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ON PERIODIC SOLVABLE GROUPS HAVING AUTOMORPHISMS WITH NILPOTENT FIXED POINT GROUPS

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ABSTRACT

Let p be a prime, G a periodic solvable p'-group acted on by an elementary group V of order p^2. We show that if $C_G(v)$ is abelian for each $v \in V^{\#}$ then G has nilpotent derived group, and if $p = 2$ and $C_G(v)$ is nilpotent for each $v \in V^{\#}$ then G is metanilpotent. Earlier results of this kind were known only for finite groups.

Introduction

In [12] J.N. Ward has considered finite solvable groups admitting an elementary automorphism group of order p^2. He has proved that if G is a finite solvable p'-group, V its elementary automorphism group of order p^2 with $C_G(v)$ being nilpotent for every $v \in V^{\#}$ then G is metanilpotent. Earlier he has shown (see [11, Theorem 2]) that in case $p = 2$ the derived group of G is nilpotent. An interesting question is whether results of this kind hold for periodic solvable groups. As it was shown in [10] some cases of this question can be answered in affirmative. In this paper we shall prove

THEOREM A: *Let p be a prime, G a periodic solvable p'-group admitting an elementary automorphism group V of order p^2. Suppose that $C_G(v)$ is abelian for every $v \in V^{\#}$. Then the derived group of G is nilpotent.*

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THEOREM B: *Let G be a periodic solvable $2'$-group admitting a four-group of automorphisms V. Suppose that $C_G(v)$ is nilpotent for every $v \in V^{\#}$. Then G is metanilpotent.*

It can be easily seen from the proof that if G is of derived length k and satisfies the conditions of Theorem A then G' is nilpotent of class at most

$$\sum_{i=0}^{k(p+1)-1} (h+1)^i,$$

where $h = h(p)$ is the function of Higman restricting class of any nilpotent group which admits a fixed-point-free automorphism of order p [5] (see [6] and [7] for an explicit expression of h).

Also there exists a function $f(k, c)$ depending only on k and c with the following property.

Let G be as in Theorem B and of derived length k. Suppose that $C_G(v)$ is nilpotent of class at most c for every $v \in V^{\#}$. Then G has a normal subgroup N such that G/N is nilpotent of class at most c and N is so of class at most $f(k, c)$. An explicit bound for this function can be obtained with aid of [3, Theorem 7] and computations made in [8]. Note however that unlike the original result of Ward [11, Theorem 2] our technique does not enable us to prove that G' in Theorem B is nilpotent.

It remains unclear whether Theorem B is valid for odd primes. Though it seems plausible the author failed to find a proof. Our notation is standard and agrees with [2].

Lemmas

LEMMA 1: *Let π be a set of primes, G a periodic solvable π'-group admitting a finite π-group of automorphisms V. Then we have*

(i) $G = [G, V]C_G(V)$;

(ii) $C_{G/N}(V) = C_G(V)N/N$ *for any V-invariant normal subgroup N of G;*

(iii) *if V is abelian then* $G = \langle C_G(W)\colon V/W \text{ is cyclic} \rangle$.

Proof: The lemma follows immediately from the corresponding finite cases [2]. ∎

LEMMA 2: *Let G be a periodic solvable p'-group which admits an elementary abelian automorphism group V of order p^n. Let $V_1, V_2, ..., V_s$ be the maximal subgroups of V. Set $G_i = C_G(V_i)$. Then for any $v \in V$ we have*

$$[G, v] = \langle [G_i, v]; 1 \leq i \leq s \rangle.$$

Proof: Clearly, we may assume that G is finite and $G = [G, v]$. Furthermore, by [1, Lemma 2.1], we may assume that G is nilpotent. By Lemma 1 we have $G_i = [G_i, v]C_G(V)$ and $G = \langle G_1, G_2, ..., G_s \rangle$. Since $G = [G, v]$, it follows that $C_G(V) \leq G' \leq \Phi(G)$, whence $G = \langle [G_i, v]; 1 \leq i \leq s \rangle$, as required. ∎

Definition: Let G be a group, V a group of operators on G. We say that a series of V-invariant subgroups of G,

$$1 = L_0 \leq L_1 \leq \cdots \leq L_m = G,$$

is a V-complex of G if the following conditions are satisfied.

(i) L_i is normal in L_{i+1}, $0 \leq i \leq m-1$;

(ii) V induces a cyclic group of automorphisms of each factor-group L_{i+1}/L_i.

LEMMA 3: *Let p be a prime, G a periodic solvable p'-group admitting an elementary abelian automorphism group V of order p^n. Then G possesses a V-complex of finite length.*

Proof: We use induction on the derived length of G. Let $V_1, V_2, ..., V_s$ be the set of maximal subgroups of V, $G_i = C_G(V_i)$. Take a V-complex of G'

$$1 \leq L_0 \leq L_1 \leq \cdots \leq L_j = G'.$$

Put $L_{j+m} = \langle L_j, G_1, G_2, ..., G_m \rangle$; $1 \leq m \leq s$. Then, by Lemma 1(iii), $L_{j+s} = G$ and $L_0 \leq L_1 \leq \cdots \leq L_{j+s}$ is a V-complex of G. Note that if G is of derived length k and $n = 2$ then $s = p + 1$ and G possesses a V-complex of length at most $k(p+1)$. The lemma is established. ∎

Given a prime p and a positive integer k, put

$$t(p, k) = \sum_{i=0}^{k(p+1)-1} (h+1)^i,$$

where $h = h(p)$ is the function of Higman [5].

LEMMA 4: *Let G be a periodic solvable p'-group of derived length k admitting an elementary automorphism group V of order p^2 such that $C_G(v)$ is abelian for every $v \in V^{\#}$. Then there exist a V-invariant normal series*

$$1 = H_0 \leq H_1 \leq \cdots \leq H_{t(p,k)} = G$$

and a sequence $v_1, v_2, ..., v_{t(p,k)}$ of elements of V, such that

$$H_{i+1}/H_i = C_{G/H_i}([G/H_i, v_{i+1}]).$$

Proof: Let m be the minimal number such that G possesses a V-complex

$$1 = L_0 \leq \cdots \leq L_{m-1} \leq L_m = G.$$

We prove by induction on m that G has a required series of length at most $\sum_{i=0}^{m-1}(h+1)^i$, where $h = h(p)$ is the function of Higman. Assume that this is true for any group having a V-complex of length at most $m-1$. In particular this is true for $L = L_{m-1}$. Consequently there exists a series $1 = F_0 \leq F_1 \leq \cdots \leq F_r = L$, where $r = \sum_{i=0}^{m-2}(h+1)^i$, which satisfies the lemma. In order to prove the lemma using Lemma 1(ii) it is sufficient to show that there exist a sequence of subgroups $1 = H_0 \leq H_1 \leq \cdots \leq H_{h+1}$ and a sequence of elements $v_1, v_2, ..., v_{h+1}$ of V such that H_i is normal in G, $H_{i+1}/H_i = C_{G/H_i}([G/H_i, v_{i+1}])$ and $F_1 \leq H_{h+1}$.

We have $F_1 = C_L([L, v])$ for some $v \in V^{\#}$. By the definition of V-complex there exists a nontrivial element w of V which acts trivially on G/L. By Lemma 1(ii) $G = LC_G(w)$. Note that $C_G(V) \leq Z(G)$ as, by Lemma 1(iii),

$$G = \langle C_G(u); u \in V^{\#}\rangle$$

and $C_G(u)$ is abelian for every $u \in V^{\#}$. So, by Lemma 1(i) and Lemma 2, F_1 centralizes all subgroups $C_G(u)$ such that $\langle u\rangle \neq \langle v\rangle$ and $\langle u\rangle \neq \langle w\rangle$. Clearly, $F_1 \bigcap C_G(w)$ centralizes $C_G(w)$, therefore, by Lemma 2, $F_1 \bigcap C_G(w) \leq C_G([G, v])$. Put $H_1 = C_G([G, v])$. If $\langle v\rangle = \langle w\rangle$ then by Lemma 2 $[L, v] = [G, v]$, whence $F_1 = H_1$ and we are done. Let $\langle v\rangle \neq \langle w\rangle$. Consider the factor-group G/H_1. For the sake of simplicity assume that $H_1 = 1$. Then, since $F_1 \bigcap C_G(w) \leq H_1$, w acts fixed-point-freely on $Q = F_1[C_G(v), w]$. Consequently, by Higman's Theorem [5], Q is nilpotent and the class of Q does not exceed h. We observe that $Z(Q) \bigcap F_1$ centralizes $C_G(v)$ so, by Lemma 2, $Z(Q) \bigcap F_1 \leq C_G([G, w])$. Put

$$H_2/H_1 = C_{G/H_1}([G/H_1, w]),$$

$$H_{i+1}/H_i = C_{G/H_i}([G/H_i, w]); \quad i = 1, 2, ..., h.$$

Then $F_1 \leq H_{h+1}$, as required. The lemma is proved. ∎

LEMMA 5: *Let p be a prime, G a metabelian p'-group acted on by an elementary abelian group V of order p^n. Suppose that for any maximal subgroup W of V, the fixed-point group $C_G(W)$ is nilpotent. Then there exists a normal subgroup N of G such that G/N is nilpotent and $C_N(V) = 1$.*

Proof: Let $V_1, V_2, ..., V_s$ be the set of maximal subgroups of V and $G_i = C_G(V_i)$. Put $L = G'$, $H_i = LG_i$. Then H_i is normal in G and by Lemma 1(iii) $G = H_1 H_2 \cdots H_s$. By Lemma 1, $H_i/[H_i, V_i]$ is nilpotent. Since $[H_i, V_i] = [L, V_i]$, we get that $[H_i, V_i] \bigcap G_i = 1$. In particular, there exists a number r such that

$$\gamma_r(H_i) \bigcap G_i = 1$$

for any $i = 1, 2, ..., s$. Put $N = \langle \gamma_r(H_i) \colon 1 \leq i \leq s \rangle$. Then G/N is generated by its normal nilpotent subgroups and consequently, by Fitting Theorem, is nilpotent. By [4, Lemma 2.6], $C_N(V) = 1$ and the lemma is proved. ∎

Theorems

Proof of Theorem A: Let $t = t(p, k)$ denote the function introduced before Lemma 4. By Lemma 4 there exist a sequence $v_1, v_2, ..., v_t$ of elements of $V^{\#}$ and a series $1 = H_0 \leq H_1 \leq \cdots \leq H_t = G$ such that $H_{i+1}/H_i = C_{G/H_i}([G/H_i, v_{i+1}])$. By Lemma 1(i) $G/[G, v_i]$ is abelian so $G' \leq [G, v_i]$, $i = 1, 2, ..., t$. Set $Z_i = G' \bigcap H_i$. It is clear that $1 = Z_0 \leq Z_1 \leq \cdots \leq Z_t = G'$ is a central series of G'. The theorem is proved. ∎

Proof of Theorem B: Let the class of $C_G(v)$ be smaller or equal to c for any $v \in V^{\#}$. By induction on the derived length of G we shall show that $\gamma_c(G)$ is nilpotent. Let L be the metabelian term of the derived series of G. By Lemma 5 there exists a number r such that V acts fixed-point-freely on $N = \gamma_r(L)$. By Lemma 1(i) $G/[G, v]$ is nilpotent of class at most c for any $v \in V^{\#}$ and therefore $\gamma_c(G) \leq \bigcap_{v \in V^{\#}} [G, v]$. Now it follows from Lemma 10 of [9] that there exists a normal V-invariant series $1 = H_0 \leq H_1 \leq \cdots \leq H_s = N$ such that

$$H_{i+1}/H_i = C_{N/H_i}(\gamma_c(G/H_i)); \quad i = 0, 1, ..., s-1.$$

Evidently $\gamma_c(G)$ is nilpotent if and only if $\gamma_c(G/N)$ is nilpotent, so without loss of generality we may assume that $N = 1$. Then L is nilpotent. By induction $\gamma_c(G/L')$ is also nilpotent. So by [3, Theorem 7] $\gamma_c(G)$ is nilpotent, as required. ∎

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