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THE STRUCTURE OF UNIFORMLY GÂTEAUX SMOOTH BANACH SPACES

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ABSTRACT

It is shown that a Banach space X admits an equivalent uniformly Gâteaux smooth norm if and only if the dual ball of X^* in its weak star topology is a uniform Eberlein compact.

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I. Introduction

A classical result, which goes back to D. P. Milman [24] and B. J. Pettis [29], asserts that a uniformly Fréchet smooth Banach space is reflexive. P. Enflo's renorming theorem [8] states that the existence of such norms characterizes super-reflexivity. However, characterizing Banach spaces on which smooth norms exist, in various possible meanings of the word "smooth", turns out to be quite a difficult task. We refer, e.g., to [3], [7], [9], [19], and [23] for more on this subject.

The present work shows that the existence of a uniformly Gâteaux smooth norm on a given Banach space X provides a lot of information on the structure of X.

Our approach relies on the use of methods of weak compactness. In [22], J. Lindenstrauss posed a problem whether smoothness of a Banach space implies the weak compact generating (WCG) in some overspace ([22, Problem 9]). We provide a positive answer in the case of *uniform Gâteaux* smoothness. On the other hand, let us recall that there are C^∞ smooth spaces that are not subspaces of weakly compactly generated spaces. For instance, the spaces defined in [20, pp. 222 and 224] are not subspaces of WCG spaces and yet they admit equivalent C^∞-smooth norms (cf., e.g., [7, p. 194], [33]).

Our main result in this note (Theorem 2) provides a characterization of Banach spaces on which there exists an equivalent uniformly Gâteaux smooth norm. Its proof is based on showing that such a space X is $K_{\sigma\delta}$ in X^{**} in its weak* topology and therefore admits projectional resolutions of identity (PRI) for every equivalent norm ([35]; cf., e.g., [7, p. 240]). Then one can apply the result from [10]. Theorem 2 is followed by several remarks showing that it is essentially optimal.

II. Notation

We work in real Banach spaces. The notation we use in this note is classical and can be found, e.g., in [7] or [9]. In particular, the unit sphere of a Banach space X is denoted by S_X, i.e., $S_X = \{x \in X; \|x\| = 1\}$. The weak* closure of a subset S of a dual space is denoted by $\overline{S}^*$. A norm $\|\,.\,\|$ is said to be **uniformly Gâteaux smooth** (in short, UG-norm) if for every $h \in X$, the limit

$$\lim_{t\to 0}(\|x + th\| - \|x\|)/t$$

exists uniformly on $x \in S_X$. A compact set K is said to be **Eberlein** (respectively **uniform Eberlein**) if it is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for some Γ (respectively to a weakly compact subset of a Hilbert space), considered in its weak topology.

III. The results

The main result in this paper is based on the following lemma, in which some ideas from [33] and [25] are used.

LEMMA 1: *Let X be a Banach space with an equivalent uniformly Gâteaux smooth norm. Then X is a $K_{\sigma\delta}$ subset of (X^{**}, w^*), i.e., $X = \bigcap_{p\geq 1}\bigcup_{m\geq 1} K_{m,p}$, where $K_{m,p}$ are some weak* compact sets in X^{**}.*

Proof: Assume that $\|\,.\,\|$ is an arbitrary equivalent norm on X. Pick any $G \in X^{**}\backslash X$. Let $H = G^{-1}(0)$ be the subspace of X^* consisting of all the elements of X^* that vanish at G. The space H is a norming subspace of X^*, that is, there is $\mu > 0$ such that for all $x \in S_X$, $\sup\{|f(x)|;\ f \in H,\ \|f\| \leq 1\} \geq \mu$. Indeed, assume that $\|G\| = 1$ and dist $(G, X) \geq \delta$. We have $H^* = X^{**}/H^{\perp}$, and it follows from the basic duality results that for all $x \in S_X$,

$$\begin{aligned}\sup\{|f(x)|;\ f \in H,\ \|f\| \leq 1\} &= \inf\{\|x - \lambda G\|;\ \lambda \in I\!R\}\\ &= \min\{\inf_{|\lambda|\leq\frac{1}{2}}\{\|x - \lambda G\|\}, \inf_{|\lambda|\geq\frac{1}{2}}\{\|x - \lambda G\|\}\\ &\geq \min\left\{\frac{1}{2}, \inf_{\lambda\geq\frac{1}{2}}\|x - \lambda G\|\right\}\\ &\geq \min\left\{\frac{1}{2}, \inf_{|\lambda|\geq\frac{1}{2}}\left\{|\lambda|\cdot\|\frac{1}{|\lambda|}x - \frac{\lambda}{|\lambda|}G\|\right\}\right\}\\ &\geq \min\left\{\frac{1}{2}, \frac{1}{2}\text{ dist }(G, X)\right\}\\ &\geq \min\left\{\frac{1}{2}, \frac{\delta}{2}\right\}.\end{aligned}$$

Using the idea in [33] and [25], for any equivalent norm $\|\,.\,\|$ on X, we define for all $n, p \in \mathbb{N}$ the subsets $S_{n,p}(\|\,.\,\|)$ in X as follows:

$$\begin{aligned}S_{n,p}(\|\,.\,\|) = \{x \in X;\ &|f(x) - g(x)| \leq 1/p\\ &\text{whenever } f, g \in X^*, \|f\| \leq 1, \|g\| \leq 1 \text{ and } \|f + g\| > 2 - 2/n\}.\end{aligned}$$

We now assume that $\|\,.\,\|$ is an equivalent uniformly Gâteaux smooth norm on X. Its dual norm is W^*UR, i.e., $f_n - g_n \to 0$ in the weak* topology whenever $\{f_n\}$ and $\{g_n\}$ are bounded in X^* and $2\|f_n\|^2 + 2\|g_n\|^2 - \|f_n + g_n\|^2 \to 0$ (see [7, Theorem II.6.7]). Thus for any $p \in \mathbb{N}$, one has

$$\bigcup_{n\geq 1} S_{n,p}(\|\,.\,\|) = X. \tag{1}$$

We will show that

$$X = \bigcap_{p\geq 1} \bigcup_{n\geq 1} \overline{S_{n,p}(\|\cdot\|)}^{*}.$$

It follows from (1) that it suffices to prove that for any $G \in X^{**}\backslash X$ there is a $p \in \mathbb{N}$ such that

$$G \notin \bigcup_{n\geq 1} \overline{S_{n,p}(\|\cdot\|)}^{*}. \tag{2}$$

We set $H = G^{-1}(0)$, and define an equivalent norm q on X by the formula

$$q(x) = \sup\{|f(x)|;\ f \in H,\ \|f\| \leq 1\}.$$

We claim that

$$S_{n,p}(\|\cdot\|) \subset S_{n,p}(q).$$

In order to prove this claim, we observe the bipolar theorem implies that the dual unit ball B_{q^*} satisfies

$$B_{q^*} = \{f \in X^*;\ q^*(f) \leq 1\} = \overline{\{f \in H,\ \|f\| \leq 1\}}^{*}. \tag{3}$$

We note that (3) implies that the unit ball of H for the original norm $\|\ .\ \|$ is contained in the unit ball of H for the norm q^*. Since, on the other hand, one clearly has $q^* \geq \|\ .\ \|$ on X^*, it follows that the norms $\|\ .\ \|$ and q^* coincide on H, and that H is a 1-norming subspace of X^* for the norm q^*. By (3), if $q^*(f) \leq 1$, $q^*(g) \leq 1$ and $q^*(f+g) > 2 - 2/n$, there are nets (f_α) and (g_α) in H, weak* convergent to f and g respectively, such that $\|f_\alpha\| \leq 1$ and $\|g_\alpha\| \leq 1$ for all α. Since the norm q^* is weak*-lower semicontinuous, one has $q^*(f_\alpha + g_\alpha) > 2 - 2/n$ when α is large enough. The norms $\|\ .\ \|$ and q^* coincide on H, and thus $\|f_\alpha + g_\alpha\| > 2 - 2/n$ for large α.

If now $x \in S_{n,p}(\|\ .\ \|)$, we have $|f_\alpha(x) - g_\alpha(x)| \leq 1/p$ for α large enough, and hence $|f(x) - g(x)| \leq 1/p$. It thus follows that $x \in S_{n,p}(q)$. This shows our claim.

In order to prove (2), it therefore suffices to show that one has

$$G \notin \bigcup_{n\geq 1} \overline{S_{n,p}(q)}^{*}$$

for $p \in \mathbb{N}$ large enough. To this end, choose $p \in \mathbb{N}$ such that $p > 1/q^{**}(G)$. Fix $n \in \mathbb{N}$ and set for simplicity $S_{n,p}(q) = S$. We need to show that $G \notin \overline{S}^{*}$. Pick $f \in X^*$ with $q^*(f) = 1$ and $G(f) > 1/p$, and let $x \in X$ be such that $q(x) \leq 1$

and $f(x) > 1 - 1/n$. Since H is 1-norming for q^*, there is $g \in H$ with $q^*(g) \leq 1$ and $g(x) > 1 - 1/n$. We have then

$$q^*(f+g) \geq (f+g)(x) > 2 - 2/n.$$

From the definition of S, for all $z \in S$ one has $(f-g)(z) \leq 1/p$. On the other hand, $G(f-g) = G(f) > 1/p$. It follows that $G \notin \overline{S}^*$.

For every $p \in \mathbb{N}$, let the family $\{K_{m,p}\}_m$ be identical after reindexation to the collection $\{\overline{S}^*_{n,p} \cap kB_{X^{**}}\}_{n,k}$. We have shown that

$$X = \bigcap_{p\geq 1} \bigcup_{m\geq 1} K_{m,p}.$$

This concludes the proof of Lemma 1. ∎

We are now ready to state and prove our main result.

THEOREM 2: (i) *A Banach space X admits an equivalent uniformly Gâteaux smooth norm if and only if the dual unit ball B_{X^*} equipped with the weak* topology is a uniform Eberlein compact.*

(ii) *A compact space K is a uniform Eberlein compact if and only if $C(K)$ admits an equivalent UG-norm, if and only if there is a Hilbert space $\mathcal{H}$ and a bounded linear operator from $\mathcal{H}$ onto a dense set in $C(K)$.*

Proof: (i) Assume that $\|\,.\,\|$ is an equivalent UG-norm on X. Since X is a $K_{\sigma\delta}$ subset of (X^{**}, w^*), X is in particular weakly countably determined (WCD), i.e., there are weak* compact sets $\{K_n\}$ in X^{**} such that, given $x \in X$ and $G \in X^{**} \setminus X$, there is n_0 such that $x \in K_{n_0}$ and $G \notin K_{n_0}$ ([7, Chapter VI]). Therefore $(X, \|\,.\,\|)$ together with all of its subspaces has a projectional resolution of identity (see [7, Chapter VI]). Now the proof of ([10, Lemma 7]) (which uses only the existence of such a projectional resolution with respect to the UG-norm) shows that (B_{X^*}, w^*) is uniformly Eberlein. Conversely, assume that $(B_{X^*}, w^*) = K$ is uniform Eberlein. Then by [5] (cf., e.g., [16, p. 233]), there is a Hilbert space $\mathcal{H}$ and a bounded linear operator T from $\mathcal{H}$ to $C(K)$ with dense range. It follows from ([7, Theorem II.6.8]) that the space $C(K)$ has an equivalent UG-norm, and so does its subspace X.

(ii) Assume that K is a uniform Eberlein compact. By the proof of (i), there is a bounded linear operator from a Hilbert space $\mathcal{H}$ onto a dense set in $C(K)$ and thus $C(K)$ admits an equivalent UG-norm (cf., e.g., [7, Theorem II.6.8]). If $C(K)$ admits an equivalent UG-norm, the dual unit ball of $C(K)$ is a uniform Eberlein in its weak* topology by (i). Hence such is its closed subset K. ∎

IV. Remarks

Let us discuss how Theorem 2 relates to some known results in this area.

(1) H. Rosenthal showed in [30] that there exists a probability measure μ such that the space $L^1(\mu)$, which is WCG and admits a UG-norm since it contains $L^2(\mu)$ as a dense subspace ([7, Theorem II.6.8]), contains a closed subspace X_R which is not WCG. As a subspace of a space with UG-norm, X_R admits a UG-norm.

(2) If K is a uniform Eberlein compact and L is a continuous image of K, then L is a uniform Eberlein compact ([4]). Indeed, by Theorem 2 (ii), $C(K)$ has an equivalent UG-norm and so does its subspace $C(L)$. By the same theorem, L is a uniform Eberlein compact.

(3) Let E be an Eberlein but not a uniform Eberlein compact set ([5]; cf., e.g., [2, p. 417]). The space $C(E)$ is weakly compactly generated ([1]; cf., e.g., [16, p. 225]) and thus there is a reflexive space R and a linear continuous map from R to $C(E)$ with dense range ([6]; cf., e.g., [16, p. 227]). By Theorem 2, $C(E)$ is not UG-renormable and thus by ([7, Theorem II.6.8]) the space R is not UG-renormable. The first example of a reflexive not UG-renormable space was shown in [21] (cf., e.g., [7, p. 170]).

(4) If a Banach space X has an equivalent weakly uniformly rotund norm, or equivalently if its dual X^* has an equivalent UG-norm, then X^* is a subspace of a WCG space (Theorem 2). Therefore X has an equivalent norm that is with its dual norm locally uniformly rotund, and thus X has C^1 smooth partitions of unity (cf., e.g., [7, Chapter VII and VIII]). If X is a subspace of $l_\infty(\mathbf{N})$ and X has an equivalent UG-norm, then X is separable, and this applies in particular to $X = C(K)$ where K is a separable compact ([26]) or to representable spaces X ([14]). In fact, a UG-renormable Banach space which continuously and linearly injects into $l_\infty(\mathbf{N})$ is separable. Indeed, if X is a subspace of a WCG space and X^* is weak* separable, then X is separable (use [31, Theorem 2.4] or [35, Corollary 2]; see also [34] or [36, Section 3]). Hence, in particular, the two spaces defined in [20, pp. 222 and 224] do not admit equivalent UG-norms.

(5) It follows from Theorem 2 and ([13, Theorem 2.10]) that a Banach space X has an equivalent UG-norm if and only if there is a Markushevich basis $\{x_\alpha, f_\alpha\}_{\alpha\in\Gamma}$ of X such that, for every $\epsilon > 0$, there is a partition $\Gamma = \bigcup_{i=1}^{\infty} \Gamma_i^\epsilon$ and there are integers m_i^ϵ such that for every $i \geq 1$ and every $f \in B_{X^*}$,

$$\operatorname{card}\{\gamma \in \Gamma_i^\epsilon;\ |f(x_\gamma)| > \epsilon\} \leq m_i^\epsilon.$$

We recall that a Markushevich basis $\{x_\alpha, f_\alpha\}$ is a biorthogonal system in the

Banach space X such that the closed linear hull of $\{x_\alpha\}$ equals X and $\{f_\alpha\}$ separates points of X (cf., e.g., [16, Chapter 10] or [35, Section 4]).

Indeed, assume that X admits an equivalent UG-norm. Then X is a subspace of a WCG space and thus admits a Markushevich basis, say $\{x_\alpha, f_\alpha\}_{\alpha\in\Gamma}$ ([1]; cf., e.g., [16, p. 219]). As X is a subspace of WCG, the set $\{\gamma \in \Gamma; f(x_\gamma) \neq 0\}$ is at most countable for every $f \in X^*$ (cf., e.g., [34], [16, p. 249] or [36, Section 4]). The dual unit ball B_{X^*} in its weak star topology is a uniform Eberlein compact by Theorem 2. Thus [13, Theorem 2.10] can be used to derive the result. Conversely, if X admits such a Markushevich basis, then the dual unit ball B_{X^*} in its weak star topology is a uniform Eberlein compact by [13, Theorem 2.10]. Then X admits an equivalent UG-norm by Theorem 2.

(6) A WCG space X is UG-renormable if and only if there exists a reflexive UG-renormable space R and a linear continuous map $T\colon R \to X$ with dense range. Indeed, since X is WCG there is (see [7], Cor. VI.5.2) a weak*-to-weak continuous linear one-to-one map S from X^* into a space $c_0(\Gamma)$. We apply the factorization theorem ([6]) to S, and it gives that $S = AB$, where B maps X^* to a reflexive space R_0 and the operators A and B are one-to-one. It follows that $B = B_0^*$ is conjugate to a bounded linear operator B_0 from $R = R_0^*$ to X, with dense range. By ([2, Lemma 3.5]), the unit ball of the factoring space R_0 is uniformly Eberlein, and thus R is UG-renormable. Note, however, that R cannot be taken superreflexive in general, as shown by the example constructed in [17]. This example shows that there exists a reflexive space, whose unit ball is uniformly Eberlein in the weak topology, and which is therefore homeomorphic to a weakly compact subset of a superreflexive space, but which is not *affinely* weakly homeomorphic to a subset of a superreflexive space. Hence such homeomorphisms are bound to "break" the linear structure.

(7) The renorming argument from the proof of Lemma 1 shows in particular that if X is UG-renormable and Y is a norming subspace of X^*, then there exists an equivalent UG-norm on X such that Y becomes 1-norming for this new norm. This is not so when "UG" is replaced by "Fréchet smooth" or "Gâteaux smooth". In the Fréchet smooth case, one can take $X = J^*$ and $Y = J$, where J is James' space. In the Gâteaux smooth case, one can take $X = C([0, \omega_1])$ and Y the norm closed linear hull of the Dirac measures $\{\delta_\alpha; \alpha < \omega_1\}$ ([11]).

(8) The sets $\overline{S_{n,p}(\|\cdot\|)}^*$ from the proof of Lemma 1 cannot in general be replaced by the similar but larger weak* closed subsets $H_{n,p}$ of X^{**} defined by

$$H_{n,p} = \{F \in X^{**};\ |F(f-g)| \leq 1/p \quad \text{whenever } f, g \in X^*, \|f\| \leq 1, \|g\| \leq 1 \text{ and } \|f+g\| > 2 - 2/n\}.$$

Indeed, by [18], there exists a dual weakly uniformly rotund norm $\|\,.\,\|$ on James' space J. If $(X, \|\,.\,\|)$ is the predual of J equipped with the predual norm to $\|\cdot\|$, then X^{**} is UG-smooth ([7, Theorem II.6.7]) and one has for all $p \in \mathbf{N}$ that

$$\bigcup_{n\geq 1} H_{n,p} = X^{**}.$$

(9) There exists a direct construction of PRI from the uniform Gâteaux smoothness (without using the concept of WCD). Indeed, recall that a multivalued mapping Φ from the dual X^* into X is called a *projectional generator* on X if $\Phi(x^*)$ is a nonempty at most countable subset of X for every $x^* \in X^*$ and if for every set $\Gamma \subset X^*$, closed under linear combinations with rational coefficients, the following identity holds:

$$(*) \qquad \Phi(\Gamma)^{\perp} \cap \overline{B_{X^*} \cap \Gamma}^{*} = \{0\}.$$

Once X admits such a Φ, then a PRI with respect to any equivalent norm on X can be constructed. See [27], [28]; cf., e.g., [9, Chapter 6].

If X has uniformly Gâteaux smooth norm, then we can put for $x^* \in X^*$

$$\Phi(x^*) = \bigcup_{n,p=1}^{\infty} \Phi_{n,p}(x^*),$$

where $\Phi_{n,p}(x^*)$ is an at most countable subset of the set $S_{n,p}$ such that

$$\sup \langle x^*, S_{n,p} \rangle = \sup \langle x^*, \Phi_{n,p}(x^*) \rangle.$$

The proof that this Φ satisfies $(*)$ goes similarly as the proof of Lemma 1.

(10) Any Banach space on which there exists a uniformly Gâteaux smooth function with bounded nonempty support has an equivalent UG-norm ([32]; see also [12]). However, it is not known whether the $c_0(\Gamma)$ spaces have UG-smooth partitions of unity.

References

[1] D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Annals of Mathematics **88** (1968), 35–44.

[2] S. Argyros and V. Farmaki, *On the structure of weakly compact subsets of Hilbert spaces and applications to the geometry of Banach spaces*, Transactions of the American Mathematical Society **289** (1985), 409–427.

[3] S. Argyros and S. Mercourakis, *On weakly Lindelöf Banach spaces*, Rocky Mountain Journal of Mathematics **23** (1993), 395–446.

[4] Y. Benyamini, M. E. Rudin and M. Wage, *Continuous images of weakly compact subsets of Banach spaces*, Pacific Journal of Mathematics **70** (1977), 309–324.

[5] Y. Benyamini and T. Starbird, *Embedding weakly compact sets into Hilbert spaces*, Israel Journal of Mathematics **23** (1976), 137–141.

[6] W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński, *Factoring weakly compact operators*, Journal of Functional Analysis **17** (1974), 311–327.

[7] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monographs **64**, Longman, London, 1993.

[8] P. Enflo, *Banach spaces which can be given an equivalent uniformly convex norm*, Israel Journal of Mathematics **13** (1972), 281–288.

[9] M. Fabian, *Gâteaux Differentiability of Convex Functions and Topology. Weak Asplund Spaces*, Wiley, New York, 1997.

[10] M. Fabian, P. Hájek and V. Zizler, *Uniform Eberlein compacta and uniformly Gâteaux smooth norms*, Serdica Mathematical Journal **23** (1997), 351–362.

[11] M. Fabian, V. Montesinos and V. Zizler, *Pointwise semicontinuous smooth norms*, to appear.

[12] M. Fabian and V. Zizler, *On uniformly Gâteaux smooth $C^{(n)}$-smooth norms on separable Banach spaces*, Czechoslovak Mathematical Journal **49** (1999), 657–672.

[13] V. Farmaki, *The structure of Eberlein, uniformly Eberlein and Talagrand compact spaces in $\Sigma(\mathbb{R}^\Gamma)$*, Fundamenta Mathematicae **128** (1987), 15–28.

[14] J. Frontisi, *Representable Banach spaces and uniformly Gâteaux smooth norms*, Serdica Mathematical Journal **22** (1996), 33–38.

[15] G. Godefroy, *Renormings of Banach spaces*, in *Handbook on Banach Spaces* (W. B. Johnson and J. Lindenstrauss, eds.), Elsevier, Amsterdam, to appear.

[16] P. Habala, P. Hájek and V. Zizler, *Introduction to Banach Spaces I, II*, Matfyzpress, Prague, 1996.

[17] P. Hájek, *Polynomials and injections of Banach spaces into superreflexive spaces*, Archiv der Mathematik **63** (1994), 39–44.

[18] P. Hájek, *Dual renormings of Banach spaces*, Commentationes Mathematicae Universitatis Carolinae **37** (1996), 241–253.

[19] R. Haydon, *Trees in renorming theory*, Proceedings of the London Mathematical Society **78** (1999), 541–584.

[20] W. B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces*, Israel Journal of Mathematics **17** (1974), 219–230 and **32** (1979), 382–383.

[21] D. Kutzarova and S. Troyanski, *Reflexive Banach spaces without equivalent norms which are uniformly convex or uniformly differentiable in every direction,* Studia Mathematica **72** (1982), 92–95.

[22] J. Lindenstrauss, *Weakly compact sets, their topological properties and the Banach spaces they generate,* Annals of Mathematics Studies **69**, Princeton University Press, 1972, pp. 235–276.

[23] S. Mercourakis and S. Negrepontis, *Banach spaces and Topology II,* in *Recent Progress in General Topology* (M. Hušek and J. van Mill, eds.), Elsevier Science Publishers B.V., Amsterdam, 1992.

[24] D. P. Milman, *On some criteria for the regularity of spaces of type (B),* Doklady Akademii Nauk SSSR **20** (1938), 243–246.

[25] A. Moltó, V. Montesinos, J. Orihuela and S. Troyanski, *Weakly uniformly rotund Banach spaces,* Commentationes Mathematicae Universitatis Carolinae **39** (1998), 749–753.

[26] A. Moltó and S. Troyanski, *On uniformly Gâteaux differentiable norms in C(K),* Mathematika **41** (1994), 233–238.

[27] J. Orihuela, *On weakly Lindelöf Banach spaces,* in *Progress in Functional Analysis* (K. D. Bierstedt, J. Bonet, J. Horváth and M. Maestre, eds.), Elsevier Science Publishers, Amsterdam, 1992, pp. 279–291.

[28] J. Orihuela and M. Valdivia, *Projective generators and resolutions of identity in Banach spaces,* Revista Matemática de la Universidad Complutense de Madrid **2**, Suppl. Issue, 1990, pp. 179–199.

[29] B. J. Pettis, *A proof that every uniformly convex space is reflexive,* Duke Mathematical Journal **5** (1939), 249–253.

[30] H. P. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces,* Compositio Mathematica **28** (1974), 83–111.

[31] M. Talagrand, *Espaces de Banach faiblement $\mathcal{K}$-analytiques,* Annals of Mathematics **110** (1979), 407–438.

[32] W. K. Tang, *Uniformly differentiable bump functions,* Archiv der Mathematik **68** (1997), 55–59.

[33] S. Troyanski, *On uniform rotundity and smoothness in every direction in nonseparable Banach spaces with unconditional basis,* Comptes Rendus de l'Académie Bulgare des Sciences **30** (1977), 1243–1246.

[34] J. Vanderwerff, J. H. M. Whitfield and V. Zizler, *Markushevich bases and Corson compacta in duality,* Canadian Journal of Mathematics **46** (1994), 200–211.

[35] L. Vašák, *On one generalization of weakly compactly generated Banach spaces,* Studia Mathematica **70** (1981), 11–19.

[36] V. Zizler, *Nonseparable Banach spaces,* in *Handbook on Banach Spaces* (W. B. Johnson and J. Lindenstrauss, eds.), Elsevier, Amsterdam, to appear.