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POLYNOMIAL ALGEBRAS ON CLASSICAL BANACH SPACES

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ABSTRACT

For the classical Banach spaces $X = \ell_p$, $C(K)$ we identify all n such that every polynomial of degree $n+1$ on X is uniformly approximable on the unit ball by elements of the algebra generated by all polynomials of degree up to n on X.

The classical Stone–Weierstrass theorem claims that the algebra of all real polynomials on a finite-dimensional real Banach space X is dense, in the topology of uniform convergence on bounded sets (we will always consider this topology, unless otherwise stated), in the space of continuous real functions on X.

On the other hand ([12]), on every infinite-dimensional Banach space X there exists a uniformly continuous real function not approximable by continuous polynomials. Moreover, on some spaces (e.g. ℓ_p — see [12], [5]) a new phenomenon occurs; the closure of the algebra generated by polynomials of degree at most n ($\mathcal{A}_n$) does not contain all polynomials of higher degree.

In our paper we completely clarify this situation for the classical Banach spaces. We also present some partial answers in the general case. With exception of $C(K)$ Asplund spaces, our results are new.

Our strategy rests on the same basic idea, used to obtain the previous partial results in [12], [5], that the polynomial $P\big((x_i)\big) = \sum_{i=1}^{\infty} x_i^n$ on ℓ_2 is not approximable by polynomials from $\mathcal{A}_m(\ell_2)$ for many values $n, m \in \mathbb{N}$. However, in order to obtain a precise characterization, we develop a new finite-dimensional method to handle polynomial approximations.

Roughly speaking, we pass from approximation to precise equality by proving that if $\sum_{i=1}^{\infty} x_i^n \in \overline{\mathcal{A}_{n-1}(\ell_2)}$, then for some $k \in \mathbb{N}$ and a certain finitely generated

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algebra $\mathcal{A}$ of polynomials on $\mathbb{R}^k$, $\sum_{i=1}^k x_i^n \in \mathcal{A}$. We show that this leads to a contradiction, due to the algebraic independence of $\mathcal{A}$ and $\{\sum_{i=1}^k x_i^n\}$. The method is based on a generalization of the well-known algebraic theory of symmetric polynomials on $\mathbb{R}^k$.

ACKNOWLEDGEMENT: Before we pass to the mathematical part of our note, we would like to take the opportunity to thank R. Aron for bringing this problem to our attention, as well as for his exciting lectures at the Paseky Spring School 1996. It was he who had the right intuition that $\sum_{i=1}^\infty x_i^n$ does not belong to $\overline{\mathcal{A}_{n-1}(\ell_2)}$ for any $n \in \{2,3,\dots\}$, pointing towards the general solution.

By a subsymmetric polynomial on $\mathbb{R}^n$ we mean a real polynomial P satisfying $P(x) = P(y)$ for every pair $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$ of elements of $\mathbb{R}^n$ such that the sequences formed by all nonzero coordinates of x and y coincide (e.g. $x = (2,0,0,1.5,\pi,0)$, $y = (0,2,1.5,0,0,\pi)$). By $H_k(\mathbb{R}^n)$, $1 \le k \le n$, we denote the finite-dimensional vector space consisting of all subsymmetric homogeneous polynomials on $\mathbb{R}^n$ of degree k. $H_k(\mathbb{R}^n)$ has a basis consisting of subelementary polynomials, denoted by $(\alpha_1,\dots,\alpha_m)$ where $\alpha_i \in \mathbb{N}$, $\sum_{i=1}^m \alpha_i = k$. We define

$$(\alpha_1,\dots,\alpha_m)(x_1,\dots,x_n) = \sum_{i_1<\dots<i_m} x_{i_1}^{\alpha_1}\cdot\ldots\cdot x_{i_m}^{\alpha_m}.$$

Note that every subsymmetric polynomial on $\mathbb{R}^n$ can be written uniquely as a linear combination of subelementary polynomials (a standard form of a subsymmetric polynomial). The set of polynomials $\bigcup_{l=1}^k H_l(\mathbb{R}^n)$ generates (using pointwise addition and multiplication, as well as scalar multiplication) an algebra $S_k(\mathbb{R}^n)$, which is a subalgebra of the algebra of all polynomials on $\mathbb{R}^n$.

Analogously, we say that a polynomial P is symmetric on $\mathbb{R}^n$ if $P(x) = P(y)$ for every pair $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$ such that for some permutation π of $\{1,\dots,n\}$, $(x_{\pi(1)},\dots,x_{\pi(n)}) = (y_1,\dots,y_n)$.

By $\mathrm{Sym}_k(\mathbb{R}^n)$ we denote the algebra generated by symmetric polynomials of degree less than or equal to k. Important examples of homogeneous symmetric polynomials on $\mathbb{R}^n$ are σ_k, $1 \le k \le n$. By definition,

$$\sigma_k(x_1,\dots,x_n) = \sum_{i_1<i_2<\dots<i_k} x_{i_1}\cdot x_{i_2}\cdot\ldots\cdot x_{i_k}.$$

By classical results, for every $\phi \in \mathrm{Sym}_k(\mathbb{R}^n)$, $k \le n$, there exists a unique polynomial $P(y_1,\dots,y_k)$ in k variables, such that

$$\phi(x_1,\dots,x_n) = P\big(\sigma_1(x_1,\dots,x_n),\sigma_2(x_1,\dots,x_n),\dots,\sigma_k(x_1,\dots,x_n)\big).$$

In order to generalize this result for the case $S_k(\mathbb{R}^n)$, $k \leq n$, let us define the notion of an algebraic basis of a given algebra $\mathcal{A}$ over $\mathbb{R}$. The set $B \subseteq \mathcal{A}$ forms an algebraic basis of $\mathcal{A}$ if for every $a \in \mathcal{A}$ there exists a unique finite subset $b_1, \dots, b_k$ of B and a unique polynomial $P(y_1, \dots, y_k)$ such that

$$P(b_1, \dots, b_k) = a.$$

The set $B \subset \mathcal{A}$ is called algebraically independent if for no finite subset $b_1, \dots, b_k$ of B and no nontrivial polynomial $P(y_1, \dots, y_k)$ we have $P(b_1, \dots, b_k) = 0$. In what follows, we will also use the fact that $H_k(\mathbb{R}^k)$ and $H_k(\mathbb{R}^n)$, $n > k$, are canonically isomorphic (via the standard form, or equivalently by restriction of elements of $H_k(\mathbb{R}^n)$ onto the first k coordinates). Thus, the elements $H_k(\mathbb{R}^k) \subseteq S_k(\mathbb{R}^k)$ will be assumed to be considered (via the canonical extension using the standard form) elements of $H_k(\mathbb{R}^n) \subseteq S_k(\mathbb{R}^n)$, $n > k$, and vice versa.

For every Banach space X, we denote by $\mathcal{P}_n(X)$, $n \geq 1$ the space of all n-homogeneous real polynomials on X. Let us point out that polynomials from $\mathcal{P}_n(X)$ in general involve an infinite number of indeterminates, as is usual in Banach space theory. More precisely, $p(x) \in \mathcal{P}_n(X)$ if there exists an n-linear bounded functional $m(x_1, \dots, x_n)$ on X^n such that $p(x) = m(x, \dots, x)$ on X. By $\mathcal{P}(X)$ we denote the space of all polynomials f on X (i.e. functions of the form $f = f_1 + \cdots + f_n$ where $f_i \in \mathcal{P}_i$) and by $\mathcal{A}_n(X)$ we denote the algebra generated by elements from $\bigcup_{i=1}^n \mathcal{P}_i(X)$.

For classical results on symmetric polynomials we refer to [15]. Results on real analytic functions (or its holomorphic counterparts) are contained in [8], [9]. Facts about subsymmetric polynomials can be found in [6], [12], [5].

LEMMA 1: *$S_n(\mathbb{R}^n)$ has an algebraic basis $B_n = \{b_1, \dots, b_{k(n)}\}$ consisting of standard polynomials. Moreover, $\sigma_1, \dots, \sigma_n \in B_n$.*

Proof: By induction. For $n = 1$, we put $b_1 = \sigma_1$.

INDUCTION STEP FROM n TO $(n+1)$. We will assume that $b_i \in \{b_1, \dots, b_{k(n)}\}$ are homogeneous, $\deg(b_i) \leq n$, $b_i = (\alpha_1^i, \alpha_2^i, \dots, \alpha_{k_i}^i)$, $\sigma_1, \dots, \sigma_n \in B_n$.

For every $f \in S_n(\mathbb{R}^n)$ there exists a unique polynomial $P(y_1, \dots, y_{k(n)})$ such that $f(x_1, \dots, x_n) = P(b_1(x_1, \dots, x_n), \dots b_{k(n)}(x_1, \dots, x_n))$. In particular, there exists no nontrivial polynomial $P(y_1, \dots, y_{k(n)})$ such that

$$P(b_1(x_1, \dots, x_n), b_2(x_1, \dots, x_n), \dots, b_{k(n)}(x_1, \dots, x_n)) \equiv 0 \tag{1}$$

on $\mathbb{R}^n$.

Therefore, b_i are algebraically independent as elements of $S_{n+1}(\mathbb{R}^{n+1})$, since for any nontrivial polynomial $P(y_1,\dots,y_{k(n)})$ there exists some $(x_1,\dots,x_n,0)$ such that

$$P(b_1(x_1,\dots,x_n,0),\dots)\neq 0.$$

We will extend the set $B_n\subset S_{n+1}(\mathbb{R}^{n+1})$ into an algebraic basis B_{n+1} of $S_{n+1}(\mathbb{R}^{n+1})$ as follows: Put

$$M_{n+1}=\{b_1^{\alpha_1}\cdot b_2^{\alpha_2}\cdot\dots\cdot b_{k(n)}^{\alpha_{k(n)}},\alpha_i\in\mathbb{N}\text{ are such that }\sum\alpha_i\cdot\deg(b_i)=n+1\}.$$

Clearly, M_{n+1} is a finite set of homogeneous polynomials of degree $(n+1)$ from $S_{n+1}(\mathbb{R}^{n+1})$. Elements of M_{n+1} are linearly independent as vectors from $H_{n+1}(\mathbb{R}^{n+1})$.

Choose subelementary polynomials $b_{k(n)+1}=(\alpha_1^{k(n)+1},\dots),\dots,b_{k(n+1)}=(\alpha_1^{k(n+1)},\dots)$ such that $M_{n+1}\cup\{b_{k(n)+1},\dots,b_{k(n+1)}\}$ is a vector space basis of $H_{n+1}(\mathbb{R}^{n+1})$. (Later we will show that we may choose $b_{k(n)+1}:=\sigma_{n+1}$, so, in particular, M_{n+1} is not a basis of $H_{n+1}(\mathbb{R}^{n+1})$.)

It is clear that B_n is an algebraic basis of $S_n(\mathbb{R}^{n+1})$. Therefore, B_{n+1} generates $S_{n+1}(\mathbb{R}^{n+1})$. We want to prove that B_{n+1} is an algebraic basis of $S_{n+1}(\mathbb{R}^{n+1})$. Assume the contrary, i.e. there is a nontrivial $P(y_1,\dots,y_{k(n+1)})$ such that

$$P(b_1(x_1,\dots,x_{n+1}),\dots,b_{k(n+1)}(x_1,\dots,x_{n+1}))\equiv 0$$

on $\mathbb{R}^{n+1}$.

We may assume that for some $1\le j\le k(n+1)$,

$$\frac{\partial P}{\partial y_j}(y_1^0,\dots,y_{k(n+1)}^0)\neq 0,$$

where $y_j^0=b_j(x_1^0,\dots,x_n^0)$ for some $(x_1^0,\dots,x_n^0)\in\mathbb{R}^{n+1}$.

Indeed, otherwise we would choose $\partial P/\partial y_j$ in place of P. By repeated choices we would get that $\partial P/\partial y_i,\partial y_j,\dots\equiv 0$ on $\mathbb{R}^{n+1}$ for all choices of $y_i,y_j,\dots$ and so $P(y_1,\dots,y_{k(n+1)})=0$ on $\mathbb{R}^{k(n+1)}$.

By the real analytic implicit function theorem, in some neighbourhood of the point $y_i^0=b_i(x_1^0,\dots,x_n^0)$, we have

$$y_j=\Phi(y_1,\dots,y_{j-1},y_{j+1},\dots)$$

where Φ is real analytic.

Therefore, in some neighbourhood of $(x_1^0, \dots, x_n^0)$, using the Taylor expansion of Φ, we have

$$\begin{aligned} b_j(x_1, \dots, x_n) &= \Phi(b_1(x_1, \dots, x_n), \dots) \\ &= \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_{n+1}=0}^{\infty} \beta_{\alpha_1, \dots, \alpha_{n(k+1)}} \cdot b_1^{\alpha_1} \cdot b_2^{\alpha_2} \cdot \dots \cdot b_{n(k+1)}^{\alpha_{n+1}}. \end{aligned}$$

Note that due to the form of Φ, $b_j(x_1, \dots, x_n)$ is excluded from the right hand side of the above equality. On both sides, we have real analytic functions in variables $x_1, \dots, x_n$. Thus the corresponding coefficients must be equal. So

$$\sum_{\sum \alpha_i \cdot \deg(b_i) = \deg(b_j)} \beta_{\alpha_1, \dots} b_1^{\alpha_1} \cdots b_{n(k+1)}^{\alpha_{n+1}} = b_j.$$

This is a contradiction. Indeed, if $\deg(b_j) < n+1$, we would have that B_n is not algebraically independent, and if $\deg(b_j) = n+1$, $M_{n+1} \cup \{b_{k(n)+1}, \dots\}$ would not be linearly independent.

We have established the existence of the algebraic basis for $S_{n+1}(\mathbb{R}^{n+1})$.

Before we proceed further, let us make the following easy observation. Suppose that, for $1 \le i \le l$, there are $\rho_i \in \mathbb{N}$ such that $\sum_{i=1}^{l} \rho_i = \rho < n$. Consider the differential operator

$$D = \frac{\partial^\rho}{\partial x_n^{\rho_1} \partial x_{n-1}^{\rho_2} \cdots \partial x_{n-l+1}^{\rho_l}}$$

acting from $S_n(\mathbb{R}^n)$ into $\mathcal{P}(\mathbb{R}^n)$.

By putting $x_n = 0, \dots, x_{n-\rho+1} = 0$ we may consider an operator $\tilde{D}$: $S_n(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^{n-\rho})$,

$$\tilde{D}(p)(x_1, \dots, x_{n-\rho}) = D(p)(x_1, \dots, x_{n-\rho}, 0, 0, \dots, 0).$$

OBSERVATION: *$\tilde{D}$ sends polynomials of degree d to polynomials of degree at most $d - \rho$.*

$\tilde{D}(S_n(\mathbb{R}^n)) \subseteq S_{n-\rho}(\mathbb{R}^{n-\rho})$.

$\tilde{D}(\mathrm{Sym}_n(\mathbb{R}^n)) \subseteq \mathrm{Sym}_{n-\rho}(\mathbb{R}^{n-\rho})$.

Proof: The first part is well known. To show that $\tilde{D}(P)$ is a subsymmetric polynomial for every $p \in S_n(\mathbb{R}^n)$, it is enough to show this for any subelementary polynomial $p = (\alpha_1, \dots, \alpha_m)$, $\sum \alpha_i \le n$. However, for such p

$$\tilde{D}(p) = \begin{cases} \rho_1! \cdot \rho_2! \cdot \dots \cdot \rho_l! \cdot (\alpha_1, \dots, \alpha_{m-l}) & \text{iff } \alpha_m = \rho_1, \alpha_{m-1} = \rho_2, \dots, \\ & \qquad \alpha_{m-l+1} = \rho_l, \\ 0 & \text{otherwise.} \end{cases}$$

The symmetric case is similar. ∎

We proceed by showing that $b_{k(n)+1}$ can be chosen to be σ_{n+1}. This is equivalent to σ_{n+1} being linearly independent of the set M_{n+1}. Assume, by contradiction, that this is not the case, i.e.

$$\sigma_{n+1} = \sum_{\sum \alpha_i \cdot \deg(b_i) = n+1} \beta_{\alpha_1,\dots,\alpha_{k(n)}} b_1^{\alpha_1} \cdots b_{k(n)}^{\alpha_{k(n)}}.$$

By classical results, $\sigma_1, \dots, \sigma_{n+1}$ form an algebraic basis of the space of symmetric polynomials on $\mathbb{R}^{n+1}$. Thus, there exists some $b_j \in B_n$, which is not symmetric, and $\alpha_1^0, \dots \alpha_{k(n)}^0$ such that $\sum \alpha_i^0 \deg(b_i) = n+1$, $\alpha_j^0 \geq 1$ and $\beta_{\alpha_1^0,\dots,\alpha_{k(n)}^0} \neq 0$.

We may assume WLOG that there is no nonsymmetric b_l having the same property and such that $\deg(b_l) > \deg(b_j)$. Also, suppose that α_j^0 is the maximal possible. Let us rewrite the right hand side as follows:

$$\begin{aligned}\sigma_{n+1} = & \sum_{\substack{\sum \alpha_i \cdot \deg(b_i) = n+1 \\ \alpha_j < \alpha_j^0}} \beta_{\alpha_1,\dots,\alpha_{k(n)}} b_1^{\alpha_1} \cdots b_{k(n)}^{\alpha_{k(n)}} + b_j^{\alpha_j^0} \\ & \sum_{\substack{\sum \alpha_i \cdot \deg(b_i) = \\ n+1-\deg(b_j)\cdot \alpha_j^0}} \beta_{\alpha_1,\dots,\alpha_{k(n)}} b_1^{\alpha_1} \cdots b_{k(n)}^{\alpha_{k(n)}} \\ = & \sum_{\substack{\sum \alpha_i \cdot \deg(b_i) = n+1 \\ \alpha_j < \alpha_j^0}} \beta_{\alpha_1,\dots,\alpha_{k(n)}} b_1^{\alpha_1} \cdots b_{k(n)}^{\alpha_{k(n)}} + b_j^{\alpha_j^0} \cdot Q(b_1, \dots, b_{k(n)}),\end{aligned}$$

where $Q\big(b_1(x_1, \dots, x_n), \dots, b_{k(n)}(x_1, \dots, x_n)\big)$ is a homogeneous polynomial of degree $n + 1 - \deg(b_j) \cdot \alpha_j^0$. Suppose

$$Q\big(b_1(x_1, \dots, x_n), \dots\big) = \sum_{\substack{\beta_i > 0 \\ \sum \beta_i = n+1-\deg(b_j)\cdot\alpha_j^0}} \gamma_{\beta_1,\dots,\beta_l} \cdot (\beta_1, \dots, \beta_l)$$

is the standard form for $Q\big(b_1(x_1, \dots, x_n), \dots\big)$. Let $\gamma_{\beta_1^0,\dots,\beta_l^0} \neq 0$. Consider the differential operator

$$D = \frac{\partial^{n+1-\deg(b_j)\cdot\alpha_j^0}}{\partial x_n^{\beta_l^0} \partial x_{n-1}^{\beta_{l-1}^0} \cdots \partial x_{n-l+1}^{\beta_1^0}}.$$

We have

$$\tilde{D}\sigma_{n+1} = \tilde{D}\Bigg(\sum_{\substack{\sum \alpha_i \cdot \deg(b_i) = n+1 \\ \alpha_j < \alpha_j^0}} \beta_{\alpha_1,\dots,\alpha_{k(n)}} b_1^{\alpha_1} \cdots b_{k(n)}^{\alpha_{k(n)}} + b_j^{\alpha_j^0} \cdot Q(b_1, \dots, b_{k(n)}) \Bigg),$$

$$\tilde{D}\sigma_{n+1} = \sum_{\substack{\sum \alpha_i \cdot \deg(b_i) = n+1 \\ (\sum_{i=1}^{l} \beta_i^j > 0) \text{ or} (\alpha_j < \alpha_j^0)}} \beta_{\alpha_1,\ldots,\alpha_{k(n)}} \cdot$$

$$\cdot \left(\sum_{\substack{\sum_{\substack{1 \le p \le k(n) \\ 1 \le q \le l}} \beta_q^p = n+1-\deg(b_j)\cdot\alpha_j^0 \\ \beta_q^p \ge 0}} \frac{\partial^{\sum_{i=1}^{l} \beta_i^1} (b_1^{\alpha_1})}{\partial^{\beta_l^1} x_n \cdot \partial^{\beta_{l-1}^1} x_{n-1} \cdot \cdots \cdot \partial^{\beta_1^1} x_{n-l+1}} \cdot \right.$$

$$\cdot \frac{\partial^{\sum_{i=1}^{l} \beta_i^2} (b_2^{\alpha_2})}{\partial^{\beta_l^2} x_n \cdot \partial^{\beta_{l-1}^2} x_{n-1} \cdot \cdots \cdot \partial^{\beta_1^2} x_{n-l+1}} \cdots$$

$$\left. \cdot \frac{\partial^{\sum_{i=1}^{l} \beta_i^{k(n)}} (b_{k(n)}^{\alpha_{k(n)}})}{\partial^{\beta_l^{k(n)}} x_n \cdot \partial^{\beta_{l-1}^{k(n)}} x_{n-1} \cdot \cdots \cdot \partial^{\beta_1^{k(n)}} x_{n-l+1}} \right)$$

$$+ b_j^{\alpha_j^0} \cdot \tilde{D}(Q).$$

Note that D was chosen in order that $\tilde{D}(Q) = \gamma_{\beta_1^0,\ldots,\beta_l^0} \cdot \beta_1^0! \cdot \cdots \cdot \beta_l^0! = c \neq 0$ is constant. On the left hand side we have a symmetric polynomial $\tilde{D}\sigma_{n+1}$ expressible in terms of $\sigma_1, \ldots, \sigma_{\deg(b_j)\cdot\alpha_j^0}$ as $P_1(\sigma_1, \ldots, \sigma_{\deg(b_j)\cdot\alpha_j^0})$. It follows from our construction that if we express $\tilde{D}$(*right hand side*) in terms of the elements of $B_{\deg(b_j)\cdot\alpha_j^0}$ as $P_2(b_1, \ldots, b_{\deg(b_j)\cdot\alpha_j^0})$, it will contain the term $c \cdot b_j^{\alpha_j^0}$. In particular,

$$P_1(\sigma_1, \ldots, \sigma_{\deg(b_j)\cdot\alpha_j^0}) - P_2(b_1, \ldots, b_{\deg(b_j)\cdot\alpha_j^0}) = 0$$

which is a contradiction to the algebraic independence of $B_{\deg(b_j)\alpha_j^0}$. This ends the proof. ∎

From what we proved, it easily follows that B_n forms an algebraic basis for every $S_n(\mathbb{R}^m)$, $m > n$. In particular, there exists no element $f \in S_n(\mathbb{R}^m)$ such that

$$f(x_1, \ldots, x_m) = \sigma_{n+1}(x_1, \ldots, x_m).$$

We will now strengthen this statement in the sense of approximation.

LEMMA 2: *For every $n, m \in \mathbb{N}$, $m \geq k(n)+1$, there exists $\varepsilon > 0$ such that*

$$\sup_{\sum_1^m |x_i| \leq 1} |f(x_1, \ldots, x_m) - \sigma_{n+1}(x_1, \ldots, x_m)| \geq \varepsilon$$

for every $f \in S_n(\mathbb{R}^m)$.

Proof: WLOG we may assume that $m = k(n)+1$. Consider the mapping $M: \mathbb{R}^m \to \mathbb{R}^m$ defined as

$$M(x_1, \ldots, x_m) =$$
$$(b_1(x_1, \ldots, x_m), b_2(x_1, \ldots, x_m), \ldots, b_{k(n)}(x_1, \ldots, x_m), b_m(x_1, \ldots, x_m)).$$

(Remember, $b_m = \sigma_{n+1}$, $\{\sigma_1, \ldots, \sigma_n\} \subset \{b_1, \ldots, b_m\}$). By a standard argument, there exists an open subset $O \subseteq \{(x_1, \ldots, x_m); \sum |x_i| \leq 1\}$ such that the rank r of the Jacobi matrix $J_M(\partial b_i / \partial x_j)$ is constant on O. In case $r = m$, using the inverse function theorem we obtain that there exists an open set $U \subset O$ such that $M(U)$ is an open set in $\mathbb{R}^m$. Choose a pair of points $p^1, p^2 \in M(U)$, $p^1 = (p_1^1, \ldots, p_{m-1}^1, p_m^1)$, $p^2 = (p_1^1, \ldots, p_{m-1}^1, p_m^2)$, $|p_m^1 - p_m^2| = 2\varepsilon \neq 0$. Put $x^1 = M^{-1}(p^1)$, $x^2 = M^{-1}(p^2)$. Then, for every polynomial $P(y_1, \ldots, y_{k(n)})$ we have

$$P(b_1(x^1), b_2(x^1), \ldots, b_{k(n)}(x^1)) = P(b_1(x^2), b_2(x^2), \ldots, b_{k(n)}(x^2)).$$

However, $|b_m(x^1) - b_m(x^2)| = 2\varepsilon$. Thus, for every $f \in S_n(\mathbb{R}^m)$,

$$(f = P(b_1, \ldots, b_{m-1}))$$

we have either

$$|f(x^1) - \sigma_{n+1}(x^1)| \geq \varepsilon$$

or

$$|f(x^2) - \sigma_{n+1}(x^2)| \geq \varepsilon.$$

In case $r < m$, by the real-analytic rank theorem, we have that for some $1 \leq j \leq m$,

$$b_j = \Phi(b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_m)$$

where Φ is real-analytic. Using the fact that b_i are actually polynomials in $x_1, \ldots, x_n$ (as in the proof of Lemma 1), we conclude that Φ may be chosen to be polynomial. This is a contradiction with the algebraic independence of $B_{n+1}(\mathbb{R}^{n+1})$. ∎

Denote $s_n(x_1, \ldots, x_m) = \sum_{i=1}^m x_i^n \in \mathrm{Sym}_n(\mathbb{R}^m)$. For future use in an infinite dimensional setting we will need the following Corollary.

COROLLARY 3: *For every $n, m \in \mathbb{N}$, $m \geq k(n) + 1$, there exists $\varepsilon > 0$ such that*

$$\sup_{\sum |x_i| \leq 1} |f(x_1, \dots, x_m) - s_{n+1}(x_1, \dots, x_m)| \geq \varepsilon$$

for every $f \in S_n(\mathbb{R}^m)$.

Proof: This follows immediately from the previous theorem and Newton's formulas:

$$s_n - s_{n-1}\sigma_1 + s_{n-2}\sigma_2 - \cdots (-1)^n \cdot n \cdot \sigma_n = 0$$

valid on $\mathbb{R}^m$.

Indeed, arbitrary close approximations of s_{n+1} would produce via the Newton's formula arbitrary close approximations of σ_{n+1}. ∎

We remark that it follows from Newton's formulas that $\{s_1, \dots, s_n\}$ forms another algebraic basis of $\mathrm{Sym}_n(\mathbb{R}^m)$, $m \geq n$.

THEOREM 4: *Given an ℓ_p space, $1 \leq p < \infty$, we have the following:*

$$\overline{\mathcal{A}_1(\ell_p)} = \cdots = \overline{\mathcal{A}_{n-1}(\ell_p)} \subsetneq \overline{\mathcal{A}_n(\ell_p)} \subsetneq \overline{\mathcal{A}_{n+1}(\ell_p)} \subsetneq \cdots$$

where $n - 1 < p \leq n$.

Proof: It was shown in [3] that every polynomial of degree $m < p$ is weakly sequentially continuous on ℓ_p. By results of [1, 2] this implies its presence in $\overline{\mathcal{A}_1(\ell_p)}$.

In case $m \geq p$, consider the polynomial $P(x) = \sum_{i=1}^{\infty} x_i^m$. It is well-known ([6, 12]) that, if this polynomial is approximable by elements from $\mathcal{A}_{m-1}(\ell_p)$, it is approximable by subsymmetric polynomials from $\mathcal{A}_{m-1}(\ell_p)$. This leads to a contradiction with Corollary 3. ∎

COROLLARY 5: *Given $X = L_p[0,1]$, $1 \leq p < \infty$, we have the following:*

$$\overline{\mathcal{A}_1(X)} \subsetneq \overline{\mathcal{A}_2(X)} \subsetneq \cdots .$$

Proof: By classical results, if $p > 1$, ℓ_2 is isomorphic to a complemented subspace of $L_p[0,1]$. If $p = 1$, ℓ_1 is isomorphic to a complemented subspace of $L_1[0,1]$. Thus the results follows from Theorem 4. ∎

In order to obtain similar results for other classical Banach spaces, we state the following Lemma.

LEMMA 6: *Given a Banach space X, suppose there exists a noncompact bounded linear operator $T\colon X \to \ell_p$, $1 \le p < \infty$. Then*

$$\overline{\mathcal{A}_1(X)} \underset{\neq}{\subset} \overline{\mathcal{A}_n(X)} \underset{\neq}{\subset} \overline{\mathcal{A}_{n+1}(X)} \underset{\neq}{\subset} \cdots$$

where $n \ge p$.

Proof: We present the proof in case $p > 1$, since the necessary adjustments in case $p = 1$ are only minor.

Let $\{x_i\}_{i=1}^{\infty} \subset B_X$ be such that $\{Tx_i\}_{i=1}^{\infty}$ forms a ε-separated set in ℓ_p. In what follows, we use the standard Schauder basis technique as in [10]. By Rosenthal's theorem we may assume that $\{Tx_i\}_{i=1}^{\infty}$ is weakly convergent. By passing to a subsequence we may assume that $\{Tx_{2i} - Tx_{2i-1}\}_{i=1}^{\infty}$ is weakly null and there exists a block sequence $\{b_i\}_{i=1}^{\infty}$ in ℓ_p such that $\sum_{i=1}^{\infty} \|b_i - T(x_{2i} - x_{2i+1})\| < \infty$. Finally, we may assume without loss of generality that $\{T(x_{2i} - x_{2i+1})\}_{i=1}^{\infty}$ forms a basic sequence in ℓ_p which is equivalent to the canonical ℓ_p-basis, and which spans a complemented subspace of ℓ_p. By composing T with the corresponding projection P, we obtain the following:

$\tilde{T} = P \circ T$ maps X into ℓ_p, and

$\tilde{T}y_i = e_i$ where $y_i = x_{2i} - x_{2i+1}$, and e_i are the basic vectors in ℓ_p.

A similar procedure based on Rosenthal's theorem applied to $\{y_i\}_{i=1}^{\infty}$ yields the following. We may assume that either $\{y_i\}_{i=1}^{\infty}$ is equivalent to the canonical basis of ℓ_1, or $\{y_i\}_{i=1}^{\infty}$ is weakly null (by passing to differences if necessary). In the former case, using the proof of Theorem 4 we obtain that the polynomial $\tilde{P}$ defined on X by $\tilde{P}(x) = P(\tilde{T}x)$, where P is a polynomial $P(x) = \sum_{i=1}^{\infty} x_i^n$ on ℓ_p, $n \ge p$, satisfies $\tilde{P} \notin \overline{\mathcal{A}_{n-1}(X)}$. In the latter case, we adopt the technique from [5], which uses the spreading model ideas. We suppose, by contradiction, that the above defined polynomial $\tilde{P}$ can be approximated by $Q \in \mathcal{A}_{n-1}(X)$, $\sup_{x \in B_X} |P(x) - Q(x)| < \varepsilon/4$ where ε comes from Lemma 2. Adopting the spreading model ideas, we obtain a finite sequence $\{y_{i_1}, \dots, y_{i_{k(n)+1}}\}$ such that $Q|_{\mathrm{span}\{y_{i_1}, \dots, y_{i_{k(n)+1}}\}}$ can be approximated by $\tilde{Q} \in S_{n-1}(\mathbb{R}^{k(n)+1})$ within $\varepsilon/4$. Thus

$$\sup_{\substack{x \in B_X \\ x \in \mathrm{span}\{y_{i_1}, \dots, y_{i_{k(n)+1}}\}}} |\tilde{P} - \tilde{Q}| \le \frac{\varepsilon}{2},$$

a contradiction with Lemma 2.

For details on the procedure, we refer the reader to [5] and references therein. ∎

Lemma 6 provides the following.

COROLLARY 7: *Let $\ell_1 \hookrightarrow X$ (in particular, $X = C(K)$, K non-scattered). Then*

$$\overline{\mathcal{A}_1(X)} \underset{\neq}{\subset} \overline{\mathcal{A}_2(X)} \underset{\neq}{\subset} \cdots .$$

Proof: By classical results [7, 14], $\ell_1 \hookrightarrow X$ implies $L_1[0,1] \hookrightarrow X^*$, in particular $\ell_2 \hookrightarrow X^*$. Thus ℓ_2 is a quotient of X and Lemma 6 applies. ∎

For completeness, we state the following known result.

PROPOSITION 8: *Let X be a Banach space with the Dunford–Pettis property, $\ell_1 \not\hookrightarrow X$ (in particular $X = C(K)$, K scattered). Then*

$$\overline{\mathcal{A}_1(X)} = \overline{\mathcal{A}_2(X)} = \cdots .$$

Proof: By [13], members of $\mathcal{P}(X)$ are weakly sequentially continuous. For spaces not containing a copy of ℓ_1, this implies that members of $\mathcal{P}(X)$ are weakly uniformly continuous on bounded sets ([1]). [2] finishes the proof. ∎

As a last example, we have the following proposition.

PROPOSITION 9: *Let X be an infinite dimensional Banach space with nontrivial type (in particular, every superreflexive space). Then for $n >$ cotype(x) we have*

$$\overline{\mathcal{A}_1(X)} \underset{\neq}{\subset} \overline{\mathcal{A}_n(X)} \underset{\neq}{\subset} \overline{\mathcal{A}_{n+1}(X)} \underset{\neq}{\subset} \cdots .$$

Proof: In [4], the authors prove that every Banach space with nontrivial type is polynomially Schur. In the course of their proof, they produce a normalized subspace $\{y_n\}$ in X^* which has upper p-estimate for some $1 < p < \infty$, i.e. $\|\sum \alpha_n y_n\| \leq (\sum |\alpha_n|^p)^{1/p}$ for any scalars α_n. In fact, since X has a type, given $\varepsilon > 0$, p can be chosen to be

$$\frac{\text{cotype}(x)}{\text{cotype}(x) - 1} - \varepsilon$$

([11]). Thus, $T\colon \ell_p \to X^*$, $T(e_n) = y_n$ is a noncompact bounded linear operator. Since T is weakly compact, $T^*\colon X \to \ell_{p'}$, $1/p + 1/p' = 1$ is a noncompact operator. Now Lemma 6 applies (put $n = [p'] + 1$). ∎

References

[1] R. M. Aron, C. Hervés and M. Valdivia, *Weakly continuous mappings on Banach spaces*, Journal of Functional Analysis **52** (1984), 189–204.

[2] R. M. Aron and J. B. Prolla, *Polynomial approximation of differentiable functions on Banach spaces*, Journal für die reine und angewandte Mathematik **313** (1980), 195–216.

[3] R. Bonic and J. Frampton, *Smooth functions on Banach manifolds*, Journal of Mathematical Mechanics **15** (1966), 877–898.

[4] J. Farmer and W. B. Johnson, *Polynomial Schur and polynomial Dunford–Pettis properties*, Colloquium Mathematicum **144** (1993), 95–105.

[5] R. Gonzalo, *Multilinear forms, subsymmetric polynomials, and spreading models on Banach spaces*, Journal of Mathematical Analysis and Applications **202** (1996), 379–397.

[6] P. Habala and P. Hájek, *Stabilization of polynomials*, Comptes Rendus de l'Académie des Sciences, Paris **320** (1995), 821–825.

[7] J. Hagler, *Some more Banach spaces which contain ℓ_1*, Studia Mathematica **46** (1973), 35–42.

[8] F. John, *Partial Differential Equations*, Springer-Verlag, Berlin, 1982.

[9] S. G. Krantz, *Function Theory of Several Complex Variables*, Pure and Applied Mathematics, Wiley, New York, 1982.

[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Springer-Verlag, Berlin, 1977.

[11] V. D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Mathematics **1200**, Springer-Verlag, Berlin, 1986.

[12] A. S. Nemirovski and S. M. Semenov, *On polynomial approximation of functions on Hilbert spaces*, Mathematics of the USSR-Sbornik **21** (1973), 255–277.

[13] A. Pelczynski, *A property of multilinear operations*, Studia Mathematica **16** (1957), 173–182.

[14] A. Pelczynski, *On Banach spaces containing $L_1(\mu)$*, Studia Mathematica **30** (1968), 231–246.

[15] B. L. van der Waerden, *Modern Algebra*, Vol. 1, Ungar Publishing, 1953.