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ESTIMATES FOR THE HYPERBOLIC METRIC OF THE PUNCTURED PLANE AND APPLICATIONS

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ABSTRACT

The hyperbolic metric h_Ω of the twice punctured complex plane Ω is studied. A new recursive algorithm for computing the density λ_Ω of h_Ω is given. For a proper subdomain G of Ω we answer a question of G. Martin concerning quasiconformal mappings of G that can be extended to the complement of G as the identity map.

1. Introduction

The hyperbolic metric h_Ω of a domain $\Omega \subset \overline{\mathbb{C}}$, with $\operatorname{card}(\overline{\mathbb{C}} \setminus \Omega) \geq 3$, introduced by H. Poincaré and developed by R. Nevanlinna (principle of hyperbolic metric) provides a convenient tool for the study of the intrinsic geometry of Ω and holomorphic mappings of plane domains. The conformal invariant h_Ω is a Riemannian metric with a density function λ_Ω. For $\Omega = \Delta, \Delta = \{z \in \mathbb{C}: |z| < 1\}$, h_Ω is the usual non-Euclidean metric of the unit disk Δ. Regarding the size of $\partial\Omega$, the two extreme cases are $\Omega = \Delta$ and $\Omega = \mathbb{C}(-1,1)$, $\mathbb{C}(-1,1) = \overline{\mathbb{C}} \setminus \{-1,1,\infty\}$. In the latter case we have no simple explicit expression for $h_{\mathbb{C}(-1,1)}$, but several

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estimates for $\lambda_{\mathbb{C}(-1,1)}$ are given in [BePom]. For domains between these two extreme cases less seems to be known, but various estimates have been proved in the literature; see, e.g., [W2, H].

In this paper we review various estimates for the hyperbolic metric and prove new estimates based on symmetrization and polarization transformations. Some applications to meromorphic functions and quasiconformal mappings will be given as well.

The paper is organized as follows. Section 2 is expository; it contains two parts. In the first one we survey known properties and estimates of the hyperbolic metric. Here we start with Nevanlinna's principle of the hyperbolic metric and then discuss symmetrization type results of A. Bermant, A. Weitsman, D. Minda, and A. Yu. Solynin. In the second part of Section 2 we discuss applications of the hyperbolic metric to quasiconformal mappings. The first such application was found by O. Teichmüller [T]. Others are due to J. Krzyz, S. B. Agard and F. W. Gehring, and G. Martin.

In Sections 3 and 4 we obtain new estimates for the hyperbolic metric of the twice punctured sphere and twice punctured disk, respectively. Besides, in Section 3, making use of an idea from [SolV], we establish an identity for the density $\lambda_{\mathbb{C}(-1,1)}$, which enables one to compute $\lambda_{\mathbb{C}(-1,1)}$ in terms of a recursive procedure — we have not seen such a procedure elsewhere. This result is given in Lemma 3.19. Section 3 contains also two applications of our estimates — one for meromorphic functions (Theorem 3.23) that strengthens Bermant's theorem in [Ber] and the second for quasiconformal mappings (Theorem 3.28). Section 5 contains one more application — here we prove an analog of Schottky's theorem for bounded analytic functions.

Studying quasiconformal self-mappings of $\mathbb{C}(-1,1)$, keeping the boundary pointwise fixed and homotopic to the identity, O. Teichmüller proved that for prescribed $a, b \in \mathbb{C}(-1,1)$, among all such maps, there is one, f, with the maximal dilatation $K(f) = (1/2)\log h_{\mathbb{C}(-1,1)}(a,b)$, with the property that $f(a) = b$ where the expression for $K(f)$ is sharp. F. W. Gehring raised the question of a similar estimate for the unit disk Δ, and this question was solved by J. Krzyz [Kr]. Very recently G. Martin [M] raised the same question for a general plane domain. We will provide an answer in Theorem 6.1 that is our main result in Section 6. In Theorem 6.4 we show that an extremal quasiconformal self-mapping of Ω has the maximal possible dilatation, denoted by $h_\Omega(K)$ below, if Ω corresponds to a Fuchsian group of the second kind. In Theorem 6.9 we prove that $h_\Omega(K)$ is invariant under conformal mappings.

Some open questions which we find interesting are discussed in Sections 3, 5, and 6.

2. Preliminaries

2.1. Estimates for the hyperbolic metric. Every domain $D \subset \bar{\mathbb{C}}$ having at least three boundary points (throughout this paper we consider such domains only) has the unit disk Δ as the universal covering surface. If $\pi\colon \Delta \to D$ is a universal covering mapping, then the density of the hyperbolic metric (or the Poincaré metric) λ_D of D is defined by the equation

$$\lambda_D(\pi(z))|\pi'(z)| = 1/(1-|z|^2) \quad (z \in \Delta,\ w = \pi(z) \in D,\ w \neq \infty). \tag{2.2}$$

In the case $w = \infty$ we set $\lambda_D(\infty) = \lambda_{D^{-1}}(0)$, where $D^{-1} = \{w\colon w^{-1} \in D\}$. The hyperbolic metric of an open disconnected set is defined by components. As is well known, the density of the hyperbolic metric may be characterized as the maximal solution of the differential equation

$$\Delta \log \lambda = 4\lambda^2 \tag{2.3}$$

in the domain D [H, p. 13]; therefore it has a constant curvature -4. The hyperbolic distance between two points $z_1, z_2 \in D$ is defined by

$$h_D(z_1, z_2) = \inf \kappa_D(\gamma) \quad \text{with } \kappa_D(\gamma) = \int_\gamma \lambda_D(z)|dz|,$$

where the infimum is taken among all rectifiable arcs $\gamma \subset D$ joining z_1 and z_2. As is well known, the definition given above for the hyperbolic distance is equivalent to the following one,

$$h_D(z_1, z_2) = \min_{\zeta_1, \zeta_2} h_\Delta(\zeta_1, \zeta_2) = \min_{\zeta_2} h_\Delta(\zeta_1, \zeta_2), \tag{2.4}$$

where the first minimum is taken over all $\zeta_1, \zeta_2 \in \Delta$ such that $\pi(\zeta_1) = z_1$, $\pi(\zeta_2) = z_2$; in the second case, $\zeta_1 \in \Delta$ is fixed such that $\pi(\zeta_1) = z_1$ and the minimum is taken over all $\zeta_2 \in \Delta$ such that $\pi(\zeta_2) = z_2$.

In Complex Analysis the hyperbolic metric provides one of the most powerful tools — the so-called

2.5. Nevanlinna's principle of the hyperbolic metric ([N, pp. 49–50]). Let D and Ω be domains on $\mathbb{C}$ and let $w = f(z)$ be holomorphic in D and take values in Ω. Then

$$\lambda_\Omega(f(z))|dw| \le \lambda_D(z)|dz| \quad \forall z \in D, \tag{1}$$

and

$$h_\Omega(f(z_1), f(z_2)) \le h_D(z_1, z_2) \quad \forall z_1, z_2 \in D. \tag{2}$$

As (2) shows, holomorphic mappings are contractions in the hyperbolic metric. If f is a conformal mapping onto Ω, then equalities hold in (1) and (2). Below we give a more general condition of the invariance of the hyperbolic distances.

Let D be a domain on $\overline{\mathbb{C}}$. A subdomain $\Omega \subset D$ will be called h_D-convex (hyperbolically convex w.r.t. D) if for every pair of points $z_1,\ z_2 \in \Omega$ the domain Ω contains at least one of the shortest hyperbolic geodesics joining z_1 and z_2. This easily implies that an h_D-convex subdomain Ω contains all the shortest geodesics joining z_1 and z_2. Indeed, let Ω be h_D-convex. Assume that there is a shortest geodesic $l \not\subset \Omega$ joining points $z_1, z_2 \in \Omega$. Since Ω is open there is a point $z' \in l \setminus \{z_1, z_2\}$ that splits l into two subarcs l_1 and l_2 such that $l_1 \not\subset \Omega$ joins z_1 and z' and $l_2 \subset \Omega$ joins z' and z_2. Since Ω is h_D-convex and $z_1, z' \in \Omega$ there is a shortest geodesic $l' \subset \Omega$ joining z_1 and z'. Let $l^* = l' \cup l_2$. Then $l^* \subset \Omega$ and

$$\kappa_D(l^*) = \kappa_D(l') + \kappa_D(l_2) \le \kappa_D(l_1) + \kappa_D(l_2) = \kappa_D(l).$$

Thus l^* is one of the shortest geodesics joining z_1 and z_2 in D. Each geodesic, being an image of a circular arc under a universal covering mapping, is analytic. Therefore l^* coincides with l since they are analytic arcs having a common subarc l_2. This contradicts our assumption that $l \not\subset \Omega$ and therefore D contains every shortest geodesic joining z_1 and z_2 if $z_1, z_2 \in D$.

The latter implies that the intersection of two h_D-convex subdomains is h_D-convex. There is a nice result of V. Jørgensen [Jø] saying that every disk or half plane contained in D is h_D-convex.

We shall use the following definition from [H]. We say that $w = f(z)$ maps a domain $D_1 \subset \mathbb{C}$ *onto* a domain $D_2 \subset \mathbb{C}$ if f can be continued as an analytic function throughout D_1 in such a way that all the values $w = f(z)$ remain in D_2 and, if similar, is true for the inverse function $z = f^{-1}(w)$, i.e., f^{-1} can be continued as an analytic function throughout D_2 in such a way that all values $z = f^{-1}(w)$ remain in D_1.

2.6. THEOREM: *Suppose that $w = f(z)$ maps D_1 onto D_2. Then*

$$\lambda_{D_2}(f(z))|f'(z)| = \lambda_{D_1}(z) \quad \forall z \in D_1. \tag{1}$$

If, additionally, $\Omega_k \subset D_k$ is h_{D_k}-convex and if $w = f(z)$ maps Ω_1 one-to-one onto Ω_2, then

$$h_{D_1}(z_1, z_2) = h_{D_2}(f(z_1), f(z_2)) \quad \forall z_1, z_2 \in \Omega_1. \tag{2}$$

In order to apply the principle of the hyperbolic metric in quantitative estimates, we need to have a set of domains with known or well-controlled hyperbolic density. Often it is enough to consider the sphere $\bar{\mathbb{C}}$, the disk $\Delta_r = \{z\colon |z| < r\}$, $\Delta = \Delta_1$, or more generally, the disk $\Delta_r(z_0) = \{z\colon |z - z_0| < r\}$, and the annulus $K(r_1, r_2) = \{z\colon r_1 < |z| < r_2\}$ punctured at a finite (perhaps zero) number of points. In what follows $D(z_1, \ldots, z_n)$ stands for the domain $D \subset \bar{\mathbb{C}}$ punctured at $z_1, \ldots, z_n \in D$. By $\lambda(z; a, b)$ and $h(z_1, z_2; a, b)$ we shall denote the density of the hyperbolic metric of $\mathbb{C}(a, b)$ and the corresponding hyperbolic distance in $\mathbb{C}(a, b)$; $\lambda(z) = \lambda(z; -1, 1)$, $h(z_1, z_2) = h(z_1, z_2; -1, 1)$. One can consider $\lambda(z; a, b)$ either as a function of z or as a function of the punctures a and b, depending on the problem studied.

The following lemma by Bermant presents one of the first results concerning the monotonicity of $\lambda(z; a, b)$ as a function of the parameters a and b.

2.7. Lemma ([Ber]): (1) *The function $\lambda(0; 1, te^{i\theta})$ with $t = 1$ increases in θ for $0 < \theta \leq \pi$;*

(2) *with $\theta = 0$, $\lambda(0; 1, te^{i\theta})$ increases in $-\infty < t < 0$;*

(3) *with $\theta = 0$, $\lambda(0; 1, te^{i\theta})$ decreases in $0 < t < 1$.*

Applying this lemma, Bermant [Ber] solved some interesting problems on the meromorphic and holomorphic functions. The following theorem is one of them.

2.8. Theorem ([Ber]): *Let $f \in \mathcal{R}$, where $\mathcal{R}$ is the class of functions holomorphic in Δ with the normalization $f(0) = 0$, $|f'(0)| = 1$. If w_1, w_2, $\arg w_2 = \pi + \arg w_1$, are exceptional for f in Δ, that is, $w_1, w_2 \in \mathbb{C} \setminus f(\Delta)$, then*

$$\frac{|w_1| + |w_2|}{2} \geq 4\pi^2 \Gamma^{-4}(1/4) = 0.228\ldots,$$

where $\Gamma(\cdot)$ stands for the Euler gamma function. Equality occurs here iff $f(z) = e^{i\alpha} f_0(e^{i\beta} z) \in \mathcal{R}$, where $\alpha, \beta \in \mathbb{R}$ and f_0 is a covering mapping of $\mathbb{C}(-A_1, A_1)$ with $A_1 = 4\pi^2\Gamma^{-4}(1/4)$.

In [W1, W2] A. Weitsman developed the theory of symmetrization for the hyperbolic metric; this provides a very efficient method of estimating the hyperbolic distances.

2.9. Theorem ([W2], see also [Sol, Theorem 13]): *Let a domain D be circularly symmetric with respect to the ray $\mathbf{R}_0 = \{z\colon \Im z = 0, \Re z \geq 0\}$. Let $D \cap \mathbb{T}_r = \{z\colon |z| = r, |\arg z| < \theta_0(r)\}$, $r > 0$, $0 < \theta_0(r) \leq \pi$, where $\mathbb{T}_r = \partial \Delta_r$; in the case $\mathbb{T}_r \subset D$, $\theta_0(r) = \pi$ and $D \cap \mathbb{T}_r = \{z\colon |z| = r, |\arg z| \leq \theta_0(r)\}$. Then $\lambda_D(re^{i\theta})$ increases strictly in θ, $0 \leq \theta \leq \theta_0(r)$, except the cases when $D = \Delta_R$, $D =$*

$\bar{\mathbb{C}} \setminus \overline{\Delta}_R$, and $D = K(r_1, r_2)$, where $0 \le r_1 < r_2 \le \infty$. In the exceptional cases $\lambda_D(re^{i\theta})$ is constant on $\mathbb{T}_r$.

2.10. Corollary: *Let D satisfy the assumptions of Theorem 2.9 and let us assume that D is not exceptional for the equality cases of this theorem. Let $r_1 \ge 0$, $r_2 \ge 0$, $0 \le \theta \le \pi$, $0 \le \varphi \le \pi - \theta$. Then the function $h_D(r_1 e^{i\theta}, r_2 e^{i(\varphi+\theta)})$ increases strictly in θ for $0 \le \min\{\theta_0(r_1), \theta_0(r_2) - \varphi\}$ and increases strictly in φ for $0 \le \varphi \le \theta_0(r_2) - \theta$. In particular,*

$$h_D(z_1, z_2) \ge h_D(|z_1|, |z_2|) \quad \textit{for all } z_1, z_2 \in D.$$

If $z_1 \ne z_2$, then equality is attained here iff $z_1, z_2 \in \mathbf{R}_0$.

In [Sol] Solynin found another method for proving the symmetrization results of Weitsman [W2] with a complete description of the equality cases. Solynin's approach is based on the *polarization* transformation introduced by V. Wolontis [Wo] in 1952.

By a **polarization** of a set $E \subset \bar{\mathbb{C}}$ w.r.t. $\mathbb{R}$ we mean the set

$$P_{\mathbb{R}}E = ((E \cup E^*) \cap \overline{\mathbb{H}}) \cup ((E \cap E^*) \cap \mathbb{H}_-), \tag{2.11}$$

where $E^* = \{z \colon \bar{z} \in E\}$, $\mathbb{H} = \{z \colon \Im z > 0\}$, and $\mathbb{H}_- = \mathbb{C} \setminus \overline{\mathbb{H}}$. Sometimes we also use a similar notation for a symmetric image A^* of a set A with respect to an arbitrary line L.

The polarization of E w.r.t. an arbitrary oriented circle $L \subset \bar{\mathbb{C}}$ is defined by

$$P_L E = \varphi^{-1}(P_{\mathbb{R}}(\varphi(E))),$$

where φ is an arbitrary Möbius mapping that takes L onto $\mathbb{R}$ and preserves the orientation.

The following theorem due to D. Minda [Min] deals with the hyperbolic densities of a domain that is already polarized.

2.12. Theorem ([Min]): *Let L be a straight line and for a given $z \in \mathbb{C}$ let z^* denote the point that is symmetric to z w.r.t. L. Let D be a domain on $\bar{\mathbb{C}}$, $z, z^* \in D$, and let $D(z)$ and $D(z^*)$ denote the connected components of $D \setminus L$ containing z and z^*, respectively. If $D(z) \subset (D(z^*))^*$ then*

$$\lambda_D(z) \le \lambda_D(z^*).$$

Equality here occurs iff $D(z) = D(z^*)$, i.e., iff D is symmetric w.r.t. L.

The above symmetrization results follow from Solynin's polarization theorem [Sol] that is formulated here for the polarization w.r.t. circles on $\bar{\mathbb{C}}$.

2.13. THEOREM ([Sol]): *Let L be a circle centered at $z_0 \in \mathbb{C}$ with radius R, $0 < R < \infty$. Let L^+ and L^- stand for L oriented in the positive and negative direction, respectively. For a given domain D, let D_1 and D_2 denote the polarization of D w.r.t. L^+ and w.r.t. L^-, respectively. Then*

$$(1) \qquad \lambda_{D_1}(z) \le \min\{\lambda_D(z),\ \lambda_D(z^*)R^2/|z-z_0|^2\} \quad \forall z \in D_1 \cap \overline{\Delta_R(z_0)},$$

$$(2) \qquad \lambda_{D_2}(z) \le \min\{\lambda_D(z),\ \lambda_D(z^*)|z-z_0|^2/R^2\} \quad \forall z \in D_2 \setminus \Delta_R(z_0),$$

$$(3) \quad \lambda_{D_1}(z)\lambda_{D_1}(z^*) = \lambda_{D_2}(z)\lambda_{D_2}(z^*) \le \lambda_D(z)\lambda_D(z^*) \quad \forall z \quad \text{s.t.} \quad z, z^* \in D.$$

Equality in any one of these inequalities occurs only in the cases $D = D_1$ or $D = D_2$.

If L is a straight line, then (1) – (3) are true if we set $|z - z_0|/R = 1$ in (1) and (2).

2.14. THEOREM: (1) *The assertion of Lemma 2.7 (1) holds for every fixed $t > 0$;*

(2) *The assertion of Lemma 2.7 (2) holds for every fixed θ, $|\theta - \pi| \le \pi/2$;*

(3) *The assertion of Lemma 2.7 (3) holds for every fixed $\theta \in \mathbb{R}$.*

Proof: (1) Let $t > 0$ be fixed and let $\theta_1 < \theta_2 \le \pi$, where $\theta_1 \ge 0$ if $t \ne 1$ and $\theta_1 > 0$ if $t = 1$. Evidently, the domain $\mathbb{C}(1, te^{i\theta_1})$ coincides with the polarization of $\mathbb{C}(1, te^{i\theta_2})$ w.r.t. the straight line $L = L(\pi + (\theta_1 + \theta_2)/2)$, where $L(\varphi) = \{z = te^{i\varphi}\colon\ -\infty < t < \infty\}$. Therefore, Theorem 2.13 (1) implies the strict inequality $\lambda(0; 1, te^{i\theta_2}) > \lambda(0; 1, te^{i\theta_1})$ since the equality conditions of that theorem evidently are not satisfied.

(2) Fix θ, $\pi/2 \le \theta \le 3\pi/2$. Let $0 < t_1 < t_2 < \infty$ and let L denote the circle centered at $z_0 = re^{i\theta}$, $r < t_1$ that passes through $z = 1$. Next, we choose the parameter r satisfying the equation

$$(t_1 - r)(t_2 - r) = |z_0 - 1|^2.$$

It is easy to see that under these conditions, the point $z = 0$ lies inside the circle L and the domain $\mathbb{C}(1, t_2e^{i\theta})$ coincides with the polarization of $\mathbb{C}(1, t_1e^{i\theta})$ w.r.t. L if L has positive orientation. Therefore the desired assertion again follows from Theorem 2.13 (1). The monotonicity is strict because the equality conditions of Theorem 2.13 (1) are not satisfied.

Applying the polarization w.r.t. a circle $\mathbb{T}_r$ with a suitable r we prove the third assertion of the theorem.

2.15. HYPERBOLIC METRIC AND QUASICONFORMAL MAPPINGS. O. Teichmüller was the first who used the hyperbolic metric in the study of quasiconformal mappings. The following theorem appeared in [T].

2.16. THEOREM ([T]): *Let $K > 1$. If f is a K-quasiconformal self-mapping of $\bar{\mathbb{C}}$, normalized by the conditions $f(-1) = -1$, $f(1) = 1$, $f(\infty) = \infty$, then*

$$h(z_0, f(z_0)) \leq (1/2)\log K$$

for each $z_0 \in \mathbb{C}(-1,1)$. Moreover, given any pair z_0, $w_0 \in \mathbb{C}(-1,1)$ satisfying $h(z_0, w_0) \leq (1/2)\log K$ there exists a normalized K-quasiconformal self-mapping of $\bar{\mathbb{C}}$ homotopic to the identity on $\mathbb{C}(-1,1)$ such that $f(z_0) = w_0$.

In fact, Teichmüller's Theorem 2.16 deals with the characteristic $h_\Omega(K)$ of a given domain $\Omega \subset \bar{\mathbb{C}}$ defined as follows. Let $K \geq 1$ and let $\Omega(K)$ denote the class of K-quasiconformal self-mappings f of Ω with boundary values given by the identity mapping that is homotopic to the identity on Ω. That is, there is a continuous mapping $g\colon \overline{\mathbb{C}} \times [0,1] \to \overline{\mathbb{C}}$ such that $g(z,t) = z$ for all $z \in \overline{\mathbb{C}} \setminus \Omega$, $0 \leq t \leq 1$, and $g(z,0) = z$ for all $z \in \Omega$ and $g(z,1) = f(z)$ for all $z \in \Omega$. Then $h_\Omega(K)$ is defined by

$$h_\Omega(K) = \sup_{z\in\Omega, f\in\Omega(K)} h_\Omega(z, f(z)). \tag{2.17}$$

It follows from Teichmüller's theorem that

$$h_{\mathbb{C}(-1,1)}(K) = (1/2)\log K. \tag{2.18}$$

J. Krzyz [Kr] considered the problem posed by F. W. Gehring concerning the value of $h_\Delta(K)$. Below we shall use the following standard notation [LV],

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K),$$

where $K > 0$, $0 < r < 1$, and

$$\mu(r) = \frac{\pi}{2}\frac{\mathfrak{K}'(r)}{\mathfrak{K}(r)}$$

is the modulus of the unit disk Δ slit along the segment $[0, r]$; this doubly connected domain is usually called the Grötzsch ring. Here $\mathfrak{K}(r)$ denotes the complete elliptic integral of modulus r, $\mathfrak{K}'(r) = \mathfrak{K}(r')$, where $r' = \sqrt{1-r^2}$ is the complement of r [AVV].

2.19. THEOREM ([Kr]): *Let $K > 1$ and $f \in \Delta(K)$. Then for every $z \in \Delta$*

$$h_\Delta(z, f(z)) \leq d(K), \tag{2.20}$$

where

$$d(K) = \frac{1}{2}\log\frac{1+a(K)}{1-a(K)} \quad \text{and} \quad a(K) = \mu^{-1}(2\mathrm{arcoth}\sqrt{K}).$$

The bound in (2.20) is best possible, and for every $K > 1$ and $z_0, w_0 \in \Delta$ with $h_\Delta(z_0, w_0) = d(K)$ there is a unique function $f_K(z, z_0, w_0) \in \Delta(K)$ such that $f_K(z_0, z_0, w_0) = w_0$.

Thus, in our notation

$$h_\Delta(K) = d(K). \tag{2.21}$$

Theorem 2.19 without the uniqueness assertion was proved by J. Krzyz [Kr]. The proof in [Kr] is essentially based on the Teichmüller construction in [T] of the extremal function denoted here by $f_K = f_K(z, z_0, w_0)$.

To prove the uniqueness assertion we may assume that $z_0 = 0$ and $w_0 = -\rho$, $0 < \rho < 1$. The function f_K is constructed in [T] in such a way that $(f_K)_{\bar z}/(f_K)_z = k\overline{\varphi(z)}/|\varphi(z)|$, where $k = (K-1)/(K+1)$ and φ is holomorphic in Δ except for a simple pole at $z = 0$. Moreover, φ has no zeros in Δ. This implies that the function

$$\tilde{f}_K(z) = \sqrt{\frac{f_K(z^2)+\rho}{1+\rho f_K(z^2)}}$$

is a K-quasiconformal self-mapping of Δ such that

$$\frac{(\tilde{f}_K)_{\bar z}}{(\tilde{f}_K)_z} = k\frac{\overline{\varphi_1(z)}}{|\varphi_1(z)|}$$

with some function φ_1 that is holomorphic in Δ. Therefore, by the uniqueness theorem of K. Strebel [St], $\tilde{f}_K$ is the unique extremal Teichmüller mapping for its boundary values. Now if $f \in \Delta(K)$ and $f(0) = -\rho$, then the function $\tilde{f}(z) = \sqrt{(f(z^2)+\rho)/(1+\rho f(z^2))}$ is a K-quasiconformal self-mapping of Δ that has the same boundary values as $\tilde{f}_K$. Therefore $\tilde{f}$ must coincide with $\tilde{f}_K$ and the uniqueness assertion follows.

It will be useful to know that

$$h_{\mathbb{C}(-1,1)}(K) < h_\Delta(K) \quad \forall K > 1. \tag{2.22}$$

Indeed, in view of (2.18) and (2.21), the inequality (2.22) is equivalent to the inequality

$$\frac{K-1}{K+1} < \mu^{-1}(2\operatorname{arcoth}\sqrt{K}). \tag{2.23}$$

The function $\mu(r)$ decreases in $0 < r < 1$. Hence, after some calculations, we see that (2.23) is equivalent to

$$\mu\Big(\frac{K-1}{K+1}\Big) > \log\frac{\sqrt{K}+1}{\sqrt{K}-1}. \tag{2.24}$$

Setting $r = (K-1)/(K+1)$ we reduce (2.24) to the inequality

$$\mu(r) > \log\frac{1+r'}{r},$$

which is known to be true; cf. [LV, p. 61, (2.10)].

Some interesting applications of the hyperbolic metric in the study of quasiconformal mappings were found by S. B. Agard and F. W. Gehring [AG]. They used the following estimate for the hyperbolic distances in $\mathbb{C}(-1,1)$.

2.25. Lemma ([AG]): *If $z_1, z_2 \in \mathbb{C}(-1,1)$, then*

$$h(z_1,z_2) \ge h(i\cot\alpha_1, i\cot\alpha_2),$$

where

$$\alpha_k = \arcsin\left(\frac{2}{|z_k+1|+|z_k-1|}\right), \quad k=1,2.$$

Lemma 2.25 has the following geometric meaning. For a curve $\gamma \in \mathbb{C}(-1,1)$ let γ^* mean the projection of γ onto the positive imaginary axis, taken along an ellipse that has the foci at $z=-1$ and $z=1$, i.e.,

$$\gamma^* = \{it, t \ge 0, \text{ such that there is } z \in \gamma \text{ with } |z-1|+|z+1| = 2\sqrt{1+t^2}\}.$$

Then the hyperbolic length of γ is bigger than the hyperbolic length of γ^*.

Agard and Gehring [AG] gave also the explicit expression for the lower bound in Lemma 2.25:

$$h(i\cot\alpha_1, i\cot\alpha_2) = \frac{1}{2}\log\left(\frac{\mu(\sin\frac{1}{2}\alpha_2)}{\mu(\sin\frac{1}{2}\alpha_1)}\right) \quad \text{if } 0<\alpha_2\le\alpha_1\le\pi/2. \tag{2.26}$$

It was shown by Agard and Gehring that Theorem 2.16, Lemma 2.25 and equation (2.26) imply the following so-called angle distortion theorem. Note that the function φ_K in [AG] is the same as our $\varphi_{1/K}$.

2.27. THEOREM ([AG]): *Suppose that f is a K-quasiconformal mapping of $\bar{\mathbb{C}}$, $f(\infty)=\infty$. Then for each triple of distinct finite points z_0, z_1, z_2,*

$$\sin\frac{1}{2}\beta \geq \varphi_{1/K}\Big(\sin\frac{1}{2}\alpha\Big),$$

where

$$\alpha = \arcsin\Big(\frac{|z_1-z_2|}{|z_1-z_0|+|z_2-z_0|}\Big),$$
$$\beta = \arcsin\Big(\frac{|f(z_1)-f(z_2)|}{|f(z_1)-f(z_0)|+|f(z_2)-f(z_0)|}\Big).$$

This inequality is sharp.

Very interesting results linking the quasiconformal mappings and the hyperbolic metric were recently obtained by G. Martin [M], who applied a new technique based on the holomorphic motions. The following theorem is among them.

2.28. THEOREM ([M]): *Let Ω be a planar domain. Suppose that $z, w \in \Omega$ and $h_\Omega(z,w) \leq (1/2)\log K$. Then there is a K-quasiconformal mapping $f \in \Omega(K)$ such that $f(z)=w$.*

In [M] the homotopy condition was not mentioned, but analyzing the proof of [M, Theorem 5.1] we see that the mapping f constructed satisfies the condition $f \in \Omega(K)$.

The paper [M] ends with the following question: Is it true that $h_\Omega(z, f(z)) \leq (1/2)\log K$ for a K-quasiconformal mapping f with boundary values given by the identity mapping?

The relations (2.18) and (2.22) show that the answer is "no" if Ω is the disk. Moreover, it follows from Theorem 3 and some remarks in [KK, Ch. 4], that the answer is "no" for all domains of the form $\bar{\mathbb{C}}(a_1,\dots,a_n)$ if a_i, $i=1,\dots,n$, are distinct and $n>3$. It will be shown in Section 6 that the correct upper bound in Martin's question is $d(K)$.

In this section we mentioned only results linking quasiconformal mappings and the hyperbolic metric that we plan to discuss later in this paper. For the general theory we refer the reader to the book of Krushkal and Kühnau [KK] and to the references therein.

3. The hyperbolic distances in $\mathbb{C}(-1,1)$

First we sum up some simple properties of the distance $h(z_1,z_2)$. Let $I_a = \{z\colon \Re z > a,\ \Im z > 0\}$, $a \in \mathbb{R}$.

3.1. LEMMA: (1) *If* $z_1, z_2 \in \bar{I}_0 \setminus \{1, \infty\}$, *then*

$$h(z_1, -z_2) \geq h(z_1, \bar{z}_2) \geq h(z_1, z_2), \quad h(z_1, -z_2) \geq h(z_1, -\bar{z}_2) \geq h(z_1, z_2).$$

Equality in the first and fourth of these inequalities occurs iff $\Re z_1 \cdot \Re z_2 = 0$; *equality in the second and third ones occurs iff* $\Im z_1 \cdot \Im z_2 = 0$.

(2) *Every disk or half plane in* $\mathbb{C}(-1,1)$ *and every disk* $\Delta_r(1)$ *with* $r \leq 2$ *is* h*-convex.*

Every image of an h*-convex domain under a Möbius automorphism of* $\mathbb{C}(-1,1)$ *is* h*-convex.*

The assertions of this lemma are easy consequences of Lemma 2.12. Assertion (2) of Lemma 3.1 implies the h-convexity of every disk $\Delta_{r(a)}(a)$ with $a > 1$, $r(a) = \sqrt{a^2 - 1}$ and of every half disk $\Delta^+_{r(a)}(a) = \Delta_{r(a)}(a) \cap \mathbb{H}$. In particular, the quadrant I_0 is h-convex.

Further we shall show that every inequality for the hyperbolic distance $h(z_1, z_2)$ generates several new estimates. To this end we use a self-mapping of two-sheeted surface over $\mathbb{C}$. Such an approach was applied in our previous paper [SolV] in studying the Teichmüller extremal capacity.

Consider the Riemann surface $\mathcal{R}_1$ consisting of two sheets over the domain $\mathbb{C}(-1,1)$ branched at the points -1 and ∞; and let $\mathcal{R}_2$ be a similar Riemann surface branched at the points -1 and 1 (cf. Fig. 1).

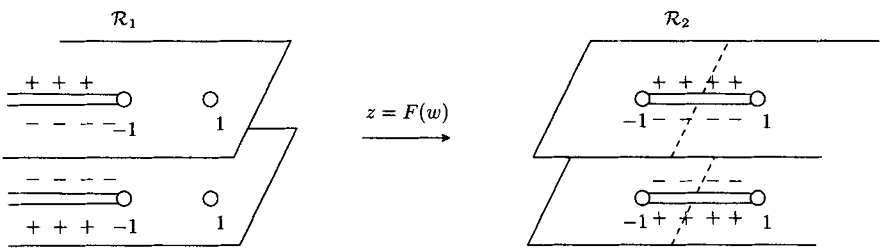

Figure 1.

The function

$$F(w) = \frac{1}{2}\Big(\sqrt{\frac{w+1}{2}} + \sqrt{\frac{2}{w+1}}\Big), \tag{3.2}$$

where $\sqrt{(w+1)/2} > 0$ when $w > -1$, maps the Riemann surface $\mathcal{R}_1$ conformally onto $\mathcal{R}_2$. Thus, $z = F(w)$ maps $\mathbb{C}(-1,1)$ onto itself in the sense of Theorem 2.6.

The inverse mapping given by

$$w = g(z) = -1 + 2(z + \sqrt{z^2 - 1})^2 \quad \text{with } \sqrt{z^2 - 1} > 0 \quad \text{for } z > 1 \tag{3.3}$$

maps the quadrant I_0 in one-to-one way onto the domain

$$D_0 = \{w \in \mathbb{H}\colon |w + 1| > 2\}.$$

By Lemma 3.1 the domains I_0 and D_0 are h-convex. Combining this result with Theorem 2.6 we get

3.4. Lemma: *The following equalities hold true:*

$$\lambda(g(z)) = \lambda(z)\frac{\sqrt{|z^2-1|}}{|\sqrt{z+1} + \sqrt{z-1}|^4} \quad \forall z \in \overline{\mathbb{H}} \setminus \{-1, 1, \infty\},$$
$$h(g(z_1), g(z_2)) = h(z_1, z_2) \quad \forall z_1, z_2 \in \overline{I_0} \setminus \{1, \infty\}.$$

Let $\mathcal{L}(\rho) = \{z \in \overline{I_0}\colon |z-1| + |z+1| = 2\rho\}$, $\rho > 1$, be an arc of ellipse, $\mathcal{L}(1) = [0, 1)$. The function $g(z)$ takes the arc $\mathcal{L}(\rho)$, $\rho \geq 1$, to the semicircle $T^+_{r(\rho)}(-1) = \{w\colon |w+1| = r(\rho)\} \cap \mathbb{H}$, where $r(\rho) = 2(\rho + \sqrt{\rho^2 - 1})^2$ (see Fig. 2).

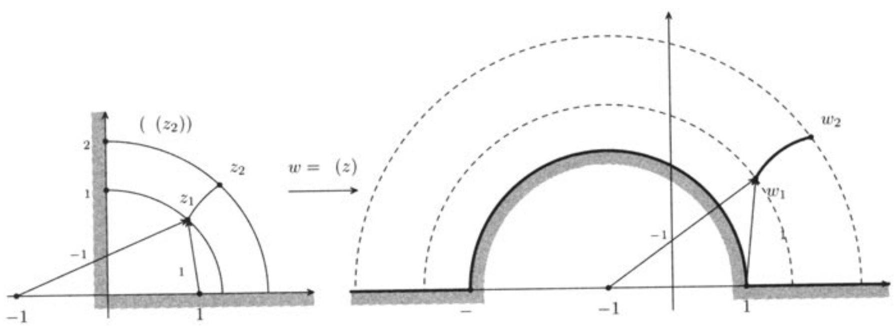

Figure 2.

Therefore, the projection along the ellipse (cf. the Agard–Gehring lemma) corresponds under the mapping $g(z)$ to the circular projection onto the ray $\{w\colon \Im w = 0, \Re w \leq -1\}$. We remark that the computation below in (3.7) and (3.8) shows that $|\sqrt{z+1} + \sqrt{z-1}|$ is a constant on the ellipses with foci at $-1, 1$. These facts immediately lead to

3.5. Corollary: *If $z = x + iy$ moves on $\mathcal{L}(\rho)$ so that y increases, then the function $\lambda(z)\sqrt{|z^2-1|}$, and therefore $\lambda(z)$, decreases.*

This Corollary shows that the Agard–Gehring estimate in Lemma 2.25 is a consequence of the symmetrization results and some simple properties of the Joukowski function.

Next we combine Lemma 3.4 with known estimates for the hyperbolic metric in order to get some new ones. Let $r_1 = r_1(z) = |z-1|$, $r_{-1} = r_{-1}(z) = |z+1|$, $R_1 = R_1(z) = |g(z)-1|$, $R_{-1} = R_{-1}(z) = |g(z)+1|$. The parameters R_1 and R_{-1} can be easily expressed in terms of r_1, and r_{-1}. Indeed, from (3.3) we have

$$\begin{aligned} R_1 &= 2\sqrt{|z^2-1|}|\sqrt{z+1}+\sqrt{z-1}|^2 \\ &= 2\sqrt{r_1 r_{-1}}(r_1 + r_{-1} + 2\sqrt{r_1 r_{-1}}\cos(\varphi/2)), \end{aligned} \tag{3.6}$$

where φ is the angle in $(0,\pi]$ formed by the segments $[z,1]$ and $[z,-1]$ at z. Applying (3.6) and the Law of Cosines, we find

$$R_1 = 2\sqrt{r_1 r_{-1}}\left(r_1 + r_{-1} + \sqrt{(r_1+r_{-1})^2-4}\right). \tag{3.7}$$

Similarly,

$$R_{-1} = (1/2)\left(r_1 + r_{-1} + \sqrt{(r_1+r_{-1})^2-4}\right)^2. \tag{3.8}$$

Now the Weitsman symmetrization result for $h(z_1,z_2)$ (see Corollary 2.10) may be written in the following form.

3.9. Lemma: *The function $h(z_1,z_2)$ admits the following lower estimates:*

$$h(1+r_1(z_1), 1+r_1(z_2)), \quad h(1+r_{-1}(z_1), 1+r_{-1}(z_2)), \quad \forall z_1, z_2 \in \overline{\mathbb{H}} \setminus \{-1,1,\infty\};$$
$$h(1+R_1(z_1), 1+R_1(z_2)), \quad h(1+R_{-1}(z_1), 1+R_{-1}(z_2)), \quad \forall z_1, z_2 \in \overline{I}_0 \setminus \{1,\infty\}.$$

The first and third estimates become equality for all $z_1 > 1$, $z_2 > 1$, the second for all $z_1 < -1$, $z_2 < -1$, and the fourth becomes equality for all z_1, z_2 such that $\Re z_1 = \Re z_2 = 0$, $\Im z_1 \geq 0$, $\Im z_2 \geq 0$.

As we have mentioned above, the fourth estimate of this lemma coincides with that of Agard–Gehring. Sometimes, but not always, the new estimates of Lemma 3.9 are better than those known before.

For explicit estimates we need to express $h(a,b)$ with $1 < a < b < \infty$ in terms of simple functions. Below we get such an expression in terms of the function $\mu(t)$.

3.10. Lemma: *If $1 < a < b < \infty$, then*

$$h(a,b) = h(-a,-b) = \frac{1}{2}\log\left(\frac{\mu\left(\sqrt{\frac{a-1}{a+1}}\right)}{\mu\left(\sqrt{\frac{b-1}{b+1}}\right)}\right). \tag{3.11}$$

Proof: It is known (see [Ne, pp. 318, 319]) that the function

$$\zeta = i\frac{\mathfrak{K}'(k)}{\mathfrak{K}(k)} \quad \text{with } k^2 = (1/2)(z+1) \tag{3.12}$$

maps the upper half plane $\mathbb{H}$ conformally onto the domain

$$\{\zeta\colon 0 < \Re\zeta < 1, |\zeta - 1/2| > 1/2\}$$

such that $\zeta(-1) = \infty$, $\zeta(\infty) = 1$, $\zeta(1) = 0$.

Using the formulas

$$\mathfrak{K}(ir/r') = r'\mathfrak{K}(r) \quad \text{and} \quad \mathfrak{K}'(ir/r') = r'[\mathfrak{K}'(r) - i\mathfrak{K}(r)]$$

(cf. [GrRy, p. 908]), we find

$$\zeta(-c) = i\frac{\mathfrak{K}'(i\sqrt{(c-1)/2})}{\mathfrak{K}(i\sqrt{(c+1)/2})} = 1 + i(2/\pi)\mu(\sqrt{(c-1)/(c+1)}) \tag{3.13}$$

with $c > 1$.

The function (3.12) can be extended to the conformal mapping of the universal covering surface over $\mathbb{C}(-1,1)$ onto $\mathbb{H}$. Therefore $h(z_1,z_2) = h_{\mathbb{H}}(\zeta(z_1),\zeta(z_2))$. Now using (3.13), after simple computation we find the desired relation.

The function (3.12) leads to the following well-known explicit expression for the hyperbolic density (cf. [Ne]):

$$\lambda(z) = \frac{|\zeta'(z)|}{2\Im\zeta(z)}. \tag{3.14}$$

For $z = x > 1$, (3.13) and (3.14) imply

$$\lambda(x) = \pi\left[8(x-1)\mathfrak{K}'\Big(\sqrt{\frac{x-1}{x+1}}\Big)\mathfrak{K}\Big(\sqrt{\frac{x-1}{x+1}}\Big)\right]^{-1}, \tag{3.15}$$

which expresses $\lambda(x)$ as a function of a real variable x. Formula (3.15) together with the well-known relations

$$\mathfrak{K}'(1) = \pi/2, \quad \mathfrak{K}(r) = \log(4/r') + \alpha(r),$$

where $\alpha(r) \to 0$ as $r \to 1$, give the following asymptotic equality:

$$\log\lambda(x) = -\log x - \log\log x - \log 2 + \alpha(x) \tag{3.16}$$

with $\alpha(x) \to 0$ as $x \to \infty$.

Lemma 3.4 is suitable for computing the hyperbolic density. A similar computational method for the Teichmüller capacity $p(z)$ was used in our paper [SolV]. The idea of such a computation is classical: we will use the identity that is satisfied for λ in order to bring our computation to the region with a known asymptotic behaviour of λ. Following [SolV] consider the sequence z_n, where $z_0 = z \in \bar{I}_0 \setminus \{1, \infty\}$ and

$$z_{n+1} = \begin{cases} g(z_n) & \text{if } \Re g(z_n) \geq 0, \\ -\overline{g(z_n)} & \text{otherwise.} \end{cases}$$

To show that $z_n \to \infty$ as $n \to \infty$, we consider the function $u(z) = \log|g(z)/z|$, which is evidently harmonic in the domain $\Omega_\delta = I_0 \setminus \overline{\Delta}_\delta(1)$, where $\delta > 0$ is small enough.

Consider the boundary behaviour of $u(z)$: if $x \geq 1 + \delta$, then

$$u(x) \geq \log(4x - 3) > \log(1 + \delta);$$

if $0 \leq x \leq 1 - \delta$, then

$$u(x) \geq \log(1/x) > \log(1 + \delta);$$

if $y \geq 0$, then

$$u(iy) \geq \log(4(y + 1/y)) \geq \log 8.$$

Let $z = 1 + \delta e^{i\theta}$, $0 \leq \theta \leq \pi$. Then

$$\begin{aligned} |g(z)/z| &= |4z + 4\sqrt{z^2 - 1} - 3/z| \\ &= |1 + 4\sqrt{2\delta} e^{i\theta/2} + 7\delta e^{i\theta}|(1 + o(\delta)) \\ &= \left[(1 + 8\sqrt{2\delta}\cos\theta/2 + 32\delta + 14\delta\cos\theta\right]^{1/2}(1 + o(\delta)) \\ &\geq (1 + 18\delta)^{1/2}(1 + o(\delta)) \geq 1 + \delta \end{aligned}$$

for all $\delta > 0$ small enough.

Clearly, $|g(z)/z| \to \infty$ as $z \in \overline{I}_0$ tends to ∞. Therefore the maximum principle shows that $u(z) > \log(1 + \delta)$ for all $z \in \Omega_\delta$. Hence $|z_{n+1}| \geq (1 + \delta)|z_n|$ for all n if $z = z_0 \in \overline{\Omega}_\delta$. Thus $|z_n| \to \infty$ for all $z \in \overline{I}_0 \setminus \{1, \infty\}$.

By Lemma 3.4 and the symmetry property of the hyperbolic density,

$$\lambda(z_n) = \lambda(z_{n-1}) \frac{\sqrt{|z_{n-1}^2 - 1|}}{|\sqrt{z_{n-1} + 1} + \sqrt{z_{n-1} - 1}|^4}.$$

Therefore,

$$\log \lambda(z) = \log \lambda(z_n) + \sum_{k=0}^{n-1} \log \frac{|\sqrt{z_k+1} + \sqrt{z_k-1}|^4}{\sqrt{|z_k^2-1|}}. \tag{3.17}$$

The asymptotic behaviour of $\lambda(z)$ for z tending to ∞ is known (see, for example, [N, p. 246]):

$$\log \lambda(z) = -\log|z| - \log\log|z| + C_\infty + \varepsilon(1/z), \tag{3.18}$$

where $C_\infty = -\log 2$ by (3.16). Now (3.17) and (3.18) give

3.19. Lemma: *The asymptotic formula*

$$\log \lambda(z) = -\log 2 - \log|z_n| - \log\log|z_n| + \sum_{k=0}^{n-1} \log \frac{|\sqrt{z_k+1} + \sqrt{z_k-1}|^4}{\sqrt{|z_k^2-1|}} + \alpha_n$$

holds true with $\alpha_n \to 0$ *as* $n \to \infty$.

By the Hartogs result (see [Ne, p. 328]),

$$\lambda(0) = 4\pi^2\Gamma^{-4}(1/4) = 0.228\ldots. \tag{3.20}$$

Using our formula with an initial point $z = 0$ and $n = 8$, we get the value $\lambda(0) = 0.229\ldots$ which agrees with (3.20). Some other computational results making use of the formula in Lemma 3.19 are shown in Fig. 3 and Fig. 4.

Now, we give one more estimate, also connected with the symmetrization result of A. Weitsman [W2]. Let λ_1 and h_1 be the hyperbolic density and hyperbolic distance in $\mathbb{C}(-1,1,0)$, respectively. By Weitsman's theorem, $\lambda_1(1+re^{i\theta})$ increases in $0 \le \theta \le \pi$. Thus,

$$h_1(\zeta_1,\zeta_2) \ge h_1(1+r_1(\zeta_1), 1+r_1(\zeta_2)) \quad \text{for all } \zeta_1,\zeta_2 \in \mathbb{C}(-1,1,0). \tag{3.21}$$

For $z_1, z_2 \in \overline{I}_0 \setminus \{1,\infty\}$ let $\zeta_1, \zeta_2 \in \mathbb{H}$ denote the preimages of z_1, z_2 under Joukowski's mapping $z = (1/2)(\zeta + 1/\zeta)$, which takes $\mathbb{C}(-1,1,0)$ onto the surface $\mathcal{R}_2$ in the sense of Theorem 2.6. Next, let $1+s_1$ and $1+s_2$, respectively, be the images of the points $1+r_1(\zeta_1)$ and $1+r_1(\zeta_2)$ under this mapping. With this notation, Lemma 3.9 and the estimate (3.21) imply

3.22. Lemma: *If* $z_1, z_2 \in \mathbb{C}(-1,1)$, $z_1 \ne z_2$, *then*

$$h(z_1,z_2) \ge h(1+s_1, 1+s_2).$$

Equality here occurs iff $z_1, z_2 \in (1, \infty)$.

After simple calculations one sees that $s_k = s(z_k)$, $k = 1, 2$, where

$$s = \delta^2/(2(1+\delta))$$

with

$$\delta = \delta(z) = |\zeta(z) - 1| = \sqrt{r_1/2}\left(\sqrt{r_1 + r_{-1} + 2} + \sqrt{r_1 + r_{-1} - 2}\right).$$

Now we give a couple of applications of our estimates of the hyperbolic metric. The first of them gives a stronger version of Bermant's theorem for the class $\mathcal{R}$ introduced in Theorem 2.8.

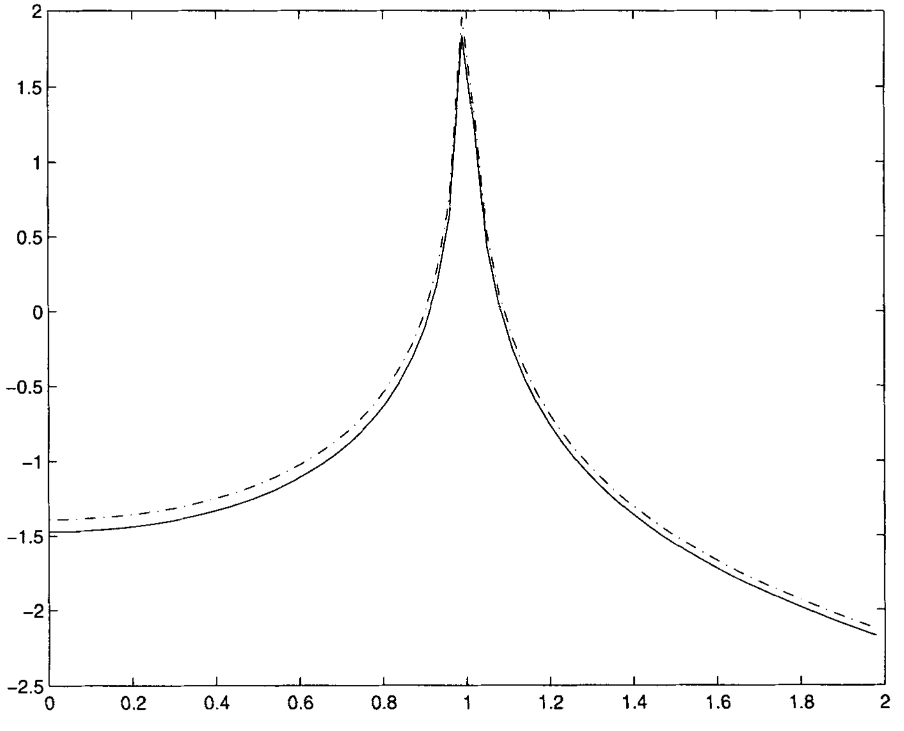

Figure 3. The function $\log \lambda(x)$ is plotted on the real axis. The formula in Lemma 3.19 with $n = 4$ (dash-dotted line) and $n = 8$ (solid line) is used.

3.23. THEOREM: *Under the hypotheses of Theorem 2.8,*

$$\sqrt{|w_1||w_2|} \geq 4\pi^2\Gamma^{-4}(1/4).$$

Equality occurs only for the functions $f(z) = e^{i\alpha} f_0(e^{i\beta} z)$ defined in Theorem 2.8.

Proof: We may assume that $|w_2| \geq |w_1|$. Let $\pi\colon \Delta \to \mathbb{C}(w_1, w_2)$ be a universal covering mapping with $\pi(0) = 0$. By the principle of subordination (see, for example, [Ne, p. 227]),

$$\lambda_{\mathbb{C}(w_1,w_2)}(0) = 1/|\pi'(0)| \leq 1/|f'(0)| = 1. \tag{3.24}$$

By performing a linear mapping, we see that (3.24) is equivalent to

$$\lambda\left(\frac{|w_2| - |w_1|}{|w_2| + |w_1|}\right) \leq \frac{|w_2| + |w_1|}{2}.$$

This together with Corollary 3.5 implies

$$\lambda(0) \leq \frac{|w_2| + |w_1|}{2}\sqrt{1 - \left(\frac{|w_2| - |w_1|}{|w_2| + |w_1|}\right)^2}, \tag{3.25}$$

and the desired inequality follows from (3.20).

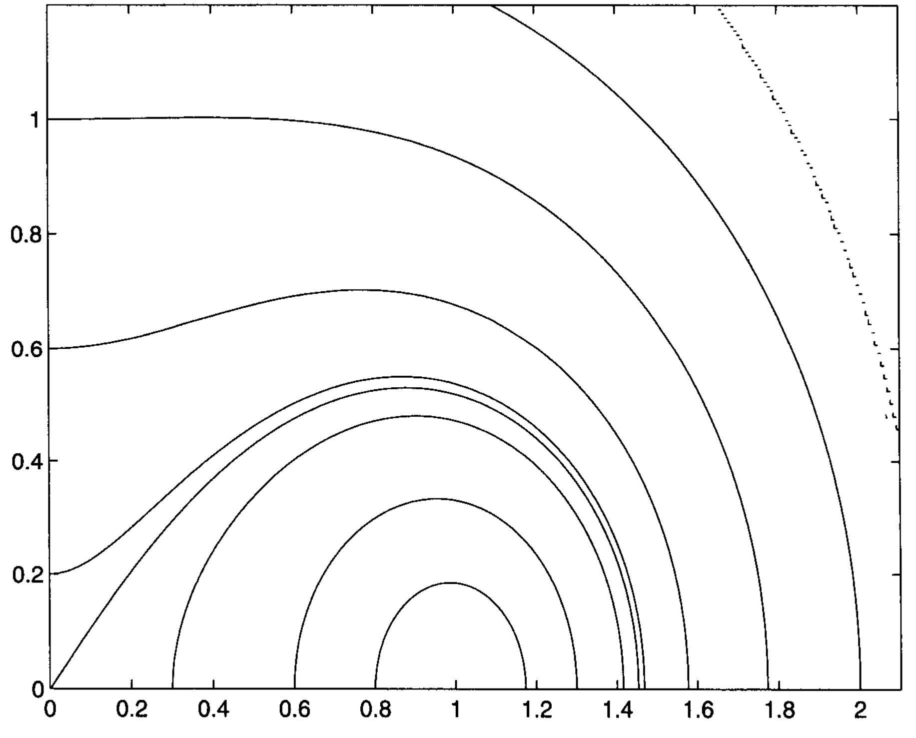

Figure 4. Some level curves of the function $\log \lambda(z)$ are plotted. For the computation, the formula in Lemma 3.19 with $n = 8$ is used.

Equality in (3.24) occurs iff $f(z)$ has the form $\pi(e^{i\beta}z)$; equality in (3.25) holds only if $|w_1| = |w_2|$. The desired equality assertion is immediate.

Let the complex numbers $w_k = r_k(f)e^{2\pi ik/n}$, $r_k(f) > 0$, $k = 0, \ldots, n-1$, be exceptional for a function f such that $f(0) = 0$, $|f'(0)| = 1$. If f is univalent in Δ, then

$$\prod_{k=0}^{n-1} r_k(f) \geq 1/4 \quad \forall n \in \mathbb{N}. \tag{3.26}$$

The case $n = 1$ is known as the Koebe 1/4-theorem; the case $n = 2$ is due to G. Szegö [Sz]; the case $n = 3$ was treated by G. M. Goluzin [G] and also by E. Reich and M. Schiffer [RS]. In the general case, this inequality was established by V. N. Dubinin [D].

3.27. Open problem. It seems interesting to prove inequalities similar to (3.26) for the class $\mathcal{R}$ and for the class $\mathcal{M}$ of all normalized functions f that are meromorphic in Δ. In Theorem 3.23 we have taken the first step in this direction.

The following theorem provides an example of an application of our estimates of the hyperbolic metric to quasiconformal mappings. This result is similar to the Agard–Gehring Theorem 2.27.

3.28. Theorem: *Suppose that f is a K-quasiconformal mapping of $\mathbb{C}$, $f(\infty) = \infty$. Then for each triple z_0, z_1, z_2*

$$\frac{|f(z_2) - f(z_1)|}{|f(z_2) - f(z_0)| + |f(z_1) - f(z_0)|} \geq \varphi_{1/K}^2\left(\sqrt{\frac{|z_2 - z_1|}{|z_2 - z_0| + |z_1 - z_0|}}\right). \tag{3.29}$$

This inequality is sharp.

Proof: Without loss of generality we may assume that $z_1 = f(z_1) = -1$, $z_2 = f(z_2) = 1$. Suppose also that $|f(z_0) - 1| + |f(z_0) + 1| < |z_0 - 1| + |z_0 + 1|$, for otherwise (3.29) follows trivially, since $\varphi_{1/K}(t) \leq t$. Now, f is a normalized K-quasiconformal mapping of $\mathbb{C}(-1, 1)$. Hence Theorem 2.16 implies that

$$h(z_0, f(z_0)) \leq \frac{1}{2} \log K. \tag{3.30}$$

From Lemmas 3.9 and 3.10 we obtain

$$h(z_0, f(z_0)) \geq \frac{1}{2} \log\left(\frac{\mu\left(\sqrt{\frac{|f(z_0)-1|}{|f(z_0)+1|}}\right)}{\mu\left(\sqrt{\frac{|z_0-1|}{|z_0+1|}}\right)}\right). \tag{3.31}$$

The desired inequality now follows from (3.30) and (3.31).

The sharpness of inequality (3.29) easily follows from the equality assertions of Theorem 2.16 and Lemma 3.9.

4. Twice-punctured disk

Let D be a simply connected domain in $\bar{\mathbb{C}}$ and let $a, b \in D$, $a \neq b$. The domain $D(a,b)$ equipped with the hyperbolic metric $\lambda_{D(a,b)}$ gives an example of a hyperbolic space depending on one real parameter. The unit disk Δ punctured at the points $\pm k$, $0 < k < 1$, may be considered as a model of such a space. Let $\lambda_k(z)$ and $h_k(z_1, z_2)$ denote the hyperbolic density and hyperbolic distance in $\Delta(-k,k)$. In this section we show that λ_k and h_k are subject to the relations similar to those of λ and h.

4.1. LEMMA: (1) *If $z_1, z_2 \in \Delta \cap \bar{I}_0 \setminus \{k\}$, $z_1 \neq z_2$, then*

$$h_k(z_1, -z_2) \geq h_k(z_1, \bar{z}_2) \geq h_k(z_1, z_2), \quad h_k(z_1, -z_2) \geq h_k(z_1, -\bar{z}_2) \geq h_k(z_1, z_2).$$

Equality in any one of these inequalities occurs iff the corresponding pairs of points coincide up to symmetry w.r.t. the corresponding coordinate axis.

(2) *Non-Euclidean disks and non-Euclidean half planes in $\Delta(-k,k)$, punctured non-Euclidean disks $\Delta_r^{(h)}(k) \setminus \{k\}$ with the non-Euclidean radius $r \leq \log((1+k)/(1-k))$, punctured discs $\Delta_r \setminus \{-k, k\}$ with $\sqrt{k} \leq r < 1$, and the punctured half disk $\{z \in \Delta(-k,k) : \Re z > 0\}$ are h_k-convex.*

The next lemma follows from Corollary 2.10 via an auxiliary conformal mapping $z \mapsto (z-k)/(1-kz)$.

4.2. LEMMA: *Let $z_1, z_2 \in \Delta(-k,k)$, $z_1 \neq z_2$, and let $z^{(k)}$ denote the projection of a point $z \in \Delta(-k,k)$ onto the segment $[k,1]$ along the non-Euclidean circle centered at k. Then*

$$h_k(z_1, z_2) \geq h_k(z_1^{(k)}, z_2^{(k)}).$$

Equality here is attained iff $z_j = z_j^{(k)}$, $j = 1, 2$.

Now we construct an analog of function (3.2) for the domain $\Delta(-k,k)$. Let

$$z = F_1(\zeta) = f_1(f_2(\zeta)),$$

where

$$f_1(w) = k\,\mathrm{sn}(w, k^2), \quad f_2(\zeta) = 1 + (i/\pi)\log\zeta, \tag{4.3}$$

and $\log\zeta$ means the principal branch of the logarithm and sn is the Jacobian elliptic function sn. The function $F_1(\zeta)$ maps the suitable annulus $K(t^{-1},t) = \{\zeta\colon t^{-1} < |\zeta| < t\}$, $t > 1$, punctured at $\zeta = 1$ conformally onto the domain $\Delta(-k,k)$. Due to the conformal invariance of the modulus of a doubly-connected domain the parameters t and k are subject to the relation

$$\log t = \mu(k^2). \tag{4.4}$$

It is known (see, for example, [Ku, p. 167]) that $F_1(\zeta)$ takes every semicircle $T_r^+ = T_r \cap \mathbb{H}$ with $1 < r < t$ onto the arc $\mathcal{L}_k^{(h)}(\kappa) \cap I_0$ of the non-Euclidean ellipse $\mathcal{L}_k^{(h)}(\kappa) = \{z\colon h_\Delta(z,-k) + h_\Delta(z,k) = \kappa\}$ with $\kappa = h_\Delta(F_1(r),-k) + h_\Delta(F_1(r),k)$. Therefore the projection along the non-Euclidean ellipses in the z-plane corresponds to the circular projection in the ζ-plane.

It is also easy to see that $F_1(\zeta)$ takes the domain $K(1,t) \cap \mathbb{H}$ conformally onto $\Delta \cap I_0$. By Minda's Lemma 2.12, $K(1,t) \cap \mathbb{H}$ is hyperbolically convex w.r.t. $K(t^{-1},t)$. Therefore,

$$h_k(z_1,z_2) = h_{K(t^{-1},t)\setminus\{1\}}(F_1^{-1}(z_1), F_1^{-1}(z_2)),$$

by Theorem 2.6 and Lemma 4.1. Combining these results and Corollary 2.10, we get

4.5. Lemma: *If $z_1, z_2 \in \Delta(-k,k)$, $z_1 \neq z_2$, then*

$$h_k(z_1,z_2) \geq h_k(i\alpha_1, i\alpha_2),$$

where $i\alpha_k$ stands for the projection of z_k onto the imaginary positive axis along the corresponding ellipse $\mathcal{L}_k^{(h)}(\kappa)$.

Equality here occurs iff $z_1 = i\alpha_1$, $z_2 = i\alpha_2$ or $z_1 = -i\alpha_1$, $z_2 = -i\alpha_2$.

4.6. Remark: There is no complete analogy between our results concerning the distances $h(z_1,z_2)$ and $h_k(z_1,z_2)$ because we cannot iterate F_1^{-1} in the same way as we do with $g(z)$ defined by (3.3). So there is no asymptotic formula for the hyperbolic density λ_k similar to such a formula for λ.

5. Schottky's theorem for bounded functions

Schottky's theorem is concerned with the rate of growth of holomorphic mappings, or, more generally, quasiregular mappings of the unit disk Δ into the complex plane that omit two finite values, which we assume to be 0 and 1. Thus we define

$$\mathcal{A}(r,M,K) = \{f\colon \Delta \to \Delta_M \setminus \{0,1\}\colon f \text{ is } K\text{-quasiregular}, |f(0)| = r\},$$

with $0 < r < M \le \infty$, $M > 1$, $K \ge 1$. As is well known [LV], in the case $K = 1$ the functions in the class $\mathcal{A}(r, M) = \mathcal{A}(r, M, 1)$ are holomorphic. The classical case deals with the class $\mathcal{A}(r) = \mathcal{A}(r, \infty)$. Schottky's problem for the class $\mathcal{A}(r, M, K)$ consists of finding the following extremal function:

$$\psi_{\mathcal{A}}(t, r, M, K) = \sup\{|f(z)|: \ f \in \mathcal{A}(r, M, K), \ |z| = t\}. \tag{5.1}$$

In the special cases we shall use the abbreviations $\psi_{\mathcal{A}}(t, r, M) = \psi_{\mathcal{A}}(t, r, M, 1)$ and $\psi_{\mathcal{A}}(t, r) = \psi_{\mathcal{A}}(t, r, \infty)$.

In our next lemma we prove the existence of a special function that is extremal for the Schottky problem in the class $\mathcal{A}(r, M)$.

5.2. Lemma: *Let $0 < r < M$, $1 < M \le \infty$. There is a function $F(z; r, M) \in \mathcal{A}(r, M)$ that maps the disk Δ onto the universal covering surface over the disk Δ_M punctured at the points 0 and 1 such that the point $w = F(z; r, M)$ runs monotonically along the segment $[-M, -r]$ when z runs through the segment $[-1, 0]$ and such that $F(0; r, M) = -r$.*

Proof: We construct $F(z; r, M)$ as follows. Choose parameters $0 < \delta < 1$ and $\tau > 0$ and consider the Möbius mapping

$$f_1(z) = -i\tau \frac{z(1 - i\delta) + i(1 + i\delta)}{z(1 + i\delta) - i(1 - i\delta)} - (1 + \tau), \tag{5.3}$$

which takes the unit disk Δ onto the upper half plane $\mathbb{H}$ such that

$$f_1(1) = -1, \ f_1(-1) = -(1 + 2\tau), \ f_1(0) = -(1 + \tau) + \tau e^{i(\frac{\pi}{2} + 2 \arctan \delta)},$$
$$f_1(-r) = -(1 + \tau) + \tau e^{i(\frac{\pi}{2} + 2 \arctan \frac{1+\delta}{r+\delta})}.$$

Note that f_1 takes the segment $[-1, 1]$ onto the semicircle

$$\{z = -(1 + \tau) + \tau e^{i\theta}: 0 \le \theta \le \pi\}.$$

Consider the circular quadrangle

$$Q(\tau) = \mathbb{H} \setminus \bar{\Delta} \setminus \bar{\Delta}_\tau(1 + \tau) \setminus \bar{\Delta}_\tau(-(1 + \tau)).$$

Let f_2 map $Q(\tau)$ conformally onto $\mathbb{H}$ such that

$$f_2(1 + 2\tau) = -f_2(-(1 + 2\tau)) = M, \ f_2(1) = -f_2(-1) = 1.$$

The function f_2 can be continued by symmetry to a universal covering mapping from $\mathbb{H}$ onto the domain $\Omega(M) = \mathbb{C} \setminus (-\infty, -M] \setminus [M, \infty) \setminus \{-1, 1\}$.

Next we consider the function

$$f_3(\zeta) = \frac{\zeta + M^2 - \sqrt{(M^2-1)(M^2-\zeta^2)}}{\zeta+1}, \tag{5.4}$$

which maps the domain $\mathbb{C} \setminus (-\infty, -M] \setminus [M, \infty)$ conformally onto Δ_M such that

$$f_3(1) = 1, \ f_3(-1) = 0, \ f_3(M) = -f_3(-M) = M.$$

Finally, the function $w = F(z)$ with

$$F = f_3 \circ f_2 \circ f_1$$

gives a universal covering mapping from Δ onto $\Delta_M(0,1)$. It is easy to see that we can choose the parameter δ, $0 < \delta < 1$ in (5.3) so that $F(0) = -r$. Besides, by our construction, the segment $[-1, 0]$ goes to the segment $[-M, -r]$.

5.5. THEOREM: *For all r, M, and t, satisfying the conditions under consideration,*

$$\psi_{\mathcal{A}}(t, r, M) = -F(-t; r, M).$$

The equality $|f(z_0)| = \psi_{\mathcal{A}}(r, M)$ for some z_0, $|z_0| = t$, occurs iff

$$f(z) = F((-z|z_0|/z_0); r, M).$$

Proof: Let $f \in \mathcal{A}(r, M)$. By the principle of the hyperbolic metric,

$$h_{\Delta_M(0,1)}(f(0), f(z_0)) \le h_\Delta(0, z_0). \tag{5.6}$$

The domain $\Delta_M(0,1)$ is circularly symmetric w.r.t. the negative real axis. Hence by Corollary 2.10,

$$\begin{aligned} h_{\Delta_M(0,1)}(f(0), f(z_0)) &\ge h_{\Delta_M(0,1)}(-r, -|f(z_0)|) \\ &= h\text{-}length([-r, -|f(z_0)|]) \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} h_{\Delta_M(0,1)}(-r, F(-t; r, M)) &= h\text{-}length([-r, F(-t; r, M]) \\ &= h_\Delta(0, z_0), \end{aligned} \tag{5.8}$$

where *h-length* denotes the hyperbolic length in $\Delta_M(0,1)$. The second equality in (5.8) follows from the construction of the function $F(z; r, M)$. The relations (5.6) – (5.8) together with the monotonicity property of the function *h-length*$([-r, -t])$ lead to the desired equality.

By the equality statement of Corollary 2.10 the equality in (5.7) can occur only if $f(0) = -r$ and $-M < f(z_0) < -r$. This together with the monotonicity property of $h_{\Delta_M(0,1)}(-s,-r)$ gives

$$f(0) = -r, \ f(z_0) = F(-t; r, M),$$

if $|f(z_0)| = \psi_{\mathcal{A}}(t,r,M)$. The function $\omega = F^{-1}(f(z); r, M)$ with $F^{-1}(-r; r, M) = 0$ satisfies the Schwarz lemma. Since $\omega(z_0) = F^{-1}(f(z_0); r, M) = -t = -|z_0|$ then $F^{-1}(f(z); r, M) = -z(|z_0|/z_0)$, and the equality assertion follows.

Theorem 5.5 combined with the K-quasiconformal Schwarz lemma leads to Schottky's theorem for bounded K-quasiregular mappings.

5.9. THEOREM: *For all r, M, K, and t satisfying the conditions under consideration,*

$$\psi_{\mathcal{A}}(t,r,M,K) = -F(-\varphi_K(t); r, M).$$

The equality $|f(z_0)| = \psi_{\mathcal{A}}(t,r,M,K)$ for some z_0, $|z_0| = r$, holds true only for the function $F(-\omega_K(z(|z_0|/z_0)); r, M)$, where $\omega_K(\cdot)$ is the K-quasiconformal function that is extremal for the K-quasiconformal Schwarz lemma (see [LV]).

In the classical case the function $\psi_{\mathcal{A}}(t,r)$ can be expressed in terms of the well-known distortion functions of quasiconformal theory as follows from [Hem], [M]. For $0 < r < 1$, $0 < K < \infty$, let

$$\psi_{\mathcal{A}}\Big(\frac{K-1}{K+1}, r\Big) = \eta_K(r),$$

where

$$\eta_K(r) = \Big[\frac{\varphi_K(t)}{\varphi_{1/K}(t')}\Big]^2 \quad \text{with } r = (t/t')^2.$$

The function $\psi_{\mathcal{A}}(t,r)$ admits the following asymptotically sharp bounds [Hem],

$$\psi_{\mathcal{A}}(t,r) < \frac{1}{16}\,(re^{\pi})^{(1+t)/(1-t)} \quad \text{for } r \geq 1,$$
$$\psi_{\mathcal{A}}(t,r) < \frac{1}{16}\,(e^{\pi})^{(1+t)/(1-t)} \quad \text{for } r \leq 1.$$

More refined estimates occur in [AnVam]. The best estimates known to us are due to S.-L. Qiu [Q], who proved that for $K \geq 1$, $t > 0$,

$$16\eta_K(t) \leq \min\{16t + A^K - A, (16t+8)^K - 8\}, \quad A = \exp(2\mu(1/\sqrt{1+t})).$$

5.10. OPEN PROBLEM. It would be interesting to find similar explicit upper bounds for $\psi_{\mathcal{A}}(t, r, M, K)$ in the general case. The main question here is to find a suitable explicit expression for the function f_2 defined in the proof of Lemma 5.2 that maps the circular quadrangle onto the upper half plane.

6. Distortion of the interior of a domain under K-quasiconformal self-mapping

In this section we study some properties of the function $h_\Omega(K)$ introduced in Section 2. The following theorem relative to Martin's question mentioned in Section 2 gives sharp bounds for $h_\Omega(K)$.

6.1. THEOREM: *If Ω is a planar domain and $K > 1$, then*

$$\frac{1}{2}\log K \le h_\Omega(K) \le d(K),$$

where $d(K)$ is defined in Theorem 2.19. These bounds are sharp.

Proof: We need to prove the second inequality only. The first one follows from Martin's Theorem 2.28.

The ingredients of the proof presented below are known and belong to L. Ahlfors, L. Bers, and others. We shall adopt those to our situation. Here we follow closely the exposition in [B2, pp. 16, 17] and in [Gar, Ch. 3].

The domain Ω may be identified with the quotient Δ/G, where G is a torsion free Fuchsian group acting on Δ. Let Λ be the limit set of G. Then either $\Lambda = \mathbb{T}$, or Λ is a closed nowhere dense subset of $\mathbb{T}$.

Let $\varphi\colon \Delta \to \Omega$ be a universal covering mapping. It is known that every mapping $f\colon \Omega \to \Omega$ homotopic to the identity lifts to a mapping $p\colon \Delta \to \Delta$ commuting with every $g \in G$ and then

$$\varphi \circ p = f \circ \varphi. \tag{6.2}$$

Vice versa, every such mapping p satisfies (6.2) with some mapping $f\colon \Omega \to \Omega$ homotopic to the identity. Moreover, the mapping p is bijective and K-quasiconformal if f is. If so, p extends by continuity to $\mathbb{T}$. Since p commutes with all elements of G, it leaves each point of Λ fixed. Besides, it leaves $\mathbb{T} \setminus \Lambda$ pointwise fixed if and only if f is homotopic to the identity modulo $(\mathbb{T} \setminus \Lambda)/G$.

Thus, if $\Lambda = \mathbb{T}$, then $p \in \Delta(K)$. For $z_0 \in \Omega$ let $\zeta_0 \in \Delta$ satisfy $\varphi(\zeta_0) = z_0$ and let $w_0 = p(\zeta_0)$. By (2.4),

$$h_\Omega(z_0, f(z_0)) = \min_{w:\varphi(w)=f(z_0)} h_\Delta(\zeta_0, w) \le h_\Delta(\zeta_0, w_0) \le d(K). \tag{6.3}$$

The second inequality in (6.3) follows from the Krzyz Theorem 2.19.

Let now $\Lambda \neq \mathbb{T}$ and let $b \in \mathbb{T} \setminus \Lambda$. Then there exists a Dirichlet fundamental domain D' for the group G which has an open boundary arc $\alpha \subset \partial D'$ such that $b \in \alpha \subset \mathbb{T} \setminus \Lambda$. Since D' is a simply connected Jordan domain, φ maps D' univalently onto some simply connected domain $\Omega' \subset \Omega$; it takes α onto a boundary set $\alpha' \subset \partial\Omega'$ in the sense of the boundary correspondence.

Note that if a point a belongs to a boundary point $A \in \alpha'$ (that might be non-singleton), then $a \in \partial\Omega$. Indeed, assume that $a \in \Omega$ and choose a sequence $z_n \to a$, $z_n \in \Omega'$. Clearly, $h_\Omega(z_n, a) \to 0$ as $n \to \infty$. Since φ is univalent in D' we can consider a univalent inverse mapping φ^{-1} from Ω' onto D'. By our assumptions, $\varphi^{-1}(z_n) \to b \in \alpha$. On the other hand, there exists $b' \in \Delta \cap \partial D'$ such that $\varphi(b') = a$. Since G acts properly discontinuously on Δ, then $\min_{\zeta\colon \varphi(\zeta)=z_n} h_\Delta(\zeta, b') \geq c$ with some constant $c > 0$ for all sufficiently big n. Therefore by (2.4), $h_\Omega(z_n, a) \geq c > 0$ for such n and we get a contradiction.

The assumptions of the theorem for f imply that the sequences $z_n \to a$ and $f(z_n) \to a$ define the same boundary point A. Therefore, the sequences $\varphi^{-1}(z_n)$ and $\varphi^{-1}(f(z_n))$ also define the same boundary point $b \in \alpha$. The latter means that p keeps the unit circle $\mathbb{T}$ to be pointwise fixed. Thus, $p \in \Delta(K)$ and (6.3) holds true, which prove the theorem in the case under consideration.

As was noted in [KK, Ch. 4], the lower bound in Theorem 6.1 is achieved only for the thrice punctured sphere. If the Fuchsian group G corresponding to Ω is of the second kind, the proof of Theorem 6.1 may be adapted in order to show that $h_\Omega(K)$ is equal to the upper bound in the inequality of Theorem 6.1. Thus, this upper bound is achieved for a rather wide class of domains.

6.4. THEOREM: *Let the Fuchsian group G corresponding to Ω be of the second kind. Then*

$$h_\Omega(K) = d(K)$$

for every $K > 1$.

Proof: The notation φ, G, D', α used in what follows was introduced in the proof of the previous theorem. Choose a sequence $\zeta_n \in D'$ such that $\zeta_n \to b \in \alpha$. Let $\rho_n > 0$ be the biggest number such that the non-Euclidean disk $\Delta^{(h)}_{2\rho_n}(\zeta_n)$ lies in D'. Clearly, $\rho_n \to \infty$ as $n \to \infty$. The Möbius mapping

$$\psi_n = (\zeta + \zeta_n)/(1 + \bar{\zeta}_n\zeta)$$

takes $\Delta^{(h)}_{\rho_n}(\zeta_n)$ onto the disk Δ_{r_n} with $r_n \to 1$ as $n \to \infty$.

Next, we define a K-quasiconformal mapping $p_n(\zeta)$ as follows. For $\zeta \in \overline{D'}$, we set

$$p_n(\zeta) = \begin{cases} \psi_n^{-1}(r_n f_K(\psi_n(\zeta)/r_n, 0, w_0)) & \text{if } \zeta \in \overline{\Delta}_{\rho_n}(\zeta_n), \\ \zeta & \text{otherwise.} \end{cases}$$

Here $f_K(\cdot, 0, w_0)$ denotes the function that is extremal for Theorem 2.19 for a suitable w_0. Then we continue $p_n(\zeta)$ to the function defined on the whole disk Δ that commutes with every element $g \in G$. It follows from Lemma 2 in [B1] that $p_n \in \Delta(K)$.

The arguments given in the proof of Theorem 6.1 show that for every p_n there exists $f_n \in \Omega(K)$ satisfying (6.2). Let $z_n = \varphi(\zeta_n)$. We have $h_\Delta(\zeta_n, p_n(\zeta_n)) < \rho_n$ by the definition of p_n, and $h_\Delta(\zeta_n, \zeta) \geq 2\rho_n$ for all $\zeta \neq p(\zeta_n)$ such that $\varphi(\zeta) = f_n(z_n)$, by the assumption $\Delta_{2\rho_n}^{(h)}(\zeta_n) \subset D'$. Therefore,

$$h_\Omega(z_n, f_n(z_n)) = \min_{\zeta:\, \varphi(\zeta) = f_n(z_n)} h_\Delta(\zeta_n, \zeta) = h_\Delta(\zeta_n, p_n(\zeta_n)). \tag{6.5}$$

By the definition of p_n,

$$h_\Delta(\zeta_n, p_n(\zeta_n)) = h_\Delta(0, \psi_n(p_n(\zeta_n))) = h_\Delta(0, r_n f_K(0, 0, w_0)), \tag{6.6}$$

where

$$h_\Delta(0, r_n f_K(0, 0, w_0)) \to h_\Delta(0, f_K(0, 0, w_0)) = d(K) \tag{6.7}$$

since $r_n \to 1$ as $n \to \infty$.

Now (6.5)–(6.7) imply the desired assertion.

The assumptions of Theorem 6.4 may be interpreted in geometrical terms. By a free boundary continuum of Ω we mean a non-degenerate continuum $\gamma \subset \partial\Omega$ that is a boundary arc of some simply connected subdomain $\Omega' \subset \Omega$. It is known (and follows also from the proof of Theorem 6.1) that Ω has a free boundary continuum if and only if the Fuchsian group of Ω is of the second kind. Thus, $h_\Omega(K) = d(K)$ for every domain having a free boundary continuum.

Actually, in the proof of Theorem 6.4 we only used the fact that the covering map φ is injective on a hyperbolic disc which has an arbitrary large radius. This is always true for Fuchsian groups of the second kind. There are also Fuchsian groups of the first kind having the same property. Therefore, the equality $h_\Omega(K) = d(K)$ is still true for every domain Ω that corresponds to such a Fuchsian group of the first kind.

6.8. Remark: Let $z_0, w_0 \in \Omega$ be such that $h_\Omega(z_0, w_0) = d(K)$. If $f \in \Omega(K)$ and $f(z_0) = w_0$, then the function p defined by (6.2) belongs to $\Delta(K)$ and satisfies the relation

$$h_\Delta(z_0, p(z_0)) = d(K).$$

Therefore, by the uniqueness assertion of Theorem 2.19, p must be the Teichmüller mapping $f_K(\zeta, \zeta_0, \omega_0)$ for some ζ_0 and ω_0 such that $\varphi(\zeta_0) = z_0$ and $\varphi(\omega_0) = w_0$. It follows from the Teichmüller construction [Teich44] that $f_K(\zeta, \zeta_0, \omega_0)$ does not commute with any Möbius self-mapping of Δ except the identity mapping. This implies that the Fuchsian group G relative to Ω is trivial and therefore Ω must be simply connected.

We finish this paper by showing that $h_\Omega(K)$ is a conformal invariant.

6.9. THEOREM: *If a domain $\Omega_1 \subset \overline{\mathbb{C}}$ is conformally equivalent to Ω_2, then*

$$h_{\Omega_1}(K) = h_{\Omega_2}(K) \quad \forall K \geq 1.$$

Proof: We may assume that Ω is not exceptional. Let F be a conformal mapping from Ω_1 onto Ω_2. Note that Ω_1 and Ω_2 have the same uniformizing Fuchsian group G.

Let φ_1: $\Delta \to \Omega_1$ be a universal covering mapping. Then $\varphi_2 = F \circ \varphi_1$ is a universal covering mapping of Ω_2.

As we have seen in the proof of Theorem 6.1, every $f \in \Omega_1(K)$ lifts to $p \in \Delta(K)$ commuting with every element of G such that

$$\varphi_1 \circ p = f \circ \varphi_1$$

and every such p may be regarded as a lift of some $q \in \Omega_2(K)$:

$$\varphi_2 \circ p = q \circ \varphi_2.$$

This implies that every $f \in \Omega_1(K)$ may be represented in the form

$$f = F^{-1} \circ q \circ F \tag{6.10}$$

with $q \in \Omega_2(K)$, Vice versa, every $q \in \Omega_2(K)$ may be represented as

$$q = F \circ f \circ F^{-1}. \tag{6.11}$$

Since the hyperbolic metric is invariant under conformal mappings, (6.10) and (6.11) imply the assertion of the theorem in the case under consideration.

6.12. OPEN QUESTIONS. It is an immediate consequence of the definition that $h_\Omega(K)$ increases in K. It seems plausible that $h_\Omega(K)$ decreases in Ω, but we were unable to prove it.

Another natural question concerns the density of the set of all values $h_\Omega(K)$. For given $K > 1$, $t > 0$, and $(1/2)\log K < t < d(K)$, does there exist a domain Ω such that $h_\Omega(K) = t$?

Our last question concerns an intriguing extremal problem with n-fold symmetric conjectural extremal configuration. Let Ω_n denote the Riemann sphere punctured at n distinct points; Ω_n^* is the Riemann sphere punctured at the roots of unity $a_k = e^{2\pi ik/n}$, $k = 0, \ldots, n-1$.

6.13 CONJECTURE: *$h_{\Omega_n}(K) \leq h_{\Omega_n^*}(K)$ for all $K > 1$.*

We expect the case of equality in (6.13) only for the images of Ω_n^* under the Möbius mappings. This problem may be reformulated in terms of fundamental polygons of the Fuchsian group relative to Ω_n.

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