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A NOTE ON ENTROPY AND INNER FUNCTIONS

BY
JOSÉ L. FERNÁNDEZ
Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

ABSTRACT
This paper constructs an inner function with infinite entropy.

1. The boundary values of an inner function f determine a measurable transformation from the torus $\mathbf{T}$ into itself and if $f(0)=0$ then f preserves Lebesgue measure on $\mathbf{T}$. Moreover, under this assumption Aaronson showed in [2] that f must be exact and so, in particular, ergodic, unless f is a rotation or the identity. See also, [1], [8], [9].

Exact endomorphisms always have positive entropy, and in our case we do have $h(f) \geqq c(1-|f'(0)|)$ where $h(f)$ denotes the entropy of f with respect to normalized Lebesgue measure on $\mathbf{T}$ and c is an absolute constant (see [11]).

On the other hand, it was shown in [7] that if B is a finite Blaschke product and $B(0)=0$ then

$$h(B)=\int_0^{2\pi} \log |B'(e^{i\theta})| \frac{d\theta}{2\pi}. \tag{1}$$

In particular, $h(B)<\infty$.

Pommerenke asked in [11] whether or not inner functions must have finite entropy. The purpose of this note is to answer this question by explicitly constructing a Blaschke product with infinite entropy.

For background on entropy we refer the reader to [13] or [4]. If $E \subset \mathbf{R}$ (or $E \subset \mathbf{T}$) is a measurable set then $|E|$ denotes its Lebesgue measure. Also, c, $C(M)$ denote various constants which depend only on the parameter displayed.

Section 2 contains an extension, probably known, of a theorem of Rohlin providing the key ingredient of our construction which is carried out in Section 3.

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2. A lemma on the entropy of F-expansions

Let Φ be a C^1-mapping from $(0,1)$ to $(-\infty,\infty)$ which is strictly increasing and satisfies $\Phi(0^+) = -\infty$ and $\Phi(1^-) = +\infty$.

Define $F(x) = \{\Phi(x)\}$ where $\{\ \}$ denotes fractional part. Thus F is a transformation from $(0,1)$ into $(0,1)$. (Strictly speaking a set of measure zero has to be removed.)

For each $n \in \mathbf{Z}$ define x_n by $\Phi(x_n) = n$ and set $I_n = (x_n, x_{n+1})$. Then we have the following

LEMMA 1. *If F preserves Lebesgue measure then*

$$h(F) \geqq \int_0^1 \log |F'(x)|\,dx.$$

In particular, if $|F'(x)| \geqq \delta |I_n|^{-1}$ on I_n, with δ independent of n, then

$$h(F) \geqq \sum_{n \in \mathbf{Z}} |I_n| \log \frac{1}{|I_n|} + \log \delta.$$

This lemma should be compared with [12], [3] and [4, Chapters 7 and 10]. Lemma 1 is a corollary of the following more general result.

LEMMA 2. *Let $(X, \mathcal{B}, m)$ be a Lebesgue probability space and let $T: X \to X$ be a measure preserving, countable to one (in the sense of [10, pp. 106–107]) transformation, then*

$$\int_X \log\left(\frac{d(m \circ T)}{dm}\right) dm \leqq h(T)$$

($d(m \circ T)/dm$ is defined in [10, p. 108]).

PROOF. Let $\alpha = \{A_k\}$ be a countable partition of X so that T is 1–1 on each A_k and let $V_k : T(A_k) \to A_k$ be the inverse of T on $T(A_k)$.

Then for every k:

$$m(A_k \mid T^{-1}(\mathcal{B})) = (\chi_{T(A_k)} \circ T) \frac{d(m \circ V_k)}{dm} \circ T$$

$$= \chi_{T(A_k)} \circ T \left(\frac{d(m \circ T)}{dm}\right)^{-1}$$

and hence we have that

$$
\begin{aligned}
I(\alpha \mid T^{-1}(\mathscr{B})) &= -\sum_k \chi_{A_k} \log m(A_k \mid T^{-1}(\mathscr{B})) \\
&= -\sum_k \chi_{A_k} \log \chi_{T(A_k)} \circ T + \log \frac{d(m \circ T)}{dm} \\
&= \log \frac{d(m \circ T)}{dm} .
\end{aligned}
$$

Clearly,

$$\lim_{n \to \infty} I(\alpha_n \mid T^{-1}(\mathscr{B})) = I(\alpha \mid T^{-1}(\mathscr{B})), \quad \text{a.e.}$$

and so, by Fatou's lemma,

$$
\begin{aligned}
\int_X \log \frac{d(m \circ T)}{dm}\, dm &= \int_X I(\alpha \mid T^{-1}(\mathscr{B}))dm \\
&\leqq \varliminf_{n \to \infty} \int_X I(\alpha_n \mid T^{-1}(\mathscr{B}))dm.
\end{aligned}
$$

Moreover, for every n,

$$
\begin{aligned}
\int_X I(\alpha_n \mid T^{-1}(\mathscr{B}))dm &= H(\alpha_n \mid T^{-1}(\mathscr{B})) \\
&\leqq H\left(\alpha_n \,\middle|\, \bigvee_{k=1}^{\infty} T^{-k}(\alpha_n)\right) \\
&= h(T, \alpha_n) \leqq h(T)
\end{aligned}
$$

and thus

$$\int_X \log \left(\frac{d(m \circ T)}{dm}\right) dm \leqq h(T).$$

3. A Blaschke product of infinite entropy

We need the following result of Frostman [5, Th. 6].

LEMMA A. *Let B be a Balschke product with zeros $\{a_n\}_{n=1}^{\infty}$. If*

$$\sum_{n=1} \frac{1-|a_n|^2}{|e^{i\theta} - a_n|^2} < \infty$$

then B' has radial limit at $e^{i\theta}$ and

$$\lim_{r \to 1} |B'(re^{i\theta})| = \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|e^{i\theta} - a_n|^2} .$$

To start the construction we select a decreasing sequence of positive numbers $\{s_n\}_{n=1}^{\infty}$ satisfying

(i) $\sum_{n=1}^{\infty} s_n = \frac{1}{2}$,

(ii) $\sum_{n=1}^{\infty} s_n \log(1/s_n) = \infty$,

(iii) if $p \leqq q \leqq rq$, $r \geqq 1$ then $s_p/s_q \leqq c(r)$.

We also define $s_{-n} = s_n$ for $n \leqq -1$.

For $j \geqq 1$, let $L_j = \{e^{i\theta} : \theta \in (t_j, t_j + 2\pi s_j)\}$ where $t_j = 2\pi \sum_{k=1}^{j-1} s_k$, if $j > 1$, and $t_1 = 0$. Also if $j \leqq -1$ we define $L_j = \overline{L_{-j}}$.

For $j \in \mathbf{Z}$, let a_j be the point in the unit disk Δ satisfying $|a_j|^2 = 1 - s_j$ and $a_j |a_j|^{-1} =$ center of L_j. Note that the a_j accumulate only at -1.

Our Blaschke product is

$$B(z) = z \prod_{j=1}^{\infty} \left\{ \frac{a_j - z}{a - \bar{a}_j z} \frac{\bar{a}_j - z}{1 - a_j z} \right\}.$$

Since $B(0) = 0$ we know that B preserves Lebesgue measure on $\mathbf{T}$ and is ergodic with respect to it (see [11]).

It is easy to see that B extends analytically to a simply connected region U containing $\partial\Delta - \{-1\}$ where B does not vanish (see, e.g., [6]). Let $\log B = \psi + i\Phi$ be the analytic branch of the logarithm of B in U satisfying $\Phi(1) = 0$. Then on $\partial\Delta \setminus \{-1\}$ we have $B(e^{i\theta}) = e^{i\Phi(e^{i\theta})}$ and using Lemma B we see that

$$\Phi'(e^{i\theta}) = \sum_{\substack{n \in Z \\ n \neq 0}} \frac{1 - |a_n|^2}{|e^{i\theta} - a_n|^2} + 1. \tag{1}$$

Also, $\lim_{\theta \to \pi^-} \Phi(e^{i\theta}) = \infty$, $\lim_{\theta \to -\pi^+} \Phi(e^{i\theta}) = -\infty$, and Φ is strictly increasing.

From Lemma A it now follows that if $e^{i\theta} \in L_n$ then

$$|B'(e^{i\theta})| > \frac{1 - |a_n|^2}{|e^{i\theta} - a_n|^2}$$

but also $|e^{i\theta} - a_n| \leqq C(1 - |a_n|^2)$, and so

$$|B'(e^{i\theta})| \geqq \frac{C}{s_n} \qquad (e^{i\theta} \in L_n),$$

and so from Lemma 1 we get that

$$h(B) = \infty.$$

4. It is natural to conjecture that if f is an inner function then f has finite entropy if and only if f' is in the Nevanlinna class and that if this is the case then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f'(e^{i\theta})| \, d\theta = h(f).$$

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