*

(b)′ $f_q(B_s(y)) = f_q(y) + A.B_s(0)$ *for all* $q \in Q$, $y \in B$ *and* $s \in]0,r]$.

(c)′ *The mapping* $\psi\colon Q \times V \to B$, $\psi(q,v) := \phi_q^{-1}(v)$ *is* $C^k_{\mathbb{K}}$. *Furthermore,* $\theta\colon Q \times B \to Q \times V$, $\theta(q,y) := (q, f(q,y))$ *is a* $C^k_{\mathbb{K}}$*-diffeomorphism, with inverse given by* $\theta^{-1}(q,v) = (q, \psi(q,v))$.

Proof: Define

$$c := \min\left\{\frac{b-1}{\|A^{-1}\|}, \frac{1-a}{\|A^{-1}\|}\right\} < \frac{1}{\|A^{-1}\|}, \quad \text{where } A := f_p'(x).$$

Then $1-c\|A^{-1}\| \geq a$ and $1+c\|A^{-1}\| \leq b$. In the situation of (i), let $e_1, \ldots, e_n$ be a basis of E. The mappings $P \times U \to E$, $(q,y) \mapsto d_2 f(q,y,e_i)$ being continuous for $i = 1, \ldots, n$, also the map

$$P \times U \to L(E), \quad (q,y) \mapsto f_q'(y) = d_2 f(q,y,\bullet) \tag{29}$$

is continuous. In the situation of (ii), the map in (29) is continuous as well, by Lemma 3.5. In either case, since $\mathrm{GL}(E)$ is open in $L(E)$ and $f_p'(x) \in \mathrm{GL}(E)$, after replacing P and U with smaller open neighbourhoods of p and x, respectively, we may assume that $f_q'(y) \in \mathrm{GL}(E)$ for all $(q,y) \in P \times U$, and $\|f_q'(y) - f_p'(x)\| \leq c/2$. Using Lemma 4.5 (resp., Lemma 3.5), we find an open neighbourhood $Q \subseteq P$ of p and $r > 0$ such that $B := B_r(x) \subseteq U$ and

$$\frac{\|f_q(z) - f_q(y) - f_q'(x).(z-y)\|}{\|z-y\|} \leq \frac{c}{2} \tag{30}$$

for all $y \neq z \in B$. As a consequence,

$$\begin{aligned}&\frac{\|f_q(z) - f_q(y) - f_p'(x).(z-y)\|}{\|z-y\|}\\ &\quad\leq \frac{\|f_q(z) - f_q(y) - f_q'(x).(z-y)\|}{\|z-y\|} + \|f_p'(x) - f_q'(x)\| \leq c\end{aligned}$$

for all $q \in Q$ and $z \neq y \in B$, entailing that

$$\sup\left\{\frac{\|f_q(z) - f_q(y) - f_p'(x).(z-y)\|}{\|z-y\|} : q \in Q, z \neq y \in B\right\} \leq c < \frac{1}{\|A^{-1}\|}. \tag{31}$$

Thus Lemma 6.2 applies to $f|_{Q\times B}$, whence $f_q(B)$ is open in E and $\phi_q := f_q|_B^{f_q(B)}$ a homeomorphism onto its image, for each $q \in Q$; the set $W := \bigcup_{q\in Q}\{q\} \times f_q(B)$

is open in $P \times E$ and thus in $Z \times E$, and ψ: $W \to B$, $\psi(q,z) := \phi_q^{-1}(z)$ is continuous; the map θ: $Q \times B \to W$, $\theta(q,y) := (q, f(q,y))$ is a homeomorphism, with inverse given by $\theta^{-1}(q,z) = (q, \psi(q,z))$.

Furthermore, in view of (31), Lemma 6.1 applies to $f_q|_B$, for all $q \in Q$, showing that (b) holds. By the Inverse Function Theorem for SC^k-maps, the map ϕ_q: $B \to f_q(B)$ is an SC^k-diffeomorphism, for all $q \in Q$. Thus (a) holds.

(d) is easily established: we set $\delta := ar/2$. After shrinking Q, we may assume $\|A^{-1}(f(q,x) - f(p,x))\| < \delta$ for all $q \in Q$. Then, using (b) with $y := x$ and $s := r$, we get $\{q\} \times f_q(B) \supseteq \{q\} \times (f_q(x) + A.B_{ar}(0)) \supseteq \{q\} \times (f_p(x) + A.B_\delta(0))$, for all $q \in Q$. Thus (d) holds.

To see that the continuous map ψ is of class $C^k_{\mathbb{K}}$ (which will entail the validity of (c)), in view of Proposition 1.15, it suffices to show that, for all smooth maps c: $\mathbb{K}^{k+1} \to W$, the composition $\psi \circ c$: $\mathbb{K}^{k+1} \to E$ is of class $C^k_{\mathbb{K}}$. Since $W \subseteq Q \times E$, we have $c = (c_1, c_2)$ with smooth mappings c_1: $\mathbb{K}^{k+1} \to Q \subseteq Z$, c_2: $\mathbb{K}^{k+1} \to E$. Define h: $\mathbb{K}^{k+1} \times B \to V$, $h(t,y) := f(c_1(t), y)$. Then h is a $C^k_{\mathbb{K}}$-map in the situation of (i) and hence SC^k (Remark 5.4). In the situation of (ii), h is of class $C^{k+1}_{\mathbb{K}}$ and hence SC^k (see Remark 5.3). Given $t \in \mathbb{K}^{k+1}$, abbreviate $h_t := h(t, \bullet)$: $B \to E$; by the above, h_t has open image and is a homeomorphism onto its image. Since $h_t'(y) = d_2 f(c_1(t), y, \bullet) \in \mathrm{GL}(E)$ for all $(t,y) \in \mathbb{K}^{k+1} \times B$, we deduce from the Implicit Function Theorem for SC^k-maps (Theorem 7.4) that κ: $W \to B$, $\kappa(t,z) := h_t^{-1}(z)$ is an SC^k-map and hence of class $C^k_{\mathbb{K}}$. Now $\psi(c_1(t), c_2(t)) = \kappa(t, c_2(t))$ for all $t \in \mathbb{K}^{k+1}$ shows that $\psi \circ c$ is of class $C^k_{\mathbb{K}}$, as required. Thus ψ is $C^k_{\mathbb{K}}$.

If E is an ultrametric Banach space, we establish (a)–(c) as just described, choosing however Q so small that $\|A^{-1}(f(q,x) - f(p,x))\| < r$ for all $q \in Q$ (we might actually replace $c/2$ with c in (30) now). Then $f_q(B) = f_q(y) + A.B_r(0) = f_p(y) + A.B_r(0) =: V$ for all $q \in Q$ (by Lemma 6.1(b), applied as at the end of 7.5), and hence $W = Q \times V$. Furthermore, again by Lemma 6.1(b), $f_q(B_s(y)) = f_q(y) + A.B_s(0)$ for all $q \in Q$, $y \in B$, and $s \in]0, r]$. Thus (a)′–(c)′ hold. ∎

Remark 8.2: Three cases described in the table given in the introduction still remain to be discussed.

(a) Suppose we retain the hypotheses of Theorem 8.1, with $k = 1$, except that we let Z be an arbitrary topological $\mathbb{K}$-vector space now (which need not be metrizable). Suppose we are in the situation of (i). Then the proof of Theorem 8.1 shows that the following weakened conclusions of the theorem remain valid: (a), (b) and (d) will hold unchanged; ψ in (c) and β will be continuous; θ in (c) will be a homeomorphism (likewise for (a)′–(c)′).

(b) Suppose we retain the hypotheses of Theorem 8.1, with $k = 1$, except that we let $\mathbb{K}$ be an arbitrary valued field and Z be an arbitrary topological $\mathbb{K}$-vector space. Suppose we are in the situation of (ii). Then the proof of Theorem 8.1 shows that the following weakened conclusions remain valid: (a), (b) and (d) will hold unchanged; ψ in (c) and β will be continuous; θ in (c) will be a homeomorphism.

(c) Suppose we retain the hypotheses of Theorem 8.1, with $k = 1$, except that we let $\mathbb{K}$ be a subfield of $\mathbb{R}$ now, equipped with the absolute value obtained by restricting the usual absolute value on $\mathbb{R}$. Suppose we are in the situation of (ii). Then (a)–(d) and their proof remain valid verbatim, and β is C^1. In fact, Proposition 1.15 remains valid when $\mathbb{R}$ is replaced with arbitrary subfields of $\mathbb{R}$ (the proof given in [2] applies without changes).

9. Applications

In this section, we sketch how our results can be used to prove the following:

1. Smoothness of inversion in diffeomorphism groups over local fields.
2. Existence of stable manifolds for dynamical systems over ultrametric fields, and their smooth dependence on parameters.[4]

We concentrate entirely on those aspects of the proofs which illustrate our current results. Full proofs (and more details) can be found in [16] and [18], respectively.

Smoothness of inversion in diffeomorphism groups. Let M be a finite-dimensional, paracompact smooth manifold over a local field $\mathbb{K}$, with valuation ring $\mathbb{O}$. Let $\mathrm{Diff}(M)$ be the group of all C^∞-diffeomorphisms of M. Then M is a disjoint union $M = \coprod_{i\in I} B_i$ of balls, i.e., open subsets $B_i \subseteq M$ diffeomorphic to $\mathbb{O}^d$, where d is the dimension of the modelling space of M. In [16], the Lie group structure on $\mathrm{Diff}(M)$ is constructed as follows: First, each $\mathrm{Diff}(B_i)$ is made a Lie group. Then, the weak direct product

$$\prod^*_{i\in I} \mathrm{Diff}(B_i) := \{(\gamma_i)_{i\in I} \in \prod_{i\in I} \mathrm{Diff}(B_i)\colon \gamma_i = \mathrm{id}_{B_i} \text{ for all but finitely many } i\}$$

is given its natural Lie group structure. Here $\prod^*_{i\in I} \mathrm{Diff}(B_i)$ can be identified with a subgroup of $\mathrm{Diff}(M)$ in an apparent way. In a third step, one verifies

4 The construction of pseudo-stable manifolds is much more complicated; it requires specialized implicit function theorems for sequence spaces. The interested reader is referred to [19].

that $\mathrm{Diff}(M)$ can be given a Lie group structure making $\prod_{i\in I}^{*}\mathrm{Diff}(B_i)$ an open subgroup.

The results of the present article enter into the first step. To explain their use, we may therefore assume now that $M = \mathbb{O}^d$. Then $P := \mathrm{Diff}(M)$ is an open subset of the metrizable locally convex space $C^\infty(M,\mathbb{K}^d)$, equipped with the initial topology with respect to the maps $C^\infty(M,\mathbb{K}^d) \to C(M^{[k]},\mathbb{K}^d)$, $\gamma \mapsto \gamma^{[k]}$ for $k \in \mathbb{N}_0$, where the spaces on the right hand side are given the topology of compact convergence (see [16, Prop. 13.2]). The inclusion map

$$i\colon P \to C^\infty(M,\mathbb{K}^d), \quad \gamma \mapsto \gamma$$

being smooth, the exponential law [16, Prop. 12.2(a)] ensures that also

$$f\colon P\times M \to \mathbb{K}^d, \quad f(\gamma,x) := i^\wedge(\gamma,x) := i(\gamma)(x) = \gamma(x)$$

is smooth, using that M is finite-dimensional. Note that $f_\gamma := f(\gamma,\bullet) = \gamma$ for each $\gamma \in P$. Hence $f'_\gamma(x) = \gamma'(x) \in \mathrm{GL}(\mathbb{K}^d)$ for each $x \in M$ in particular. Thus Theorem 8.1(c)$'$ entails that the map

$$g\colon P\times M \to M, \quad g(\gamma,x) := (f_\gamma)^{-1}(x) = \gamma^{-1}(x)$$

is smooth. Using the other direction of the exponential law ([16, La. 12.1(a)]), we deduce that the map

$$\mathrm{Diff}(M) \to C^\infty(M,\mathbb{K}^d), \quad \gamma \mapsto g^\vee(\gamma) := g(\gamma,\bullet) = \gamma^{-1}$$

is smooth. But this is the inversion map of the group $\mathrm{Diff}(M)$.

Existence and parameter-dependence of stable manifolds. We now describe how Irwin's method can be used to construct stable manifolds around hyperbolic fixed points of dynamical systems over ultrametric fields. The method applies to all of the SC^k-, smooth, and analytic cases (see[18]); for simplicity, we restrict attention to the smooth case here.

Throughout the remainder of this section, E is an ultrametric Banach space over a complete ultrametric field $(\mathbb{K},|.|)$, and $a \in]0,1]$.

Definition 9.1: A (bicontinuous) linear automorphism $\alpha \in \mathrm{GL}(E)$ is called ***a*-hyperbolic** if $E = E_1 \oplus E_2$ for certain α-invariant closed vector subspaces E_1, E_2;

$$\|x_1 + x_2\| = \max\{\|x_1\|, \|x_2\|\} \quad \text{for all } x_1 \in E_1 \text{ and } x_2 \in E_2$$

holds for an ultrametric norm $\|.\|$ on E equivalent to the original norm; and $\alpha = \alpha_1 \oplus \alpha_2$ with $\|\alpha_1\| < a$ and $\|\alpha_2^{-1}\|^{-1} > a$.

Remark 9.2: If $\mathbb{K}$ is a local field here and $\dim_{\mathbb{K}}(E) < \infty$, then α is a-hyperbolic if and only if $a \neq |\lambda|$ for each eigenvalue $\lambda \in \overline{\mathbb{K}}$ of $\alpha \otimes_{\mathbb{K}} \mathrm{id}_{\overline{\mathbb{K}}} \in \mathrm{GL}(E \otimes_{\mathbb{K}} \overline{\mathbb{K}})$, where $\overline{\mathbb{K}}$ is an algebraic closure of $\mathbb{K}$.

Throughout the following, let $\alpha \in \mathrm{GL}(E)$ be a-hyperbolic, say $E = E_1 \oplus E_2$ with E_1 and E_2 as in Definition 9.1. Let $r > 0$ and $f\colon U \to E$ be a smooth map on $U := B_r^E(0) = U_1 \times U_2$, where $U_j := B_r^{E_j}(0)$, such that $f(0) = 0$ and $f'(0) = \alpha$. We define the **a-stable set of** f via

$$W_{s,a} := \{z \in U\colon f^n(z) \text{ defined for all } n \in \mathbb{N}_0, a^{-n}\|f^n(z)\| < r \,\&\, f^n(z) = o(a^n)\}.$$

To emphasize f, we also write $W_{s,a}(f) := W_{s,a}$. Clearly $f(W_{s,a}) \subseteq W_{s,a}$. The goal is to see that $W_{s,a}$ is a submanifold of E. Note that if $z \in W_{s,a}$, then the orbit $\omega := (f^n(z))_{n\in\mathbb{N}_0}$ is an element of the Banach sequence space

$$\mathcal{S}_a(E) := \{z = (z_n)_{n\in\mathbb{N}_0} \in E^{\mathbb{N}_0}\colon z_n = o(a^n)\}$$

with norm $\|z\|_a := \max\{a^{-n}\|z_n\|\colon n \in \mathbb{N}_0\}$. Let $\mathcal{U} := \{z \in \mathcal{S}_a(E)\colon \|z\|_a < r\}$ and $\tilde{f} := f - \alpha$. Since f is strictly differentiable, after shrinking r we may assume that $\tilde{f}$ is (globally) Lipschitz continuous, with arbitrarily small Lipschitz constant $\mathrm{Lip}(\tilde{f}) := \sup\{\|\tilde{f}(z_2) - \tilde{f}(z_1)\| \cdot \|z_2 - z_1\|^{-1}\colon z_1 \neq z_2 \in U\}$. We require that

$$\mathrm{Lip}(\tilde{f}) < \min\{1, \|\alpha_2^{-1}\|^{-1}\}. \tag{32}$$

THEOREM 9.3 (Ultrametric Stable Manifold Theorem): *$W_{s,a}$ is the graph of a smooth map $\phi\colon U_1 \to U_2$ such that $\phi(0) = 0$ and $\phi'(0) = 0$. Thus $W_{s,a}$ is a smooth submanifold of E which is tangent to the a-stable subspace E_1 at 0.*

Sketch of proof: Elements of $\mathcal{S}_a(E)$ will be written in the form $z = (z_n)_{n\in\mathbb{N}_0} = (x_n, y_n)_{n\in\mathbb{N}_0}$ now, with $x_n \in E_1$, $y_n \in E_2$. In terms of components, $f = (f_1, f_2)$ and $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$. We define a map $g\colon \mathcal{U} \to \mathcal{S}_a(E)$ via

$$g(z)_n := \begin{cases} (0, \alpha_2^{-1}(y_1 - \tilde{f}_2(z_0))) & \text{if } n = 0; \\ (f_1(z_{n-1}), \alpha_2^{-1}(y_{n+1} + \tilde{f}_2(z_n))) & \text{if } n \geq 1. \end{cases}$$

Then $g(0) = 0$. It can be shown (with considerable effort) that g is smooth. (32) implies that g is Lipschitz continuous, with $\mathrm{Lip}(g) < 1$, whence $\|g'(z)\| < 1$ for each $z \in \mathcal{U}$. Thus Lemma 6.1(b) (and the proof of Lemma 7.2) entail that

$$G\colon \mathrm{id}_{\mathcal{U}} - g\colon \mathcal{U} \to \mathcal{U}$$

is an isometry and a C^∞-diffeomorphism. Then

$$w\colon U_1 \to \mathcal{U}, \quad w(x) := G^{-1}((x,0),(0,0),(0,0),\ldots)$$

is a smooth map such that $(\mathrm{id}_{\mathcal{U}} - g)(w(x)) = G(w(x)) = ((x,0),(0,0),\ldots)$ for all $x \in U_1$ and thus

$$w(x) = ((x,0),(0,0),\ldots) + g(w(x)). \tag{33}$$

Comparing the 0-th component of both sides of (33), we see that

$$w(x)_0 = (x,0) + (0, \underbrace{\alpha_2^{-1}(\mathrm{pr}_2(w(x)_1) - \tilde{f}_2(w(x)_0))}_{=:\phi(x)}),$$

where $\mathrm{pr}_2 : E_1 \oplus E_2 \to E_2$ is the projection. Then $\phi\colon U_1 \to U_2$ is smooth, and $w(x)_0 = (x, \phi(x))$. Comparing also the other components of both sides of (33), it can be shown that $w(x)$ is the orbit of $(x,\phi(x)) = w(x)_0$, and $(x,\phi(x)) \in W_{s,a}$. Conversely, given $z = (z_n)_{n\in\mathbb{N}} \in W_{s,a}$ it is easily checked that (33) holds with z in place of $w(x)$, whence $G(z) = ((x,0),(0,0),\ldots) = G(w(x))$ and thus $z = w(x)$. Hence $W_{s,a}$ is the graph of ϕ. See [18] for details, and the proof of $\phi'(0) = 0$.

Smooth dependence of $W_{s,a}(f)$ on the non-linearity $\tilde{f}$.

9.4. Let Z be a metrizable topological $\mathbb{K}$-vector space, $P \subseteq Z$ be open and $f\colon P \times U \to E$ be a smooth map such that $f(p,0) = 0$ and $f_p'(0) = \alpha$ (as above), for each $p \in P$. Set $\tilde{f}(p,x) := f(p,x) - \alpha(x)$. Lemma 3.5 ensures that, after shrinking r and passing to a neighbourhood of a given point in P, we can assume that $\mathrm{Lip}(\tilde{f}_p) < \min\{1, \|\alpha_2^{-1}\|^{-1}\}$ for each p. Repeating the proof of Theorem 9.3, we obtain functions $g\colon P \times \mathcal{U} \to \mathcal{S}_a(E)$ and $G\colon P \times \mathcal{U} \to \mathcal{U}$ depending now also on the parameter p; as shown in [18], g (and hence G) is smooth. Now our Inverse Function Theorem with Parameters (Theorem 8.1(c)) shows that $w\colon P \times U_1 \to \mathcal{U}$, $w(p,x) := G_p^{-1}((x,0),(0,0),\ldots)$ is smooth, and hence so is $\phi\colon P \times U_1 \to U_2$ with $w(p,x)_0 = (x, \phi(p,x))$. *Thus*

$$W_{s,a}(f_p) = \mathrm{graph}(\phi_p) \quad \textit{for each } p \in P,$$

for a smooth map $\phi\colon P \times U_1 \to U_2$.

9.5. If $\mathbb{K}$ is a local field and $\dim_{\mathbb{K}}(E) < \infty$, then

$$P := \{\tilde{f} \in Z\colon \mathrm{Lip}(\tilde{f}) < \min\{1, \|\alpha_2^{-1}\|^{-1}\}\}$$

is an open 0-neighbourhood in the metrizable locally convex space

$$Z := \{\tilde{f} \in C^\infty(U,E)\colon \tilde{f}(0) = 0 \text{ and } \tilde{f}'(0) = 0\}.$$

The evaluation map $C^\infty(U,E) \times U \to E$, $(\gamma, x) \mapsto \gamma(x)$ being smooth [16, Prop. 11.1], also $f\colon P \times U \to E$, $f(\tilde{f}, x) := \alpha(x) + \tilde{f}(x)$ is smooth. Applying 9.4 to this map f and using the exponential law [16, La. 12.1(a)], we deduce:

PROPOSITION 9.6: *In the situation of 9.5, the map*

$$C^\infty(U,E) \supseteq P \to C^\infty(U_1, E_2), \quad \tilde{f} \mapsto \phi_{\tilde{f}}$$

taking $\tilde{f} \in P$ to the smooth map $\phi_{\tilde{f}}$ with $\operatorname{graph}(\phi_{\tilde{f}}) = W_{s,a}(\alpha + \tilde{f})$ *is smooth.*

Appendix: FC^k-maps vs. SC^k-maps in the real case

THEOREM A.7: *Let E be a normed vector space over $\mathbb{R}$, F be a real locally convex space, $U \subseteq E$ be open, $f\colon U \to F$ be a map, and $k \in \mathbb{N}_0 \cup \{\infty\}$. If f is FC^k, then f is SC^k.*

Proof: We may assume that $k \in \mathbb{N}_0$. The proof is by induction. The case $k = 0$ is trivial, and the case $k = 1$ is a standard fact (see [6, 2.3.3], cf. also [8, Thm. 3.8.1]).

Induction step. Suppose that $k \geq 2$, and suppose that every FC^{k-1}-map is SC^{k-1}. Let $f\colon E \supseteq U \to F$ be an FC^k-map. Then f is SC^{k-1} and hence SC^1 in particular. Then, f being SC^{k-1}, so is $f^{[1]}$ on $\{(x,y,t) \in U^{[1]}\colon t \neq 0\}$. It therefore only remains to show that, for every $x_0 \in U$ and $y_0 \in E$, the map $f^{[1]}$ is SC^{k-1} on some open neighbourhood of $(x_0, y_0, 0)$. There is $r > 0$ such that $B_{2r}(x_0) \subseteq U$. Choose $\delta \in]0, r[$ such that $\delta < r/(2(\|y_0\| + 1))$. Then $V := B_r(x_0) \times B_1(y_0) \times]-\delta, \delta[\subseteq U^{[1]}$, and we have

$$\begin{aligned} f^{[1]}(x,y,t) &= \int_0^1 df(x + sty, y)\,ds \\ &= \int_0^1 h((x,y,t),s)\,ds \quad \text{for all } (x,y,t) \in V, \end{aligned} \tag{34}$$

where $h\colon V \times]-2\delta, 2\delta[\to F$, $h((x,y,t),s) := df(x + sty, y)$ is an FC^{k-1}-map. In view of (34), we inductively deduce from [9, 8.11.2] that $f^{[1]}|_V$ is an FC^{k-1}-map, if F is a *Banach space.* If, more generally, F is a *complete* locally convex space, then $F = \varprojlim F_i$ is a projective limit in the category of locally convex spaces of some projective system of Banach spaces, and thus $L(H, F) = \varprojlim L(H, F_i)$ (which again is a complete locally convex space), for every normed space H. A simple inductive argument now shows that a map g from an open subset of a normed space to a projective limit $F = \varprojlim F_i$ is an FC^k-map if and only if $\pi_i \circ g$ is FC^k for each i, where $\pi_i\colon F \to F_i$ are the limit maps. In the situation we are interested in, $\pi_i \circ f^{[1]}|_V = (\pi_i \circ f)^{[1]}|_V$ maps into a Banach space and hence is an FC^{k-1}-map, by what has already been shown, whence $f^{[1]}|_V$ is an FC^{k-1}-map to the projective limit F. In the *general case*, when F is not necessarily complete, the preceding shows that $f^{[1]}|_V$ is FC^{k-1} as a mapping into the completion $\widetilde{F}$ of F. Since $(f^{[1]})'(x) = d(f^{[1]})(x, \bullet)$ for $x \in V$ actually is a map into F (not only into $\widetilde{F}$), and likewise for the higher order differentials, we deduce that $f^{[1]}|_V$ is FC^{k-1} as a map into F also in the fully general case.

Now $f^{[1]}|_V$ being an FC^{k-1}-map, it is an SC^{k-1}-map, by induction. Thus f is SC^1 with $f^{[1]}$ an SC^{k-1}-map, and hence f is SC^k. ∎

The author does not know whether, conversely, every SC^k-map is FC^k. For $k = 1$, this is well-known [6, 2.3.3], but the generalization to higher k does not seem to be clear.

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