THE FITTING LENGTH OF SOLVABLE H_{p^n}-GROUPS

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ABSTRACT

For a (finite) group G and some prime power p^n, the H_{p^n}-subgroup $H_{p^n}(G)$ is defined by $H_{p^n}(G)=\langle x \in G \mid x^{p^n} \neq 1\rangle$. A group $H \neq 1$ is called a H_{p^n}-group, if there is a finite group G such that H is isomorphic to $H_{p^n}(G)$ and $H_{p^n}(G) \neq G$. It is known that the Fitting length of a solvable H_{p^n}-group cannot be arbitrarily large: Hartley and Rae proved in 1973 that it is bounded by some quadratic function of n. In the following paper, we show that it is even bounded by some linear function of n. In view of known examples of solvable H_{p^n}-groups having Fitting length n, this result is "almost" best possible.

1. Introduction

The concept of the generalized Hughes H_{p^n}-subgroup $H_{p^n}(G)=\langle x \in G \mid x^{p^n} \neq 1\rangle$ of a finite group $G(p$ a prime, $n \geqq 1)$ is a direct generalization of the (Hughes-) H_p-subgroup $H_p(G)$ defined by Hughes in [11], $H_p(G)$ being just $H_{p^1}(G)$. To have a picture, consider a nonabelian dihedral group G, then $H_2(G)$ is the cyclic subgroup of index 2 in G. The subgroup $H_{p^n}(G)$ was first introduced and investigated by Kurzweil ([15]) and by Gallian ([2]), and it seems natural to look for theorems about H_{p^n}-subgroups generalizing the known theorems about H_p-subgroups.

In the beginning Hughes and others were concerned with the possible index of $H_p(G)$ in G, if it is a *proper* subgroup of G, see [2]. But soon afterwards the subgroups themselves were investigated, and Hughes and Thompson ([12]) and Kegel ([14]) showed that groups occurring as the *proper* H_p-group of some finite group are nilpotent. This fact, of course, is related to the nilpotency of a finite group admitting a fixed-point-free automorphism of prime order. As for the generalized H_{p^n}-subgroups: the problem of "determining" their structure should

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be related to the problem of describing groups admitting a fixed-point-free automorphism of order p^n.

Now it is known that finite solvable groups admitting a fixed-point-free automorphism of order p^n have Fitting length bounded by a linear function of n (in fact, this function is n itself, see [1]).

In the following we shall determine a function $f(n)$, such that the Fitting length of a finite solvable H_{p^n}-group (i.e. a group isomorphic to the proper H_{p^n}-subgroup of a finite solvable group) is bounded by $f(n)$. And while in [10] Hartley and Rae gave a quadratic function, this one is linear.

Our proof uses induction on n, and the induction step is based on an investigation of the representation of the semidirect product $(F_{i+1}(G)/F_i(G))A$ on $F_i(G)/F_{i-1}(G)$, where $F_j(G)$ denotes the j-th term of the upper Fitting series of G and A is some cyclic subgroup of G not contained in $H_{p^n}(G)$. Thus the proof parallels the proofs of the corresponding theorems about fixed-point-free action of cyclic p-groups — but some care is needed, since in our case the action of the p-elements is not really fixed-point-free, and G is not necessarily a semidirect product. As a consequence of the somehow more complicated representation theory for $p = 2$, we get a worse result for $p = 2$ than for odd primes. To make it precise: the function is $f(n) = 2n$ for odd primes p, while it is $f(n) = 4n$ for $p = 2$ (Theorem 2.7, Theorem 2.9).

Nonsolvable groups are not considered in this paper. There is no hope to prove the solvability of H_{p^n}-groups in general: there is a group H with $A_6 \lneqq H \lneqq \mathrm{Aut}(A_6)$, such that $A_6 = H_8(H)$. Note that H is neither isomorphic to Σ_6-nor to $PGL_2(9)$.

Notation is taken from [13]; the Fitting length of a solvable group G, i.e. the length of its upper Fitting series $\{F_i(G)\}$, is denoted by $h(G)$. A critical subgroup of a finite p-group P is a characteristic subgroup C of P such that every p'-automorphism of P acts nontrivially on C and C has the properties described in ([3], 5.3.11 and 5.3.13). An element x of order m is said to act exceptionally on the module V, if the degree of the minimal polynomial of x on V is smaller than m.

2. The results

The following obvious facts are frequently used in the reduction of minimal counterexamples.

(2.1) LEMMA. *Let G be a finite group with $H_{p^n}(G) \neq G$, $x \in G \backslash H_{p^n}(G)$. Then:*

(i) $H_{p^n}(U) \subseteq H_{p^n}(G) \cap U$ *for every subgroup* U *of* G.

(ii) $H_{p^n}(G/N) \subseteq H_{p^n}(G)N/N$ *for every normal subgroup* N *of* G.

(iii) $C_G(xu)$ *is a* p*-group for every* $u \in H_{p^n}(G)$.

(iv) $(xu)^{p^n} = 1$ *for every* $u \in H_{p^n}(G)$.

(v) *If* $U \leqq G$, $U^x = U$ *then* $H_{p^n}(\langle U, x\rangle) \neq \langle U, x\rangle$.

PROOF. (i) to (iv) follow directly from the definition, while (v) is a consequence of (i).

(2.2) LEMMA. *Let* G *be a finite group with* $H_{p^n}(G) \neq G$, *and* $x \in G \backslash H_{p^n}(G)$ *an element of order* p^n. *Let* $V = M/N$ *be an elementary abelian* $\langle x\rangle$*-invariant section of* $H_{p^n}(G)$. *Then*:

(i) x *acts exceptionally on* V,

(ii) *if* K *is some normal subgroup of* G *such that* V *is* K*-invariant and* $V = V_1 + V_2 + \cdots + V_k$ *is the direct sum of its homogeneous* K*-components, then* $z := x^{p^{n-1}}$ *fixes every* V_i.

PROOF. (i) Assume there is an element $v \in M$ such that

$$w = v^{x^{p^n-1}} v^{x^{p^n-2}} \cdots v^{x^2} v^x v \notin N,$$

then $(xv)^{p^n} = x^{p^n} w = w \notin N$, contradicting (2.1)(iv).

(ii) Assume by way of contradiction that $x^{p^{n-1}}$ does not fix the submodule V_1. Then the sum of the submodules $V_1^{x^j}$, $j = 1, \ldots, p^n$, is a direct sum and $\langle x\rangle$ is represented regularly on it contradicting (i).

(2.3) LEMMA. *Let* $G = QP$ *be a finite group with* $1 \neq Q = O_q(G)$ *and* $P = \langle x\rangle$ *cyclic of order* p^n, $q \neq p$ *primes,* p *odd or* $p = 2$, $p^n = 4$. *Let* G *act faithfully and irreducibly on the* GF(r)*-module* V, r *a prime, and let* $[Q, x^{p^{n-1}}] \neq 1$. *Then* $C_Q(x) \neq 1$ *provided one of the following conditions holds*:

(i) $r = p$, *and* x *acts exceptionally on* V,

(ii) $r \neq p$, *and* x *acts fixed-point-freely on* V.

PROOF. Consider the case $p \neq 2$. Assume condition (ii) holds, then x cannot be fixed-point-free on Q by [5] Theorem 2, so $C_Q(x) \neq 1$. But if condition (i) holds, then by a Hall–Higman-reduction we may assume that $Q = [Q, x^{p^{n-1}}]$, Q/Q' is irreducible under the action of $\langle x\rangle$, $c(Q) \leqq 2$, p a Fermat prime and $q = 2$. Now the arguments of [5], pp. 1442–1445 go through to show $C_Q(x) \neq 1$.

So let $p^n = 4$; then (after the usual reductions), if V is a Q-homogeneous module, $Q = [Q, x^2]$ is extraspecial, $Z(Q) \subseteq Z(G)$ and so $C_Q(x) \neq 1$.

If, however, V is decomposed into the sum of four Q-homogeneous components permuted by $\langle x\rangle$, then $\langle x\rangle$ is represented regularly on V. And if the number of components is two, then x^2 does not induce a central automorphism on both components since $Q = [Q, x^2]$ and therefore again $\langle x \rangle$ is represented regularly on V. In both cases neither (i) or (ii) can be satisfied.

REMARK. Gross's example ([5], p. 1441) shows that (2.3) does not hold for powers of 2 greater than 4.

The following fundamental property of the upper Fitting series of a finite solvable group will be used in (2.6) and (2.8). To have an easy reference we put it as a lemma; of course, it is well-known, see for instance [7], Lemma 2.2.

(2.4) LEMMA. *Let G be a group, p a prime and $i \geqq 1$. Then $O_p(G/F_i(G))$ operates faithfully on $O_{p'}(F_i(G)/F_{i-1}(G))$.*

PROOF. By induction we may assume $i = 1$. Then if L denotes the subgroup of G such that $L/F(G) = C_{O_p(G/F(G))}O_{p'}(F(G))$, then L is a normal nilpotent subgroup of G, thus contained in $F(G)$.

The next lemma is more or less implicit in the proofs of [6] Theorem 2.2 and Theorem 2.4. For the second statement an induction proof similar to ours was given by Hartley in [9] Lemma 1.

(2.5) LEMMA. *Let V be a nondegenerate symplectic space over* GF(r), $r = 2^m - 1$ *a Mersenne prime, and let g be a symplectic transformation of V of order 2^m such that the dimension of the subspace $[V, g]$ is at most 2. Let Q be a q-group of symplectic transformations of V normalized by g, q a prime different from r, and let $h = g^{2^{m-2}}$, $z := g^{2^{m-1}}$. Then the following holds*:
(i) $[Q, z] = 1$ *if q is odd,*
(ii) $[Q, h, h] = 1$ *if $q = 2$.*

PROOF. Since z inverts elementwise the subspace $[V, z]$, the whole space V splits into the orthogonal sum of $[V, z]$ and $C_V(z)$. Therefore $[V, z]$ is itself a nondegenerate symplectic space with the restriction of the symplectic form, and so $[V, z] = [V, g]$ is two-dimensional.

We use induction on $\dim_{GF(r)} V + |Q|$.

Since by hypothesis q is different from r, under the action of $\langle Q, g \rangle$ the module V splits into a direct sum of irreducible submodules V_i. And since $[V, g]$ is irreducible for $\langle g \rangle$, we may assume that $[V, g]$ is contained in V_1. But now if V_1 is properly contained in V, by induction we get $[Q, z] \subseteq C_Q(V_1)$ if q is odd and $[Q, h, h] \subseteq C_Q(V_1)$ if $q = 2$, since the irreducible module V_1 is not totally isotropic and so is nondegenerate. Thus the result follows if V_1 is properly contained in V, since h centralizes the V_i, $i \neq 1$, and therefore $[Q, z] \subseteq$

$\bigcap_{i \neq 1} C_Q(V_i) \cap C_Q(V_1) = 1$ for q odd and $[Q, h, h] \subseteq \bigcap_{i \neq 1} C_Q(V_i) \cap C_Q(V_1) = 1$ for $q = 2$.

Thus we may assume V to be irreducible for $\langle Q, g \rangle$.

Assume q is an odd prime.

Then we may also assume that Q is a q-group of exponent q and class at most two and $Q = [Q, z]$, by taking a critical subgroup C of Q and applying the induction hypothesis to $[C, z]$ if it is a proper subgroup of Q. Let $V_Q = U_1 + \cdots + U_k$ be the decomposition of the irreducible $\langle Q, g \rangle$-module V into its Q-homogeneous components. Then $k \leqq 2$, since the U_i are permuted transitively by $\langle g \rangle$ and $\dim[V, g] = 2$; and if $k = 2$, then z fixes U_1 and U_2, and so $[V, g] \neq [V, z]$, contradicting the hypothesis. Thus V is the direct sum of isomorphic Q-modules, and therefore Q is either cyclic or extraspecial. Next we tensor the GF(r)-module V with an extension $\mathcal{K}$ of GF(r) which is a splitting field for all the subgroups of $\langle Q, g \rangle$, then the module $V^* = V \otimes \mathcal{K}$ is a direct sum of irreducible $\mathcal{K}$ $\langle Q, g \rangle$-modules W_i, and since $C_V(Q) \otimes \mathcal{K} = C_{V^*}(Q) = 0$ and $\dim_{GF(r)}[V, g] = \dim_{\mathcal{K}}[V^*, g] = 2$, there are at most two of these. As above, one easily finds that each W_i is Q-homogeneous and therefore Q-irreducible by [17] Theorem A. But since the dimension of an irreducible $\mathcal{K}Q$-module is a power of q, and dimension V^* is even, we get $V^* = W_1 + W_2$. Now z does not centralize W_1 or W_2, so the commutator modules $[W_i, g] = [W_i, z]$ are one-dimensional for $i = 1, 2$.

If Q is extraspecial, this is impossible by [13], Theorem 17.13. If Q is abelian, the modules W_i themselves are one-dimensional and V^* is inverted elementwise by z. This gives $Q = [Q, z] \subseteq C_Q(V) = 1$, a final contradiction. This contradiction finishes the case q odd.

Assume $q = 2$.

Let $x \in Q$ be an element such that $[x, h, h] \neq 1$. Then by minimality of $|Q|$ we can assume that $\langle Q, g \rangle = \langle x, g \rangle$.

Let A be a maximal elementary abelian normal subgroup of $G := \langle x, g \rangle$ and take N a normal subgroup of G containing A minimal such that V viewed as a N-module is the sum of two-dimensional irreducible N-modules. Clearly such N exists. Let $V_N = W_1 + \cdots + W_k$ be the decomposition of the irreducible G-module V into the N-homogeneous components. Then as above g has to fix all of them, so $\langle x \rangle$ permutes the W_i transitively, and since without loss of generality $[V, g] \subseteq W_1$ and g centralizes the W_i for $i \neq 1$, we must have $k = 1$, since otherwise $[h, x]$ would commute with h in the action on V. So V is a homogeneous N-module and we may decompose V into a direct sum of isomorphic irreducible N-modules U_j, which are two-dimensional. And since N

is faithful on V and the submodules U_j are isomorphic N-modules, N operates faithfully on each U_j. Thus N is isomorphic to a subgroup of a Sylow 2-subgroup of $GL_2(r)$. We want to show that A is cyclic.

So assume that A is not cyclic, then A is a proper, elementary abelian normal subgroup of N and by the structure of $GL_2(r)$, N must be a dihedral group of order 8, and A one of the two Klein fours groups contained in N. Since both N and A are normal in G, h normalizes both fours groups in N and so $z = h^2$ centralizes N. But now N operates on $[V, z]$ and we may assume $[V, z] = U_1$. Thus U_1 is a nondegenerate symplectic space and N is isomorphic to a subgroup of $Sp(2, r) = SL_2(r)$ and A has order two, contradicting the assumption. So we know that every maximal elementary abelian normal subgroup G is cyclic and therefore by [3], 5.4.10 G is either cyclic, dihedral, semidihedral or generalized quaternion and by the structure of these groups there is only one involution in G which is also a square. Clearly this involution is z, and so $z \in Z(G)$ and $[V, z]$ is G-invariant. This implies $V = [V, z]$ and G is isomorphic to a subgroup of a Sylow 2 subgroup of $Sp(2, r) = SL_2(r)$ which is of order $2(r + 1) = 2^{m+1}$. Thus the index of $\langle g \rangle$ in G is at most two and $\langle g \rangle$ is normal in G. Clearly this contradicts the assumption $|x, h, h| \neq 1$.

(2.6) THEOREM. *Let G be a finite solvable group; p an odd prime, $n \geqq 1$; or $p^n = 4$. Then $z := x^{p^{n-1}} \in F_2(G)$ for every $x \in G \backslash H_{p^n}(G)$.*

PROOF. Assume false and choose G a minimal counterexample, $x \in G \backslash H_{p^n}(G)$ with $z := x^{p^{n-1}} \notin F_2(G)$. We may assume that G is not a p-group. First we prove the following property of $F(G)$:

(*) $F(G) = F(H_{p^n}(G))$ is the unique minimal normal subgroup of G.

Let N be a minimal normal subgroup of G. Then if $N \not\subseteq H_{p^n}(G)$ we get

$$[N, H_{p^n}(G)] \subseteq N \cap H_{p^n}(G) = 1,$$

whence there are elements $y \in N \backslash H_{p^n}(G)$ such that $C_G(y) \supseteq H_{p^n}(G)$ is not a p-group, contradicting (2.1)(iii). Thus every minimal normal subgroup of G is contained in $H_{p^n}(G)$. Let $N_1 \neq N_2$ be two minimal normal subgroups of G. Then $|G/N_i| < |G|$ for $i = 1, 2$ and since by (2.1)(ii)

$$H_{p^n}(G/N_i) \subseteq H_{p^n}(G)N_i/N_i = H_{p^n}(G)/N_i,$$

induction gives $zN_i \in F_2(G/N_i)$. Let $E_i/N_i = F_2(G/N_i)$, then $z \in E_i$ for $i = 1, 2$; and since $E_1 \cap E_2 = F_2(G)$ by $N_1 \cap N_2 = 1$ we have $z \in F_2(G)$ contradicting the choice of x. Thus G has exactly one minimal normal subgroup. Assume

$\phi(F(G)) \neq 1$ then we can again use induction to $F/\phi(F(G))$. But $F(G/\phi(F(F))) = F(G)/\phi(F(G))$ and so the induction gives $z \in F_2(G)$, contradicting the hypothesis. Thus $\phi(F(G)) = 1$, and $F(G)$ is a product of minimal normal subgroups of G, and hence is the unique minimal normal subgroup of G. Of course, $F(G) = F(H_{p^n}(G))$. Put $\bar{G} := G/F(G)$. Then $\bar{G}$ acts as a group of linear transformations on the irreducible module $F(G)$.

Induction to $G/F(G)$ gives $z \in F_3(G)$, and by (2.4) there is some $\langle x \rangle$-invariant q-subgroup $Q = [Q, z] \neq 1$ of $F(\bar{G}) = F_2(G)/F(G)$.

Let V be an irreducible $\langle \bar{x}, Q \rangle$-submodule of $F(G)$ not centralized by Q, then (2.3) gives $C_Q(\bar{x}) \neq 1$, since the hypotheses for (2.3) are given by (2.2)(i) and (2.1)(iii). But now q divides the order of $C_G(x)$, contradicting (2.1)(iii).

(2.7) THEOREM. *Let G be a finite solvable group with $G \neq H_{p^n}(G)$ for some odd prime $p, n \geqq 1$. Then $h(G) \leqq 2n$.*

PROOF. Use induction on n, the case $n = 1$ being clear by the Hughes–Kegel–Thompson Theorem.

We may assume $F_2(G) \subset H_{p^n}(G)$, since if $y \in F_2(G) \backslash H_{p^n}(G)$, then $[G, y] \subseteq F_2(G)$ and $G/F_2(G)$ is a p-group by (2.1(iii)). Let $x \in G \backslash H_{p^n}(G)$ be an element of order p^n. Then $x^{p^{n-1}} \in F_2(G)$ by (2.6) and therefore if $\bar{G} = G/F_2(G)$, every element $\bar{x}$ in $\bar{G} \backslash \overline{H_{p^n}(G)}$ has order at most p^{n-1}. Thus $H_{p^{n-1}}(\bar{G}) \subseteq \overline{H_{p^n}(G)} \neq \bar{G}$ and induction gives $h(\bar{G}) \leqq 2(n-1)$. So $h(G) \leqq 2 + 2(n-1) = 2n$ and the result follows.

(2.8) THEOREM. *Let G be a finite solvable group, $n \geqq 1$. Then $x^{2^{n-1}} \in F_4(G)$ for every $x \in G \backslash H_{2^n}(G)$.*

PROOF. Assume false and take a counterexample G of minimal order. Let $x \in G \backslash H_{2^n}(G)$ such that $z := x^{2^{n-1}} \notin F_4(G)$. Since the case $n = 1$ is well known (we have $h(G) \leqq 2$ if $G \neq H_2(G)$), we may assume $n \geqq 2$.

Then as in (2.6)(*) we can show that $F(G) = F(H_{2^n}(G))$ is the unique minimal normal subgroup of G, and therefore by (2.1)(ii) we have $H_{2^n}(G/F(G)) \subseteq H_{2^n}(G)/F(G)$; by minimality of G we get $z \in F_5(G)$. Thus z acts nontrivially on $O_{2'}(F_4(G)/F_3(G))$ by (2.4) and so $z \notin F_4(\langle F_4(G), x \rangle)$. Since by (2.1)(i) $H_{2^n}(\langle F_4(G), x \rangle) \subseteq H_{2^n}(G)$ we know that $\langle F_4(G), x \rangle$ is a counterexample to the theorem, and so $G = \langle F_4(G), x \rangle$ by minimality of G.

But also $x \notin H_{2^n}(\langle [F_4(G), z], x \rangle)$ by (2.1) and so if $\langle [F_4(G), z], x \rangle < G$ we have $z \in F_4(\langle [F_4(G), z], x \rangle)$ by minimality of G. So there is a positive integer i such that

$$[F_4(G), \underbrace{z, \ldots, z}_{i}] \subseteq [F_4(G), z] \cap F_3(\langle [F_4(G), z], x \rangle) \subseteq F_3([F_4, z]) \subseteq F_3(G)$$

since $[F_4(G), z]$ is a normal subgroup of $G = \langle F_4(G), x\rangle$. But now z centralizes $O_{2'}(F_4(G)/F_3(G))$ and by (2.4) we get $z \in F_4(G)$, contradicting the choice of x. Thus we have $G = \langle [F_4(G), z], x\rangle$.

Next we show that z centralizes the Frattini factor group of

$$O_{2}(F_3(G)/F_2(G)) =: S.$$

So assume there is $y \in S$ such that $[y, z] \notin \phi(S)$. Then $[y, h, h] \neq 1$ for $h := x^{2^{n-2}}$, since otherwise $[y, z] = [y, h^2] = [y, h]^2$ is a square in S, and so an element of $\phi(S)$.

Since S operates faithfully on $O_{2'}$, $(F_2(G)/F(G))$ by (2.4), there is an odd prime r such that $[y, h, h]$ does not centralize a critical subgroup R of $O_r(F_2(G)/F(G))$. Put $\bar{G} = G/F(G)$, then $\bar{G}$ acts irreducibly on $V = F(G)$, and since $R \lhd \bar{G}$, the module V splits into the direct sum of its homogeneous R-components $V_R = V_1 + \cdots + V_k$. Now z fixes every V_i by (2.2)(ii) and therefore $[F_4(G), z] = F_4(G)$ does as well.

But even h fixes every V_i, since if h does not fix V_j for some j, then the minimal polynomial of x on V has degree at least 2^{n-1} times the degree of the minimal polynomial of z on V_j, whence z has to induce a scalar multiplication on V_j by (2.2)(i). But now $F_4(G) = [F_4(G), z]$ centralizes V_j and so $C_V(F_4(G)) = V$ by the irreducible action of $\bar{G}$. This is clearly impossible and so h fixes every V_i. Let g be an element of $\langle x\rangle$ of maximal order, say 2^m, fixing V_1. Then g fixes every V_i and acts exceptionally on every V_i since these are permuted transitively by $\langle x\rangle$.

But now if the (isomorphic) groups $R/C_R(V_i)$ are elementary abelian, they are cyclic and $[S, h] \subseteq \bigcap_{i=1}^{k} C_S(R/C_R(V_i)) = C_S(R)$ contradicting $[y, h, h] \notin C_S(R)$. Thus $R/C_R(V_i)$ is extraspecial for $i = 1, \ldots, k$. Now if z centralizes some $R/C_R(V_i)$, then z centralizes the whole of R and S does as well, since $F_4(G) = [F_4(G), z]$, contradicting $[y, h, h] \notin C_S(R)$. Thus we can apply [6] Theorem 2.2 by restricting $\langle R/C_R(V_i), g\rangle$ to some irreducible submodule of V_i.

As is well-known, $2^m - 1 = r^d$ implies $d = 1$, and so if we view the Frattini factor group of $R/C_R(V_i)$ as a nondegenerate symplectic space over GF(r), the commutator module of $\langle g\rangle$ on it has dimension of $2d = 2$. Now (2.5)(ii) is applicable to show $[y, h, h] \in C_S(R/C_R(V_i))$ and hence $[y, h, h] \in \bigcap_{i=1}^{k} C_S(R/C_R(V_i)) = C_S(R)$, contradicting the choice of r. Thus we have shown (**).

Now since we know that z operates nontrivially on $O_{2'}(F_4(G)/F_3(G))$ we may choose an odd prime q, such that z does not centralize $O_q(F_4(G)/F_3(G))$. Let $Q = [O_q(F_4(G)/F_3(G)), z]$; then $Q \neq 1$ operates faithfully on $O_{q'}(F_3(G)/F_2(G))$

and there is a prime $r \neq q$, such that Q does not centralize $O_r(F_3(G)/F_2(G))$. But as $Q = [Q, z]$ and z centralizes the Frattini factor group of $O_2(F_3(G)/F_2(G))$, we know that Q centralizes $O_2(F_3(G)/F_2(G))$ and therefore $r \neq 2$. Let R be a critical subgroup of $O_r(F_3(G)/F_2(G))$, then $[Q, R] \neq 1$ and we may choose a G-composition factor V of $F_2(G)/F(G)$ such that $[Q, R] \not\subseteq C_R(V)$.

Then let $V_R = V_1 + \cdots + V_k$ be the decomposition of the G-irreducible module V into its homogeneous R-components, then z fixes every V_i by (2.2)(ii) and therefore $Q = [Q, z]$ acts on every $R/C_R(V_i)$ and we may assume that $[Q, R] \not\subseteq C_R(V_1)$. Let g be an element of maximal order 2^m in $\langle x \rangle$ fixing V_1, then again g is exceptional on V_1. Also $R/C_R(V_1)$ is not cyclic, since z and $Q = [Q, z]$ act nontrivially on $R/C_R(V_1)$, and therefore $R/C_R(V_1)$ is extraspecial. Now as above [6] Theorem 2.2 gives that the commutator module of $\langle g \rangle$ on the Frattini factor group of $R/C_R(V_1)$ viewed as a nondegenerate symplectic module over GF(r) has dimension 2. But this obviously contradicts (2.6)(i).

(2.9) Theorem. *Let G be a finite solvable group with $G \neq H_{2^n}(G)$ for some $n \geqq 1$. Then $h(G) \leqq 4n$.*

Proof. The proof follows strictly the proof of (2.7).

We use induction over n; the case $n = 1$ is well-known, so $n \geqq 2$. Assume first that $F_4(G) \not\subseteq H_{2^n}(G)$. Then for some $y \in F_4(G) \backslash H_{2^n}(G)$ we have $[G, y] \subseteq F_4(G)$, and by (2.1)(iii) $G/F_4(G)$ is a p-group. Thus $h(G) \leqq 5 \leqq 2n$ and the result follows since $n \geqq 2$.

So we may assume $F_4(G) \subseteq H_{2^n}(G)$. By (2.8) $x^{2^{n-1}} \in F_4(G)$ for every $x \in G \backslash H_{2^n}(G)$, and so if $\bar{G} = G/F_4(G)$, every element of $\bar{G} \backslash \overline{H_{2^n}(G)}$ has order at most 2^{n-1}. Thus $H_{2^{n-1}}(\bar{G}) \subseteq \overline{H_{2^n}(G)} \neq \bar{G}$ and by induction $h(\bar{G}) \leqq 4(n-1)$. This, of course, gives the desired result $h(G) \leqq 4 + 4(n-1) = 4n$.

As we indicated in the introduction, the Fitting length of solvable H_{p^n}-groups "should be" the same as the Fitting length of a solvable group admitting a fixed-point-free cyclic automorphism group of order p^n which is n. The following example shows that the Fitting length of a solvable H_{p^n}-group is at least n.

Example. The semidirect product of a solvable group H of Fitting length n admitting a cyclic fixed-point-free automorphism group A of order p^n (such groups can easily be constructed) with this fixed-point-free automorphism group A has the following property:

$$H_{p^n}(HA) = HA^p \subsetneqq HA \text{ and } h(H) = n.$$

But to our knowledge there are no examples of H_{p^n}-groups known with Fitting height bigger than n. So perhaps the bound on the Fitting length of G in

Theorems (2.7) and (2.9) could be proven to be $n+1$. As a test for that "conjecture" we look at the case $p^n=4$.

(2.10) THEOREM. *Let G be a finite solvable group with $G\neq H_4(G)$. Then $h(G)\leqq 3$.*

PROOF. Assume false and let G be a minimal counterexample. As in (2.6)(*) we may assume that $F(G)=F(H_4(G))$ is the unique minimal normal subgroup of G and by (2.6) $x^2\in F_2(G)$ for every $x\in G\backslash H_4(G)$.

If $F(G)=O_2(G)$, then 2 does not divide the order of $F_2(G)/F(G)$ and $x^2\in F(G)$ for every $x\in G\backslash H_4(G)$. If we put $\bar{G}:=G/F(G)$, every element in $\overline{G\backslash H_4(G)}$ is an involution, and by the nilpotence of H_2-groups we get $h(\bar{G})\leqq 2$ and $h(G)\leqq 3$. Thus we may assume that $V=F(G)=O_p$ for some odd prime p. As above $F(\bar{G})\subseteq \overline{H_4(G)}$. Now $S=O_2(\bar{G})\neq 1$, and $C_V(S)=0$ by the faithful and irreducible action of $\bar{G}$ on V. Let $V_S=V_1+\cdots+V_S$ be the decomposition of V into its homogeneous S-components, then $Z(S/C_S(V_i))$ operates fixed-point-freely on V_i, so for every i there is an element d_i in S inverting V_i elementwise. Therefore, if for $x\in G\backslash H_4(G)$ we have $C_V(x^2)\neq 0$, then since $\bar{x}^2\in S$, there is an index i such that either x or xd_i has a nontrivial fixed point on V_i contradicting (2.1)(iii). Therefore $[\bar{G},\bar{x}^2]\subseteq C_{\bar{G}}(V)=1$ and $\bar{x}^2\in Z(\bar{G})$ for every $x\in G\backslash H_4(G)$. But now if $Z/F(G)=Z(\bar{G})$, we have $Z\leqq H_4(G)$ and for $\tilde{G}=G/Z$ every element of $\widetilde{G\backslash H_4(G)}$ is an involution. Therefore $\tilde{G}\neq H_2(\tilde{G})$ and $h(\tilde{G})\leqq 2$. But, of course, $h(\tilde{G})=h(\bar{G})$, and so $h(G)=1+h(\bar{G})\leqq 3$, a final contradiction.

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