EMBEDDING l_p^k IN SUBSPACES OF L_p FOR $p > 2$

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ABSTRACT

The aim of the present paper is to estimate in a precise manner the integer $k = k(p,m,n,\epsilon)$ so that an arbitrary m-dimensional subspace X of the space l_p^n; $p > 2$, contains an $(1+\epsilon)$-isomorph of l_p^k. The main argument of the proof consists of a probabilistic selection which uses a lemma of Slepian. The same method also shows that any system of normalized functions in L_p; $p \geq 2$, which is equivalent to the unit vector basis of l_p^n, contains, for any $\epsilon > 0$, a subsystem of size h proportional to n, which is $(1+\epsilon)$-equivalent to the unit vector basis of l_p^h.

0. Introduction

The geometry of finite dimensional subspaces of the space L_p has been studied intensively and quite precise numerical estimates have been proved in many cases. For instance, it was shown that any m-dimensional subspace X of l_∞^n of dimension $m \geq n^\delta$, for some $\delta > 0$, contains a well isomorphic copy of l_∞^k, with $k \geq C(\delta)m^{1/2}$, where $C(\delta)$ is a constant depending on δ only (cf. [9] in the case when m is proportional to n and [4] for a general δ). It was also noticed in [9] that, for a random subspace X of l_∞^n, the above estimates are best possible.

The aim of this paper is to consider the similar problem in l_p^n with $p > 2$, i.e., to estimate the maximal k so that a well isomorphic copy of l_p^k can be found in any subspace X of l_p^n; $p > 2$, of a fixed dimension m. The dual case was considered in [6]. There it has been proved that if X is an m-dimensional subspace of l_p^n, $p > 2$, then its dual X^* contains a well isomorphic copy of $l_{p'}^k$, with $p' = p/(p-1)$ and k of order of magnitude $(m/n^{2/p})^{p/(p-2)}$. This estimate is best pos-

†The authors were supported by Grant No. 87-0079 from BSF.
Received May 31, 1990

sible. We shall see in the sequel that one cannot expect to prove such a strong assertion for X itself.

Fix $p > 2$ and $1 \le m < n$, and let $\mathcal{G}_{n,m}$ denote the Grassman manifold of all m-dimensional subspaces of L_p^n. It follows from [13] that any $X \in \mathcal{G}_{n,m}$ contains a 2-Hilbertian subspace of dimension about m/d_X^2, where d_X stands for the euclidean distance of X. Since, by [1], this number cannot exceed a multiple of $n^{2/p}$ we conclude that $d_X \ge cm^{1/2}/n^{1/p}$, for some constant $c > 0$. By using the isoperimetric inequality in the usual way, one can show that, for random elements X of $\mathcal{G}_{n,m}$, the euclidean distance is minimal (i.e. d_X is of order of magnitude $m^{1/2}/n^{1/p}$) and even that

$$\|x\|_{L_p^n} \le C\frac{m^{1/2}}{n^{1/p}}\|x\|_{L_2^n},$$

for all $x \in X$ and some constant $C = C_p < \infty$ (see, e.g., [15]).

For a sequence $x = (a_1, a_2, \dots, a_n)$ in $\mathbf{R}^n$, the reader should distinguish between the norm $\|x\|_p = (\sum_{j=1}^n |a_j|^p)^{1/p}$ in l_p^n and $\|x\|_{L_p^n} = (\sum_{j=1}^n |a_j|^p/n)^{1/p}$ in L_p^n.

Suppose now that X is a random element of $\mathcal{G}_{n,m}$ and let $k = k_p(X,K)$ be the maximal dimension of a subspace of X which is K-isomorphic to l_p^k. The case $m \le n^{2/p}$ is trivial since, under this assumption, X is Hilbertian. If $m > n^{2/p}$ choose $2 < r < p$ such that $m = n^{2/r}$ and note that the above considerations show that on X the L_2^n and L_r^n norms are equivalent and, therefore, also that

$$\max_{\substack{x\in X\\ x\neq 0}} \|x\|_{L_r^n}/\|x\|_{L_1^n} = C < \infty,$$

for some constant $C = C_r < \infty$, depending on r only.

Let E be a subspace of X for which there exists an invertible operator $T: l_p^k \to E$ such that $\|T\|\,\|T^{-1}\| \le K$ and let E_q stand for E when E is considered as a subspace of L_q^n; $1 \le q \le \infty$. Denote by $I_{q,s}$ the formal identity map from L_q^n into L_s^n and, by $i_{q,s}$, that from l_q^n into l_s^n. Consider the following diagram:

$$l_\infty^k \overset{i_{\infty,p}}{\to} l_p^k \overset{T}{\to} E_p \overset{I_{p,\infty}}{\to} E_\infty \overset{I_{\infty,1}}{\to} E_1 \overset{I_{1,r}}{\to} E_r \overset{I_{r,p}}{\to} E_p \overset{T^{-1}}{\to} l_p^k \overset{i_{p,\infty}}{\to} l_\infty^k$$

and note that the 1-summing norm $\pi_1(l_\infty^k)$ of the identity operator on l_∞^k satisfies the inequalities

$$\begin{aligned} k \le \pi_1(l_\infty^k) &\le \|i_{\infty,p}\|\,\|T\|\,\|I_{p,\infty}\|\,\pi_1(I_{\infty,1})\,\|I_{1,r|X}\|\,\|I_{r,p}\|\,\|T^{-1}\| \\ &\le KCk^{1/p}n^{1/p}n^{1/r-1/p} = KCk^{1/p}n^{1/r}. \end{aligned}$$

It follows that

$$k_p(X,K) \le (KC)^{p'}m^{p'/2}$$

i.e., k cannot exceed a multiple of $m^{p'/2}$. Besides the example of random elements of $\mathcal{G}_{n,m}$, one can consider the following type of subspaces of L_p^n. Fix again $n^{2/p} < m < n$, take $j = (m/n^{2/p})^{p/(p-2)}$ and $N = n/j$ (under the assumption that all these numbers are integers which can be made without any loss of generality). Since L_p^N contains a K-isomorph of $l_2^{N^{2/p}}$, for some $K < \infty$, it follows that L_p^n contains a subspace X which is K-isomorphic to the direct sum in the sense of l_p^j of j copies of $l_2^{N^{2/p}}$. It is easily checked that $k_p(X,K) = j = (m/n^{2/p})^{p/(p-2)}$ while $\dim X = jN^{2/p} = j^{1-2/p}n^{2/p} = m$.

By combining the two estimates obtained above, we conclude that, for each value of $n^{2/p} < m < n$ and $p > 2$,

$$k_{\max}(n,m,p,K) = \max\{k_p(X,K); X \in \mathcal{G}_{n,m}\}$$

satisfies the inequality

$$k_{\max}(n,m,p,K) \leq M \min(m^{p'/2}, (m/n^{2/p})^{p/(p-2)}),$$

for some constant $M = M(p,K) < \infty$.

It turns out that this is, up to a constant, the right order of magnitude.

THEOREM 0.1. *For every $p > 2$ and $\epsilon > 0$, there exists a constant $c = c(p,\epsilon) > 0$ such that, whenever $m < n$ and X is an m-dimensional subspace of L_p^n, then X contains a subspace E of dimension*

$$k \geq c \min(m^{p'/2}, (m/n^{2/p})^{p/(p-2)})$$

which is $(1+\epsilon)$-isomorphic to l_p^k.

REMARK. By using [5] Theorem 4.3, one can strengthen Theorem 0.1 so as to assert that E is also c^{-1}-complemented in L_p^n.

The above result can be used to provide a negative solution to the following question raised by V. Milman in [14].

PROBLEM 0.2. Does there exist, for every $K > 1$ and $\epsilon > 0$, a constant $c(K,\epsilon) > 0$ such that, whenever X and Y are finite dimensional spaces with $\dim X = \dim Y = n$ and $d(X,Y) \leq K$, then one can find subspaces X_ϵ of X and Y_ϵ of Y for which $d(X_\epsilon, Y_\epsilon) < 1 + \epsilon$ and $\dim X_\epsilon = \dim Y_\epsilon \geq c(K,\epsilon)n$?

Indeed, fix $\epsilon > 0$ and $K > (1+\epsilon)^2$, and suppose that the problem above has a solution $c = c(K,\epsilon) > 0$. Fix $p_0 > 2$, let n be an integer satisfying $n^{1/2-1/p_0} > K$, and select $2 < p < p_0$ so that $n^{1/p-1/p_0} = K$. Then the spaces $X = l_{p_0}^n$ and $Y = l_p^n$ clearly satisfy the condition $d(X,Y) = K$. Let now X_0 be a subspace of X of dimension $\geq cn$. By using Theorem 0.1, we conclude the existence of a subspace X_1 of X_0 such that

$$d = \dim X_1 \geq a n^{p_0'/2} \quad \text{and} \quad d(X_1, l_{p_0}^d) < 1 + \epsilon,$$

for some $a = a(K, \epsilon, p_0) > 0$. It follows that, for any subspace Y_1 of Y with $\dim Y_1 = d$, we have that

$$(1+\epsilon)d(X_1, Y_1) \geq d(l_{p_0}^d, Y_1) \geq d^{1/p - 1/p_0} \geq a^{1/p-1/p_0} K^{p_0'/2} \geq a^{(\log K/\log n)} K^{1/2},$$

which shows that $d(X_1, Y_1)$ remains bounded away from 1, as $n \to \infty$. ■

Before presenting the proof of Theorem 0.1 in detail, we describe briefly its components.

Given an m-dimensional subspace X of L_p^n, for some $p > 2$, one constructs first a system of about m normalized vectors $\{\varphi_i\}_{i=1}^m$ in X whose upper p-estimate is $m^{1/2-1/p}$ and, after a change of density, is orthogonal and has a square function which is pointwise bounded by $n^{1/p}$. Each of these functions is then truncated at the level $\lambda = k^{1/p}$, where $k = \min(m^{p'/2}, (m/n^{2/p})^{p/(p-2)})$ is basically the dimension of the space l_p^k that we want to embed in X.

The main ingredient of the proof consists of a probabilistic selection result which can be used to lower the upper p-estimate of a system of uniformly bounded functions in L_p; $p > 2$, by passing to a suitable subsystem, provided one has a good estimate on the square function and the pointwise bound of these functions. This selection theorem allows us to show that each random subset $\sigma \subset \{1, 2, \ldots, m\}$ of cardinality about k contains a subset σ' of size $|\sigma'| \geq 9|\sigma|/10$ so that the flat parts $\{\varphi_{i,\lambda}\}_{i \in \sigma'}$ of $\{\varphi_i\}_{i \in \sigma'}$ have a bounded upper p-estimate.

The peak parts $\varphi_i^\lambda = \varphi_i - \varphi_{i,\lambda}$; $1 \leq i \leq m$ are then considered separately and, by an argument using multilinear interpolation, one shows again that a generic subset σ of $\{1, 2, \ldots, m\}$ of cardinality k contains a subset σ'' with $|\sigma''| \geq 9|\sigma|/10$ such that $\{\varphi_i^\lambda\}_{i \in \sigma''}$, also has a bounded upper p-estimate. A standard argument of restricted invertibility (cf. [5]) proves then the existence of a subset η of $\{1, 2, \ldots, m\}$ of cardinality proportional to k such that $\{\varphi_i\}_{i \in \eta}$ is equivalent to the unit vector basis of $l_p^{|\eta|}$.

In order to pass to an almost isometric copy of l_p^k, we prove the general fact that any sequence $\{f_i\}_{i=1}^h$ of functions in L_p; $2 \leq p < \infty$, which is equivalent to the unit vector basis of l_p^h contains, for any $\epsilon > 0$, a subsequence $\{f_i\}_{i \in \tau}$, with $|\tau|$ proportional to h, which is $(1+\epsilon)$-equivalent to the natural basis of $l_p^{|\tau|}$. This is done by basically applying the previous part of the proof to the functions $\{f_i\}_{i=1}^h$ instead of $\{\varphi_i\}_{i=1}^m$ except that this case is handled easier.

1. Random selections

We start by proving the main part of the argument needed in the proof of Theorem 0.1.

PROPOSITION 1.1. *For every $q > 2$, there exists a constant $C = C(q) < \infty$ such that, whenever $2 < p < q$ and $\{\psi_i\}_{i=1}^m$ is a set of functions in the space $L_p(\mu)$, for some probability measure μ, so that*

(i)
$$\left\| \sum_{i=1}^{m} a_i \psi_i \right\|_{L_p} \le \Gamma \| a \|_p,$$

for all $a = (a_1, a_2, \ldots, a_m) \in l_p^m$,

(ii)
$$\left\| \left(\sum_{i=1}^{m} |\psi_i|^2 \right)^{1/2} \right\|_{L_q} \le B,$$

for some $B < \infty$, and

(iii)
$$\| \psi_i \|_{L_\infty} \le \lambda,$$

for some $\lambda < \infty$, then, for any choice of $0 < \delta < 1$, a random subset σ of $\{1,2,\ldots,m\}$ of cardinality $|\sigma| = [\delta m]$ contains in turn a subset σ' such that $|\sigma'| \ge 9|\sigma|/10$ and

(iv)
$$\left\| \sum_{i \in \sigma'} a_i \psi_i \right\|_{L_p} \le C(\delta^{1/p'}\Gamma + \delta^{1/2-1/p} B m^{-1/p} + (\delta m)^{-1/p}\lambda) \| a \|_p,$$

for any $a = (a_i)_{i \in \sigma'} \in l_p^{|\sigma'|}$.

PROOF. Fix p, q, Γ, B, λ and consider a system of functions $\{\psi_i\}_{i=1}^m$ that satisfies (i), (ii), and (iii), as above. Put $r = q'$, $\tau = m^{1/p'}/\Gamma$ and consider

$$\mathcal{G}_r = \{x = (f, \tau\langle f, \psi_1\rangle, \tau\langle f, \psi_2\rangle, \ldots, \tau\langle f, \psi_m\rangle); f \in L_r(\mu)\}$$

as a subspace of the direct sum $L_r(\mu) \oplus L_r^m$. Then, for an arbitrary subset η of $\{1,2,\ldots,m\}$, define the operator $T_\eta : \mathcal{G}_r \to l_1^{|\eta|}$ by setting

$$T_\eta x = (\tau\langle f, \psi_i\rangle)_{i \in \eta}; \qquad x \in \mathcal{G}_r.$$

Fix $0 < \delta < 1$, let $\{\xi_i\}_{i=1}^m$ be a sequence of $\{0,1\}$-valued independent random variables of mean δ over some probability space (Ω, Σ, ν) and put

$$\eta(\omega) = \{1 \le i \le m; \xi_i(\omega) = 1\}; \qquad \omega \in \Omega.$$

Then

$$\int_\Omega \|T_{\eta(\omega)}\|\,d\nu = \tau \int_\Omega \sup_{\substack{x\in\mathcal{G}_r\\ \|x\|\le 1}} \left[\sum_{i=1}^m \xi_i(\omega)|\langle f,\psi_i\rangle|\right] d\nu$$

$$\le \tau\delta \sup_{\substack{x\in\mathcal{G}_r\\ \|x\|\le 1}} \left[\sum_{i=1}^m |\langle f,\psi_i\rangle|\right] + \tau\sqrt{2\pi}\int_{\Omega'}\int_\Omega \sup_{\substack{x\in\mathcal{G}_r\\ \|x\|\le 1}} \left[\left|\sum_{i=1}^m \xi_i(\omega)g_i(\omega')|\langle f,\psi_i\rangle|\right|\right] d\nu\, d\nu',$$

where $\{g_i\}_{i=1}^m$ is a sequence of mean zero independent Gaussian random variables over a probability space (Ω',Σ',ν') which are normalized in $L_2(\nu')$. By using Slepian's inequality (see, e.g., [7] Lemma 1.5 or [8]), we conclude that

$$\int_\Omega \|T_{\eta(\omega)}\|\,d\nu \le \delta m + D_1\tau\int_{\Omega'}\int_\Omega \sup_{\substack{x\in\mathcal{G}_r\\ \|x\|\le 1}} \left[\left|\left\langle \sum_{i=1}^m \xi_i(\omega)g_i(\omega')\psi_i, f\right\rangle\right|\right] d\nu\, d\nu'$$

$$\le \delta m + D_1\tau \int_\Omega \left\|\left(\sum_{i=1}^m \xi_i(\omega)|\psi_i|^2\right)^{1/2}\right\|_{L_q} d\nu,$$

for some constant $D_1 < \infty$. The integral can be easily estimated by repeating the argument above (cf. [6]) and we obtain that

$$J = \int_\Omega \left\|\left(\sum_{i=1}^m \xi_i(\omega)|\psi_i|^2\right)^{1/2}\right\|_{L_q} d\nu$$

$$\le \delta^{1/2}\left\|\left(\sum_{i=1}^m |\psi_i|^2\right)^{1/2}\right\|_{L_q} + D_2\int_\Omega \left\|\left(\sum_{i=1}^m \xi_i(\omega)|\psi_i|^4\right)^{1/4}\right\|_{L_q} d\nu$$

$$\le \delta^{1/2}B + D_2 J^{1/2}\left\|\max_{1\le i\le m}|\psi_i|\right\|_{L_q}^{1/2} \le \delta^{1/2}B + D_2 J^{1/2}\lambda^{1/2},$$

for some $D_2 < \infty$. Hence,

$$\int_\Omega \|T_{\eta(\omega)}\|\,d\nu \le D_3(\delta m + \tau\delta^{1/2}B + \tau\lambda),$$

for some numerical constant $D_3 < \infty$, which yields that, for a random subset $\sigma \subset \{1,2,\ldots,m\}$ of cardinality $|\sigma| = [\delta m]$, we have that

$$\tau\sum_{i\in\sigma}|\langle f,\psi_i\rangle| \le 2D_3(\delta m + \tau\delta^{1/2}B + \tau\lambda)\left(\|f\|_{L_r} + \tau m^{-1/r}\left(\sum_{i=1}^m |\langle f,\psi_i\rangle|^r\right)^{1/r}\right),$$

for all $f \in L_r(\mu)$. Therefore, by using a standard argument involving p'-stable variables (see, e.g., [5] Proposition 3.7), we get that the adjoint T_σ^* of T_σ is p-summing when considered as acting from $L_\infty^{|\sigma|}$ into $L_p(\mu) \oplus L_p^m$ and

$$\pi_p(T_\sigma^* : L_\infty^{|\sigma|} \to L_p(\mu) \oplus L_p^m) \le D(q) \| T_\sigma : L_r(\mu) \oplus L_r^m \to L_1^{|\sigma|} \|$$
$$\le 2D_3 D(q)(\delta m)^{-1}(\delta m + \tau \delta^{1/2} B + \tau\lambda),$$

where $D(q)$ is a constant depending on q only. Then, by the factorization theorem of Pietsch (cf. [16]), one concludes that, for some $\sigma' \subset \sigma$ of cardinality $|\sigma'| \ge 9|\sigma|/10$ and some constant $C(q)$, depending on q only, we have that

$$\| T_{\sigma'}^* : L_p^{|\sigma|} \to L_p(\mu) \oplus L_p^m \| \le C(q)(1 + \tau\delta^{-1/2} m^{-1} B + (\delta m)^{-1} \tau\lambda),$$

i.e., that

$$\tau(\delta m)^{-1/p'} \left(\sum_{i \in \sigma'} |\langle f, \psi_i \rangle|^{p'} \right)^{1/p'}$$
$$\le C(q)(1 + \tau\delta^{-1/2} m^{-1} B + (\delta m)^{-1}\tau\lambda) \left(\| f \|_{L_{p'}} + \tau m^{-1/p'} \left(\sum_{i=1}^m |\langle f, \psi_i \rangle|^{p'} \right)^{1/p'} \right),$$

for all $f \in L_{p'}(\mu)$.

However, it is easily verified that

$$\left(\sum_{i=1}^m |\langle f, \psi_i \rangle|^{p'} \right)^{1/p'} \le \Gamma \| f \|_{L_{p'}}; \qquad f \in L_{p'}(\mu),$$

which yields, in view of the choice of τ, that

$$\left(\sum_{i \in \sigma'} |\langle f, \psi_i \rangle|^{p'} \right)^{1/p'} \le C(q)(\delta^{1/p'} \Gamma + \delta^{1/2 - 1/p} m^{-1/p} B + (\delta m)^{-1/p} \lambda) \| f \|_{L_{p'}},$$

for all $f \in L_{p'}(\mu)$. This, of course, implies our assertion. ∎

2. Proof of Theorem 0.1

The object of this section is to present the proof of Theorem 0.1 in detail. To this end, we study systems of vectors $\{\varphi_i\}_{i=1}^m$ in an $L_p(\mu)$-space, for $p > 2$ and some probability measure μ, which satisfy the following conditions:

$$(1°) \qquad \left\| \sum_{i=1}^m a_i \varphi_i \right\|_{L_p} \le m^{1/2 - 1/p} \| a \|_p,$$

for any sequence $a = (a_1, a_2, \ldots, a_m) \in l_p^m$,

$$(2°)\qquad \left\| \left(\sum_{i=1}^{m} |\varphi_i|^2 \right)^{1/2} \right\|_\infty \le n^{1/p},$$

$$(3°)\qquad \|\varphi_i\|_{L_p} \le 1,$$

for all $1 \le i \le m$.

The "flat" parts of the functions $\{\varphi_i\}_{i=1}^m$ are considered first in the next result.

Proposition 2.1. *For every $p > 2$, there exists a constant $C_1 = C_1(p) < \infty$ such that, whenever $\{\varphi_i\}_{i=1}^m$ is a sequence of functions in an $L_p(\mu)$-space which satisfies the conditions* 1°, 2° *and* 3° *above, then a random subset $\sigma \subset \{1, 2, \ldots, m\}$ of cardinality*

$$k = \min(m^{p'/2}, (m/n^{2/p})^{p/(p-2)})$$

contains in turn a subset σ' of cardinality $|\sigma'| \ge 9|\sigma|/10$ such that the truncations $\psi_i = \varphi_i \chi_{[|\varphi_i| \le k^{1/p}]}$; $i \in \sigma'$, of $\{\varphi_i\}_{i\in\sigma'}$ satisfy the inequality

$$\left\| \sum_{i\in\sigma'} a_i \psi_i \right\|_{L_p} \le C_1(p) \|a\|_p,$$

for any $a = (a_i)_{i\in\sigma'} \in l_p^{|\sigma'|}$.

Proof. Put $\lambda = k^{1/p}$ and $\varphi_i^\lambda = \varphi_i - \psi_i$; $1 \le i \le m$. Then notice that

$$\lambda \left\| \sum_{i=1}^{m} \chi_{[|\varphi_i| > \lambda]} \right\|_\infty^{1/2} \le \left\| \left(\sum_{i=1}^{m} |\varphi_i|^2 \right)^{1/2} \right\|_\infty \le n^{1/p},$$

which yields

$$\left\| \sum_{i=1}^{m} \chi_{[|\varphi_i| > \lambda]} \right\|_\infty \le \lambda^{-2} n^{2/p}.$$

Hence, by Hölder's inequality, we get that

$$\left| \sum_{i=1}^{m} a_i \varphi_i^\lambda \right| \le \sum_{i=1}^{m} |a_i \varphi_i^\lambda| \le (\lambda^{-2} n^{2/p})^{1/p'} \left(\sum_{i=1}^{m} |a_i \varphi_i^\lambda|^p \right)^{1/p}$$

$$\le (n/k)^{2/pp'} \left(\sum_{i=1}^{m} |a_i \varphi_i|^p \right)^{1/p}$$

and, further, that

for all $a \in l_p^m$. It follows that

$$\left\| \sum_{i=1}^{m} a_i \psi_i \right\|_{L_p} \leq (m^{1/2-1/p} + (n/k)^{2/pp'}) \| a \|_p,$$

for all $a \in l_p^m$, again. Thus, by Proposition 1.1 applied to the functions $\{\psi_i\}_{i=1}^m$ with $\Gamma = m^{1/2-1/p} + (n/k)^{2/pp'}$, $B = n^{1/p}$, $\lambda = k^{1/p}$ and $\delta = k/m$, we get that a random subset $\sigma \subset \{1,2,\ldots,m\}$ of cardinality k contains a subset σ' such that $|\sigma'| \geq 9|\sigma|/10$ and

$$\left\| \sum_{i \in \sigma'} a_i \psi_i \right\|_{L_p} \leq C_1 [(k/m)^{1/p'} (m^{1/2-1/p} + (n/k)^{2/pp'}) + (k/m)^{1/2-1/p} (n/m)^{1/p} + 1] \leq 4C_1,$$

for some constant C_1, depending on p, and all $a \in l_p^{|\sigma'|}$ of norm ≤ 1. ■

We like to prove a similar estimate for the "peak" parts $\{\varphi_i^\lambda\}_{i=1}^m$ of $\{\varphi_i\}_{i=1}^m$. To this end, we present first the following lemma.

Lemma 2.2. *For every integer h, there exists a constant $C_2 = C_2(h) < \infty$ such that, whenever $\{\varphi_i\}_{i=1}^m$ is a sequence of functions in an $L_p(\mu)$-space; $p > 2$, that satisfies conditions 1°, 2° and 3° above, $k = \min(m^{p'/2}, (m/n^{2/p})^{p/(p-2)})$, $\lambda = k^{1/p}$, $\varphi_i^\lambda = \varphi_i \chi_{[|\varphi_i| > \lambda]}$; $1 \leq i \leq m$, then a random subset $\sigma \subset \{1,2,\ldots,m\}$ of cardinality k contains a subset σ'' of cardinality $|\sigma''| > 9|\sigma|/10$ such that, for all $1 \leq j \leq h$,*

$$\sum_{i_1,i_2,\ldots,i_h \in \sigma''}^{(j)} \int |\varphi_{i_1}^\lambda \varphi_{i_2}^\lambda \cdots \varphi_{i_h}^\lambda|^{p/h} \, d\mu \leq C_2(h); \qquad i_j \in \sigma'',$$

where $\sum_{i_1,i_2,\ldots,i_h \in \sigma''}^{(j)}$ denotes summation over all indices $i_1, i_2, \ldots, i_h \in \sigma''$, except i_j, which remains fixed.

Proof. In order to simplify our notation, put

$$M_{i_1,i_2,\ldots,i_h} = \int |\varphi_{i_1}^\lambda \varphi_{i_2}^\lambda \cdots \varphi_{i_h}^\lambda|^{p/h} \, d\mu,$$

for any $1 \leq i_1, i_2, \ldots, i_h \leq m$. Next, observe that, with $\delta = k/m$,

$$\int \sum_{i=1}^{m} [illegible] \qquad \int \sum_{i=1}^{m} [illegible] = m = \delta^{-1} k$$

Also, if $s > \max(p/2, h)$ then we have that

$$\int \left(\sum_{i=1}^{m} |\varphi_i^\lambda|^{p/s} \right)^s d\mu \le \int \left(\sum_{i=1}^{m} |\varphi_i^\lambda|^{p/s-2} |\varphi_i|^2 \right)^s d\mu$$

$$\le \lambda^{p-2s} \int \left(\sum_{i=1}^{m} |\varphi_i|^2 \right)^s d\mu \le k^{1-2s/p} n^{2s/p} = (n/k)^{2s/p} k \le \delta^{-s} k,$$

in view of the fact that $k \le (m/n^{2/p})^{p/(p-2)}$. Hence, by Hölder's inequality, we conclude that, for each $1 \le t \le s$,

$$\int \left(\sum_{i=1}^{m} |\varphi_i^\lambda|^{p/t} \right)^t d\mu \le \delta^{-t} k.$$

Then, with $\{\xi_i\}_{i=1}^m$ being again a sequence of $\{0,1\}$-valued independent random variables of mean δ over some probability space (Ω, Σ, ν) and $l(i_1, i_2, \ldots, i_h)$ denoting the exact number of distinct indices in the tuple $(i_1, i_2, \ldots, i_h)$, it follows that

$$\sum_{1 \le i_1, i_2, \ldots, i_h \le m} \int \xi_{i_1}(\omega) \xi_{i_2}(\omega) \cdots \xi_{i_h}(\omega) \, d\nu \, M_{i_1, i_2, \ldots, i_h}$$

$$\le \sum_{l=1}^{h} \delta^l \sum_{\substack{1 \le i_1, i_2, \ldots, i_h \le m \\ l(i_1, i_2, \ldots, i_h) = l}} M_{i_1, i_2, \ldots, i_h}.$$

Now, for a fixed l and fixed decomposition of h as a sum $h = h_1 + h_2 + \cdots + h_l$ of integers $\{h_j\}_{j=1}^l$, consider all the tuples $(i_1, i_2, \ldots, i_h)$ which contain exactly l distinct integers $j_1, j_2, \ldots, j_l$ having $h_1, h_2, \ldots, h_l$ as their respective multiplicities. Then the sum Σ' over these tuples only can be estimated by

$$\Sigma' M_{i_1, i_2, \ldots, i_h} \le \int \prod_{j=1}^{l} \left(\sum_{i=1}^{m} |\varphi_i^\lambda|^{ph_j/h} \right) d\mu$$

$$\le \prod_{j=1}^{l} \left[\int \left(\sum_{i=1}^{m} |\varphi_i^\lambda|^{ph_j/h} \right)^{h/h_j} d\mu \right]^{h_j/h} \le \prod_{j=1}^{l} (\delta^{-1} k^{h_j/h}) = \delta^{-l} k$$

(use Hölder's inequality with the indices $\{h/h_j\}_{j=1}^l$ which clearly satisfy

$$1/(h/h_1) + 1/(h/h_2) + \cdots + 1/(h/h_l) = 1).$$

Therefore, we conclude that

$$\sum_{i \le i_1, i_2, \ldots, i_h \le m} \int \xi_{i_1}(\omega) \xi_{i_2}(\omega) \cdots \xi_{i_h}(\omega) \, d\nu \, M_{i_1, i_2, \ldots, i_h} \le C_0 k,$$

for some constant $C_0 = C_0(h) < \infty$. Hence, for a random subset $\sigma \subset \{1,2,\ldots,m\}$ of cardinality $|\sigma| = \delta m = k$, we get that

$$\sum_{i_1,i_2,\ldots,i_h \in \sigma} M_{i_1,i_2,\ldots,i_h} \leq 2C_0 k.$$

This, of course, yields that any such σ contains in turn a subset σ'' of cardinality $|\sigma''| > 9|\sigma|/10$ such that

$$\sum_{i_1,i_2,\ldots,i_h \in \sigma''}^{(j)} M_{i_1,i_2,\ldots,i_h} \leq C_2(h),$$

for all $1 \leq j \leq h$, $i_j \in \sigma''$ and some constant $C_2(h)$, depending on h only. ∎

Proposition 2.3. *For every $p > 2$, there exists a constant $C_3 = C_3(p) < \infty$ such that, whenever $\{\varphi_i\}_{i=1}^m$ is a sequence of functions in an $L_p(\mu)$-space which satisfies the conditions 1°, 2° and 3° above, $k = \min(m^{p'/2}, (m/n^{2/p})^{p(p-2)})$, $\lambda = k^{1/p}$ and $\varphi_i^\lambda = \varphi_i \chi_{[|\varphi_i|>\lambda]}$; $1 \leq i \leq m$, then a random subset $\sigma \subset \{1,2,\ldots,m\}$ of cardinality k contains a subset σ'' of cardinality $|\sigma''| > 9|\sigma|/10$ such that*

$$\left\| \sum_{i\in\sigma''} a_i \varphi_i^\lambda \right\|_{L_p} \leq C_3 \|a\|_p,$$

for all $a = (a_1, a_2, \ldots, a_m) \in l_p^m$.

Proof. Fix an *integer* $q > p$, choose $0 < \theta < 1$ so that $1/p = \theta/q + (1-\theta)/2$ and notice that, by Hölder's inequality,

$$\sum_{i=1}^m |a_i\varphi_i^\lambda| = \sum_{i=1}^m |a_i\varphi_i^\lambda|^{\theta p/q} |a_i\varphi_i^\lambda|^{(1-\theta)p/2} \leq \left(\sum_{i=1}^m |a_i\varphi_i^\lambda|^{p/q}\right)^\theta \left(\sum_{i=1}^m |a_i\varphi_i^\lambda|^{p/2}\right)^{1-\theta},$$

from which it follows that

$$\left\| \sum_{i=1}^m a_i\varphi_i^\lambda \right\|_{L_p} \leq \left\| \sum_{i=1}^m |a_i\varphi_i^\lambda|^{p/q} \right\|_{L_q}^\theta \cdot \left\| \sum_{i=1}^m |a_i\varphi_i^\lambda|^{p/2} \right\|_{L_2}^{1-\theta},$$

for all $a \in l_p^m$.

In order to evaluate the expressions appearing on the right-hand side, let h be an integer standing for either 2 or q and observe that

$$\left\| \sum_{i=1}^m |a_i\varphi_i^\lambda|^{p/h} \right\|_{L_h}^h = \sum_{1\leq i_1,i_2,\ldots,i_h\leq m} |a_{i_1}a_{i_2}\cdots a_{i_h}|^{p/h} M_{i_1,i_2,\ldots,i_h},$$

where

$$M_{i_1,i_2,\dots,i_h}=\int|\varphi_{i_1}^\lambda\varphi_{i_2}^\lambda\cdots\varphi_{i_h}^\lambda|^{p/h}\,d\mu,$$

for $1\le i_1,i_2,\dots,i_h\le m$.

Let now σ be a random subset of $\{1,2,\dots,m\}$ having cardinality k and $\sigma''\subset\sigma$ the subset of cardinality $|\sigma''|>9|\sigma|/10$ given by Lemma 2.2 such that

$$\sum_{i_1,i_2,\dots,i_h\in\sigma''}^{(j)} M_{i_1,i_2,\dots,i_h}\le C_2,$$

for some $C_2=C_2(h)<\infty$, all $1\le j\le h$ and $i_j\in\sigma''$. Define next an h-linear form $T:\mathbf{R}^{|\sigma''|h}\to\mathbf{R}$ by setting

$$T(a^1,a^2,\dots,a^h)=\sum_{i_1,i_2,\dots,i_h\in\sigma''}a_{i_1}^1a_{i_2}^2\cdots a_{i_h}^hM_{i_1,i_2,\dots,i_h},$$

for $a^j=(a_i^j)_{i\in\sigma''}$; $1\le j\le h$. Notice that, for each $1\le j\le h$,

$$\begin{aligned}&|T(a^1,a^2,\dots,a^h)|\\&\le\sum_{i_j\in\sigma''}\sum_{i_1,i_2,\dots,i_h\in\sigma''}^{(j)}\|a^1\|_\infty\cdots\|a^{j-1}\|_\infty|a_{i_j}^j|\,\|a^{j+1}\|_\infty\cdots\|a^h\|_\infty M_{i_1,i_2,\dots,i_h}\\&\le C_2\|a^1\|_\infty\cdots\|a^{j-1}\|_\infty\|a^j\|_1\|a^{j+1}\|_\infty\cdots\|a^h\|_\infty,\end{aligned}$$

for any $a^j\in\mathbf{R}^{|\sigma''|}$; $1\le j\le h$. Therefore, by multilinear interpolation (cf., e.g., [2] p. 20), we conclude that

$$|T(a^1,a^2,\dots,a^h)|\le C_3\|a^1\|_h\|a^2\|_h\cdots\|a^h\|_h,$$

for some constant C_3, depending only on p (since h depends on p only), and all $a^j\in l_h^{|\sigma''|}$; $1\le j\le h$. Hence, for any sequence $a=(a_i)_{i\in\sigma''}$ with $\|a\|_p\le1$, we have that

$$\left\|\sum_{i\in\sigma''}|a_i\varphi_i^\lambda|^{p/h}\right\|_{L_h}^h=T(|a|^{p/h},|a|^{p/h},\dots,|a|^{p/h})\le C_3\|a^{p/h}\|_h^h\le C_3,$$

which further yields that

$$\left\|\sum_{i\in\sigma''}a_i\varphi_i^\lambda\right\|_{L_p}\le C_3. \qquad ■$$

COROLLARY 2.4. *For every $p>2$, there exists a constant $D=D(p)<\infty$ such that, whenever $\{\varphi_i\}_{i=1}^m$ is a sequence of elements in an $L_p(\mu)$-space, for some* *probability measure μ, which satisfies the conditions 1°, 2° and 3° above, then a random subset $\sigma \subset \{1,2,\dots,m\}$ of cardinality*

$$k = \min(m^{p'/2}, (m/n^{2/p})^{p/(p-2)})$$

contains in turn a subset σ_0 of cardinality $|\sigma_0| > |\sigma|/2$ such that

$$\left\| \sum_{i\in\sigma_0} a_i\varphi_i \right\|_{L_p} \le D\|a\|_p,$$

for any choice of $a = (a_i)_{i\in\sigma_0} \in l_p^{|\sigma_0|}$.

The next major step in the proof of Theorem 0.1 is to construct a suitable system of vectors in an arbitrary m-dimensional subspace X of an L_p^n-space for which Corollary 2.4 can be applied.

PROPOSITION 2.5. *For every $p > 2$, there exists a constant $c = c(p) > 0$ such that any m-dimensional subspace X of L_p^n contains a sequence $\{\varphi_i\}_{i=1}^{m'}$ of elements that satisfies conditions 1°, 2° and 3° above and, in addition, $m' > m/10$ and*

$$\|\varphi_i\|_{L_p^n} \ge c,$$

for all $1 \le i \le m'$.

PROOF. Fix $p > 2$, $1 < m \le n$, and let X be an m-dimensional subspace of L_p^n. By a result of D. Lewis [12], one may assume that, after a change of density, X_2 (i.e. X considered as a subspace of the corresponding $L_2^n(\mu)$-space) admits an $L_2^n(\mu)$-normalized orthogonal system $\{\zeta_i\}_{i=1}^m$ such that

$$\sum_{i=1}^m |\zeta_i|^2 = m.$$

Since any other orthonormal system in X_2 is obtained from $\{\zeta_i\}_{i=1}^m$ by applying a unitary transformation it follows that any $L_2^n(\mu)$-normalized orthogonal system $\{\zeta_i'\}_{i=1}^m$ satisfies the condition

$$\sum_{i=1}^m |\zeta_i'|^2 = m.$$

Next, we use a well-known result of F. John [10] in order to conclude the existence of an invertible operator u from l_2^m onto X_p (i.e. X considered as a subspace of $L_p^n(\mu)$) such that $\|u\| = 1$ and $\pi_2(u^{-1}) = m^{1/2}$.

In order to avoid confusion, we shall denote the norm in the space l_2^m by $|\cdot|_2$ to distinguish it from that in $L_2^n(\mu)$, which is denoted by $\|\cdot\|_{L_2^n}$.

The operator u is then used to construct, by induction, a sequence of vectors $\{w_j\}_{j=1}^J$ in l_2^m such that $J \geq m/3$ and

(i) $$|w_j|_2 = 1,$$

(ii) $$w_j \perp w_{j'} \qquad \text{in } l_2^m,$$

(iii) $$\|u(w_j)\|_{L_p^n} > 1/2, \qquad \text{and}$$

(iv) $$u(w_j) \perp u(w_{j'}) \qquad \text{in } L_2^n,$$

for all $1 \leq j \neq j' \leq J$.

Indeed, suppose that $\{w_j\}_{j=1}^l$ have already been constructed for some $l < m/3$ so that conditions (i)–(iv) are fulfilled and notice that the subspace H of l_2^m, defined by

$$H = [w_1, w_2, \ldots, w_l]^\perp \cap u^{-1}([u(w_1), u(w_2), \ldots, u(w_l)]^\perp),$$

is of dimension $\geq m - 2l > m/3$ and the identity map i_H on H can be written as

$$i_H = u_{|u(H)}^{-1} \circ u_{|H}.$$

Hence,

$$(m/3)^{1/2} < \pi_2(i_H) \leq \pi_2(u^{-1})\|u_{|H}\| \leq m^{1/2}\|u_{|H}\|$$

which further yields that

$$\|u_{|H}\| \geq 1/\sqrt{3}.$$

It follows that there exists a vector w_{l+1} in H so that $|w_{l+1}|_2 = 1$ and $\|u(w_{l+1})\|_{L_p^n} > 1/2$, thus completing the induction argument.

Define

$$\psi_j = u(w_j)/\|u(w_j)\|_{L_p^n}; \qquad 1 \leq j \leq J,$$

and notice that

$$\left\|\sum_{j=1}^J a_j\psi_j\right\|_{L_p^n} \leq 2\left|\sum_{j=1}^J a_j w_j\right|_2 = 2\left(\sum_{j=1}^J |a_j|^2\right)^{1/2} \leq 2m^{1/2-1/p}\left(\sum_{j=1}^J |a_j|^p\right)^{1/p},$$

for any choice of $(a_j)_{j=1}^J \in l_p^J$.

Assume now that $\{\psi_j\}_{j=1}^J$ have been reordered so as to have

$$\|\psi_1\|_{L_2^n} \leq \|\psi_2\|_{L_2^n} \leq \cdots \leq \|\psi_J\|_{L_2^n}$$

and that J is an even integer. By the inequality above and the orthogonality of the vectors $\{\psi_j\}_{j=1}^J$ in L_2^n, we have that

$$2\left(\sum_{j=J/2}^{J}|a_j|^2\right)^{1/2} \geq \left\|\sum_{j=J/2}^{J} a_j\psi_j\right\|_{L_p^n} \geq \left\|\sum_{j=J/2}^{J} a_j\psi_j\right\|_{L_2^n} \geq \left(\sum_{j=J/2}^{J}|a_j|^2\right)^{1/2}\|\psi_{J/2}\|_{L_2^n},$$

for all $(a_j)_{j=J/2}^J$. Thus, as we have explained in the introduction,

$$2\|\psi_{J/2}\|_{L_2^n}^{-1} \geq d([\psi_j]_{j=J/2}^J, l_2^{J/2}) \geq am^{1/2}/n^{1/p},$$

for some constant $1 > a > 0$, depending on p only. Consequently,

$$\|\psi_j\|_{L_2^n} \leq 2n^{1/p}/am^{1/2}; \qquad 1 \leq j \leq J/2,$$

from which one deduces that

$$\sum_{j=1}^{J/2}|\psi_j|^2 \leq \frac{4n^{2/p}}{a^2m}\sum_{j=1}^{J/2}\frac{|\psi_j|^2}{\|\psi_j\|_{L_2^n}^2} \leq \frac{4n^{2/p}}{a^2}.$$

Therefore, the system $\varphi_j = a\psi_j/2$; $1 \leq j \leq J/2$, will satisfy the conditions 1°, 2° and 3° above with $m' > m/10$. Moreover, $\|\varphi_j\|_{L_p^n} = a/2$, for all $1 \leq j \leq J/2$. ■

Proof of Theorem 0.1. Let X be an m-dimensional subspace of L_p^n, for some $p > 2$. By Proposition 2.5, construct a system $\{\varphi_j\}_{j=1}^{m'}$ of vectors in X with $m' > m/10$ which satisfies the conditions 1°, 2° and 3° above and, in addition, $\|\varphi_j\|_{L_p^n} \geq c$, for some $c = c(p) > 0$ and all $1 \leq j \leq m'$. Then, by using Corollary 2.4, conclude the existence of a subset $\sigma_0 \subset \{1, 2, \ldots, m'\}$ of cardinality $\geq bk$, for some $b = b(p) > 0$, such that

$$\left\|\sum_{j\in\sigma_0} a_j\varphi_j\right\|_{L_p^n} \leq D\|a\|_p,$$

for all $a \in l_p^{|\sigma_0|}$, where D is the constant appearing in the statement of Corollary 2.4. This upper estimate together with the fact that $\|\varphi_j\|_{L_p^n} \geq c$, for $j \in \sigma_0$, imply, by interpolation, that

$$\|\max_{j\in\sigma_0}|\varphi_j|\,\|_{L_p^n} \geq c_1|\sigma_0|^{1/p},$$

for some constant $c_1 = c_1(p) > 0$. Therefore, there exists a subset $\sigma_1 \subset \sigma_0$ of cardinality $|\sigma_1| \geq c_1^p|\sigma_0|/2$ and mutually disjoint subsets $\{\eta_j\}_{j\in\sigma_1}$ of $\{1, 2, \ldots, n\}$ such that $\|\varphi_j\chi_{\eta_j}\|_{L_p^n} \geq c_1/\sqrt{2}$; $j \in \sigma_1$. Hence, in view of Proposition 4.4 from [5], it follows that

$$\left\| \sum_{j \in \sigma_2} a_j \varphi_j \right\|_{L_p^n} \geq c_2 \| a \|_p,$$

for some constant $c_2 = c_2(p) > 0$, some subset $\sigma_2 \subset \sigma_1$ of cardinality proportional to that of σ_1 and all $a \in l_p^{|\sigma_2|}$. This proves the assertion of Theorem 0.1 for *some* $\epsilon > 0$. The passage from this particular case to the general one follows from the result presented in the next section. ∎

3. Almost isometric copies of l_p^m; $p \geq 2$

The object of this section is to show that, for $p \geq 2$, a sequence of vectors in an L_p-space, which is equivalent to the unit vector basis of l_p^m, contains in turn a subsequence of length m' proportional to m that spans an almost isometric copy of $l_p^{m'}$. The exact statement is as follows.

THEOREM 3.1. *For every $p \geq 2$, $1 < K < \infty$ and $0 < \epsilon < 1$, there exists a constant $c = c(p,K,\epsilon) > 0$ such that, whenever $\{f_i\}_{i=1}^m$ is a sequence of normalized functions in an L_p-space which satisfies the condition*

$$K^{-1} \| a \|_p \leq \left\| \sum_{i=1}^m a_i f_i \right\|_{L_p} \leq K \| a \|_p,$$

for all $a = (a_i)_{i=1}^m \in l_p^m$, then one can find a subset $\tau \subset \{1,2,\ldots,m\}$ such that $|\tau| \geq cm$ and

$$(1-\epsilon)\| a \|_p \leq \left\| \sum_{i \in \tau} a_i f_i \right\|_{L_p} \leq (1+\epsilon)\| a \|_p,$$

for any $a = (a_i)_{i \in \tau} \in l_p^{|\tau|}$.

REMARK. The assertion of Theorem 3.1 is false for $1 \leq p < 2$, as simple examples show. It is also false even, for $p \geq 2$, if the underlying space is not an L_p-space.

PROOF OF THEOREM 3.1 FOR $p = 2$. Consider the matrix $(\langle f_i, f_j \rangle)_{i,j=1}^m$ as a linear operator T acting on l_2^m. The assumptions made on the functions $\{f_i\}_{i=1}^m$ imply that T is of norm $\leq K^2$ and has 1's on the diagonal. Therefore, by [5] Theorem 1.6 (or [7] Corollary 1.2 for a sharper version), one can conclude the existence of a constant $c = c(2,K,\epsilon) > 0$ and of a subset $\tau \subset \{1,2,\ldots,m\}$ such that $|\tau| \geq cm$ and $\| R_\tau (T - I) R_\tau \| < \epsilon^2$, where R_τ denotes the restriction operator defined by

$$R_\tau\left(\sum_{i=1}^m a_i e_i\right) = \sum_{i\in\tau} a_i e_i,$$

for all $a = (a_i)_{i=1}^m \in l_2^m$.

However, for any choice of $a = (a_i)_{i\in\tau} \in l_2^{|\tau|}$ with norm equal to 1, we have that

$$\left| \left\| \sum_{i\in\tau} a_i f_i \right\|_{L_2}^2 - \|a\|_2^2 \right| = \left| \sum_{\substack{i,j\in\tau\\ i\neq j}} a_i a_j \langle f_i, f_j\rangle \right| = \left| \left\langle (T-I)\left(\sum_{i\in\tau} a_i e_i\right), \sum_{i\in\tau} a_i e_i \right\rangle \right| < \epsilon^2,$$

which, of course, completes the proof in this case. ■

Proof of Theorem 3.1 for $p > 2$. This case is considerably more complicated than the previous one and requires again the use of Proposition 1.1.

Let $\{f_i\}_{i=1}^m$ be a sequence of functions in an $L_p(\mu)$-space, for some $p > 2$ and some probability measure μ, which satisfies the assumptions of Theorem 3.1, for some $K > 1$. Then, as is easily verified,

$$\left\| \left(\sum_{i=1}^m |f_i|^2\right)^{1/2} \right\|_{L_p(\mu)} \leq K m^{1/p}.$$

Put

$$F = \left(\sum_{i=1}^m |f_i|^2\right)^{1/2} \Bigg/ \left\| \left(\sum_{i=1}^m |f_i|^2\right)^{1/2} \right\|_{L_p(\mu)},$$

define the measure ν by setting $d\nu = F^p \, d\mu$ and notice that the map $f \to f/F$ from $L_p(\mu)$ onto $L_p(\nu)$ is a linear isometry that takes the functions f_i into $g_i = f_i/F$; $1 \leq i \leq m$. Moreover,

$$\left\| \left(\sum_{i=1}^m |g_i|^2\right)^{1/2} \right\|_\infty \leq K m^{1/p}.$$

Fix now $\epsilon > 0$, $2 < s < p < q$ and $\lambda = \delta^{1/s} m^{1/p}$, where $0 < \delta < 1$ will be chosen later, and observe that

$$\lambda \left\| \sum_{i=1}^m \chi_{[|g_i| > \lambda]} \right\|_\infty^{1/2} \leq \left\| \left(\sum_{i=1}^m |g_i|^2\right)^{1/2} \right\|_\infty.$$

Thus,

$$\left\| \sum_{i=1}^m \chi_{[|g_i| > \lambda]} \right\|_\infty \leq K^2/\delta^{2/s}$$

which further yields that the functions $g_i^\lambda = g_i \chi_{[|g_i|>\lambda]}$; $1 \le i \le m$, satisfy the estimate

$$\left\| \sum_{i=1}^m a_i g_i^\lambda \right\|_{L_p(\nu)} \le (K^{2/p'}/\delta^{2/sp'}) \| a \|_p,$$

for all $a = (a_i)_{i=1}^m \in l_p^m$. It follows that the truncates $g_{i,\lambda} = g_i - g_i^\lambda$ of g_i; $1 \le i \le m$, satisfy the upper estimate

$$\left\| \sum_{i=1}^m a_i g_{i,\lambda} \right\|_{L_p(\nu)} \le [K + K^{2/p'}/\delta^{2/sp'}] \| a \|_p,$$

for $a \in l_p^m$. Therefore, by applying Proposition 1.1, we conclude the existence of a constant $C(q) < \infty$ and of a subset $\sigma \subset \{1,2,\dots,m\}$ so that $|\sigma| = [\delta m]$ and

$$\left\| \sum_{i \in \sigma} a_i g_{i,\lambda} \right\|_{L_p(\nu)} \le C(q) [K\delta^{1/p'} + K^{2/p'}\delta^{(1-2/s)/p'} + K\delta^{1/2-1/p} + \delta^{1/s-1/p}] \| a \|_p,$$

for $a = (a_i)_{i \in \sigma} \in l_p^{|\sigma|}$.

Suppose now that we have chosen $0 < \delta < 1$ so as to ensure that

$$\left\| \sum_{i \in \sigma} a_i g_{i,\lambda} \right\|_{L_p(\mu)} \le \epsilon \| a \|_p,$$

for $a \in l_p^{|\sigma|}$, and consider now the peak parts $\{g_i^\lambda\}_{i \in \sigma}$ of $\{g_i\}_{i \in \sigma}$. Put

$$B_i = [|g_i| > \lambda] = \operatorname{supp} g_i^\lambda; \qquad i \in \sigma,$$

and consider the matrix $(\int_{B_j} |g_i^\lambda|^p d\nu)_{i,j \in \sigma}$ as a linear operator V acting on the space $l_1^{|\sigma|}$. Since

$$\sum_{j \in \sigma} \int_{B_j} |g_i^\lambda|^p d\nu = \int |g_i^\lambda|^p \Big(\sum_{j \in \sigma} \chi_{B_j} \Big) d\nu \le \| g_i^\lambda \|_{L_p(\nu)}^p \left\| \sum_{i=1}^m \chi_{B_j} \right\|_\infty \le K^2/\delta^{2/s},$$

for all $i \in \sigma$, the operator V has norm $\le K^2/\delta^{2/s}$. Thus, by a result from [11] or [3], one can find a constant $c_1 = c_1(p,K,\epsilon) > 0$ and a subset $\tau \subset \sigma$ such that $|\tau| \ge c_1 |\sigma|$ and

$$\| R_\tau (V - I) R_\tau \| < \epsilon^p (\delta^{2/s}/K^2)^{p-1}.$$

This fact can be reinterpreted as to assert that

$$\left\| \sum_{i \in \tau} |a_i| |g_i^\lambda| \chi_{\bigcup_{j \in \tau, j \ne i} B_j} \right\|_{L_p(\nu)} \le (K^2/\delta^{2/s})^{1/p'} \left\| \Big(\sum_{i \in \tau} |a_i|^p |g_i^\lambda|^p \chi_{\bigcup_{j \in \tau, j \ne i} B_j} \Big)^{1/p} \right\|_{L_p(\nu)} < \epsilon \| a \|_p,$$

for all $a \in l_p^{|\tau|}$. Also notice that

$$\Big\| g_i^\lambda \chi_{B_i \sim (\bigcup_{j\in\tau, j\neq i} B_j)} \Big\|_{L_p(\nu)} \geq \| g_i^\lambda \|_{L_p(\nu)} - \Big\| g_i^\lambda \chi_{B_i \cap (\bigcup_{j\in\tau, j\neq i} B_j)} \Big\|_{L_p(\nu)}$$

$$\geq 1 - \| g_{i,\lambda} \|_{L_p(\nu)} - \Big\| g_i^\lambda \sum_{j\in\tau, j\neq i} \chi_{B_j} \Big\|_{L_p(\nu)} \geq 1 - 2\epsilon,$$

for all $i \in \tau$.

By combining these estimates, we obtain that

$$\Big\| \sum_{i\in\tau} a_i g_i^\lambda \Big\|_{L_p(\nu)} \geq \Big\| \sum_{i\in\tau} a_i g_i^\lambda \chi_{B_i \sim (\bigcup_{j\in\tau, j\neq i} B_j)} \Big\|_{L_p(\nu)} - \epsilon \| a \|_p$$

$$\geq \Big(\sum_{i\in\tau} |a_i|^p \Big\| g_i^\lambda \chi_{B_i \sim (\bigcup_{j\in\tau, j\neq i} B_j)} \Big\|_{L_p(\nu)}^p \Big)^{1/p} - \epsilon \| a \|_p \geq (1 - 3\epsilon) \| a \|_p$$

and

$$\Big\| \sum_{i\in\tau} a_i g_i^\lambda \Big\|_{L_p(\nu)} \leq \Big\| \sum_{i\in\tau} a_i g_i^\lambda \chi_{B_i \sim (\bigcup_{j\in\tau, j\neq i} B_j)} \Big\|_{L_p(\nu)} + \epsilon \| a \|_p \leq (1 + \epsilon) \| a \|_p,$$

for any $a = (a_i)_{i\in\tau} \in l_p^{|\tau|}$. This yields that

$$(1 - 4\epsilon) \| a \|_p < \Big\| \sum_{i\in\tau} a_i g_i \Big\|_{L_p(\nu)} < (1 - 2\epsilon) \| a \|_p,$$

again, for all $a \in l_p^{|\tau|}$, thus completing the proof. ∎

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